AUTOMORPHISMS OF SUPERSINGULAR $K3$ SURFACES AND SALEM POLYNOMIALS

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Dedicated to Professor Tetsuji Shioda on the occasion of his 75th birthday

Abstract. We present a method to generate many automorphisms of a supersingular $K3$ surface in odd characteristic. As an application, we show that, if p is an odd prime less than or equal to 7919, then every supersingular $K3$ surface in characteristic p has an automorphism whose characteristic polynomial on the Néron–Severi lattice is a Salem polynomial of degree 22. For a supersingular $K3$ surface with Artin invariant 10, the same holds for odd primes less than or equal to 17389.

1. Introduction

An irreducible monic polynomial $\phi(t) \in \mathbb{Z}[t]$ of even degree $2d > 0$ is called a *Salem polynomial* if $\phi(t)$ is reciprocal, $\phi(t) = 0$ has two positive real roots, and the other $2d - 2$ complex roots are located on $\{z \in \mathbb{C} \mid |z| = 1\} \setminus \{\pm 1\}$.

The notion of Salem polynomials plays an important role in the study of dynamics of automorphisms of algebraic varieties. We have the following fundamental theorem due to McMullen [10]. See also [6] and [4, Proposition 3.1].

Theorem 1.1 ([10]). *Let g be an automorphism of an algebraic $K3$ surface X defined over an algebraically closed field. Then the characteristic polynomial of the action of g on the Néron–Severi lattice S_X of X is a product of cyclotomic polynomials and at most one Salem polynomial counting with multiplicities.*

A $K3$ surface X defined over an algebraically closed field k of characteristic $p > 0$ is said to be *supersingular* if the rank of its Néron–Severi lattice S_X is 22. We say that an automorphism g of a supersingular $K3$ surface X is *of irreducible Salem type* if the characteristic polynomial of the action of g on S_X is a Salem polynomial of degree 22.

The purpose of this note is to report the following theorems, which are the results of computer-aided experiments. By a *double plane involution* of a $K3$ surface X in characteristic not equal to 2, we mean an automorphism of X of order 2 induced by the Galois transformation of a generically finite morphism $X \to \mathbb{P}^2$ of degree 2.

Theorem 1.2. *Let p be an odd prime less than or equal to* 7919*. Then every supersingular $K3$ surface X in characteristic p has a sequence of double plane involutions $\tau_1, \dots, \tau_l$ of length at most* 22 *such that their product $\tau_1 \cdots \tau_l$ is an automorphism of irreducible Salem type.*

2010 *Mathematics Subject Classification.* 14J28, 14J50, 37B40, 14Q10.
Partially supported by JSPS Grants-in-Aid for Scientific Research (C) No. 25400042.

Let X be a supersingular $K3$ surface in characteristic $p > 0$, and let $S_X^\vee$ denote the *dual lattice* $\mathrm{Hom}(S_X, \mathbb{Z})$ of S_X, into which S_X is embedded as a submodule of finite index by the intersection form of S_X. Artin [1] showed that the discriminant group $S_X^\vee/S_X$ of S_X is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{2\sigma}$, where σ is a positive integer less than or equal to 10. This integer σ is called the *Artin invariant* of X. By the result of Ogus [12, 13], the supersingular $K3$ surfaces of Artin invariant $\le \sigma$ defined over an algebraically closed field k constitute a moduli of dimension $\sigma - 1$, and a supersingular $K3$ surface $X(p)$ with Artin invariant 1 is unique up to isomorphism.

For supersingular $K3$ surfaces with Artin invariant $\sigma = 10$ in characteristic p with $11 \le p \le 17389$, we found a class of sequences of double plane involutions whose product is *frequently* of irreducible Salem type. (See Section 6 for the detail.) Using this class, we obtain the following theorem:

Theorem 1.3. *Let p be an odd prime less than or equal to* 17389*. Then every supersingular $K3$ surface X in characteristic p with Artin invariant* 10 *has a sequence of double plane involutions of length at most* 22 *such that their product is an automorphism of irreducible Salem type.*

The interest of an automorphism of irreducible Salem type stems from the following observation due to Esnault and Oguiso [4, 5]:

Theorem 1.4 ([4, 5])**.** *Let g be an automorphism of a supersingular $K3$ surface X. If the characteristic polynomial of the action of g on S_X is irreducible, then the pair (X, g) can never be lifted to characteristic* 0*.*

Hence we obtain the following corollary.

Corollary 1.5. *Let X be a supersingular $K3$ surface in odd characteristic p with Artin invariant σ. Suppose that $p \le 7919$ or ($\sigma = 10$ and $p \le 17389$). Then X has an automorphism g such that the pair (X, g) can never be lifted to characteristic* 0*.*

Recently, several authors have studied the non-liftability of automorphisms of supersingular $K3$ surfaces by means of Salem polynomials. See [2, 4, 5, 16]. In particular, the existence of a non-liftable automorphism has been established for a supersingular $K3$ surface $X(p)$ in characteristic p *with Artin invariant* 1, except for the cases $p = 7$ and 13.

Remark 1.6. In [8], the existence of a non-liftable automorphism of $X(p)$ was proved for p large enough by another method.

Our main theorems not only fill the remaining cases $X(7)$ and $X(13)$ for supersingular $K3$ surfaces with Artin invariant 1, but also suggest that this result can be extended to supersingular $K3$ surfaces with arbitrary Artin invariant, at least in odd characteristics. There exists no theoretical significance in the bounds $p \le 7919$ in Theorem 1.2 and $p \le 17389$ in Theorem 1.3. We merely stopped our computations at the 1000th prime ($p = 7919$) and the 2000th prime ($p = 17389$).

The main tool of the proof of Theorems 1.2 and 1.3 is the structure theorem of the Néron–Severi lattices of supersingular $K3$ surfaces X due to Rudakov and Shafarevich [14], which states that the isomorphism class of the lattice S_X is uniquely determined by p and the Artin invariant σ of X.

Let X be a supersingular $K3$ surface X in odd characteristic. In this paper, we present a method to generate many matrix representations on S_X of double plane involutions of X. Composing some of these involutions, we obtain an automorphism of irreducible Salem type. In order to produce double plane involutions, we have to find the nef cone in $S_X \otimes \mathbb{R}$. For this purpose, we introduce a notion of an *ample list of vectors*. (See Section 2 for the definitions.)

The results of the experiments are presented in the author's web page [19].

Thanks are due to Professors Junmyeong Jang, Toshiyuki Katsura, Jonghae Keum, Keiji Oguiso, Matthias Schütt and Hirokazu Yanagihara for stimulating discussions.

2. Lattices

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle_L : L \times L \to \mathbb{Z}$, which we call the *intersection form*. We let the group $\mathrm{O}(L)$ of isometries of L act on L from the *right*, and write the action of $g \in \mathrm{O}(L)$ on L by $x \mapsto x^g$. A lattice L is *even* if $\langle v, v\rangle_L$ is even for any vector $v \in L$. A lattice L is *hyperbolic* if its rank n is larger than 1 and the real quadratic space $L \otimes \mathbb{R}$ is of signature $(1, n-1)$.

Let L be an even hyperbolic lattice. The open subset $\{x \in L \otimes \mathbb{R} \mid \langle x, x\rangle_L > 0\}$ of $L \otimes \mathbb{R}$ has two connected components, each of which is called a *positive cone*. We choose a positive cone $\mathcal{P}_L$, and denote by $\mathrm{O}^+(L)$ the stabilizer subgroup of $\mathcal{P}_L$ in $\mathrm{O}(L)$. A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r\rangle_L = -2$. Let r be a (-2)-vector. We put

$$(r)^\perp := \{\ x \in \mathcal{P}_L \ \mid\ \langle x, r\rangle_L = 0\ \},$$

and call it a (-2)-*hyperplane*. The reflection

$$s_r : x \mapsto x + \langle x, r\rangle_L \cdot r$$

in $(r)^\perp$ is an element of $\mathrm{O}^+(L)$. We denote by $W(L)$ the subgroup of $\mathrm{O}^+(L)$ generated by all the reflections s_r in (-2)-hyperplanes, and call $W(L)$ the *Weyl group* of L. A *standard fundamental domain of* $W(L)$ is the closure in $\mathcal{P}_L$ of a connected component of

$$\mathcal{P}_L \ \setminus \bigcup_r (r)^\perp,$$

where r ranges through the set of (-2)-vectors. Note that $W(L)$ acts on the set of standard fundamental domains transitively.

Suppose that a basis of an even hyperbolic lattice L and the Gram matrix of the intersection form $\langle\ ,\ \rangle_L$ with respect to this basis are given. We have the following algorithms. See [20, Section 3] for the details.

Algorithm 2.1. Let v be a vector in $\mathcal{P}_L \cap L$. Then, for an integer a and an even integer d, the finite set $\{x \in L \mid \langle x, v\rangle_L = a, \langle x, x\rangle_L = d\}$ can be calculated. In particular, the sets

$$\mathcal{R}(v) := \{\ r \in L \ \mid\ \langle r, v\rangle_L = 0,\ \langle r, r\rangle_L = -2\ \}$$

and

$$\mathcal{F}(v) := \{\ f \in L \ \mid\ \langle f, v\rangle_L = 1,\ \langle f, f\rangle_L = 0\ \}$$

can be calculated. ∎

Algorithm 2.2. Let u and v be vectors in $\mathcal{P}_L \cap L$. Then, for a negative even integer d, the finite set $\{x \in L \mid \langle x, u\rangle_L > 0, \langle x, v\rangle_L < 0, \langle x, x\rangle_L = d\}$ can be calculated. In particular, the set

$$\mathcal{S}(u, v) := \{\ r \in L \ \mid\ \langle r, u\rangle_L > 0,\ \langle r, v\rangle_L < 0,\ \langle r, r\rangle_L = -2\ \}$$

can be calculated. ∎

We call an ordered nonempty set

$$\boldsymbol{a} := [h_0, \rho_1, \dots, \rho_K]$$

of vectors of L an *ample list of vectors* if $h_0 \in \mathcal{P}_L \cap L$ and, for any $r \in \mathcal{R}(h_0)$, there exists a member ρ_i of $\{\rho_1, \dots, \rho_K\}$ such that $\langle r, \rho_i \rangle_L \neq 0$.

Example 2.3. (1) If vectors $\rho_1, \dots, \rho_K$ of L span the linear space $L \otimes \mathbb{Q}$ over $\mathbb{Q}$, then $[h_0, \rho_1, \dots, \rho_K]$ is an ample list of vectors for any vector $h_0 \in \mathcal{P}_L \cap L$.

(2) If a vector $h_0 \in \mathcal{P}_L \cap L$ satisfies $\mathcal{R}(h_0) = \emptyset$, then the list $[h_0]$ is an ample list of vectors.

(3) If $[h_0, \rho_1, \dots, \rho_K]$ is an ample list of vectors, then $[h_0, \rho_1, \dots, \rho_K, \rho_{K+1}]$ is an ample list of vectors for any $\rho_{K+1} \in L$.

Let $\boldsymbol{a} = [h_0, \rho_1, \dots, \rho_K]$ be an ample list of vectors. We define $D(\boldsymbol{a})$ to be the unique standard fundamental domain of $W(L)$ such that

$$\boldsymbol{a}_\varepsilon := h_0 + \varepsilon\rho_1 + \cdots + \varepsilon^K \rho_K$$

is contained in the interior of $D(\boldsymbol{a})$, where ε is a sufficiently small positive real number. For $x \in \mathcal{P}_L$, we write

$$\langle \boldsymbol{a}, x \rangle_L > 0$$

if the real vector

$$(\ \langle h_0, x \rangle_L,\ \langle \rho_1, x \rangle_L,\ \dots,\ \langle \rho_K, x \rangle_L\ ) \in \mathbb{R}^{K+1}$$

is nonzero and its leftmost nonzero entry is positive; that is, $\langle \boldsymbol{a}_\varepsilon, x \rangle_L \in \mathbb{R}$ is positive for a sufficiently small positive real number ε. For $x_1, x_2 \in \mathcal{P}_L$, we write

$$\langle \boldsymbol{a}, x_1 \rangle_L > \langle \boldsymbol{a}, x_2 \rangle_L$$

if $\langle \boldsymbol{a}, x_1 - x_2 \rangle_L > 0$. We put

$$\mathcal{R}^+(\boldsymbol{a}) := \{\ r \in \mathcal{R}(h_0)\ \mid\ \langle \boldsymbol{a}, r \rangle_L > 0\ \}.$$

Note that $\mathcal{R}(h_0)$ is the disjoint union of $\mathcal{R}^+(\boldsymbol{a})$ and $-\mathcal{R}^+(\boldsymbol{a})$. Then $D(\boldsymbol{a})$ is the unique standard fundamental domain of $W(L)$ that contains h_0 and is contained in the region

$$\{\ x \in \mathcal{P}_L\ \mid\ \langle x, r \rangle_L \geq 0\ \text{ for any vector }\ r \in \mathcal{R}^+(\boldsymbol{a})\ \}.$$

The following lemma is obvious.

Lemma 2.4. *A vector $v \in \mathcal{P}_L \cap L$ is contained in $D(\boldsymbol{a})$ if and only if $\mathcal{S}(h_0, v) = \emptyset$ and $\langle v, r \rangle_L \geq 0$ for any vector $r \in \mathcal{R}^+(\boldsymbol{a})$.*

Let d be an even positive integer. Suppose that a vector $v \in \mathcal{P}_L \cap L$ satisfies $\langle v, v \rangle_L = d$. From v, we can find a vector h_v in $D(\boldsymbol{a}) \cap L$ satisfying $\langle h_v, h_v \rangle_L = d$ by the following method. First we calculate the union

$$\mathcal{S}(h_0, v) \cup \mathcal{R}' = \{r_1, \dots, r_M\},$$

where

$$\mathcal{R}' := \{\ r \in \mathcal{R}^+(\boldsymbol{a})\ \mid\ \langle v, r \rangle_L < 0\ \}.$$

Note that we have $\langle v, r_i\rangle_L < 0$ and $\langle \boldsymbol{a}, r_i\rangle_L > 0$ for each $r_i \in \mathcal{S}(h_0, v) \cup \mathcal{R}'$. Note also that, if a (-2)-vector r satisfies $\langle v, r\rangle_L < 0$ and $\langle \boldsymbol{a}, r\rangle_L > 0$, then r belongs to $\mathcal{S}(h_0, v) \cup \mathcal{R}'$. We put

$$\boldsymbol{t}_i := \frac{-1}{\langle v, r_i\rangle_L} \left(\langle h_0, r_i\rangle_L,\ \langle \rho_1, r_i\rangle_L,\ \dots,\ \langle \rho_K, r_i\rangle_L \right) \in \mathbb{R}^{K+1}.$$

If $\boldsymbol{t}_i = \boldsymbol{t}_j$ holds for some distinct indices i and j, then we choose a random vector $\rho_{K+1} \in L$ and replace $\boldsymbol{a}$ by a new ample list of vectors

$$[h_0, \rho_1, \dots, \rho_K, \rho_{K+1}].$$

(Note that this replacement of $\boldsymbol{a}$ does not change $D(\boldsymbol{a})$.) Repeating this process, we can assume that $\boldsymbol{t}_1, \dots, \boldsymbol{t}_M$ are distinct. We sort the vectors $r_1, \dots, r_M$ of $\mathcal{S}(h_0, v) \cup \mathcal{R}'$ in such a way that, if $i > j$, then the leftmost nonzero entry of $\boldsymbol{t}_i - \boldsymbol{t}_j$ is positive. Consider the half-line ℓ in $\mathcal{P}_L$ given by

$$\boldsymbol{a}_\varepsilon + tv \quad (t \in \mathbb{R}_{\geq 0}),$$

where ε is a sufficiently small positive real number. Then ℓ is not contained in any (-2)-hyperplane, the (-2)-hyperplanes $(r_1)^\perp, \dots, (r_M)^\perp$ intersect ℓ at distinct points, and any (-2)-hyperplane intersecting ℓ is one of $(r_1)^\perp, \dots, (r_M)^\perp$. Moreover, the values t_i of the parameter t of ℓ at which ℓ intersects $(r_i)^\perp$ satisfy

$$t_1 > \cdots > t_M > 0,$$

because, if $\boldsymbol{t}_i = (t_{i,0}, t_{i,1}, \dots, t_{i,K}) \in \mathbb{R}^{K+1}$, then we have

$$t_i = t_{i,0} + \varepsilon\, t_{i,1} + \cdots + \varepsilon^K\, t_{i,K}.$$

Therefore, if we denote by $s_i \in W(L)$ the reflection in $(r_i)^\perp$, then the vector

$$h_v := v^{s_1 \cdots s_M} \tag{2.1}$$

belongs to $D(\boldsymbol{a}) \cap L$.

3. Polarizations of degree 2

Let X be a $K3$ surface defined over an algebraically closed field k of characteristic not equal to 2, and let S_X denote the Néron–Severi lattice of X with the intersection form $\langle\ ,\ \rangle_S$. Suppose that $\operatorname{rank} S_X$ is larger than 1. Then S_X is an even hyperbolic lattice. We let the automorphism group $\operatorname{Aut}(X)$ act on X from the left and act on S_X from the right by the pull-back. Let $\mathcal{P}(X)$ denote the positive cone of S_X that contains an ample class. We put

$$N(X) := \{\ x \in \mathcal{P}(X) \ \mid\ \langle x, [C]\rangle_S \geq 0 \ \text{ for any curve } \ C \subset X\ \},$$

where $[C] \in S_X$ is the class of a curve C on X. It is well known that $N(X)$ is a standard fundamental domain of the Weyl group $W(S_X)$. A vector $h \in S_X$ with $\langle h, h\rangle_S = 2$ is called a *polarization of degree* 2 if the complete linear system $|\mathcal{L}_h|$ of a line bundle $\mathcal{L}_h \to X$ whose class is h is fixed-component free. By [11], we have the following criterion.

Proposition 3.1. *A vector $h \in S_X$ with $\langle h, h\rangle_S = 2$ is a polarization of degree 2 if and only if $h \in N(X)$ and $\mathcal{F}(h) = \emptyset$.*

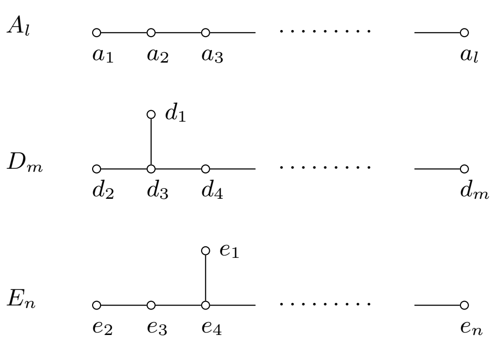

FIGURE 3.1. Indecomposable ADE-configurations

Suppose that $h \in S_X$ is a polarization of degree 2. Then, by [15], the complete linear system $|\mathcal{L}_h|$ is base-point free, and hence defines a generically finite morphism $\Phi_h : X \to \mathbb{P}^2$ of degree 2. Let

$$X \stackrel{\psi_h}{\longrightarrow} Y_h \stackrel{\pi_h}{\longrightarrow} \mathbb{P}^2$$

be the Stein factorization of Φ_h, and let $B_h \subset \mathbb{P}^2$ be the branch curve of the double covering π_h. Then $\psi_h : X \to Y_h$ is a contraction of smooth rational curves, and B_h is a curve of degree 6 with only simple singularities. For each singular point P of B_h, the curves contracted to P by Φ_h form an indecomposable ADE-configuration of smooth rational curves. We put

$$\mathcal{E}_P(h) \;:=\; \{\; [C] \;\mid\; C \text{ is a smooth rational curve on } X \text{ contracted to } P \text{ by } \Phi_h \;\},$$

and label the elements of $\mathcal{E}_P(h)$ in such a way that their dual graph is indicated in Figure 3.1.

We denote by $\tau(h) \in \mathrm{Aut}(X)$ the involution of X induced by the Galois transformation of the double covering π_h, and call it a *double plane involution*. Suppose that a basis of S_X and the Gram matrix of $\langle\ ,\ \rangle_S$ with respect to this basis are given. Suppose also that we have an ample list of vectors $\boldsymbol{a}$ such that

$$D(\boldsymbol{a}) = N(X)$$

holds. Then we can calculate the matrix representation $M(h)$ of the action of $\tau(h)$ on S_X by the following method. It is well known that there exists a successive blowing up $\beta_h : F_h \to \mathbb{P}^2$ of $\mathbb{P}^2$ at (possibly infinitely near) points of the singular locus of B_h such that Φ_h factors as

$$X \stackrel{q_h}{\longrightarrow} F_h \stackrel{\beta_h}{\longrightarrow} \mathbb{P}^2,$$

where q_h is the quotient morphism by $\tau(h)$. Let S_F denote the Néron–Severi lattice of the smooth rational surface F_h. Then the pull-back q_h^* by q_h identifies $S_F \otimes \mathbb{Q}$ with the eigenspace of $\tau(h)$ in $S_X \otimes \mathbb{Q}$ with eigenvalue 1, and hence $\tau(h)$ acts on the orthogonal complement of $q_h^* S_F \otimes \mathbb{Q}$ in $S_X \otimes \mathbb{Q}$ as the scalar multiplication by -1. On the other hand, the subspace $q_h^* S_F \otimes \mathbb{Q}$ is generated by h and the vectors of the form $r + r^{\tau(h)}$, where $r \in \mathcal{E}_P(h)$ and $P \in \mathrm{Sing}(B_h)$. The action of $\tau(h)$ on $\mathcal{E}_P(h)$ is as follows:

- If P is of type A_l, then $a_i^{\tau(h)} = a_{l+1-i}$ for $i = 1, \dots, l$.
- If P is of type D_{2k}, then $\tau(h)$ acts on $\mathcal{E}_P(h)$ as the identity.
- If P is of type D_{2k+1}, then $d_1^{\tau(h)} = d_2$, $d_2^{\tau(h)} = d_1$, and $d_i^{\tau(h)} = d_i$ for $i = 3, \dots, 2k+1$.
- If P is of type E_6, then $e_1^{\tau(h)} = e_1$, and $e_i^{\tau(h)} = e_{8-i}$ for $i = 2, \dots, 6$.
- If P is of type E_7 or E_8, then $\tau(h)$ acts on $\mathcal{E}_P(h)$ as the identity.

Hence, in order to calculate the matrix representation $M(h)$ of $\tau(h)$ on S_X, it is enough to calculate the sets $\mathcal{E}_P(h)$.

We put

$$\mathcal{E}(h) := \bigcup_{P \in \mathrm{Sing}(B_h)} \mathcal{E}_P(h).$$

First we calculate the finite set

$$\mathcal{R}^+(h) := \{\, r \in \mathcal{R}(h) \;\mid\; \langle \boldsymbol{a}, r \rangle_S > 0 \,\}.$$

Note that, since $D(\boldsymbol{a})$ is equal to $N(X)$ and any $r \in \mathcal{E}(h)$ is the class of a curve, we have $\langle \boldsymbol{a}, r \rangle_S > 0$ for any vector $r \in \mathcal{E}(h)$. Moreover, any vector $r' \in \mathcal{R}^+(h)$ is the class of an effective divisor, each irreducible component of which is a smooth rational curve contracted by Φ_h. Therefore, we have $\mathcal{E}(h) \subset \mathcal{R}^+(h)$. Moreover, a vector $r' \in \mathcal{R}^+(h)$ is a linear combination with nonnegative integer coefficients of vectors in $\mathcal{E}(h)$. Consequently, a vector $r' \in \mathcal{R}^+(h)$ does *not* belong to $\mathcal{E}(h)$ if and only if r' can be written as a linear combination with nonnegative integer coefficients of vectors r'' in $\mathcal{R}^+(h)$ satisfying $\langle \boldsymbol{a}, r'' \rangle_S < \langle \boldsymbol{a}, r' \rangle_S$. Thus, starting from the vector r_0 of $\mathcal{R}^+(h)$ with the smallest $\langle \boldsymbol{a}, r_0 \rangle_S$, we can successively detect the elements of $\mathcal{E}(h)$ in $\mathcal{R}^+(h)$. We connect two distinct elements r, r' of $\mathcal{E}(h)$ by an edge if and only if $\langle r, r' \rangle_S = 1$. Then the vertices of each connected component of $\mathcal{E}(h)$ form the set $\mathcal{E}_P(h)$.

Remark 3.2. This method of calculating the action of $\tau(h)$ on S_X was also used in finding a finite set of generators of $\mathrm{Aut}(X)$ by Borcherds method in [9] and [18], and in the study of projective models of the supersingular $K3$ surface $X(5)$ in characteristic 5 with Artin invariant 1 in [20].

4. Néron–Severi lattices of supersingular $K3$ surfaces

Rudakov and Shafarevich [14] proved the following theorems. For the proof of Theorem 4.1, see also [3, Chapter 15].

Theorem 4.1. *Let p be an odd prime, and let σ be a positive integer less than or equal to* 10. *Then there exists a lattice $\Lambda^-_{p,\sigma}$, unique up to isomorphism, with the following properties.* (i) *$\Lambda^-_{p,\sigma}$ is an even hyperbolic lattice of rank* 22. (ii) *The discriminant group $(\Lambda^-_{p,\sigma})^\vee / \Lambda^-_{p,\sigma}$ of $\Lambda^-_{p,\sigma}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{2\sigma}$.*

Theorem 4.2. *Let X be a supersingular $K3$ surface in odd characteristic p with Artin invariant σ. Then its Néron–Severi lattice S_X is isomorphic to $\Lambda^-_{p,\sigma}$.*

An explicit method of constructing $\Lambda^-_{p,\sigma}$ is also given in [14] (see also [17]). We use the following construction, which is slightly different from the one given in [14]. The ingredients of the construction are the following lattices.

(i) Let U and $U^{(p)}$ be the even hyperbolic lattices of rank 2 with the Gram matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix}, \tag{4.1}$$

respectively.

(ii) Let q be a prime satisfying

$$q \equiv 3 \bmod 8 \quad \text{and} \quad \left(\frac{-q}{p}\right) = -1,$$

and let γ be an integer satisfying $\gamma^2 + p \equiv 0 \bmod q$. Let $H^{(-p)}$ be the even *negative* definite lattice of rank 4 with the Gram matrix

$$(-1)\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & (q+1)/2 & 0 & \gamma \\ 0 & 0 & p(q+1)/2 & p \\ 0 & \gamma & p & 2(p+\gamma^2)/q \end{bmatrix}.$$

Then the discriminant group of $H^{(-p)}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. See [7] and [17].

(iii) Let E_8 denote the root lattice of type E_8, which is an even unimodular positive definite lattice of rank 8. Then E_8 has a *standard basis* $e_1, \dots, e_8$, whose dual graph is given in Figure 3.1. Let $E_8^{(-1)}$ be the lattice obtained from E_8 by multiplying the intersection form by -1, and let $E_8^{(-p)}$ be the lattice obtained from $E_8^{(-1)}$ by multiplying the intersection form by p. Then the discriminant group of $E_8^{(-p)}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^8$.

Then $\Lambda^-_{p,\sigma}$ is isomorphic to the following lattices:

$$\begin{array}{ll}
U \oplus H^{(-p)} \oplus E_8^{(-1)} \oplus E_8^{(-1)} & \text{if } \sigma = 1, \\
U^{(p)} \oplus H^{(-p)} \oplus E_8^{(-1)} \oplus E_8^{(-1)} & \text{if } \sigma = 2, \\
U \oplus H^{(-p)} \oplus H^{(-p)} \oplus H^{(-p)} \oplus E_8^{(-1)} & \text{if } \sigma = 3, \\
U^{(p)} \oplus H^{(-p)} \oplus H^{(-p)} \oplus H^{(-p)} \oplus E_8^{(-1)} & \text{if } \sigma = 4, \\
U \oplus H^{(-p)} \oplus E_8^{(-1)} \oplus E_8^{(-p)} & \text{if } \sigma = 5, \\
U^{(p)} \oplus H^{(-p)} \oplus E_8^{(-1)} \oplus E_8^{(-p)} & \text{if } \sigma = 6, \\
U \oplus H^{(-p)} \oplus H^{(-p)} \oplus H^{(-p)} \oplus E_8^{(-p)} & \text{if } \sigma = 7, \\
U^{(p)} \oplus H^{(-p)} \oplus H^{(-p)} \oplus H^{(-p)} \oplus E_8^{(-p)} & \text{if } \sigma = 8, \\
U \oplus H^{(-p)} \oplus E_8^{(-p)} \oplus E_8^{(-p)} & \text{if } \sigma = 9, \\
U^{(p)} \oplus H^{(-p)} \oplus E_8^{(-p)} \oplus E_8^{(-p)} & \text{if } \sigma = 10.
\end{array}$$

Let $\langle\ ,\ \rangle_\Lambda$ denote the intersection form of $\Lambda^-_{p,\sigma}$. Note that $\Lambda^-_{p,\sigma}$ has the form of the orthogonal direct sum

$$U' \oplus N,$$

where U' is U or $U^{(p)}$ according to the parity of σ, and N is an even negative definite lattice with the intersection form $\langle\ ,\ \rangle_N$. We put

$$p' := \begin{cases} 1 & \text{if } U' \text{ is } U, \\ p & \text{if } U' \text{ is } U^{(p)}. \end{cases}$$

We choose a vector $n \in N$ randomly. If $2 - \langle n, n\rangle_N$ is divisible by $2p'$, then we can find a vector $u \in U'$ such that $v := u + n \in \Lambda^-_{p,\sigma}$ satisfies $\langle v, v\rangle_\Lambda = 2$. By this method, we can generate many vectors of $\Lambda^-_{p,\sigma}$ with square-norm 2.

5. Generating double plane involutions

We fix an odd prime p and a positive integer σ less than or equal to 10. Let X be a supersingular $K3$ surface in characteristic p with Artin invariant σ. We make a set $\mathcal{M}$ of matrix representations on S_X of double plane involutions $\tau(h) \in \mathrm{Aut}(X)$ associated with polarizations $h \in S_X$ of degree 2.

(0) We set $\mathcal{M} = \{\}$.

(1) We construct a Gram matrix of the lattice $\Lambda^-_{p,\sigma}$ by the result in Section 4.

(2) We find a vector $h_0 \in \Lambda^-_{p,\sigma}$ such that $\langle h_0, h_0\rangle_\Lambda > 0$. Let $\mathcal{P}_\Lambda$ be the positive cone of $\Lambda^-_{p,\sigma}$ containing h_0.

(3) We calculate $\mathcal{R}(h_0)$, and choose an ample list of vectors

$$\boldsymbol{a} := [h_0, \rho_1, \dots, \rho_K].$$

(4) By Theorem 4.2, there exists an isomorphism $\iota : \Lambda^-_{p,\sigma} \xrightarrow{\sim} S_X$ of lattices. Multiplying ι by -1 if necessary, we can assume that ι maps $\mathcal{P}_\Lambda$ to $\mathcal{P}(X)$. Composing ι with an element of $W(S_X)$ if necessary, we can further assume that ι maps $D(\boldsymbol{a})$ to $N(X)$. From now on, we identify $\Lambda^-_{p,\sigma}$ with S_X, and $D(\boldsymbol{a})$ with $N(X)$ by the isometry ι.

(5) We make a finite set $\mathcal{V}$ of vectors $v \in \Lambda^-_{p,\sigma}$ with $\langle v, v\rangle_\Lambda = 2$ by the method described in Section 4.

(6) For each $v \in \mathcal{V}$, we execute the following calculations.

- (6-1) If $\langle v, h_0\rangle_\Lambda < 0$, then we replace v with $-v$, so that we can assume that $v \in \mathcal{P}_\Lambda$.
- (6-2) We calculate $\mathcal{F}(v)$. If $\mathcal{F}(v) \neq \emptyset$, we proceed to the next element of $\mathcal{V}$. If $\mathcal{F}(v) = \emptyset$, we go to Step (6-3).
- (6-3) From v, we construct the vector $h_v \in \Lambda^-_{p,\sigma}$ with $\langle h_v, h_v\rangle_\Lambda = 2$ that belongs to $D(\boldsymbol{a})$ by the method described in Section 2. Since h_v and v are related by (2.1), we have $\mathcal{F}(h_v) = \emptyset$. By the identification of $D(\boldsymbol{a})$ with $N(X)$, we see that h_v is nef. Therefore, by Proposition 3.1, we see that h_v is a polarization of degree 2.
- (6-4) We then calculate the matrix representation $M(h_v)$ of the double plane involution $\tau(h_v) \in \mathrm{Aut}(X)$ by the method described in Section 3, and append $M(h_v)$ to $\mathcal{M}$.

Once we make a sufficiently large set

$$\mathcal{M} = \{M(h_1), \dots, M(h_N)\}$$

of 22×22 matrices representing the action of double plane involutions of X on S_X, we make a product

$$M := M(h_{i_1}) \cdots M(h_{i_\nu})$$

of randomly chosen elements of $\mathcal{M}$, and calculate its characteristic polynomial $\phi_M(t)$. By Theorem 1.1, if $\phi_M(t)$ is irreducible in $\mathbb{Z}[t]$ and not equal to the cyclotomic polynomial $(t^{23}-1)/(t-1)$, then $\phi_M(t)$ is a Salem polynomial.

By this method, we confirm that, if p is an odd prime ≤ 7919, then $\mathrm{Aut}(X)$ contains an automorphism of irreducible Salem type that is a product of at most 22 double plane involutions.

Remark 5.1. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{22}$ be a basis of $\Lambda^-_{p,\sigma}$, and let $\boldsymbol{e}_1^\vee, \dots, \boldsymbol{e}_{22}^\vee$ be the dual basis. Note that $p\boldsymbol{e}_i^\vee \in \Lambda^-_{p,\sigma}$ holds for $i=1,\dots,22$. Hence, in Step (3), we can choose $[h_0, p\boldsymbol{e}_1^\vee, \dots, p\boldsymbol{e}_{22}^\vee]$ as an ample list of vectors.

6. Supersingular $K3$ surfaces with Artin invariant 10

We consider a supersingular $K3$ surface X in characteristic $p \ge 11$ with Artin invariant 10. We have

$$\Lambda^-_{p,10} = U^{(p)} \oplus H^{(-p)} \oplus E_8^{(-p)} \oplus E_8^{(-p)}.$$

Let u_1, u_2 be the basis of $U^{(p)}$ with the Gram matrix (4.1), and let $e_1, \dots, e_8$ (resp. $e'_1, \dots, e'_8$) be the standard basis of the first $E_8^{(-p)}$ (resp. the second $E_8^{(-p)}$). In particular, each e_ν or e'_ν is of square-norm $-2p$. For $v \in H^{(-p)}$ and $a \in \mathbb{Z}$, we denote by

$$(a, 1, v) \in U^{(p)} \oplus H^{(-p)}$$

the vector $au_1 + u_2 + v$. Then the square-norm of $(a,1,v)$ is $2pa + \langle v, v\rangle_H$, where $\langle\ ,\ \rangle_H$ is the intersection form of $H^{(-p)}$. Note that, if $(a,1,v) \in U^{(p)} \oplus H^{(-p)}$ is of square-norm 2, then the vectors $(a+1,1,v)+e_\nu$ and $(a+1,1,v)+e'_\nu$ of $\Lambda^-_{p,10}$ are also of square-norm 2 for $\nu = 1,\dots,8$.

For p with $11 \le p \le 17389$, we have found six vectors $v_k \in H^{(-p)}$ and six positive integers $a_k \in \mathbb{Z}$ with the following properties (i)–(v).

(i) The vector $h_k := (a_k, 1, v_k)$ is of square-norm 2 for $k = 1, \dots, 6$.

We put

$$h_{6+\nu} := (a_k+1, 1, v_k) + e_\nu, \quad h_{14+\nu} := (a_k+1,1,v_k) + e'_\nu,$$

for $\nu = 1, \dots, 8$. Then $h_7, \dots, h_{22}$ are also of square-norm 2.

(ii) $\langle h_1, h_i\rangle_\Lambda > 0$ for $i = 2, \dots, 22$.
(iii) $\mathcal{S}(h_1, h_i) = \emptyset$ for $i = 2, \dots, 22$.
(iv) $\mathcal{R}(h_i) = \emptyset$ and $\mathcal{F}(h_i) = \emptyset$ for $i = 1, \dots, 22$.

Since $R(h_1) = \emptyset$, there exists a unique standard fundamental domain $D([h_1])$ of the Weyl group $W(\Lambda^-_{p,\sigma})$ that contains h_1 in its interior. Since $\mathcal{S}(h_1, h_i) = \emptyset$ for $i = 2, \dots, 22$, we see that $h_1, \dots, h_{22}$ are also contained in $D([h_1])$. Hence, under a suitable isometry $\Lambda^-_{p,10} \xrightarrow{\sim} S_X$, we can assume that each h_i is a nef vector in S_X. Since $\mathcal{F}(h_i) = \emptyset$ for $i = 1, \dots, 22$, we see that each h_i is a polarization of degree 2 on X. Moreover, since $\mathcal{R}(h_i) = \emptyset$, the branch curve $B_{h_i} \subset \mathbb{P}^2$ of the double plane involution $\tau(h_i)$ is smooth. Hence $\tau(h_i)$ acts on h_i trivially, and on the orthogonal compliment of h_i as the multiplication by -1.

(v) The product $g := \tau(h_1) \cdots \tau(h_{22})$ is of irreducible Salem type.

This observation and a computer-aided calculation give the proof of Theorem 1.3.

Example 6.1. Consider the case $p = 17389$. Then $H^{(-p)}$ has a Gram matrix

$$\begin{bmatrix} -2 & -1 & 0 & 0 \\ -1 & -30 & 0 & -4 \\ 0 & 0 & -521670 & -17389 \\ 0 & -4 & -17389 & -590 \end{bmatrix}$$

under a certain basis $\eta_1, \dots, \eta_4$ of $H^{(-p)}$. The vectors

$$\begin{array}{lcl} h_1 &=& [\ 1,\ 1,\ 15,\ 31,\ 0,\ -3\ ], \\ h_2 &=& [\ 1,\ 1,\ 9,\ 18,\ -1,\ 25\ ], \\ h_3 &=& [\ 1,\ 1,\ 51,\ 4,\ 0,\ -7\ ], \\ h_4 &=& [\ 1,\ 1,\ 30,\ 29,\ 0,\ 3\ ], \\ h_5 &=& [\ 1,\ 1,\ 55,\ -4,\ 0,\ 7\ ], \\ h_6 &=& [\ 2,\ 1,\ 19,\ 23,\ -2,\ 56\ ], \end{array}$$

of $U^{(p)} \oplus H^{(-p)}$ written with respect to the basis $u_1, u_2, \eta_1, \dots, \eta_4$ satisfies the properties (i)–(v). The characteristic polynomial on S_X of the automorphism g obtained from these six vectors has a real root $4.2539\ldots \times 10^{100}$.

Remark 6.2. Let g_p be the automorphism of a supersingular $K3$ surface X with Artin invariant 10 in characteristic p obtained by the method described in this section, let ρ_p be the real root > 1 of the characteristic polynomial of g_p on S_X, and let $\lambda_p := \log \rho_p$ be the *entropy* of g_p. Then, for $11 \le p \le 17389$, we have

$$\lambda_p \sim 19.1 + 21.8 \log p.$$

See Figure 6.1.

7. An example with Artin invariant 1

We denote by $X(p)$ a supersingular $K3$ surface in characteristic p with Artin invariant 1, which is unique up to isomorphism by the result of Ogus [12, 13]. The existence of an automorphism $g \in \mathrm{Aut}(X(p))$ of irreducible Salem type was established by Blanc and Cantat [2] for $p = 2$, by Esnault and Oguiso [4] for $p = 3$, and by Esnault, Oguiso, and Yu [5] for $p = 11$ or $p \ge 17$. On the other hand, in [16], Schütt showed that, if p is odd and satisfies $p \equiv 2 \bmod 3$, then there exists a non-liftable automorphism of $X(p)$ whose characteristic polynomial on $S_{X(p)}$ is divisible by a Salem polynomial of degree 20.

We consider the supersingular $K3$ surface $X(7)$, which has not yet been treated by the previous works. The lattice $\Lambda^-_{7,1} = U \oplus H^{(-7)} \oplus E_8^{(-1)} \oplus E_8^{(-1)}$ has a basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{22}$ such that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ form a basis of U with the Gram matrix (4.1), $\boldsymbol{e}_3, \dots, \boldsymbol{e}_6$ form a basis of $H^{(-7)}$ with the Gram matrix

$$\begin{bmatrix} -2 & -1 & 0 & 0 \\ -1 & -6 & 0 & -2 \\ 0 & 0 & -42 & -7 \\ 0 & -2 & -7 & -2 \end{bmatrix},$$

and $\boldsymbol{e}_7, \dots, \boldsymbol{e}_{14}$ (resp., $\boldsymbol{e}_{15}, \dots, \boldsymbol{e}_{22}$) form the standard basis of the first $E_8^{(-1)}$ (resp., the second $E_8^{(-1)}$). We put

$$h_0 := [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \in \Lambda^-_{7,1},$$

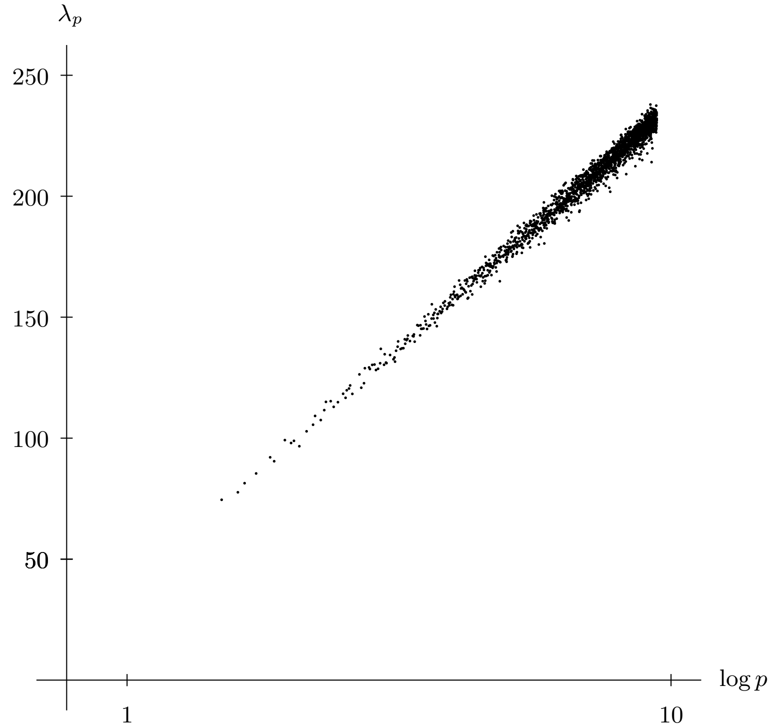

FIGURE 6.1. Growth of the entropy

which is of square-norm 2. The set $\mathcal{R}(h_0)$ consists of 486 vectors. The list

$$\boldsymbol{a} := [h_0, 7\boldsymbol{e}_1^{\vee}, \dots, 7\boldsymbol{e}_{22}^{\vee}]$$

is an ample list of vectors. We identify $\Lambda_{7,1}^{-}$ with $S_{X(7)}$ by an isometry $\Lambda_{7,1}^{-} \xrightarrow{\sim} S_{X(7)}$ that maps $D(\boldsymbol{a})$ to $N(X(7))$. (Since $\mathcal{F}(h_0) \neq \emptyset$, the vector h_0 is *not* a polarization of degree 2.)

We consider the three vectors

$$\begin{aligned} h_1 &:= [5,5,-2,3,2,-11,-12,-8,-16,-24,-20,-15,-10, \\ &\qquad -5,-8,-5,-10,-15,-12,-9,-6,-3], \\ h_2 &:= [5,5,-1,0,0,-2,-13,-9,-17,-25,-20,-15,-10, \\ &\qquad -5,-11,-7,-14,-21,-17,-13,-9,-5], \\ h_3 &:= [3,6,-2,2,2,-9,-5,-4,-7,-10,-8,-6,-4,-2,0,0,0,0,0,0,0,0], \end{aligned}$$

of square-norm 2. By means of Lemma 2.4, we can confirm that h_1, h_2, h_3 are located in $D(\boldsymbol{a}) = N(X(7))$. Moreover we have $\mathcal{F}(h_1) = \mathcal{F}(h_2) = \mathcal{F}(h_3) = \emptyset$. Hence these h_i are polarizations of degree 2, and induce double plane involutions $\tau(h_i)$. The type of the singularities of the branch curve B_{h_i} is

$$A_4 + A_5 + A_7, \quad 2A_1 + A_7 + A_9, \quad A_2 + D_7 + E_8,$$

respectively. The matrix representations $M(h_i)$ of $\tau(h_i)$ on $S_{X(7)}$ are given in Figures 7.1–7.3. (Recall that $\mathrm{O}(S_X)$ acts on S_X from the right. Hence $M(h_i)$

$$\begin{bmatrix}
24 & 24 & -10 & 15 & 10 & -55 & -57 & -38 & -76 & -114 & -95 & -71 & -48 & -24 & -40 & -25 & -50 & -75 & -60 & -45 & -30 & -15 \\
24 & 24 & -10 & 15 & 10 & -55 & -57 & -38 & -76 & -114 & -95 & -72 & -48 & -24 & -40 & -25 & -50 & -75 & -60 & -45 & -30 & -15 \\
5 & 5 & -3 & 3 & 2 & -11 & -12 & -8 & -16 & -24 & -20 & -15 & -10 & -5 & -8 & -5 & -10 & -15 & -12 & -9 & -6 & -3 \\
30 & 30 & -12 & 17 & 12 & -66 & -72 & -48 & -96 & -144 & -120 & -90 & -60 & -30 & -48 & -30 & -60 & -90 & -72 & -54 & -36 & -18 \\
-35 & -35 & 14 & -21 & -15 & 77 & 84 & 56 & 112 & 168 & 140 & 105 & 70 & 35 & 56 & 35 & 70 & 105 & 84 & 63 & 42 & 21 \\
10 & 10 & -4 & 6 & 4 & -23 & -24 & -16 & -32 & -48 & -40 & -30 & -20 & -10 & -16 & -10 & -20 & -30 & -24 & -18 & -12 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 5 & -2 & 3 & 2 & -11 & -10 & -7 & -14 & -20 & -16 & -12 & -8 & -4 & -8 & -5 & -10 & -15 & -12 & -9 & -6 & -3 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -3 & -2 & -4 & -6 & -5 & -4 & -3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 5 & -2 & 3 & 2 & -11 & -12 & -8 & -16 & -24 & -20 & -15 & -10 & -5 & -9 & -6 & -12 & -18 & -15 & -12 & -8 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

FIGURE 7.1. $M(h_1)$

$$\begin{bmatrix}
6 & 6 & 0 & 0 & 0 & -3 & -15 & -10 & -20 & -29 & -24 & -18 & -12 & -6 & -12 & -8 & -16 & -24 & -20 & -16 & -12 & -6 \\
6 & 6 & 0 & 0 & 0 & -3 & -15 & -10 & -20 & -30 & -24 & -18 & -12 & -6 & -12 & -8 & -16 & -24 & -20 & -16 & -12 & -6 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 6 & 1 & -1 & 0 & -2 & -16 & -10 & -20 & -30 & -24 & -18 & -12 & -6 & -12 & -8 & -16 & -24 & -20 & -16 & -12 & -6 \\
21 & 21 & 0 & 0 & -1 & -7 & -56 & -35 & -70 & -105 & -84 & -63 & -42 & -21 & -42 & -28 & -56 & -84 & -70 & -56 & -42 & -21 \\
6 & 6 & 0 & 0 & 0 & -3 & -16 & -10 & -20 & -30 & -24 & -18 & -12 & -6 & -12 & -8 & -16 & -24 & -20 & -16 & -12 & -6 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -4 & -6 & -5 & -4 & -3 & -2 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -3 & -2 & -4 & -6 & -5 & -4 & -3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & -1 & -5 & -4 & -7 & -10 & -8 & -6 & -4 & -2 & -5 & -4 & -7 & -10 & -8 & -6 & -4 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

FIGURE 7.2. $M(h_2)$

$$\begin{bmatrix}
14 & 27 & -9 & 9 & 9 & -42 & -30 & -24 & -42 & -60 & -48 & -36 & -24 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 14 & -5 & 5 & 5 & -23 & -15 & -12 & -21 & -30 & -24 & -18 & -12 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 9 & -4 & 3 & 3 & -14 & -10 & -8 & -14 & -20 & -16 & -12 & -8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 39 & -13 & 12 & 13 & -60 & -40 & -32 & -56 & -80 & -64 & -48 & -32 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-49 & -84 & 28 & -28 & -29 & 133 & 105 & 84 & 147 & 210 & 168 & 126 & 84 & 42 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & -1 & 1 & 1 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 6 & -2 & 2 & 2 & -9 & -8 & -5 & -10 & -15 & -12 & -9 & -6 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Figure 7.3. $M(h_3)$

satisfies $M(h_i) \cdot G_\Lambda \cdot {}^t M(h_i) = G_\Lambda$, where G_Λ is the Gram matrix of $\Lambda^-_{7,1}$ with respect to $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{22}$.) The characteristic polynomial of the product

$$M := M(h_1)M(h_2)M(h_3)$$

is a Salem polynomial

$$\begin{aligned}
&t^{22} - 993\,t^{21} - 1152\,t^{20} - 123\,t^{19} + 924\,t^{18} + 584\,t^{17} - 500\,t^{16} - 1022\,t^{15} \\
&-661\,t^{14} + 105\,t^{13} + 476\,t^{12} + 878\,t^{11} + 476\,t^{10} + 105\,t^{9} - 661\,t^{8} \\
&-1022\,t^{7} - 500\,t^{6} + 584\,t^{5} + 924\,t^{4} - 123\,t^{3} - 1152\,t^{2} - 993\,t + 1,
\end{aligned}$$

which has a positive real root $994.15889\dots$.

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