BORCHERDS' METHOD FOR ENRIQUES SURFACES

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Abstract. We classify all primitive embeddings of the lattice of numerical equivalence classes of divisors of an Enriques surface with the intersection form multiplied by 2 into an even unimodular hyperbolic lattice of rank 26. These embeddings have a property that facilitates the computation of the automorphism group of an Enriques surface by Borcherds' method.

1. Introduction

First we fix notation about lattices. A *lattice* is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form $\langle\, ,\, \rangle\colon L \times L \to \mathbb{Z}$. A lattice L is *even* if $\langle v, v\rangle \in 2\mathbb{Z}$ for all $v \in L$. An embedding $S \hookrightarrow L$ of a lattice S into a lattice L is *primitive* if the cokernel is torsion-free. A lattice L of rank $n > 1$ is *hyperbolic* if $L \otimes \mathbb{R}$ is of signature $(1, n-1)$. For a lattice L and a non-zero integer m, let $L(m)$ denote the lattice with the same underlying $\mathbb{Z}$-module as L and with the symmetric bilinear form being the m times that of L. The automorphism group of a lattice L is denoted by $\mathrm{O}(L)$, and an element of $\mathrm{O}(L)$ is called an *isometry*. We let act $\mathrm{O}(L)$ on L from the *right*. A lattice L is embedded by $\langle\, ,\, \rangle$ into the *dual lattice* $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$ as a submodule of finite index. The cokernel $A(L) := L^\vee/L$ is called the *discriminant group* of L. We say that L is *unimodular* if $A(L)$ is trivial. For an integer k, a vector v of a lattice is called a *k-vector* if $\langle v, v\rangle = k$. Let $\mathcal{R}(L)$ denote the set of (-2)-vectors of a lattice L. A *negative-definite root lattice* is a negative-definite lattice L generated by $\mathcal{R}(L)$. It is well-known that negative-definite root lattices are classified by their ADE-types (see, for example, Chapter 1 of [8]). Let L_n be an even unimodular hyperbolic lattice of rank n. It is also well-known (for example, see Section V of [20]) that L_n exists if and only if $n \equiv 2 \bmod 8$, and that, if $n \equiv 2 \bmod 8$, then L_n is unique up to isomorphism.

The lattice theory is a very strong tool in the study of $K3$ and Enriques surfaces. Let Z be a $K3$ or an Enriques surface defined over an algebraically closed field. We denote by S_Z the lattice of numerical equivalence classes of divisors on Z. Note that S_Z is even, and if $\operatorname{rank} S_Z > 1$, then S_Z is hyperbolic. *Borcherds' method* [2, 3] is a procedure to calculate the automorphism group $\mathrm{Aut}(X)$ of a $K3$ surface X by embedding S_X primitively into L_{26}, and applying Conway's result [5] on $\mathrm{O}(L_{26})$. After the work of Kondo [12], this method has been applied to many $K3$ surfaces, and automatized for computer calculation (see [22] and the references therein).

Let Y be an Enriques surface in characteristic $\neq 2$ with the universal covering $\pi\colon X \to Y$. Then we have a primitive embedding $\pi^*\colon S_Y(2) \hookrightarrow S_X$. Note that S_Y

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No.	name	rt	m4	og
1	12A	D_8	1376	$2^{29} \cdot 3^7 \cdot 5^3 \cdot 7^2$
2	12B	A_7	1824	$2^{23} \cdot 3^6 \cdot 5^2 \cdot 7^2$
3	20A	$D_4 + D_5$	1760	$2^{25} \cdot 3^7 \cdot 5^2 \cdot 7$
4	20B	$2D_4$	1888	$2^{29} \cdot 3^4 \cdot 5 \cdot 7$
5	20C	$10A_1 + D_6$	1632	$2^{28} \cdot 3^6 \cdot 5^3 \cdot 7$
6	20D	$A_3 + A_4$	2016	$2^{16} \cdot 3^6 \cdot 5^3 \cdot 7$
7	20E	$5A_1 + A_5$	1952	$2^{20} \cdot 3^7 \cdot 5^3$
8	20F	$2A_3$	2080	$2^{23} \cdot 3^4 \cdot 5^2$
9	40A	$4A_1 + 2A_3$	2016	$2^{25} \cdot 3^5 \cdot 5$
10	40B	$8A_1 + 2D_4$	1760	$2^{30} \cdot 3^6 \cdot 5 \cdot 7$
11	40C	$6A_1 + A_3$	2080	$2^{20} \cdot 3^5 \cdot 5 \cdot 7$
12	40D	$12A_1 + D_4$	1888	$2^{28} \cdot 3^5 \cdot 5^2$
13	40E	$2A_1 + 2A_2$	2144	$2^{16} \cdot 3^6 \cdot 5^2$
14	96A	$8A_1$	2144	$2^{28} \cdot 3^3$
15	96B	$16A_1$	2016	$2^{31} \cdot 3^5$
16	96C	$4A_1$	2208	$2^{22} \cdot 3^5$
17	infty		2272	$2^{26} \cdot 3^2 \cdot 5 \cdot 7$

TABLE 1.1. Primitive embeddings

is isomorphic to L_{10}. Hence, to extend Borcherds' method to Enriques surfaces, it is important to study the primitive embeddings of $L_{10}(2)$ into L_{26}.

We identify $\mathrm{O}(L_{10}(2))$ with $\mathrm{O}(L_{10})$. We say that two embeddings ι and ι' of $L_{10}(2)$ into L_{26} are *equivalent up to the action of* $\mathrm{O}(L_{10})$ *and* $\mathrm{O}(L_{26})$ if there exist isometries $g \in \mathrm{O}(L_{10})$ and $g' \in \mathrm{O}(L_{26})$ such that, for all $v \in L_{10}(2)$, one has $\iota(v)^{g'} = \iota'(v^g)$. Our first main result is as follows.

Theorem 1.1. *Up to the action of* $\mathrm{O}(L_{10})$ *and* $\mathrm{O}(L_{26})$*, there exist exactly* 17 *equivalence classes of primitive embeddings of* $L_{10}(2)$ *into* L_{26}*, and they are given in Table 1.1.*

Explanation of Table 1.1. Let R_ι denote the orthogonal complement of the image of a primitive embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ in L_{26}. Note that R_ι is a negative-definite even lattice of rank 16 with $A(L_{10}(2)) \cong (\mathbb{Z}/2\mathbb{Z})^{10}$. The item `rt` is the ADE-type of the negative-definite root lattice generated by $\mathcal{R}(R_\iota)$. For the embedding `infty`, the lattice R_ι contains no (-2)-vectors. The item `m4` is the number of (-4)-vectors in R_ι. The item `og` is the order of the group $\mathrm{O}(R_\iota)$.

These 17 embeddings have a remarkable property, which is very useful for the calculation of the automorphism group of an Enriques surface. In order to state this property, we need to explain the notion of *tessellation by chambers*. Let L be an even hyperbolic lattice. A *positive cone* $\mathcal{P}(L)$ is one of the two connected components of the subspace of $L \otimes \mathbb{R}$ consisting of vectors $x \in L \otimes \mathbb{R}$ such that $\langle x, x\rangle > 0$. We fix a positive cone $\mathcal{P}(L)$ of L, and denote by $\mathrm{O}(L, \mathcal{P})$ the stabilizer subgroup of $\mathcal{P}(L)$ in $\mathrm{O}(L)$, which is of index 2 in $\mathrm{O}(L)$. A *rational hyperplane* $(v)^\perp$ is a subspace of $\mathcal{P}(L)$ defined by $\langle x, v\rangle = 0$, where $v \in L \otimes \mathbb{Q}$ is a vector satisfying $\langle v, v\rangle < 0$. Let $\mathcal{F}$ be a locally finite family of rational hyperplanes of $\mathcal{P}(L)$. A closed subset D of $\mathcal{P}(L)$ is said to be an *$\mathcal{F}$-chamber* if D is the closure in $\mathcal{P}(L)$ of a connected component of the complement

$$\mathcal{P}(L) \setminus \bigcup_{H \in \mathcal{F}} H.$$

We say that a subset N of $\mathcal{P}(L)$ has a *tessellation by $\mathcal{F}$-chambers* if N is a union of $\mathcal{F}$-chambers. For example, if $\mathcal{F}'$ is a subfamily of $\mathcal{F}$, then every $\mathcal{F}'$-chamber has a tessellation by $\mathcal{F}$-chambers.

Definition 1.2. Note that $\mathcal{P}(L)$ has a tessellation by $\mathcal{F}$-chambers. We say that this tessellation of $\mathcal{P}(L)$ is *simple* if there exists a subgroup of $\mathrm{O}(L, \mathcal{P})$ that preserves the family $\mathcal{F}$ of hyperplanes (and hence the set of $\mathcal{F}$-chambers) and acts on the set of $\mathcal{F}$-chambers transitively.

Definition 1.3. We say that $\mathcal{F}$-chambers D and D' are *isomorphic* if there exists an isometry $g \in \mathrm{O}(L, \mathcal{P})$ such that $D^g = D'$. The automorphism group of an $\mathcal{F}$-chamber is defined to be

$$\mathrm{O}(L, D) := \{g \in \mathrm{O}(L, \mathcal{P}) \mid D^g = D\}.$$

Definition 1.4. Let D be an $\mathcal{F}$-chamber, and $\overline{D}$ the closure of D in $L \otimes \mathbb{R}$. We say that D is *quasi-finite* if $\overline{D} \setminus D$ is contained in a union of at most countably many half-lines $\mathbb{R}_{\geq 0} v_i \subset \partial\, \overline{\mathcal{P}}(L)$, where v_i are non-zero vectors of $L \otimes \mathbb{R}$ satisfying $\langle v_i, v_i\rangle = 0$, $\overline{\mathcal{P}}(L)$ is the closure of $\mathcal{P}(L)$ in $L \otimes \mathbb{R}$, and $\partial\, \overline{\mathcal{P}}(L) := \overline{\mathcal{P}}(L) \setminus \mathcal{P}(L)$.

Each (-2)-vector $r \in \mathcal{R}(L)$ defines the *reflection* $s_r \in \mathrm{O}(L, \mathcal{P})$ into the mirror $(r)^\perp$, which is defined by $x^{s_r} := x + \langle x, r\rangle r$. Let $W(L)$ denote the subgroup of $\mathrm{O}(L, \mathcal{P})$ generated by reflections s_r, where r runs through $\mathcal{R}(L)$.

Example 1.5. We put $\mathcal{R}(L)^\perp := \{(r)^\perp \mid r \in \mathcal{R}(L)\}$, which is a locally finite family of rational hyperplanes. Then an $\mathcal{R}(L)^\perp$-chamber $D_{\mathcal{R}}$ is a standard fundamental domain of the action on $\mathcal{P}(L)$ of $W(L)$. Hence the tessellation of $\mathcal{P}(L)$ by $\mathcal{R}(L)^\perp$-chambers is simple. Note that we have $\mathrm{O}(L, \mathcal{P}) = W(L) \rtimes \mathrm{O}(L, D_{\mathcal{R}})$.

Definition 1.6. The shape of an $\mathcal{R}(L_n)^\perp$-chamber was determined by Vinberg [31] for $n = 10$ and 18, and by Conway [5] for $n = 26$. Hence we call an $\mathcal{R}(L_{10})^\perp$-chamber a *Vinberg chamber*, and an $\mathcal{R}(L_{26})^\perp$-chamber a *Conway chamber*.

It is known that Vinberg chambers and Conway chambers are quasi-finite.

Definition 1.7. Let D be an *$\mathcal{F}$-chamber*. A *wall* of D is a closed subset w of D disjoint from the interior of D satisfying the following; there exists a hyperplane $(v)^\perp \in \mathcal{F}$ such that w is equal to $D \cap (v)^\perp$ and that w contains a non-empty open subset of $(v)^\perp$. We say that $v \in L \otimes \mathbb{Q}$ *defines a wall* w of D if w is equal to $D \cap (v)^\perp$ and $\langle x, v\rangle \geq 0$ holds for all $x \in D$.

Example 1.8. Let $D_{\mathcal{R}}$ be as in Example 1.5. Then the group $W(L)$ is generated by reflections with respect to the (-2)-vectors defining walls of $D_{\mathcal{R}}$.

Definition 1.9. Let D be an $\mathcal{F}$-chamber, and w a wall of D. Then there exists a unique $\mathcal{F}$-chamber D' such that $D \cap D' = w$. We call D' the $\mathcal{F}$-chamber *adjacent to D across the wall w*.

Let $\iota\colon S \hookrightarrow L$ be an embedding of an even hyperbolic lattice S, $\mathcal{P}(S)$ the positive cone of S that is mapped to $\mathcal{P}(L)$ by $\iota \otimes \mathbb{R}$, and $\iota_{\mathcal{P}}\colon \mathcal{P}(S) \hookrightarrow \mathcal{P}(L)$ the induced inclusion. We put

$$\iota^*\mathcal{F} := \{\iota_{\mathcal{P}}^{-1}(H) \mid H \in \mathcal{F},\ \iota_{\mathcal{P}}^{-1}(H) \neq \emptyset\}.$$

Then $\iota^*\mathcal{F}$ is a locally finite family of rational hyperplanes of $\mathcal{P}(S)$, and $\mathcal{P}(S)$ has a tessellation by $\iota^*\mathcal{F}$-chambers. If all $\mathcal{F}$-chambers are quasi-finite, then so are all $\iota^*\mathcal{F}$-chambers.

In the following, we identify $\mathcal{P}(L_{10}(2))$ with $\mathcal{P}(L_{10})$. If $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ is a primitive embedding, then $\mathcal{P}(L_{10})$ has a tessellation by $\iota^*\mathcal{R}(L_{26})^{\perp}$-chambers. We call an $\iota^*\mathcal{R}(L_{26})^{\perp}$-chamber an *induced chamber* associated with the embedding ι. Note that every induced chamber is quasi-finite.

In the application of Borcherds' method for the calculation of $\mathrm{Aut}(X)$ of a $K3$ surface X, we embed S_X into L_{26} primitively and investigate the tessellation of $\mathcal{P}_X$ by induced chambers. This tessellation is usually not simple, and in these cases, the computation of $\mathrm{Aut}(X)$ becomes very hard. See, for example, the case of the singular $K3$ surface with transcendental lattice of discriminant 11 treated in [22], or the case of the supersingular $K3$ surface of Artin invariant 1 in characteristic 5 studied in [9].

Our second main result is as follows.

Theorem 1.10. *Let $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ be a primitive embedding that is not of type* `infty`. *Then the number of walls of an induced chamber D is finite, and each wall of D is defined by a (-2)-vector of L_{10}. If $r \in \mathcal{R}(L_{10})$ defines a wall $w = D \cap (r)^{\perp}$ of D, then the reflection s_r with respect to r preserves the family of hyperplanes $\iota^*\mathcal{R}(L_{26})^{\perp}$ and hence the set of induced chambers. In particular, the induced chamber adjacent to D across the wall $w = D \cap (r)^{\perp}$ is equal to D^{s_r}.*

Corollary 1.11. *If $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ is not of type* `infty`, *then the tessellation of $\mathcal{P}(L_{10})$ by induced chambers is simple.*

The data of the induced chambers D are given in Table 1.2. Before explaining the contents of Table 1.2, we recall two classical results about automorphism groups of Enriques surfaces. Let Y be an Enriques surface. We denote by $\mathcal{P}_Y$ the positive cone of $S_Y \otimes \mathbb{R}$ containing an ample class. We then put

$$N_Y := \{x \in \mathcal{P}_Y \mid \langle x, [\Gamma]\rangle \geq 0 \ \text{ for all curves } \Gamma \text{ on } Y\ \}.$$

Then N_Y has a tessellation by Vinberg chambers, because N_Y is bounded by the hyperplanes $([\Gamma])^{\perp}$ defined by the classes $[\Gamma]$ of smooth rational curves Γ on Y and every smooth rational curve on Y has the self-intersection number -2.

Let Y be a complex *generic* Enriques surface. Then we have $\mathcal{P}_Y = N_Y$. Barth and Peters [1] showed that $\mathrm{Aut}(Y)$ is canonically identified with the kernel of the mod 2-reduction homomorphism $\mathrm{O}(S_Y, \mathcal{P}) \to \mathrm{O}(S_Y, \mathcal{P}) \otimes \mathbb{F}_2$. Since a Vinberg chamber has no automorphism group, the group $\mathrm{O}(S_Y, \mathcal{P})$ is equal to the subgroup $W(S_Y)$. Since the mod 2-reduction homomorphism above is surjective (see [1] and Section 2.3 of this paper), there exists a union $\mathcal{V}$ of

$$|\mathrm{O}(S_Y, \mathcal{P}) \otimes \mathbb{F}_2| = 46998591897600 = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$$

Vinberg chambers such that (i) $\mathcal{P}(L_{10})$ is the union of $\mathcal{V}^g$, where g runs through $\mathrm{Aut}(Y)$, and (ii) if $g \in \mathrm{Aut}(Y)$ is not the identity, then the interiors of $\mathcal{V}$ and of $\mathcal{V}^g$ are disjoint.

No.	name	walls	volindex	gD	orb	isom	NK
1	12A	12	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	2^2	$2+2+2+2+4$		I
2	12B	12	$2^{12} \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3$	$6+6$		II
3	20A	20	$2^8 \cdot 3^4 \cdot 5 \cdot 7$	$2^3 \cdot 3$	$4+4+6+6$		V
4	20B	20	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	2^6	$4+8+8$		III
5	20C	20	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$5+15$	20D	VII
6	20D	20	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$2^3 \cdot 3 \cdot 5$	$5+15$	20C	VII
7	20E	20	$2^7 \cdot 3^4 \cdot 5$	$2^3 \cdot 3 \cdot 5$	$10+10$		VI
8	20F	20	$2^9 \cdot 3^2 \cdot 5$	$2^6 \cdot 5$	20		IV
9	40A	40	$2^7 \cdot 3^2 \cdot 5$	$2^7 \cdot 3$	$12+12+16$		
10	40B	40	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$2^7 \cdot 3^2$	$16+24$	40C	
11	40C	40	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$2^7 \cdot 3^2$	$16+24$	40B	
12	40D	40	$2^5 \cdot 3^2 \cdot 5$	$2^5 \cdot 3^2 \cdot 5$	$10+30$	40E	
13	40E	40	$2^5 \cdot 3^2 \cdot 5$	$2^5 \cdot 3^2 \cdot 5$	$10+30$	40D	
14	96A	96	$2^5 \cdot 3^2$	$2^{13} \cdot 3$	$32+64$		
15	96B	96	$2^3 \cdot 3^2$	$2^{12} \cdot 3^3$	96	96C	
16	96C	96	$2^3 \cdot 3^2$	$2^{12} \cdot 3^3$	96	96B	
17	infty	∞					

TABLE 1.2. Induced chambers

Kondo [11] and Nikulin [16] classified all complex Enriques surfaces with finite automorphism group. This classification was extended to odd characteristics by Martin [13]. It turns out that Enriques surfaces in characteristic $\neq 2$ with finite automorphism group are divided into 7 classes I, ..., VII. An Enriques surface Y with finite automorphism group has only a finite number of smooth rational curves Γ, and N_Y is bounded by the hyperplanes $([\Gamma])^\perp$ defined by these curves. The configurations of these smooth rational curves are explicitly depicted in [11].

Explanation of Table 1.2. The item `walls` is the number of walls of an induced chamber D. Since every wall of D is defined by a (-2)-vector of L_{10}, it follows that D is a union of Vinberg chambers. The item `volindex` shows that the number of Vinberg chambers contained in D is equal to

$$|\mathrm{O}(S_Y, \mathcal{P}) \otimes \mathbb{F}_2| / \texttt{volindex} = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 / \texttt{volindex}.$$

The item `gD` is the order of the automorphism group $\mathrm{O}(L_{10}, D)$ of D. The item `orb` describes the orbit decomposition of the set of walls under the action of $\mathrm{O}(L_{10}, D)$. The item `isom` shows that, for example, the induced chambers of the primitive embeddings `20C` and `20D` are isomorphic. The item `NK` shows that, for example, the induced chamber of the primitive embedding `12A` is, under a suitable isomorphism $L_{10} \cong S_Y$, equal to N_Y of an Enriques surface Y of type I.

Since all 7 types I, ..., VII appear in the column `NK`, our results on the induced chamber D can be applied to N_Y for an arbitrary Enriques surface Y with finite automorphism group in characteristic $\neq 2$.

Borcherds' method has been applied to Enriques surfaces in [23] and [25] without using the facts proved in this paper. These facts actually give us a big advantage in the calculation of the automorphism group $\mathrm{Aut}(Y)$ of an Enriques surface Y by Borcherds' method, as is exemplified in [27]. We can also enumerate all polarizations of Y with a fixed degree modulo $\mathrm{Aut}(Y)$ by means of the method in [24]. These applications will be treated in other papers.

For the computation, the first author used a mixture of `SageMath, PARI, GAP` [30, 29, 28], and the second author used `GAP` [28]. The explicit computational data is available at the second author's webpage [26].

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Notation. To avoid possible confusions between L_{10} and $L_{10}(2)$, we put

$$\mathbf{S} := L_{10}(2).$$

We identify the underlying $\mathbb{Z}$-modules of L_{10} and $\mathbf{S}$, and choose positive cones so that $\mathcal{P}(L_{10}) = \mathcal{P}(\mathbf{S})$. We also have a natural identification $\mathrm{O}(L_{10}, \mathcal{P}) = \mathrm{O}(\mathbf{S}, \mathcal{P})$. We denote by $\langle\, , \,\rangle_{\mathbf{S}}$, $\langle\, , \,\rangle_{10}$ and $\langle\, , \,\rangle_{26}$ the symmetric bilinear forms of $\mathbf{S}$, L_{10}, and L_{26}, respectively.

2. Proof of Theorems 1.1 and 1.10

2.1. **Discriminant form.** Let L be an even lattice. Recall that $A(L) = L^\vee/L$ is the discriminant group of L. The quadratic form

$$q(L) \colon A(L) \to \mathbb{Q}/2\mathbb{Z}$$

defined by $u \bmod L \mapsto \langle u, u\rangle \bmod 2\mathbb{Z}$ for $u \in L^\vee$ is called the *discriminant form* of L. Let $\mathrm{O}(q(L))$ denote the automorphism group of the finite quadratic form $q(L)$. Then we have a natural homomorphism

$$\eta(L) \colon \mathrm{O}(L) \to \mathrm{O}(q(L)).$$

See Nikulin [15] for the basic properties of discriminant forms. Among these properties, the following is especially important for us:

Proposition 2.1. *Let M and N be even lattices. We consider the following sets:*

(a) *the set $\mathcal{L}$ of even unimodular lattices L contained in $M^\vee \oplus N^\vee$, containing $M \oplus N$, and containing each of M and N primitively, and*

(b) *the set $\mathcal{Q}$ of isomorphisms between the finite quadratic forms $q(M)$ and $-q(N)$.*

Let ϕ be an isomorphism from $q(M)$ to $-q(N)$, let $\Gamma_\phi \subset A(M) \oplus A(N)$ denote the graph of ϕ, and let $L_\phi \subset M^\vee \oplus N^\vee$ be the pull-back of Γ_ϕ by the natural projection $M^\vee \oplus N^\vee \to A(M) \oplus A(N)$. Then the mapping $\phi \mapsto L_\phi$ gives rise to a bijection from $\mathcal{Q}$ to $\mathcal{L}$. This bijection $\mathcal{Q} \cong \mathcal{L}$ is compatible with the natural actions of $\mathrm{O}(M) \times \mathrm{O}(N)$ on $\mathcal{Q}$ and on $\mathcal{L}$. □

Suppose that $L \in \mathcal{L}$, so that N is the orthogonal complement of the primitive sublattice $M \subset L$. Let $\phi \colon q(M) \xrightarrow{\sim} -q(N)$ be the isomorphism corresponding to L, and $\mathrm{O}(\phi) \colon \mathrm{O}(q(M)) \xrightarrow{\sim} \mathrm{O}(q(N))$ the induced isomorphism. We put

$$\mathrm{O}(L, M) := \{\tilde{g} \in \mathrm{O}(L) \mid \tilde{g} \text{ preserves } M\},$$

and let $\tilde{g} \mapsto \tilde{g}|M$ and $\tilde{g} \mapsto \tilde{g}|N$ denote the restriction homomorphisms from $\mathrm{O}(L, M)$ to $\mathrm{O}(M)$ and $\mathrm{O}(N)$, respectively. We say that $\tilde{g} \in \mathrm{O}(L, M)$ is a *lift* of $g \in \mathrm{O}(M)$ if $\tilde{g}|M = g$.

Corollary 2.2. *Let g be an isometry of M. Then the homomorphism $\tilde{g} \mapsto \tilde{g}|N$ induces a bijection from the set of lifts $\tilde{g}$ of g to the set of all isometries $h \in \mathrm{O}(N)$ of N such that $\eta(M)(g) \in \mathrm{O}(q(M))$ is mapped to $\eta(N)(h) \in \mathrm{O}(q(N))$ by $\mathrm{O}(\phi)$.* □

2.2. **Kneser's neighbor method.** This method allows us to efficiently compute all lattices in a given genus. We review the basic idea. For proofs and a more complete treatment, see [10] and [19].

Recall that two lattices L and L' are *in the same genus* if we have an isomorphisms

$$L \otimes \mathbb{Z}_p \cong L' \otimes \mathbb{Z}_p \quad \text{and} \quad L \otimes \mathbb{R} \cong L' \otimes \mathbb{R}$$

of $\mathbb{Z}_p$- or $\mathbb{R}$-valued quadratic modules for every prime p, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Suppose that L and L' are in the same genus. Then, by the Hasse–Minkowski theorem, we have $L \otimes \mathbb{Q} \cong L' \otimes \mathbb{Q}$. Thus we may and will assume that $L \otimes \mathbb{Q} = L' \otimes \mathbb{Q}$. Let p be an odd prime which does not divide the determinant $\det L := |A(L)|$ of L. We say that two lattices L and L' are *p-neighbors* if

$$p = [L : L \cap L'] = [L' : L \cap L'].$$

Suppose that L and L' are p-neighbors. Then $L \otimes \mathbb{Z}_q = L' \otimes \mathbb{Z}_q$ for all primes $q \neq p$. Moreover, since p does not divide $\det L$, both $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are unimodular $\mathbb{Z}_p$-lattices isomorphic over the field of p-adic rationals $\mathbb{Q}_p$. Thus $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are in fact isomorphic. We have proved that the p-neighbors L and L' are in the same genus.

For a given genus $\mathcal{G}$, we denote by C the set of isomorphism classes $[L]$ of lattices L in this genus. Let p be an odd prime. Set

$$E := \{([L], [L']) \in C \times C \mid L \text{ and } L' \text{ are } p\text{-neighbors}\}.$$

Then (C, E) is called the *p-neighbor graph* of $\mathcal{G}$. Assume further that $L \otimes \mathbb{Z}_p$ represents 0 for a lattice L in this genus. This is certainly the case if the rank of L is at least 5. In general each connected component of this graph is the union of several so called proper spinor genera. In the case relevant to us, the genus consists of a single proper spinor genus, so this does not concern us.

For given L and $v \in L \setminus pL$ with $\langle v, v\rangle \in p^2\mathbb{Z}_p$, the lattice

$$L(v) := L_v + \mathbb{Z}(v/p) \text{ where } L_v = \{x \in L \mid \langle x, v\rangle \in p\mathbb{Z}\}$$

is called the *p-neighbor of L with respect to v*. One can show that L and $L(v)$ are indeed p-neighbors, that L_v depends only on $v \mod pL$ (as long as $\langle v, v\rangle$ stays divisible by p^2), and that every p-neighbor of L arises in this fashion.

Thus one can classify lattices in the genus $\mathcal{G}$ by iteratively computing the neighbors of the lattices in C and testing for isomorphism (see [18]). One can speed this up by computing the neighbors of a given lattice only up to the action of the orthogonal group. When we are interested only in the vertices and not in the edges, we can break the computation when we have "explored" all vertices. The *mass* of the genus $\mathcal{G}$ is defined as

$$\mathrm{mass}(\mathcal{G}) := \sum_{[L] \in \mathcal{G}} \frac{1}{|O(L)|}.$$

It can be calculated from the invariants of $\mathcal{G}$ alone as described in [6]. We can break the computation as soon as the sum of the reciprocals of $|\mathrm{O}(L)|$ reaches $\mathrm{mass}(\mathcal{G})$.

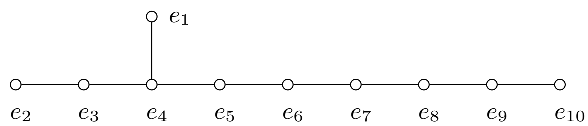

FIGURE 2.1. Basis of L_{10}

This procedure is implemented for example in `Magma` [4]. In the example relevant to us, the computation with `Magma` simply exhausted all memory available. Thus we had to resort to a modified strategy: A random walk through the neighbor graph.

2.3. **Proof of Theorems 1.1.** Let $e_1, \dots, e_{10}$ be a basis of L_{10} consisting of (-2)-vectors that form the configuration in Figure 2.1. Then

$$V := \{x \in \mathcal{P}(L_{10}) \mid \langle x, e_i\rangle_{10} \geq 0 \text{ for } i = 1, \dots, 10\} \tag{2.1}$$

is a Vinberg chamber, and each $V \cap (e_i)^\perp$ is a wall of V (see Vinberg [31]). Since the graph in Figure 2.1 has no non-trivial automorphisms, the group $\mathrm{O}(L_{10}, \mathcal{P})$ is generated by the 10 reflections $s_1, \dots, s_{10}$ with respect to $e_1, \dots, e_{10}$. Recall from the paragraph Notation at the end of Introduction that we put $\mathbf{S} := L_{10}(2)$. In $L_{10} \otimes \mathbb{Q} = \mathbf{S} \otimes \mathbb{Q}$, we have $L_{10} = L_{10}^\vee = \mathbf{S} \subset \mathbf{S}^\vee$, and the mapping $v \mapsto v/2$ gives an isomorphism $L_{10} \cong \mathbf{S}^\vee$ of $\mathbb{Z}$-modules, which gives rise to an isomorphism

$$(L_{10}/2L_{10}, q_L) \cong q(\mathbf{S}), \quad \text{where } q_L(u \bmod 2L_{10}) := \frac{1}{2}\langle u, u\rangle_{10} \bmod 2\mathbb{Z}$$

of finite quadratic forms. Hence we see that $\mathrm{O}(q(\mathbf{S}))$ is of order $2^{21}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31$ by Proposition 1.7 of [1]. Since we have explicit generators $s_1, \dots, s_{10}$ of $\mathrm{O}(L_{10}, \mathcal{P}) = \mathrm{O}(\mathbf{S}, \mathcal{P})$, we can confirm that $\eta(\mathbf{S})\colon \mathrm{O}(\mathbf{S}) \to \mathrm{O}(q(\mathbf{S}))$ restricted to $\mathrm{O}(L_{10}, \mathcal{P})$ is surjective.

Remark 2.3. The surjectivity of $\eta(\mathbf{S})$ can also be proved by Theorem 7.5 and Lemma 7.7 of Chapter VIII of [14].

Let $\iota\colon \mathbf{S} \hookrightarrow L_{26}$ be a primitive embedding, and let R_ι be the orthogonal complement of the image of ι in L_{26}. Then R_ι is of signature $(0, 16)$. By Proposition 2.1, the discriminant form $q(R_\iota)$ is isomorphic to $-q(\mathbf{S})$. Since $\eta(\mathbf{S})$ is surjective, Proposition 2.1 implies that, if a primitive embedding $\iota'\colon \mathbf{S} \hookrightarrow L_{26}$ satisfies $R_{\iota'} \cong R_\iota$, then ι' is equivalent to ι up to the action of $\mathrm{O}(\mathbf{S}) = \mathrm{O}(L_{10})$ and $\mathrm{O}(L_{26})$. Hence the proof of Theorem 1.1 is reduced to the classification of isomorphism classes of even lattices R with signature $(0, 16)$ such that $q(R) \cong -q(\mathbf{S})$. Note that these conditions on signature and discriminant form determine the genus $\mathcal{G}_R$ of R. By the mass formula [6], we see that the mass of this genus is

$$\mathrm{mass}(\mathcal{G}_R) = 64150367/28766348771328000.$$

Let $u\colon \mathbb{F}_2^2 \to \mathbb{Q}/2\mathbb{Z}$ be defined by $u(x, y) := xy$. Then one calculates that

$$-q(\mathbf{S}) \cong -u^{\oplus 5} = u^{\oplus 5} \cong q(R)$$

and that $q(D_8) \cong u$ and $q(E_8(2)) \cong u^{\oplus 4}$, where D_8 and E_8 are the negative-definite root lattices of ADE-type D_8 and E_8, respectively. Thus we have found a first lattice $L = D_8 \oplus E_8(2)$ in $\mathcal{G}_R$. To find representatives up to isomorphism, we use a variant of Kneser's neighbor method for $p = 3$. Start by inserting L into a list C. Then enter the following loop. Pick a random L in C and a random $v \in L \setminus 3L$

with $\langle v, v\rangle$ divisible by 3, replace v by $v+3w$ for $w \in L$ such that $\langle v, v\rangle$ is divisible by 9. Calculate the 3-neighbor $L(v)$ and check if it is isomorphic to any lattice in the list C. If not add it to C. Break the loop when the mass of the lattices in C matches $\mathrm{mass}(\mathcal{G}_R)$. By this computation, it turns out that $\mathcal{G}_R$ is constituted by 17 isomorphism classes in Table 1.1, and hence Theorems 1.1 follows. □

2.4. **Conway theory.** Let $\mathbf{w}$ be a non-zero primitive vector of L_{26} contained in $\partial\overline{\mathcal{P}}(L_{26})$. Note that $\langle \mathbf{w}, \mathbf{w}\rangle_{26} = 0$. We put

$$[\mathbf{w}] := \mathbb{Z}\mathbf{w}, \quad [\mathbf{w}]^\perp := \{v \in L_{26} \mid \langle v, \mathbf{w}\rangle_{26} = 0\}.$$

Then $[\mathbf{w}]^\perp/[\mathbf{w}]$ has a natural structure of an even unimodular negative-definite lattice, and hence is isomorphic to $N(-1)$, where N is one of the 24 Niemeier lattices (see, for example, Chapter 16 of [7]).

Definition 2.4. We say that $\mathbf{w}$ is a Weyl vector if $[\mathbf{w}]^\perp/[\mathbf{w}]$ is isomorphic to the negative-definite Leech lattice.

Since the Leech lattice is characterized as the unique Niemeier lattice with no roots, we can determine whether $\mathbf{w}$ is a Weyl vector or not by calculating the set $\mathcal{R}([\mathbf{w}]^\perp/[\mathbf{w}])$ of (-2)-vectors in $[\mathbf{w}]^\perp/[\mathbf{w}]$.

For a Weyl vector $\mathbf{w}$, we put

$$C(\mathbf{w}) := \{x \in \mathcal{P}(L_{26}) \mid \langle x, r\rangle_{26} \geq 0 \text{ for all } r \in \mathcal{R}(L_{26}) \text{ with } \langle r, \mathbf{w}\rangle_{26} = 1\}.$$

The following theorem is very important.

Theorem 2.5 (Conway [5])**.** *The mapping $\mathbf{w} \mapsto C(\mathbf{w})$ gives a bijection from the set of Weyl vectors to the set of Conway chambers.* □

Remark 2.6. Let $\mathbf{w}$ be a Weyl vector. Since $\mathbf{w}$ is primitive and L_{26} is unimodular, there exists a vector $\mathbf{w}'$ such that $\langle \mathbf{w}', \mathbf{w}'\rangle_{26} = 0$ and $\langle \mathbf{w}, \mathbf{w}'\rangle_{26} = 1$. Then every (-2)-vector r of L_{26} with $\langle \mathbf{w}, r\rangle_{26} = 1$ is written as

$$\alpha_\lambda \mathbf{w} + \mathbf{w}' + \lambda, \quad \text{where } \alpha_\lambda = \frac{-\langle \lambda, \lambda\rangle_{26} - 2}{2} \text{ and } \langle \mathbf{w}, \lambda\rangle_{26} = \langle \mathbf{w}', \lambda\rangle_{26} = 0.$$

Since $\langle \mathbf{w}, \lambda\rangle_{26} = \langle \mathbf{w}', \lambda\rangle_{26} = 0$ implies $\langle \lambda, \lambda\rangle_{26} \leq 0$, we see that $a_{26} := 2\mathbf{w} + \mathbf{w}'$ is an interior point of $C(\mathbf{w})$.

2.5. **Proof of Theorem 1.10.** In Section 2.3, we have calculated the 17 primitive embeddings $\iota\colon \mathbf{S} \hookrightarrow L_{26}$ explicitly. As was said in Notation, we identify $\mathcal{P}(\mathbf{S})$ and $\mathcal{P}(L_{10})$, and denote by $\iota_{\mathcal{P}}\colon \mathcal{P}(L_{10}) \hookrightarrow \mathcal{P}(L_{26})$ the induced inclusion.

Our first task is to find a Weyl vector $\mathbf{w}$ such that $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$ is an induced chamber, that is, $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$ contains a non-empty open subset of $\mathcal{P}(L_{10})$. Recall that we have fixed a basis $e_1, \dots, e_{10}$ of L_{10}. We put $a_{10} := e_1^\vee + \dots + e_{10}^\vee$, where $e_1^\vee, \dots, e_{10}^\vee$ are the basis of $L_{10}^\vee = L_{10}$ dual to $e_1, \dots, e_{10}$. Then a_{10} is an interior point of the Vinberg chamber V defined by (2.1), and we have $\langle a_{10}, a_{10}\rangle_{10} = 1240$. By direct calculation, we confirm the equality

(2.2)
$$\{r \in \mathcal{R}(L_{26}) \mid \langle r, \iota(a_{10})\rangle_{26} = 0\} \;=\; \{r \in \mathcal{R}(L_{26}) \mid \langle r, \iota(x)\rangle_{26} = 0 \text{ for all } x \in L_{10}\},$$

which means that, if a hyperplane $(r)^\perp$ of $\mathcal{P}(L_{26})$ defined by $r \in \mathcal{R}(L_{26})$ passes through $\iota(a_{10})$, then $(r)^\perp$ contains the image of $\iota_{\mathcal{P}}$. (Note that the second set in (2.2) is identified with $\mathcal{R}(R_\iota)$ by the embedding $R_\iota \hookrightarrow L_{26}$.) Therefore a_{10} is an interior point of an induced chamber D.

Definition 2.7. Let L be an even hyperbolic lattice. Suppose that v_1, v_2 are vectors of $\mathcal{P}(L) \cap (L \otimes \mathbb{Q})$. We say that a (-2)-vector $r \in \mathcal{R}(L)$ *separates* v_1 and v_2 if $\langle r, v_1 \rangle \cdot \langle r, v_2 \rangle < 0$. We can calculate the set of (-2)-vectors of L separating v_1 and v_2 by the algorithm given in Section 3.3 of [21].

We perturb a_{10} to $a'_{10} \in \mathcal{P}(L_{10}) \cap (L_{10} \otimes \mathbb{Q})$ in a general direction so that a'_{10} is also an interior point of the same induced chamber D as a_{10}, that is, the equality (2.2) remains true with $\iota(a_{10})$ replaced by $\iota(a'_{10})$ and there exist no (-2)-vectors r of L_{26} separating $\iota(a_{10})$ and $\iota(a'_{10})$. We choose an arbitrary Weyl vector $\mathbf{w}_0$ of L_{26}, and calculate a vector $a_{26} \in L_{26}$ in the interior of $C(\mathbf{w}_0)$ by Remark 2.6. We then calculate the set $\{\pm r_1, \dots, \pm r_N\}$ of (-2)-vectors of L_{26} separating $\iota(a'_{10})$ and a_{26}. We sort these (-2)-vectors $r_1, \dots, r_N$ in such a way that the line segment from a_{26} to $\iota(a'_{10})$ intersects the hyperplanes $(r_1)^\perp, \dots, (r_N)^\perp$ in this order. Since a'_{10} is a result of general perturbation, these N intersection points are distinct. Let $s_\nu \in \mathrm{O}(L_{26}, \mathcal{P})$ be the reflection with respect to r_ν. We move $\mathbf{w}_0$ by $s_1, \dots, s_N$ in this order and obtain a new Weyl vector $\mathbf{w} := \mathbf{w}_0^{s_1 \cdots s_N}$. Then $C(\mathbf{w})$ contains $a_{26}^{s_1 \cdots s_N}$ in its interior, and there exist no (-2)-vectors of L_{26} separating $\iota(a'_{10})$ and $a_{26}^{s_1 \cdots s_N}$. Therefore $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$ is the induced chamber D containing a_{10} in its interior.

Next we calculate the set of walls of the induced chamber $D = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$. We denote by $v \mapsto v_{\mathbf{S}}$ and $v \mapsto v_R$ the orthogonal projections $L_{26} \to \mathbf{S}^\vee$ and $L_{26} \to R_\iota^\vee$, and let $\langle\, , \rangle_R$ denote the symmetric bilinear form of R_ι. It turns out that $\langle \mathbf{w}_{\mathbf{S}}, \mathbf{w}_{\mathbf{S}} \rangle_{\mathbf{S}} > 0$ holds except for the case where ι is of type `infty`. Henceforth we assume that ι is not of type `infty`. We put

$$\begin{aligned} \mathcal{R}(L_{26}, \mathbf{w}) &:= \{ r \in \mathcal{R}(L_{26}) \mid \langle \mathbf{w}, r \rangle_{26} = 1 \}, \\ \mathcal{R}(L_{26}, D) &:= \{ r \in \mathcal{R}(L_{26}, \mathbf{w}) \mid \langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} < 0 \}, \\ \mathcal{R}_D &:= \{ r_{\mathbf{S}} \mid r \in \mathcal{R}(L_{26}, D) \}. \end{aligned}$$

For $r \in \mathcal{R}(L_{26})$, the hyperplane $(r)^\perp$ of $\mathcal{P}(L_{26})$ intersects the image of $\iota_{\mathcal{P}}$ if and only if $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} < 0$. Hence we have

$$D = \{ x \in \mathcal{P}(L_{10}) \mid \langle r_{\mathbf{S}}, x \rangle_{\mathbf{S}} \geq 0 \ \text{ for all } \ r_{\mathbf{S}} \in \mathcal{R}_D \}.$$

The set $\mathcal{R}_D$ can be calculated explicitly as follows. Suppose that $r \in \mathcal{R}(L_{26}, D)$. Then we have

$$\langle \mathbf{w}_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} + \langle \mathbf{w}_R, r_R \rangle_R = 1, \quad \langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} + \langle r_R, r_R \rangle_R = -2.$$

Since R_ι is negative-definite, the condition $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} < 0$ implies $-2 < \langle r_R, r_R \rangle_R \leq 0$, and if $\langle r_R, r_R \rangle_R = 0$, then we would have $r_R = 0$, $r = \iota(r_{\mathbf{S}})$ and hence $r_{\mathbf{S}} \in \mathbf{S}$, which is impossible because $\langle r, r \rangle_{26} = -2$ whereas $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} = 2 \langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{10}$ is a multiple of 4. Since $\mathbf{S}^\vee = \frac{1}{2}\mathbf{S}$, the discriminant form of $\mathbf{S}$ takes values in $\mathbb{Z}/2\mathbb{Z}$. Naturally, the same is true for $q(R_\iota) \cong -q(\mathbf{S})$. This means that $\langle v, v \rangle_R$ is integral for any $v \in R_\iota^\vee$. In particular, if $v \in R_\iota^\vee$ satisfies $-2 < \langle v, v \rangle_R < 0$, then $\langle v, v \rangle_R = -1$. Therefore we have $r_R \in V_R$ for all $r \in \mathcal{R}(L_{26}, D)$, where

$$V_R := \{ v \in R_\iota^\vee \mid \langle v, v \rangle_R = -1 \}.$$

For $v \in V_R$, we put $a(v) := 1 - \langle \mathbf{w}_R, v \rangle_R$, and let $a(V_R)$ be the set $\{ a(v) \mid v \in V_R \}$. For each $a \in a(V_R)$, we calculate

$$V_{\mathbf{S}}(a) := \{ u \in \mathbf{S}^\vee \mid \langle \mathbf{w}_{\mathbf{S}}, u \rangle_{\mathbf{S}} = a, \ \langle u, u \rangle_{\mathbf{S}} = -1 \},$$

which is finite because $\langle \mathbf{w_S}, \mathbf{w_S} \rangle_{\mathbf{S}} > 0$. We calculate the sum $u + v \in \mathbf{S}^{\vee} \oplus R_{\iota}^{\vee}$ for all pairs (v, u) of $v \in V_R$ and $u \in V_{\mathbf{S}}(a(v))$, and check whether $u + v$ is in L_{26} or not. Thus we calculate

$$\mathcal{R}(L_{26}, D) = \ L_{26} \ \cap \ \{u + v \mid v \in V_R, \ u \in V_{\mathbf{S}}(a(v))\}$$

and the set $\mathcal{R}_D$. It turns out that the hyperplanes $(r_{\mathbf{S}})^{\perp}$ for $r_{\mathbf{S}} \in \mathcal{R}_D$ are distinct, and that $\mathcal{R}_D$ spans $L_{10} \otimes \mathbb{Q}$ over $\mathbb{Q}$. We will show that each $r_{\mathbf{S}} \in \mathcal{R}_D$ defines a wall of D. We can calculate the finite group G of isometries of L_{10} that preserves the finite set $\mathcal{R}_D \subset L_{10}$. Recall that a_{10} is an interior point of D. We put $\sigma_{10} := \sum_{g \in G} a_{10}^{g}$, which is an interior point of D fixed by G. We put $t := -\langle \sigma_{10}, r_{\mathbf{S}} \rangle_{\mathbf{S}} / \langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}}$, and consider the point $\sigma_{10}' := \sigma_{10} + t \cdot r_{\mathbf{S}}$ on $(r_{\mathbf{S}})^{\perp}$. Then we have $\langle \sigma_{10}', r_{\mathbf{S}}' \rangle_{\mathbf{S}} > 0$ for all $r_{\mathbf{S}}' \in \mathcal{R}_D \setminus \{r_{\mathbf{S}}\}$, which means that σ_{10}' is an interior point of the subset $D \cap (r_{\mathbf{S}})^{\perp}$ of $(r_{\mathbf{S}})^{\perp}$. Therefore $D \cap (r_{\mathbf{S}})^{\perp}$ is a wall of D. Note that, since $\langle r_{\mathbf{S}}, r_{\mathbf{S}} \rangle_{\mathbf{S}} = -1$ for $r_{\mathbf{S}} \in \mathcal{R}_D$ and $2\mathbf{S}^{\vee} \subset \mathbf{S}$, we see that $2r_{\mathbf{S}} \in L_{10}$ and $\langle 2r_{\mathbf{S}}, 2r_{\mathbf{S}} \rangle_{10} = -2$. Therefore each wall of D is defined by a (-2)-vector $2r_{\mathbf{S}}$ of L_{10}. The group G is equal to $\mathrm{O}(L_{10}, D)$.

The assertions in Theorem 1.10 and Table 1.2 about the walls of an induced chamber are now proved for the induced chamber $D = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$ defined by this particular Weyl vector $\mathbf{w}$. The data `volindex` of D in Table 1.2 is calculated by the method given in [25].

To prove that the induced tessellation of $\mathcal{P}(L_{10})$ is simple and thus complete the proof of Theorem 1.10, it is enough to prove the following:

Proposition 2.8. *For each wall $D \cap (r)^{\perp}$ of D, where $r \in \mathcal{R}(L_{10})$, there exists an isometry $\tilde{g} \in \mathrm{O}(L_{26}, \mathcal{P})$ with the following property: the isometry $\tilde{g}$ preserves the image of $\iota \colon \mathbf{S} \hookrightarrow L_{26}$ and its restriction $\tilde{g}|\mathbf{S} \in \mathrm{O}(\mathbf{S}, \mathcal{P}) = \mathrm{O}(L_{10}, \mathcal{P})$ to $\mathbf{S}$ is equal to the reflection $s_r \in \mathrm{O}(L_{10}, \mathcal{P})$ with respect to $r \in \mathcal{R}(L_{10})$.*

Suppose that Proposition 2.8 is proved. Since $\tilde{g}$ preserves $\mathcal{R}(L_{26})$, the isometry $\tilde{g}|\mathbf{S} = s_r$ of $\mathbf{S}$ preserves the family $\iota^{*}\mathcal{R}(L_{26})^{\perp}$ of hyperplanes and hence preserves the tessellation of $\mathcal{P}(L_{10})$ by induced chambers. Since D and D^{s_r} have the common wall $D \cap (r)^{\perp}$, it follows that D^{s_r} is the induced chamber adjacent to D across the wall $D \cap (r)^{\perp}$. For any induced chamber D', there exists a chain of induced chambers

$$D = D^{(0)}, \ D^{(1)}, \ \dots, \ D^{(m)} = D'$$

such that $D^{(i-1)}$ and $D^{(i)}$ are adjacent for $i = 1, \dots, m$. By induction on the length m of the chain, we can prove that there exists an isometry $\tilde{g}' \in \mathrm{O}(L_{26}, \mathcal{P})$ preserving $\iota(\mathbf{S})$ such that the induced isometry $\tilde{g}'|\mathbf{S}$ of $\mathbf{S}$ maps D to D'. Therefore the tessellation of $\mathcal{P}(L_{10})$ by induced chambers is simple.

Proof of Proposition 2.8. The isometry $\tilde{g}$ with the hoped-for property is explicitly given in [26] for each wall of D, and thus the proof of Theorem 1.10 is completed. □

We explain the method by which we found the isometry $\tilde{g} \in \mathrm{O}(L_{26}, \mathcal{P})$. It is based on some optimistic heuristics. The isometry $\tilde{g}$ does not have to satisfy the conditions (i) and (ii) below. To our surprise, this method worked for every wall $w = D \cap (r)^{\perp}$ of the induced chamber D. We put

$$Q := \{v \in L_{26} \mid \langle v, x \rangle_{26} = 0 \ \text{ for all } \ x \in \iota_{\mathcal{P}}(w)\}.$$

No.	name	$\|V_R\|$	$d_{\mathbf{w}}$	$n_{\mathbf{w}}$	a_r	ADE-type of Σ	numb
1	12A	256	1	70	1	$2A_1+D_8$	10
					8	D_9	2
2	12B	144	2	21	1	$2A_1+A_7$	12
3	20A	160	1	22	1	$2A_1+D_4+D_5$	10
					4	$2D_5$	6
					5	D_4+D_6	4
4	20B	128	1	14	1	$2A_1+2D_4$	12
					4	D_4+D_5	8
5	20C	192	1	30	2	$8A_1+A_3+D_6$	15
					6	$10A_1+D_7$	5
6	20D	96	2	15/2	1	$2A_1+A_3+A_4$	15
					3	A_4+D_4	5
7	20E	112	1	10	1	$7A_1+A_5$	10
					2	$3A_1+A_3+A_5$	10
8	20F	80	2	5	1	$2A_1+2A_3$	20
9	40A	96	1	6	1	$6A_1+2A_3$	12
					2	$2A_1+3A_3$	12
					3	$4A_1+A_3+D_4$	16
10	40B	160	1	16	2	$6A_1+A_3+2D_4$	16
					4	$8A_1+D_4+D_5$	24
11	40C	80	1	4	1	$8A_1+A_3$	16
					2	$4A_1+2A_3$	24
12	40D	128	1	10	2	$10A_1+A_3+D_4$	30
					4	$12A_1+D_5$	10
13	40E	64	2	5/2	1	$4A_1+2A_2$	30
					2	$2A_2+A_3$	10
14	96A	64	1	2	1	$10A_1$	32
					2	$6A_1+A_3$	64
15	96B	96	1	4	2	$14A_1+A_3$	96
16	96C	48	2	1	1	$6A_1$	96
17	infty	32	1	0			

TABLE 2.1. Walls of D

Since $\dim (r)^{\perp}=9$, the even lattice Q is negative-definite of rank $26-9=17$ and contains R_ι. Let $\langle\, ,\, \rangle_Q$ denote the symmetric bilinear form of Q. We calculate the set $\mathcal{R}(Q)$ of (-2)-vectors of Q. The hyperplanes of $Q\otimes\mathbb{R}$ defined by $\langle x, r'\rangle_Q=0$, where $r'\in\mathcal{R}(Q)$, divide $Q\otimes\mathbb{R}$ into a finite number of regions, and they correspond bijectively to the Conway chambers containing $\iota_{\mathcal{P}}(w)$. We put

$$\Sigma := \{r'\in\mathcal{R}(Q) \mid \langle \mathbf{w}, r'\rangle_{26}=1\}, \tag{2.3}$$

where we regard $\mathcal{R}(Q)$ as a subset of $\mathcal{R}(L_{26})$ by the embedding $Q\hookrightarrow L_{26}$. Let P_w be an interior point of the wall $w=D\cap(r)^{\perp}$ in $(r)^{\perp}$. Locally around P_w, the Conway chamber $C(\mathbf{w})$ is defined by the inequalities $\langle x, r'\rangle_{26}\ge 0$, where r' runs through Σ. Let $C(\mathbf{w}')$ be the Conway chamber defined locally around P_w by the opposite inequalities $\langle x, r'\rangle_{26}\le 0$, where r' runs through Σ. Then $C(\mathbf{w}')$ is one of the Conway chambers that induce the induced chamber D' adjacent to D across the wall w; $\iota_{\mathcal{P}}^{-1}(C(\mathbf{w}')) = D'$. We search for isometries $\tilde{g}$ of L_{26} such that (i) $\tilde{g}$ maps Σ to $-\Sigma$, and (ii) the restriction $\tilde{g}|\langle\Sigma\rangle^{\perp}$ of $\tilde{g}$ to the orthogonal complement $\langle\Sigma\rangle^{\perp}$ in L_{26} of the sublattice $\langle\Sigma\rangle$ generated by Σ is the identity. If $\tilde{g}$ satisfies (i) and (ii), then $\tilde{g}$ fixes $\iota_{\mathcal{P}}(P_w) \in \langle\Sigma\rangle^{\perp} \otimes \mathbb{R}$ and $\tilde{g}$ maps $C(\mathbf{w})$ to $C(\mathbf{w}')$. We then check the following conditions: (iii) $\tilde{g}$ preserves the image of ι, and hence its restriction $\tilde{g}|\mathbf{S}$ to $\mathbf{S}$ maps $D = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}))$ to the adjacent chamber $D' = \iota_{\mathcal{P}}^{-1}(C(\mathbf{w}'))$, and (iv) $\tilde{g}|\mathbf{S}$ is equal to the reflection s_r. If we find an isometry $\tilde{g}$ satisfying (iii) and (iv), we are done.

Explanation of Table 2.1. We denote by $d_{\mathbf{w}}$ the minimal positive integer such that $d_{\mathbf{w}}\mathbf{w}_{\mathbf{S}} \in \mathbf{S}$, where $\mathbf{w}_{\mathbf{S}}$ is the image of $\mathbf{w}$ by the orthogonal projection $L_{26} \to \mathbf{S}^{\vee}$. We put $n_{\mathbf{w}} := \langle \mathbf{w}_{\mathbf{S}}, \mathbf{w}_{\mathbf{S}} \rangle_{10}$. For a (-2)-vector r defining a wall of D, we put $a_r := \langle \mathbf{w}_{\mathbf{S}}, r \rangle_{10}$. The root system Σ is defined by (2.3). The item `numb` is the number of walls with the described properties.

Remark 2.9. Let $\phi\colon q(\mathbf{S}) \xrightarrow{\sim} -q(R_\iota)$ be the isomorphism induced by $L_{26} \subset \mathbf{S}^{\vee} \oplus R_\iota^{\vee}$, and $\mathrm{O}(\phi)\colon \mathrm{O}(q(\mathbf{S})) \xrightarrow{\sim} \mathrm{O}(q(R_\iota))$ the isomorphism induced by ϕ. Proposition 2.8 can also be proved by showing that the image of $\eta(\mathbf{S})(s_r) \in \mathrm{O}(q(\mathbf{S}))$ by $\mathrm{O}(\phi)$ belongs to the image of $\eta(R_\iota)\colon \mathrm{O}(R_\iota) \to \mathrm{O}(q(R_\iota))$.

Example 2.10. In [17], Ohashi classified all fixed-point free involutions of the Kummer surface $X := \mathrm{Km}(\mathrm{Jac}(C))$ associated with the Jacobian variety of a generic genus-2 curve C, and showed that X has exactly $6 + 15 + 10$ fixed-point free involutions modulo conjugation in $\mathrm{Aut}(X)$. The automorphism group $\mathrm{Aut}(X)$ had been calculated by Kondo [12] by Borcherds' method. Let $\pi\colon X \to Y$ be the quotient morphism by a fixed-point free involution of X. We compose the embedding $\iota_X\colon S_X \hookrightarrow L_{26}$ used in [12] with the pull-back homomorphism $\pi^*\colon S_Y(2) \hookrightarrow S_X$, and obtain a primitive embedding $\iota_Y\colon S_Y(2) \hookrightarrow L_{26}$. We see that ι_Y is of type `20E` for 6 Enriques surfaces, of type `40A` for 15 Enriques surfaces, and of type `40C` for 10 Enriques surfaces.

See [27] for more examples, and for applications to the calculation of $\mathrm{Aut}(Y)$.

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