HOLES OF THE LEECH LATTICE AND THE PROJECTIVE MODELS OF $K3$ SURFACES

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Abstract. Using the theory of holes of the Leech lattice and Borcherds method for the computation of the automorphism group of a $K3$ surface, we give an effective bound for the set of isomorphism classes of projective models of fixed degree for certain $K3$ surfaces.

1. Introduction

Let X be a $K3$ surface defined over an algebraically closed field k, and let d be an even positive integer. Sterk [27] and Lieblich and Maulik [14] showed that, at least when the base field k is not of characteristic 2, there exist only a finite number of projective models of X with degree d up to the action of the automorphism group $\mathrm{Aut}(X)$ of X. On the other hand, by means of Borcherds method ([1], [2]), the automorphism groups of several $K3$ surfaces have been calculated ([5], [6], [8], [10], [11], [23], [25], [28]). Combining this method with the precise description of holes of the Leech lattice due to Borcherds, Conway, Parker, Queen, and Sloane ([4, Chapters 23–25]), we obtain an effective bound for the set of isomorphism classes of projective models of degree d. This bound is applicable to a wide class of $K3$ surfaces.

Our result on $K3$ surfaces is a corollary of Theorem 1.2 on the Conway chamber of the even unimodular hyperbolic lattice $\mathbf{L} := \mathrm{II}_{1,25}$ of rank 26.

We fix some terminologies and notation about lattices. A *lattice* is a free $\mathbb{Z}$-module of finite rank with a nondegenerate symmetric bilinear form that takes values in $\mathbb{Z}$, which we call the *intersection form*. Let M be a lattice with the intersection form $\langle\ ,\ \rangle_M$. We let the orthogonal group $\mathrm{O}(M)$ of M act on M from the *right*. We say that M is *hyperbolic* if its rank n is > 1 and its signature is $(1, n-1)$, whereas M is *negative-definite* if its signature is $(0, n)$.

We say that M is *even* if $\langle v, v\rangle_M \in 2\mathbb{Z}$ holds for all vectors $v \in M$. Suppose that M is even. We put

$$\mathcal{R}_M := \{\ r \in M \ \mid\ \langle r, r\rangle_M = -2\ \}.$$

The *dual lattice* $M^\vee$ of M is the $\mathbb{Z}$-module $\mathrm{Hom}(M, \mathbb{Z})$, into which M is embedded by $\langle\ ,\ \rangle_M$ as a submodule of finite index. We say that M is *unimodular* if $M = M^\vee$ holds.

Suppose that M is an even hyperbolic lattice. A *positive cone* of M is one of the two connected components of $\{x \in M \otimes \mathbb{R} \mid \langle x, x\rangle_M > 0\}$. We denote by $\mathrm{O}^+(M)$

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the stabilizer subgroup of a positive cone of M in $\mathrm{O}(M)$. We choose a positive cone $\mathcal{P}_M$. Then $\mathrm{O}^+(M)$ acts on $\mathcal{P}_M$. For each $r \in \mathcal{R}_M$, we put

$$(r)^\perp := \{\ x \in \mathcal{P}_M \ \mid\ \langle x, r\rangle_M = 0\ \},$$

and denote by s_r the element of $\mathrm{O}^+(M)$ given by

$$s_r \colon x \mapsto x + \langle x, r\rangle_M \cdot r.$$

Then s_r acts on $\mathcal{P}_M$ as the reflection in the real hyperplane $(r)^\perp$. Let W_M denote the subgroup of $\mathrm{O}^+(M)$ generated by all the reflections s_r, where r ranges through $\mathcal{R}_M$. The closure in $\mathcal{P}_M$ of a connected component of

$$\mathcal{P}_M \setminus \bigcup_{r\in\mathcal{R}_M} (r)^\perp$$

is called a *standard fundamental domain* of the action of W_M on $\mathcal{P}_M$.

Let $\mathbf{L}$ be an even unimodular hyperbolic lattice of rank 26, and let $\langle\ ,\ \rangle_{\mathbf{L}}$ denote the intersection form of $\mathbf{L}$. It is well known that $\mathbf{L}$ is unique up to isomorphism. We choose a positive cone $\mathcal{P}_{\mathbf{L}}$ of $\mathbf{L}$. By the *negative-definite Leech lattice*, we mean an even negative-definite unimodular lattice Λ^- of rank 24 with no vectors of square norm -2. It is well known that Λ^- is unique up to isomorphism. A vector $w \in \mathbf{L}$ is called a *Weyl vector* if w is a nonzero primitive vector of square norm 0 contained in the closure of $\mathcal{P}_{\mathbf{L}}$ in $\mathbf{L}\otimes\mathbb{R}$ such that the lattice $\langle w\rangle^\perp/\langle w\rangle$ is isomorphic to Λ^-, where $\langle w\rangle^\perp$ is the orthogonal complement of the submodule $\langle w\rangle := \mathbb{Z}w$ in $\mathbf{L}$. A standard fundamental domain of the action of $W_{\mathbf{L}}$ on $\mathcal{P}_{\mathbf{L}}$ is called a *Conway chamber*. For a Weyl vector w, we put

$$\mathcal{R}_{\mathbf{L}}(w) := \{\ r \in \mathcal{R}_{\mathbf{L}} \ \mid\ \langle r, w\rangle_{\mathbf{L}} = 1\ \},$$

and

$$\mathcal{D}(w) := \{\ x \in \mathcal{P}_{\mathbf{L}} \ \mid\ \langle r, x\rangle_{\mathbf{L}} \geq 0 \ \text{ for all }\ r \in \mathcal{R}_{\mathbf{L}}(w)\ \}.$$

We have the following theorem.

Theorem 1.1 (Conway [3]). *The mapping $w \mapsto \mathcal{D}(w)$ gives rise to a bijection from the set of Weyl vectors to the set of Conway chambers.*

Our main result is as follows:

Theorem 1.2. *Let $w \in \mathbf{L}$ be a Weyl vector, and let d be an even positive integer. Then, for any vector $v \in \mathcal{D}(w) \cap \mathbf{L}$ with $\langle v, v\rangle_{\mathbf{L}} = d$, we have*

$$\langle v, w\rangle_{\mathbf{L}} \leq \phi(d) := \frac{\sqrt{1081}\,(529\,d+1)}{23} = 756.20698\cdots d + 1.4295028\cdots.$$

We apply Theorem 1.2 to $K3$ surfaces X, and obtain an effective bound for the set of nef classes of self-intersection number d modulo the action of $\mathrm{Aut}(X)$ for certain $K3$ surfaces (Corollary 1.8). For this purpose, we give a review of Borcherds method ([1], [2]). See also [23] for the computational aspects of this method.

First we recall the definition of the discriminant forms. Let M be an even lattice. Then the dual lattice $M^\vee$ is equipped with a canonical $\mathbb{Q}$-valued symmetric bilinear form extending $\langle\ ,\ \rangle_M$. This $\mathbb{Q}$-valued symmetric bilinear form defines a finite quadratic form

$$q_M \colon M^\vee/M \to \mathbb{Q}/2\mathbb{Z},$$

which is called the *discriminant form* of M. (See Nikulin [15] for the basic properties of discriminant forms.) Let $\mathrm{O}(q_M)$ denote the automorphism group of the finite quadratic form q_M, and let $\eta_M\colon \mathrm{O}(M) \to \mathrm{O}(q_M)$ denote the natural homomorphism.

Let X be a $K3$ surface, and let S_X denote the Néron–Severi lattice of X with the intersection form $\langle\ ,\ \rangle_S$. Suppose that $\operatorname{rank} S_X > 1$. Then S_X is an even hyperbolic lattice. Let $\mathcal{P}(X)$ be the positive cone of S_X that contains an ample class. We let $\mathrm{Aut}(X)$ act on X from the left, and on S_X from the right by the pull-back. Hence we have a natural homomorphism

$$\mathrm{Aut}(X) \to \mathrm{O}^+(S_X).$$

Suppose that X is defined over $\mathbb{C}$ or is supersingular in characteristic $\neq 2$. Then we can use Torelli theorem (Piatetski-Shapiro and Shafarevich [20], Ogus [18], [19]) for X. We put

$$N(X) := \{\ x \in \mathcal{P}(X)\ \mid\ \langle x, C\rangle_S \ge 0\ \text{ for all curves } C \text{ on } X\ \}.$$

It is well known that $N(X)$ is a standard fundamental domain of the action of W_{S_X} on $\mathcal{P}(X)$. When X is defined over $\mathbb{C}$, we denote by H_X the unimodular lattice $H^2(X,\mathbb{Z})$ with the cup-product, by $\widetilde{G}_X$ the subgroup of $\mathrm{O}(H_X)$ consisting of isometries of H_X that preserve the 1-dimensional subspace $H^{2,0}(X)$ of $H_X \otimes \mathbb{C}$, and put

$$G_X := \{\ g \in \mathrm{O}^+(S_X)\ \mid\ g \text{ extends to an isometry } \tilde{g} \in \widetilde{G}_X\ \}.$$

When X is supersingular, we put

$$G_X := \{\ g \in \mathrm{O}^+(S_X)\ \mid\ g \text{ preserves the period of } X\ \}.$$

(See Ogus [18], [19] for the definition of the period of a supersingular $K3$ surface.) Note that, in either case, G_X is of finite index in $\mathrm{O}^+(S_X)$. By Torelli theorem, the image of the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}^+(S_X)$ is equal to

$$\{\ g \in G_X\ \mid\ N(X)^g = N(X)\ \}.$$

Suppose that we have a primitive embedding of S_X into the even unimodular hyperbolic lattice $\mathbf{L}$ of rank 26. By changing the sign of the embedding if necessary, we can assume that $\mathcal{P}(X) \subset \mathcal{P}_{\mathbf{L}}$. Let R denote the orthogonal complement of S_X in $\mathbf{L}$. Then the even unimodular overlattice $\mathbf{L}$ of $S_X \oplus R$ induces an isomorphism

$$\delta_{\mathbf{L}}\colon q_{S_X} \xrightarrow{\sim} -q_R$$

of finite quadratic forms.

Assumption 1.3. We assume that the following conditions hold.

(a) The negative-definite lattice R cannot be embedded into the negative-definite Leech lattice Λ^-.

(b) The image $\eta_{S_X}(G_X)$ of G_X by $\eta_{S_X}\colon \mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})$ is contained in the image $\eta_R(\mathrm{O}(R))$ of $\eta_R\colon \mathrm{O}(R) \to \mathrm{O}(q_R)$, where $\mathrm{O}(q_{S_X})$ and $\mathrm{O}(q_R)$ are identified by the isomorphism $\delta_{\mathbf{L}}$.

Remark 1.4. When X is defined over $\mathbb{C}$, we always have a primitive embedding of S_X into $\mathbf{L}$. See [23, Proposition 8.1].

A closed subset D of $\mathcal{P}(X)$ is said to be an *induced chamber* if there exists a Conway chamber $\mathcal{D}(w)$ such that $D = \mathcal{P}(X) \cap \mathcal{D}(w)$ holds and the interior of D in $\mathcal{P}(X)$ is nonempty. Since $\mathcal{P}_{\mathbf{L}}$ is tessellated by the Conway chambers, $\mathcal{P}(X)$ is tessellated by the induced chambers. Moreover, since $N(X)$ is bounded by hyperplanes of $\mathcal{P}(X)$ perpendicular to vectors in $\mathcal{R}_{S_X}$ and $\mathcal{R}_{S_X}$ is a subset of $\mathcal{R}_{\mathbf{L}}$ by the embedding $S_X \hookrightarrow \mathbf{L}$, it follows that $N(X)$ is also tessellated by induced chambers. We say that two induced chambers D and D' are G_X*-congruent* if there exists an element $g \in G_X$ such that $D^g = D'$. Then we have the following theorem.

Theorem 1.5 ([23]). *Suppose that S_X has a primitive embedding into $\mathbf{L}$ satisfying Assumption 1.3 and $\mathcal{P}(X) \subset \mathcal{P}_{\mathbf{L}}$. Then the following statements hold:*

(1) *Each induced chamber D is bounded by a finite number of hyperplanes of $\mathcal{P}(X)$, and the group $\mathrm{Aut}_{G_X}(D) := \{g \in G_X \mid D^g = D\}$ is finite.*
(2) *The number of G_X-congruence classes of induced chambers is finite.*

In [23], we presented an algorithm to calculate a complete set

$$\{D_0, \dots, D_{m-1}\}$$

of representatives of G_X-congruence classes of induced chambers contained in $N(X)$. We also presented an algorithm to calculate the set of hyperplanes bounding D_i and the finite group $\mathrm{Aut}_{G_X}(D_i)$ for each D_i. Then, for any vector $v \in N(X) \cap S_X$, there exist an automorphism $g \in \mathrm{Aut}(X)$ and an index i such that $v^g \in D_i$. Let $\mathrm{pr}_S \colon \mathbf{L} \to S_X^\vee$ denote the orthogonal projection. Let $w_i \in \mathbf{L}$ be a Weyl vector such that

$$D_i = \mathcal{P}_{\mathbf{L}} \cap \mathcal{D}(w_i).$$

We put

$$a_i := \mathrm{pr}_S(w_i).$$

We have $\langle a_i, a_i \rangle_S > 0$. (See Remark 5.4.) Moreover we have $\langle v, w_i \rangle_{\mathbf{L}} = \langle v, a_i \rangle_S$ for any vector $v \in S_X$. Therefore we obtain the following corollary of Theorem 1.2.

Corollary 1.6. *Suppose that S_X has a primitive embedding into $\mathbf{L}$ satisfying Assumption 1.3 and $\mathcal{P}(X) \subset \mathcal{P}_{\mathbf{L}}$. Then there exist vectors $a_0, \dots, a_{m-1}$ of $S_X^\vee$ satisfying $\langle a_i, a_i \rangle_S > 0$ such that, for any vector $v \in N(X) \cap S_X$ with $\langle v, v \rangle_S = d > 0$, there exist an automorphism $g \in \mathrm{Aut}(X)$ and an index i satisfying $\langle v^g, a_i \rangle_S \le \phi(d)$.*

Since $\langle a_i, a_i \rangle_S > 0$, the set of all vectors $v \in S_X$ satisfying $\langle v, v \rangle_S = d$ and $\langle v, a_i \rangle_S \le \phi(d)$ is finite for each $d > 0$. Therefore, provided that we have obtained, by the algorithm in [23], a set of Weyl vectors $w_0, \dots, w_{m-1}$ that give the representatives of G_X-congruence classes of induced chambers, we get an effective bound for the set of nef vectors of square norm d up to the action of $\mathrm{Aut}(X)$. Unfortunately, we do not yet have a general bound for such a set $\{w_0, \dots, w_{m-1}\}$. In some cases, however, the algorithm in [23] terminates very quickly.

Definition 1.7. Let X be a $K3$ surface that is defined over $\mathbb{C}$ or is supersingular in characteristic $\ne 2$, and let $h \in S_X \otimes \mathbb{Q}$ be an ample class. We say that (X, h) is a polarized $K3$ surface *of simple Borcherds type* if S_X admits a primitive embedding $S_X \hookrightarrow \mathbf{L}$ satisfying Assumption 1.3, $\mathcal{P}(X) \subset \mathcal{P}_{\mathbf{L}}$, and the following condition; there exists only one G_X-congruence classes of induced chambers, and it is represented by $D = \mathcal{P}_{\mathbf{L}} \cap \mathcal{D}(w)$ with $h = \mathrm{pr}_S(w)$.

Corollary 1.8. *Let (X, h) be a polarized $K3$ surface of simple Borcherds type. If $v \in S_X$ is a nef vector with $\langle v, v \rangle_S = d > 0$, then there exists an automorphism $g \in \mathrm{Aut}(X)$ such that $\langle v^g, h \rangle_S \le \phi(d)$.*

Example 1.9. The following polarized $K3$ surfaces (X, h) are of simple Borcherds type. For each of them, $\mathrm{Aut}(X)$ was determined by Borcherds method.

- The $K3$ surface X is the complex Kummer surface $\mathrm{Km}(\mathrm{Jac}(C))$ associated with the Jacobian of a generic curve C of genus 2, and h is a polarization of degree 8 that embeds X in $\mathbb{P}^5$ as a complete intersection of multi-degree $(2, 2, 2)$. We have $\operatorname{rank} S_X = 17$. See [10].
- The $K3$ surface X is the complex Kummer surface $\mathrm{Km}(E \times F)$, where E and F are generic elliptic curves, and h is a polarization of degree 28. We have $\operatorname{rank} S_X = 18$. See [8].
- The $K3$ surface X is the Fermat quartic surface in characteristic 3, and h is the class of a hyperplane section. We have $\operatorname{rank} S_X = 22$. See [11].

See Section 5 for further examples.

The problem to classify all Jacobian fibrations on a given $K3$ surface X up to the action of $\mathrm{Aut}(X)$ has been studied by many authors. For example, this classification was done for the three $K3$ surfaces in Example 1.9. See [12] for $\mathrm{Km}(\mathrm{Jac}(C))$, [13], [16], and [17] for $\mathrm{Km}(E \times F)$, and [22] for the Fermat quartic surface in characteristic 3. This problem is equivalent to the classification modulo $\mathrm{Aut}(X)$ of primitive nef vectors v satisfying $\langle v, v\rangle_S = 0$ and a certain condition corresponding to the existence of a zero section. Our problem can be regarded as an extension of this problem to the case where $\langle v, v\rangle_S > 0$.

The proof of Theorem 1.2 relies on the enumeration [4, Table 25.1, Chapter 25] of holes of Λ carried out by Borcherds, Conway, and Queen. Hence the correctness of their list is crucial for our result. Using the data we computed for the proof of Theorem 1.2, we reconfirmed the correctness of the list. See Remark 2.10. Since the whole computational data are too large to be put in the paper, we present the data only on the most important hole (the deep hole of type D_{24}), and the rest is put in the author's web page [24]. [1] For the computation, we used `GAP` [7].

The plan of this paper is as follows. In Section 2, we give a review of the theory of holes of the Leech lattice, and describe a method to obtain representatives of equivalence classes of holes. In Section 3, we define several invariants of holes, and relate them to the set of possible values of $\langle v, w\rangle_{\mathbf{L}}$, where $w \in \mathbf{L}$ is a fixed Weyl vector and v ranges through $\mathcal{D}(w) \cap \mathbf{L}$. Proposition 3.2 in this section is the principal ingredient of the proof of Theorem 1.2, which is carried out in Section 4. In Section 5, we discuss some examples, and conclude the paper by several remarks.

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2. Holes of the Leech lattice

We review the theory of holes of the Leech lattice by Borcherds, Conway, Parker, Queen, and Sloane. See [4, Chapters 23–25] for the details.

We denote by Λ the positive-definite Leech lattice with the intersection form $\langle\ ,\ \rangle_\Lambda$. Let $\Lambda_{\mathbb{R}}$ denote $\Lambda \otimes \mathbb{R}$. We use the basis of Λ given in [4, Chapter 4, Figure 4.12], and write elements of $\Lambda_{\mathbb{R}}$ as a row vector with respect to this basis. We put $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle_\Lambda}$ for $\mathbf{x} \in \Lambda_{\mathbb{R}}$, and define the function $d_\Lambda \colon \Lambda_{\mathbb{R}} \to \mathbb{R}$ by

$$d_\Lambda(\mathbf{x}) := \min\{\ \|\mathbf{x} - \lambda\| \ \mid\ \lambda \in \Lambda\ \}.$$

[1] See also Appendix B.

By the main result of [4, Chapter 23], we know that the maximum of the function d_Λ on $\Lambda_\mathbb{R}$ is $\sqrt{2}$.

Definition 2.1. A point $\mathbf{c}$ of $\Lambda_\mathbb{R}$ is called a *hole* if d_Λ attains a local maximum at $\mathbf{c}$. The *radius* $R(\mathbf{c})$ of a hole $\mathbf{c}$ is defined to be $d_\Lambda(\mathbf{c})$. We say that a hole $\mathbf{c}$ is *deep* if $R(\mathbf{c}) = \sqrt{2}$, whereas $\mathbf{c}$ is *shallow* if $R(\mathbf{c}) < \sqrt{2}$.

For $\lambda \in \Lambda$, we define the *Voronoi cell* of λ by

$$V(\lambda) := \{\ \mathbf{x} \in \Lambda_\mathbb{R} \ \mid\ \|\mathbf{x} - \lambda\| \le \|\mathbf{x} - \lambda'\| \ \text{ for all } \ \lambda' \in \Lambda \setminus \{\lambda\}\ \}.$$

Then $V(\lambda)$ is a convex polytope, and $\Lambda_\mathbb{R}$ is tessellated by these Voronoi cells. Moreover, a point $\mathbf{c}$ of $\Lambda_\mathbb{R}$ is a hole if and only if $\mathbf{c}$ is a vertex of a Voronoi cell $V(\lambda)$ for some $\lambda \in \Lambda$.

Let $\mathbf{c}$ be a hole. We put

$$P_\mathbf{c} := \{\ \lambda \in \Lambda \ \mid\ \|\lambda - \mathbf{c}\| = R(\mathbf{c})\ \} = \{\ \lambda \in \Lambda \ \mid\ \mathbf{c} \in V(\lambda)\ \},$$

and let $\overline{P}_\mathbf{c}$ denote the convex hull of $P_\mathbf{c}$ in $\Lambda_\mathbb{R}$. The following remark is important in the proof of our main result.

Remark 2.2. The affine space $\Lambda_\mathbb{R}$ is tessellated by the convex polytopes $\overline{P}_\mathbf{c}$, where $\mathbf{c}$ ranges though the set of all holes. This tessellation is dual to the tessellation of $\Lambda_\mathbb{R}$ by the Voronoi cells.

In [4, Chapter 23, Section 2], it is shown that $\|\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j\| \in \{2, \sqrt{6}, \sqrt{8}\}$ for any distinct points $\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j$ of $P_\mathbf{c}$. We define $\Delta_\mathbf{c}$ to be the graph whose set of nodes is $P_\mathbf{c}$ and whose edges are drawn by the following rule:

$$\begin{array}{lcl} \boldsymbol{\lambda}_i \text{ and } \boldsymbol{\lambda}_j \text{ are not connected} & \Longleftrightarrow & \|\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j\| = 2, \\ \boldsymbol{\lambda}_i \text{ and } \boldsymbol{\lambda}_j \text{ are connected by a single edge} & \Longleftrightarrow & \|\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j\| = \sqrt{6}, \\ \boldsymbol{\lambda}_i \text{ and } \boldsymbol{\lambda}_j \text{ are connected by a double edge} & \Longleftrightarrow & \|\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j\| = \sqrt{8}. \end{array}$$

Then each connected component of the graph $\Delta_\mathbf{c}$ is an indecomposable Coxeter–Dynkin diagram; that is, the diagram of type A_k or a_k $(k \ge 1)$, or D_k or d_k $(k \ge 4)$, or E_k or e_k $(k = 6, 7, 8)$. See [4, Chapter 23, Figure 23.1] for these diagram. We say that A_k, D_k, E_k are *extended*, and a_k, d_k, e_k are *ordinary.* (The readers are warned that this usage of the symbols A_k, D_k, E_k for extended diagrams and a_k, d_k, e_k for ordinary diagrams is not standard.) Let

$$\Delta_\mathbf{c} = \Delta_{\mathbf{c},1} \sqcup \cdots \sqcup \Delta_{\mathbf{c},m}$$

be the decomposition of $\Delta_\mathbf{c}$ into the connected components, and let

$$P_\mathbf{c} = P_{\mathbf{c},1} \sqcup \cdots \sqcup P_{\mathbf{c},m} \tag{2.1}$$

be the corresponding decomposition of the nodes. Let $\tau_{\mathbf{c},i}$ be the type of the indecomposable Coxeter–Dynkin diagram $\Delta_{\mathbf{c},i}$. We define the *hole type* $\tau(\mathbf{c})$ of $\mathbf{c}$ to be the product

$$\tau(\mathbf{c}) := \tau_{\mathbf{c},1} \cdots \tau_{\mathbf{c},m}.$$

Note that, if $\tau_{\mathbf{c},i}$ is A_k, D_k, or E_k, then $|P_{\mathbf{c},i}| = k + 1$, whereas if $\tau_{\mathbf{c},i}$ is a_k, d_k, or e_k, then $|P_{\mathbf{c},i}| = k$.

For a nonempty subset S of $\Lambda_\mathbb{R}$, we denote by $\langle S \rangle$ the minimal affine subspace of $\Lambda_\mathbb{R}$ containing S. For an affine subspace E of $\Lambda_\mathbb{R}$ and a point $\mathbf{x}$ of E, we denote by $E_\mathbf{x}$ the linear space obtained from E by regarding $\mathbf{x}$ as the origin. Then $E_\mathbf{x}$ is a linear subspace of the linear space $(\Lambda_\mathbb{R})_\mathbf{x}$.

By the classification of the deep holes in [4, Chapter 23], we obtain the following:

Theorem 2.3. *Suppose that* $\mathbf{c}$ *is deep. Then each* $\tau_{\mathbf{c},i}$ *is extended, and the convex hull* $\overline{P}_{\mathbf{c},i}$ *of each* $P_{\mathbf{c},i}$ *is an* (n_i-1)*-dimensional simplex containing* $\mathbf{c}$ *in its interior, where* $n_i := |P_{\mathbf{c},i}|$. *The linear space* $(\Lambda_{\mathbb{R}})_{\mathbf{c}}$ *is the orthogonal direct sum of the subspaces* $\langle P_{\mathbf{c},1}\rangle_{\mathbf{c}}, \dots, \langle P_{\mathbf{c},m}\rangle_{\mathbf{c}}$. *In particular, we have* $\sum_i (n_i - 1) = 24$.

By the classification of the shallow holes in [4, Chapter 25], we obtain the following:

Theorem 2.4. *Suppose that* $\mathbf{c}$ *is shallow. Then each* $\tau_{\mathbf{c},i}$ *is ordinary. Moreover, we have* $|P_{\mathbf{c}}| = 25$, *and* $\overline{P}_{\mathbf{c}}$ *is a* 24*-dimensional simplex containing* $\mathbf{c}$ *in its interior.*

We say that two holes $\mathbf{c}$ and $\mathbf{c}'$ are *equivalent* if there exists an affine isometry g of Λ such that $\mathbf{c}^g = \mathbf{c}'$. For a hole $\mathbf{c}$, we denote by $[\mathbf{c}]$ the equivalence class of holes containing $\mathbf{c}$. The equivalence classes of holes are enumerated in [4, Table 25.1, Chapter 25]. The result is summarized as follows.

Theorem 2.5. *There exist exactly* 23 *equivalence classes of deep holes, and* 284 *equivalence classes of shallow holes. Each equivalence class* $[\mathbf{c}]$ *is determined uniquely by the hole type* $\tau(\mathbf{c})$, *except for the following hole types:*

$$a_{17}a_8, \;\; d_7a_{17}a_1, \;\; d_7a_{11}a_3a_2^2, \;\; a_9^2a_4a_3. \tag{2.2}$$

For each of the hole types in (2.2), *there exist exactly two equivalence classes of holes.*

Remark 2.6. The two equivalence classes of each hole type in (2.2) can be distinguished by another method. See Remark 3.1.

We describe a method to find a representative element $\mathbf{c}$ of each equivalence class $[\mathbf{c}]$ of holes and the set $P_{\mathbf{c}}$ of vertices of $\overline{P}_{\mathbf{c}}$.

Let P and P' be finite sets of Λ. A *congruence map* from P to P' is a bijection $\gamma\colon P \xrightarrow{\sim} P'$ such that

$$\|\mathbf{v}_1 - \mathbf{v}_2\| = \|\gamma(\mathbf{v}_1) - \gamma(\mathbf{v}_2)\|$$

holds for any $\mathbf{v}_1, \mathbf{v}_2 \in P$. Suppose that $\mathbf{c}$ is a hole. Then the congruence class containing $P_{\mathbf{c}}$ is determined by $\tau(\mathbf{c})$, and hence is denoted by $[\tau(\mathbf{c})]$. If P' belongs to $[\tau(\mathbf{c})]$, then the convex hull $\overline{P'}$ of P' is circumscribed by a 23-dimensional sphere of radius $R(\mathbf{c})$, and hence $\overline{P'}$ has the circumcenter $c(P')$.

Proposition 2.7. *Let* $\mathbf{c}$ *be a hole. Suppose that* P' *belongs to* $[\tau(\mathbf{c})]$. *Then* $c(P')$ *is a hole with* $P_{c(P')} = P'$ *and* $\tau(c(P')) = \tau(\mathbf{c})$.

Proof. For the case where $\mathbf{c}$ is deep, this result follows from [4, Chapter 23, Theorem 7]. The proof for the case where $\mathbf{c}$ is shallow is almost the same. Let $\mathbf{c}$ be a shallow hole. Then $\overline{P'}$ is a 24-dimensional simplex whose circumradius R' is smaller than $\sqrt{2}$. It is enough to show that there exist no vectors $\mathbf{z} \in \Lambda$ such that $\mathbf{z} \notin P'$ and $\|\mathbf{z} - c(P')\| \le R'$. Suppose that $\mathbf{z} \in \Lambda$ satisfies $\mathbf{z} \notin P'$ and $\|\mathbf{z} - c(P')\| \le R'$. Then, for any $\mathbf{v}_i \in P'$, we have

$$4 \le \|\mathbf{z} - \mathbf{v}_i\|^2 = \|\mathbf{z} - c(P')\|^2 - 2\,\langle \mathbf{z} - c(P'), \mathbf{v}_i - c(P')\rangle_\Lambda + \|\mathbf{v}_i - c(P')\|^2,$$

where the first inequality follows from $\mathbf{z}, \mathbf{v}_i \in \Lambda$ and $\mathbf{z} \neq \mathbf{v}_i$. Since $\|\mathbf{z} - c(P')\| \le R' < \sqrt{2}$ and $\|\mathbf{v}_i - c(P')\| = R' < \sqrt{2}$, we have

$$\langle \mathbf{z} - c(P'), \mathbf{v}_i - c(P')\rangle_\Lambda < 0. \tag{2.3}$$

On the other hand, since $c(P')$ is the circumcenter of the simplex $\overline{P'}$ contained in the interior, there exist positive real numbers a_i such that

$$\sum_{\mathbf{v}_i \in P'} a_i \, (\mathbf{v}_i - c(P')) = \mathbf{0}. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain a contradiction. □

Suppose that $\mathbf{c}$ is a hole, and let P_1 and P_2 be elements of $[\tau(\mathbf{c})]$. We can determine whether the holes $c(P_1)$ and $c(P_2)$ are equivalent or not by the following method. Since P_1 and P_2 are finite, we can make the list of all congruence maps γ from P_1 to P_2. Since $\langle P_1 \rangle = \langle P_2 \rangle = \Lambda_{\mathbb{R}}$, each congruence map γ induces an affine isometry

$$\gamma_\Lambda \colon \Lambda \otimes \mathbb{Q} \xrightarrow{\sim} \Lambda \otimes \mathbb{Q}.$$

Then $c(P_1)$ and $c(P_2)$ are equivalent if and only if there exists a congruence map γ from P_1 to P_2 such that γ_Λ maps $\Lambda \subset \Lambda \otimes \mathbb{Q}$ to itself.

Remark 2.8. Let $\mathbf{c}$ be a hole. Let $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ denote the group of all congruence maps from $P_{\mathbf{c}}$ to $P_{\mathbf{c}}$, and let $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ denote the group of all affine isometries of Λ that maps $P_{\mathbf{c}}$ to $P_{\mathbf{c}}$. If the order of $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is not very large, we can calculate $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ by selecting from $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ all the congruence maps g such that g_Λ preserves Λ.

We describe a method to find a representative $\mathbf{c}$ of an equivalence class $[\mathbf{c}]$ of hole type $\tau(\mathbf{c})$. The case where $\tau(\mathbf{c}) = A_1^{24}$ is described in [4, Chapter 23] in details. Hence we assume that $\tau(\mathbf{c}) \neq A_1^{24}$. Then the graph $\Delta_{\mathbf{c}}$ contains no double edges. By an affine translation of Λ, we can assume that $P_{\mathbf{c}}$ contains the origin O of Λ. Then $P_{\mathbf{c}}$ is a subset of the set $\mathcal{N}_{\le 6} := \{O\} \cup \mathcal{N}_4 \cup \mathcal{N}_6$ of cardinality $1 + 196560 + 16773120$, where

$$\mathcal{N}_{2d} := \{\, \lambda \in \Lambda \;\mid\; \langle \lambda, \lambda \rangle_\Lambda = 2d \,\}.$$

We make the set $\mathcal{N}_{\le 6}$, and search for a subset P' of $\mathcal{N}_{\le 6}$ such that the congruence class of P' is $[\tau(\mathbf{c})]$. If $\tau(\mathbf{c})$ is not on the list (2.2), then $c(P')$ is a representative of $[\mathbf{c}]$ and $P_{c(P')}$ is equal to P' by Theorem 2.5 and Proposition 2.7. Suppose that $\tau(\mathbf{c})$ is on the list (2.2). We search for subsets $P'_1, \dots, P'_K$ of $\mathcal{N}_{\le 6}$ contained in the congruence class $[\tau(\mathbf{c})]$ until $c(P'_K)$ is not equivalent to $c(P'_1)$. Then $c(P'_1)$ and $c(P'_K)$ are representatives of the two equivalence classes of hole type $\tau(\mathbf{c})$.

Remark 2.9. For the computation, we used the standard backtrack algorithm. See [9] for the definition of this algorithm.

In the author's web page [24], we present a representative element $\mathbf{c}$ of each equivalence class $[\mathbf{c}]$ and the set $P_{\mathbf{c}}$ of vertices of $\overline{P}_{\mathbf{c}}$ in the vector representation.

Remark 2.10. The computation above relies on the enumeration [4, Table 25.1, Chapter 25] of equivalence classes of holes of Λ. In order to show that this enumeration is complete, Borcherds, Conway, and Queen used the volume formula

$$\sum_{[\mathbf{c}]} \frac{\mathrm{vol}(\overline{P}_{\mathbf{c}})}{|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)|} = \frac{1}{|\mathrm{Co}_0|}, \tag{2.5}$$

where $\mathrm{vol}(\overline{P}_{\mathbf{c}})$ is the volume of $\overline{P}_{\mathbf{c}}$, $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is defined in Remark 2.8, Co_0 is the Conway group, and the summation is taken over the set of all equivalence classes of holes. Using the sets $P_{\mathbf{c}}$ that we computed, we have reconfirmed the equality (2.5). The volume $\mathrm{vol}(\overline{P}_{\mathbf{c}})$ can be computed easily from $P_{\mathbf{c}}$. The groups $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ for deep holes $\mathbf{c}$ are studied in detail in [4, Chapters 23 and 24]. For the shallow holes, we can use the method described in Remark 2.8, except for the holes of type

$$a_5a_2^{10},\ \ d_4a_1^{21},\ \ a_3a_2^{11},\ \ a_3a_1^{22},\ \ a_1a_2^{12},\ \ a_2a_1^{23},\ \ a_1^{25}.$$

For example, for the hole $\mathbf{c} = \mathbf{c}_{303}$ of type $\tau(\mathbf{c}_{303}) = a_3a_2^{11}$, the order of $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is $2 \cdot 2^{11} \cdot 11! = 163499212800$, which is too large to be treated by this naive method. To deal with these holes, we need some consideration involving Golay codes and Mathieu groups. In particular, a characterization of Golay codes by Pless [21] plays an important role. See a note presented in the web page [24]. [2]

3. Geometry of holes and the integer points in a Conway chamber

Let $\mathbf{c}$ be a hole of radius $R(\mathbf{c})$. Suppose that $\mathbf{c}$ is shallow. Then there exists a positive rational number $s(\mathbf{c})$ that satisfies

$$R(\mathbf{c}) = \sqrt{2 - \frac{1}{s(\mathbf{c})}}. \tag{3.1}$$

When $\mathbf{c}$ is deep, we put $s(\mathbf{c}) := \infty$. It is obvious that $s(\mathbf{c})$ depends only on $[\mathbf{c}]$.

Let v be a point of $\Lambda \otimes \mathbb{Q}$. We define $m(v)$ to be the order of $v \bmod \Lambda$ in the torsion group $(\Lambda \otimes \mathbb{Q})/\Lambda \cong (\mathbb{Q}/\mathbb{Z})^{24}$. It is obvious that $m(v)$ is invariant under the action of affine isometries of Λ.

Note that $\mathbf{c}$ belongs to $\Lambda \otimes \mathbb{Q}$, because $\mathbf{c}$ is the intersection point of the bisectors of distinct two points of $P_{\mathbf{c}}$. It is obvious that $m(\mathbf{c})$ depends only on $[\mathbf{c}]$.

Remark 3.1. The invariant $m(v)$ enables us to distinguish the two equivalence classes of each hole type in (2.2).

(1) For the two equivalence classes $[\mathbf{c}_{42}]$ and $[\mathbf{c}_{43}]$ with $\tau(\mathbf{c}_{42}) = \tau(\mathbf{c}_{43}) = a_{17}a_8$, we have $m(\mathbf{c}_{42}) = 33$ and $m(\mathbf{c}_{43}) = 99$.

(2) For the two equivalence classes $[\mathbf{c}_{45}]$ and $[\mathbf{c}_{46}]$ with $\tau(\mathbf{c}_{45}) = \tau(\mathbf{c}_{46}) = d_7a_{17}a_1$, we have $m(\mathbf{c}_{45}) = 144$ and $m(\mathbf{c}_{46}) = 48$.

(3) For the two equivalence classes $[\mathbf{c}_{130}]$ and $[\mathbf{c}_{131}]$ with $\tau(\mathbf{c}_{130}) = \tau(\mathbf{c}_{131}) = d_7a_{11}a_3a_2^2$, we have $m(\mathbf{c}_{130}) = m(\mathbf{c}_{131}) = 54$. For $\nu = 130$ and 131, let v_ν^1 and v_ν^2 be the two vertices of $\overline{P}_{\mathbf{c}_\nu}$ that correspond to the two nodes of valency 1 in the Coxeter–Dynkin diagram of type a_3 in $d_7a_{11}a_3a_2^2$. For $i = 1$ and 2, let c_ν^i be the circumcenter of the 23-dimensional face of $\overline{P}_{\mathbf{c}_\nu}$ that does not contain v_ν^i. Then we have $\{m(c_{130}^1), m(c_{130}^2)\} = \{120, 240\}$ and $\{m(c_{131}^1), m(c_{131}^2)\} = \{480\}$. Therefore $\mathbf{c}_{130}$ and $\mathbf{c}_{131}$ are not equivalent.

(4) For the two equivalence classes $[\mathbf{c}_{181}]$ and $[\mathbf{c}_{182}]$ with $\tau(\mathbf{c}_{181}) = \tau(\mathbf{c}_{182}) = a_9^2a_4a_3$, we have $m(\mathbf{c}_{181}) = m(\mathbf{c}_{182}) = 60$. For $\nu = 181$ and 182, let v_ν^1 and v_ν^2 be the two vertices of $\overline{P}_{\mathbf{c}_\nu}$ that correspond to the two nodes of valency 1 in a_4. For $i = 1$ and 2, let c_ν^i be the circumcenter of the 23-dimensional face of $\overline{P}_{\mathbf{c}_\nu}$ that does not contain v_ν^i. Then we have $\{m(c_{181}^1), m(c_{181}^2)\} = \{350, 70\}$ and $\{m(c_{182}^1), m(c_{182}^2)\} = \{350\}$. Therefore $\mathbf{c}_{181}$ and $\mathbf{c}_{182}$ are not equivalent.

We then define the invariant $N(\mathbf{c})$ of $[\mathbf{c}]$ as follows. When $\mathbf{c}$ is deep, we put

$$N(\mathbf{c}) := \begin{cases} m(\mathbf{c})/2 & \text{if } m(\mathbf{c}) \text{ is even,} \\ m(\mathbf{c}) & \text{if } m(\mathbf{c}) \text{ is odd.} \end{cases}$$

[2] See also Appendix A.

When $\mathbf{c}$ is shallow, we define $N(\mathbf{c})$ to be the least positive integer such that $N(\mathbf{c})/s(\mathbf{c}) \in \mathbb{Z}$.

For a positive real number r, we put

$$\Xi(r) := \{\ \mathbf{x} \in \Lambda_{\mathbb{R}} \ \mid\ d_\Lambda(\mathbf{x}) \ge r\ \}.$$

Let $\mathbf{c}$ be a hole. We put

$$\Xi_{\mathbf{c}}(r) := \{\ \mathbf{x} \in \overline{P}_{\mathbf{c}} \ \mid\ \|\mathbf{x} - \boldsymbol{\lambda}\| \ge r \ \text{ for all } \ \boldsymbol{\lambda} \in P_{\mathbf{c}}\ \}.$$

Then we obviously have

$$\Xi(r) \cap \overline{P}_{\mathbf{c}} \ \subset\ \Xi_{\mathbf{c}}(r). \tag{3.2}$$

Note also that, if $r \le R(\mathbf{c})$, then we have $\mathbf{c} \in \Xi_{\mathbf{c}}(r)$. Let $\theta(\mathbf{c})$ be the minimal real number such that, if r satisfies $\theta(\mathbf{c}) < r \le R(\mathbf{c})$, then $\Xi_{\mathbf{c}}(r)$ is contained in the interior of $\overline{P}_{\mathbf{c}}$. For r with $\theta(\mathbf{c}) \le r \le R(\mathbf{c})$, we put

$$\sigma(\mathbf{c}, r) := \max\{\ \|\mathbf{x} - \mathbf{c}\| \ \mid\ \mathbf{x} \in \Xi_{\mathbf{c}}(r)\ \}.$$

Since $\theta(\mathbf{c})$ and $\sigma(\mathbf{c}, r)$ depend only on the congruence class of the polytope $\overline{P}_{\mathbf{c}}$, they depend only on the hole type $\tau(\mathbf{c})$, and hence only on the equivalence class $[\mathbf{c}]$. It is easy to see that $\sigma(\mathbf{c}, r)$ is a decreasing function with respect to r, and that $\sigma(\mathbf{c}, R(\mathbf{c})) = 0$. For simplicity, we put

$$\sigma(\mathbf{c}, r) := 0 \ \text{ for } r > R(\mathbf{c}).$$

In fact, the function $\sigma(\mathbf{c}, r)$ can be calculated from the real number $\theta(\mathbf{c})$ (see Section 4.1).

Using these invariants of holes, we can state our principal result. For each even positive integer d, we put

$$\rho_d(x) := \sqrt{2 - \frac{d}{x^2}},$$

which is a function defined for $x \ge \sqrt{d/2}$.

Proposition 3.2. *Let $w \in \mathbf{L}$ be a Weyl vector, and let d be an even positive integer. Let v be a point of $\mathcal{D}(w) \cap \mathbf{L}$ with $\langle v, v\rangle_{\mathbf{L}} = d$, and suppose that $b := \langle v, w\rangle_{\mathbf{L}}$ satisfies $b \ge \sqrt{d/2}$. Then there exists a hole $\mathbf{c}$ for which b satisfies one of the following conditions.*

(I) *b^2 divides $N(\mathbf{c})^2 d$, and $b^2 \le s(\mathbf{c}) d$,*

(II) *$\rho_d(b) \le \theta(\mathbf{c})$, or*

(III) *$\rho_d(b) \ge \theta(\mathbf{c})$ and $\sigma(\mathbf{c}, \rho_d(b)) \ge \dfrac{2}{m(\mathbf{c}) b}$.*

Remark 3.3. When $\mathbf{c}$ is deep, the second condition in (I) is vacuous.

For the proof of Proposition 3.2, we use the following lemma.

Lemma 3.4. *For any hole $\mathbf{c}' \in [\mathbf{c}]$, we have $N(\mathbf{c})\, \langle \mathbf{c}', \mathbf{c}'\rangle_\Lambda \in \mathbb{Z}$.*

Proof. Let $\lambda_0 \in \Lambda$ be an element of $P_{\mathbf{c}'}$, and we put $\mathbf{c}'' := \mathbf{c}' - \lambda_0$. Note that $\mathbf{c}'' \in [\mathbf{c}]$ and hence $m(\mathbf{c})\, \mathbf{c}'' \in \Lambda$. Moreover, we have $\langle \mathbf{c}'', \mathbf{c}''\rangle_\Lambda = R(\mathbf{c})^2$. Hence we have

$$\langle \mathbf{c}', \mathbf{c}'\rangle_\Lambda = R(\mathbf{c})^2 + 2\langle \mathbf{c}'', \lambda_0\rangle_\Lambda + \langle \lambda_0, \lambda_0\rangle_\Lambda.$$

Suppose that $\mathbf{c}$ is deep. Then we have $R(\mathbf{c})^2 = 2 \in \mathbb{Z}$, and $2N(\mathbf{c})\, \mathbf{c}'' \in \Lambda$. Therefore $N(\mathbf{c})\, \langle \mathbf{c}', \mathbf{c}'\rangle_\Lambda \in \mathbb{Z}$ holds. Suppose that $\mathbf{c}$ is shallow. Then we have $N(\mathbf{c}) R(\mathbf{c})^2 \in \mathbb{Z}$

by (3.1). By the list [24], we confirm that $m(\mathbf{c})$ divides $2N(\mathbf{c})$, and thus we obtain $2N(\mathbf{c})\,\langle \mathbf{c}'', \lambda_0\rangle_\Lambda \in \mathbb{Z}$. Therefore $N(\mathbf{c})\,\langle \mathbf{c}', \mathbf{c}'\rangle_\Lambda \in \mathbb{Z}$ holds. □

Proof of Proposition 3.2. Let U denote the hyperbolic plane; that is, U is the lattice of rank 2 with a basis $\mathbf{e}_1, \mathbf{e}_2$ with respect to which the Gram matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We put

$$\mathbf{L} := U \oplus \Lambda^-,$$

where Λ^- is the negative-definite Leech lattice. Then $\mathbf{L}$ is an even unimodular hyperbolic lattice of rank 26. A vector of $\mathbf{L}\otimes\mathbb{R}$ is written as $(a, b, \mathbf{v})$, where $(a, b) = a\,\mathbf{e}_1 + b\,\mathbf{e}_2 \in U\otimes\mathbb{R}$ and $\mathbf{v} \in \Lambda\otimes\mathbb{R}$. The intersection form $\langle\ ,\ \rangle_{\mathbf{L}}$ of $\mathbf{L}$ is given by

$$\langle (a, b, \mathbf{v}), (a', b', \mathbf{v}')\rangle_{\mathbf{L}} = ab' + a'b - \langle \mathbf{v}, \mathbf{v}'\rangle_\Lambda.$$

We choose the positive cone $\mathcal{P}_{\mathbf{L}}$ of $\mathbf{L}\otimes\mathbb{R}$ in such a way that the primitive vector

$$w_0 := (1, 0, \mathbf{0})$$

of square norm 0 is contained in the closure of $\mathcal{P}_{\mathbf{L}}$ in $\mathbf{L}\otimes\mathbb{R}$. Since $\langle w_0\rangle^\perp/\langle w_0\rangle$ is isomorphic to Λ^-, we see that w_0 is a Weyl vector. Since the group $\mathrm{O}^+(\mathbf{L})$ acts on the set of Weyl vectors transitively, it is enough to prove Proposition 3.2 for the Weyl vector w_0.

For $\lambda \in \Lambda$, we put

$$r_\lambda := \left(\frac{\lambda^2}{2} - 1, 1, \lambda\right) \in \mathcal{R}_{\mathbf{L}}, \quad \text{where} \quad \lambda^2 := \langle \lambda, \lambda\rangle_\Lambda.$$

Then we have $\mathcal{R}_{\mathbf{L}}(w_0) = \{ r_\lambda \mid \lambda \in \Lambda\}$, and hence

$$\mathcal{D}(w_0) = \{\ x \in \mathcal{P}_{\mathbf{L}}\ \mid\ \langle x, r_\lambda\rangle_{\mathbf{L}} \ge 0\ \text{ for all }\ \lambda \in \Lambda\ \}.$$

Let $v = (a, b, \mathbf{v})$ be an arbitrary vector of $\mathcal{D}(w_0) \cap \mathbf{L}$ satisfying $\langle v, v\rangle_{\mathbf{L}} = d$, and suppose that $b = \langle v, w_0\rangle_{\mathbf{L}}$ satisfies $b \ge \sqrt{d/2}$.

Note that a, b, and $\mathbf{v}$ satisfy the following conditions:

(i) $a, b \in \mathbb{Z}$ and $\mathbf{v} \in \Lambda$,

(ii) $\langle v, r_\lambda\rangle_{\mathbf{L}} = a + \left(\dfrac{\lambda^2}{2} - 1\right) b - \langle \mathbf{v}, \lambda\rangle_\Lambda \ge 0$ for all vectors $\lambda \in \Lambda$,

(iii) $\langle v, v\rangle_{\mathbf{L}} = 2ab - \langle \mathbf{v}, \mathbf{v}\rangle_\Lambda = d$.

By condition (iii), we have

$$\frac{a}{b} = \frac{1}{2}\left(\frac{d}{b^2} + \left\langle \frac{\mathbf{v}}{b}, \frac{\mathbf{v}}{b}\right\rangle_\Lambda\right).$$

Combining this with the assumption $b \ge \sqrt{d/2}$, we see that condition (ii) is equivalent to

$$\left\| \frac{\mathbf{v}}{b} - \lambda \right\| \ge \sqrt{2 - \frac{d}{b^2}} \quad \text{for all }\ \lambda \in \Lambda. \tag{3.3}$$

In other words, we have

$$\mathbf{v}/b\ \in\ \Xi(\,\rho_d(b)\,). \tag{3.4}$$

By Remark 2.2, there exists a hole $\mathbf{c}$ such that the convex polytope $\overline{P}_{\mathbf{c}}$ contains the point $\mathbf{v}/b$. By (3.2) and (3.4), we have

$$\frac{\mathbf{v}}{b} \in \Xi_{\mathbf{c}}(\,\rho_d(b)\,). \tag{3.5}$$

We will show that b satisfies one of conditions (I), (II) or (III) for this hole $\mathbf{c}$.

Lemma 3.5. *Suppose that $\mathbf{v}/b$ is equal to the hole $\mathbf{c}$, and let N be a positive integer such that $N\,\langle \mathbf{c},\mathbf{c}\rangle_\Lambda \in \mathbb{Z}$. Then b^2 divides $N^2 d$.*

Proof. We put $M := N\,\langle \mathbf{c},\mathbf{c}\rangle_\Lambda \in \mathbb{Z}$. By condition (iii) and the assumption $\mathbf{v}/b = \mathbf{c}$, we have

$$a = \frac{d}{2b} + \frac{Mb}{2N}.$$

Multiplying $2N$ on both sides, we obtain

$$L := \frac{Nd}{b} = 2Na - Mb \in \mathbb{Z}.$$

Moreover, we have

$$a = \frac{d}{2b} + \frac{Md}{2L}.$$

Multiplying $2L$ on both sides, we obtain

$$\frac{Ld}{b} = \frac{Nd^2}{b^2} = 2La - Md \in \mathbb{Z}.$$

This completes the proof. □

Case 1. Suppose that $\mathbf{v}/b$ is equal to the hole $\mathbf{c}$. From the case $\lambda \in P_{\mathbf{c}}$ in (3.3), we obtain $\sqrt{2 - d/b^2} \le R(\mathbf{c}) = \sqrt{2 - 1/s(\mathbf{c})}$, and hence $b^2 \le s(\mathbf{c})\,d$. By Lemmas 3.4 and 3.5, we also have that b^2 divides $N(\mathbf{c})^2\,d$. Therefore b satisfies condition (I).

Case 2. Suppose that $\mathbf{v}/b$ is not equal to $\mathbf{c}$. Then $m(\mathbf{c})\,\mathbf{v}$ and $b\,m(\mathbf{c})\,\mathbf{c}$ are distinct points of Λ by the definition of $m(\mathbf{c})$ and hence $\|m(\mathbf{c})\,\mathbf{v} - b\,m(\mathbf{c})\,\mathbf{c}\|^2 \ge 4$ holds. Therefore we have

$$\left\| \frac{\mathbf{v}}{b} - \mathbf{c} \right\| \ge \frac{2}{m(\mathbf{c})b}. \tag{3.6}$$

We assume that b does not satisfy condition (II). Then $\Xi_{\mathbf{c}}(\,\rho_d(b)\,)$ is contained in the interior of $\overline{P}_{\mathbf{c}}$. By (3.5) and the definition of $\sigma(\mathbf{c}, r)$, we have

$$\left\| \frac{\mathbf{v}}{b} - \mathbf{c} \right\| \le \sigma\left(\mathbf{c}, \rho_d(b)\,\right). \tag{3.7}$$

Combining (3.6) and (3.7), we see that b satisfies condition (III). □

4. Proof of Theorem 1.2

4.1. **Computation of the hole invariants.** The values of $s(\mathbf{c})$, $m(\mathbf{c})$, and $N(\mathbf{c})$ can be easily obtained from the set $P_{\mathbf{c}}$ of vertices of $\overline{P}_{\mathbf{c}}$. To calculate the value of $\theta(\mathbf{c})$ and the function $\sigma(\mathbf{c}, r)$, we use the following lemma.

Lemma 4.1. *Let $\mathbf{c}$ be a hole. Let $F_1, \dots, F_M$ be the 23-dimensional faces of $\overline{P}_{\mathbf{c}}$. Then each F_j is a 23-dimensional simplex.*

Proof. If $\mathbf{c}$ is shallow, then the convex polytope $\overline{P}_{\mathbf{c}}$ is a 24-dimensional simplex, and it has exactly 25 faces of dimension 23, each of which is obviously a simplex. Suppose that $\mathbf{c}$ is deep. We consider the decomposition (2.1) of $P_{\mathbf{c}}$. Note that $\overline{P}_{\mathbf{c},i}$ is an (n_i-1)-dimensional simplex in the (n_i-1)-dimensional affine space $\langle P_{\mathbf{c},i}\rangle$ containing $P_{\mathbf{c},i}$ for $i=1,\dots,m$, where $n_i=|P_{\mathbf{c},i}|$. If F is a 23-dimensional face of $\overline{P}_{\mathbf{c}}$, then the intersection $F\cap\langle P_{\mathbf{c},i}\rangle$ is an (n_i-2)-dimensional face of the simplex $\overline{P}_{\mathbf{c},i}$. Conversely, if $F^{(i)}$ is an (n_i-2)-dimensional face of the simplex $\overline{P}_{\mathbf{c},i}$ for $i=1,\dots,m$, then the convex hull F of the vertices of $F^{(1)},\dots,F^{(m)}$ is a 23-dimensional face of $\overline{P}_{\mathbf{c}}$. By Theorem 2.3, we see that the sum $\sum_i(n_i-1)$ of the numbers of the vertices of $F^{(1)},\dots,F^{(m)}$ is 24. Hence their convex hull F is a 23-dimensional simplex. □

The proof above also indicates a method to make the list of all 23-dimensional faces $F_1,\dots,F_M$ of $\overline{P}_{\mathbf{c}}$. Let $\mathbf{h}_j$ denote the point on $\langle F_j\rangle$ such that the line passing through $\mathbf{c}$ and $\mathbf{h}_j$ is perpendicular to $\langle F_j\rangle$. Then $\mathbf{h}_j$ lies in the interior of F_j, and F_j is circumscribed by a 22-dimensional sphere in the 23-dimensional affine space $\langle F_j\rangle$ with center $\mathbf{h}_j$ of radius

$$R_j := \sqrt{R(\mathbf{c})^2-\|\mathbf{h}_j-\mathbf{c}\|^2}.$$

Therefore we have

$$\theta(\mathbf{c}) = \max\{\, R_j \mid j=1,\dots,M \,\}, \tag{4.1}$$

$$\sigma(\mathbf{c},r) = \max(0,\sqrt{R(\mathbf{c})^2-\theta(\mathbf{c})^2}-\sqrt{r^2-\theta(\mathbf{c})^2}\,). \tag{4.2}$$

Example 4.2. Let $\mathbf{c}_1\in\Lambda_{\mathbb{R}}$ be the point such that

$$46\,\mathbf{c}_1=[15,-2,-1,-2,5,-1,-2,4,0,0,-6,12,-1,0,0,0,5,-4,-2,0,3,12,2,14].$$

Then $\mathbf{c}_1$ is a deep hole with $\tau(\mathbf{c}_1)=D_{24}$. We have $m(\mathbf{c}_1)=46$. The convex polytope $\overline{P}_{\mathbf{c}_1}$ is a 24-dimensional simplex, and its vertices are given in Table 4.1. The nodes of the graph $\Delta_{\mathbf{c}_1}$ correspond to these vertices in the way indicated in the graph in Table 4.1. Let F_j be the 23-dimensional face of $\overline{P}_{\mathbf{c}_1}$ that does not contain $\boldsymbol{\lambda}_j$. Then $\|\mathbf{h}_j-\mathbf{c}_1\|^2$ is calculated as in Table 4.2. Note that, by the symmetry of the simplex $\overline{P}_{\mathbf{c}_1}$, we have $\|\mathbf{h}_j-\mathbf{c}_1\|=\|\mathbf{h}_{26-j}-\mathbf{c}_1\|$ and $\|\mathbf{h}_1-\mathbf{c}_1\|=\|\mathbf{h}_2-\mathbf{c}_1\|$. Therefore we have

$$\theta(\mathbf{c}_1)^2=8647/4324.$$

In the list [24], we present the values of these invariants s, m, N, and θ^2.

4.2. **Definition of the set $\mathcal{S}(d)$.** For simplicity, we introduce three series of sets $\mathcal{S}_{\mathrm{I}}([\mathbf{c}],d)$, $\mathcal{S}_{\mathrm{II}}([\mathbf{c}],d)$, $\mathcal{S}_{\mathrm{III}}([\mathbf{c}],d)$ of positive integers, which correspond to the three possibilities in Proposition 3.2. Let $\mathbf{c}$ be a hole, and let d be an even positive integer. We put

$$\mathcal{S}_{\mathrm{I}}([\mathbf{c}],d):=\{\, b\in\mathbb{Z}_{>0} \mid b^2 \text{ divides } N(\mathbf{c})^2 d, \text{ and } b^2\le s(\mathbf{c})\,d \,\}.$$

$$
\begin{array}{rcl}
\boldsymbol{\lambda}_1 & = & [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \\
\boldsymbol{\lambda}_2 & = & [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0] \\
\boldsymbol{\lambda}_3 & = & [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1] \\
\boldsymbol{\lambda}_4 & = & [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0] \\
\boldsymbol{\lambda}_5 & = & [0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0] \\
\boldsymbol{\lambda}_6 & = & [1,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,1] \\
\boldsymbol{\lambda}_7 & = & [2,-1,-1,-1,0,0,0,0,-1,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0] \\
\boldsymbol{\lambda}_8 & = & [0,0,0,0,0,-1,-1,2,1,0,0,0,0,0,0,0,1,-1,0,0,-1,1,0,0] \\
\boldsymbol{\lambda}_9 & = & [-2,1,1,1,1,1,1,-2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1] \\
\boldsymbol{\lambda}_{10} & = & [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0] \\
\boldsymbol{\lambda}_{11} & = & [2,0,-1,-1,0,0,-1,1,0,0,-1,1,0,-1,1,0,0,-1,0,0,0,1,0,0] \\
\boldsymbol{\lambda}_{12} & = & [2,-1,0,0,-1,0,0,0,-1,0,0,0,-1,1,0,0,0,0,0,0,1,0,0,0] \\
\boldsymbol{\lambda}_{13} & = & [1,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1] \\
\boldsymbol{\lambda}_{14} & = & [0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0] \\
\boldsymbol{\lambda}_{15} & = & [-1,0,0,1,1,1,1,-1,0,-1,0,1,0,0,-1,0,0,1,-1,0,0,0,1,0] \\
\boldsymbol{\lambda}_{16} & = & [-3,1,1,0,1,0,0,1,1,1,0,0,1,-1,0,0,1,-1,0,0,-1,1,0,0] \\
\boldsymbol{\lambda}_{17} & = & [1,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1] \\
\boldsymbol{\lambda}_{18} & = & [-1,0,0,0,1,0,0,1,1,1,0,0,0,-1,0,0,0,-1,0,0,0,1,0,0] \\
\boldsymbol{\lambda}_{19} & = & [3,-1,0,-1,-1,-1,0,0,-1,-1,-1,1,0,1,0,0,0,0,1,0,0,0,0,0] \\
\boldsymbol{\lambda}_{20} & = & [-2,0,0,1,1,1,0,0,1,0,1,0,0,0,0,0,1,0,-2,0,0,0,0,1] \\
\boldsymbol{\lambda}_{21} & = & [5,-1,-2,-2,-1,-1,0,0,0,0,-1,1,0,0,0,-1,0,0,1,-1,2,-1,-1,2] \\
\boldsymbol{\lambda}_{22} & = & [-5,2,3,2,0,1,0,0,-1,1,1,-2,-1,0,0,2,-1,0,0,2,-2,2,2,-3] \\
\boldsymbol{\lambda}_{23} & = & [1,0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,-1,1] \\
\boldsymbol{\lambda}_{24} & = & [1,0,-1,-1,1,0,1,0,0,-1,-1,2,0,0,0,0,1,0,0,-2,0,0,0,1] \\
\boldsymbol{\lambda}_{25} & = & [4,-2,-2,-1,0,-1,-1,2,0,-1,-1,2,1,0,0,-2,0,0,0,0,0,0,0,1]
\end{array}
$$

Table 4.1. Vertices of $\overline{P}_{\mathbf{c}_1}$

j	1	3	4	5	6	7	8
$\lVert\mathbf{h}_j - \mathbf{c}_1\rVert^2$	1/4324	1/3312	1/2875	1/2484	1/2139	1/1840	1/1587

j	9	10	11	12	13
$\lVert\mathbf{h}_j - \mathbf{c}_1\rVert^2$	1/1380	1/1219	1/1104	1/1035	1/1012

.

Table 4.2. $\lVert\mathbf{h}_j - \mathbf{c}_1\rVert^2$

We put

$$\begin{aligned}\mathcal{T}(d) &:= \left\{ b \in \mathbb{Z}_{>0} \;\middle|\; 2-\frac{d}{b^2}<0 \right\} = \left\{ b \in \mathbb{Z}_{>0} \;\middle|\; b \le \sqrt{\frac{d}{2}} \right\}, \quad \text{and} \\ \mathcal{S}_{\mathrm{II}}([\mathbf{c}],d) &:= \mathcal{T}(d) \;\cup\; \left\{ b \in \mathbb{Z}_{>0}\setminus\mathcal{T}(d) \;\middle|\; \sqrt{2-\frac{d}{b^2}} \le \theta(\mathbf{c}) \right\} \\ &= \left\{ b \in \mathbb{Z}_{>0} \;\middle|\; b \le \sqrt{\frac{d}{2-\theta(\mathbf{c})^2}} \right\}.\end{aligned}$$

If $b \notin \mathcal{S}_{\mathrm{II}}([\mathbf{c}],d)$, then $\sigma(\mathbf{c}, \sqrt{2-d/b^2}\,)$ is defined. We put

$$\mathcal{S}_{\mathrm{III}}([\mathbf{c}],d) := \left\{ b \in \mathbb{Z}_{>0}\setminus\mathcal{S}_{\mathrm{II}}([\mathbf{c}],d) \;\middle|\; \sigma\left(\mathbf{c}, \sqrt{2-\frac{d}{b^2}}\right) \ge \frac{2}{m(\mathbf{c})b} \right\}.$$

Consider the rational function

$$\psi_{\mathbf{c}}(t) := \left(\sqrt{R(\mathbf{c})^2-\theta(\mathbf{c})^2} - \frac{2}{m(\mathbf{c})\,t}\right)^2 - \left(2-\frac{d}{t^2}-\theta(\mathbf{c})^2\right)$$

of t. By (4.2), we see that a positive real number t_0 satisfying $\sqrt{2-d/t_0^2} \ge \theta(\mathbf{c})$ satisfies

$$\sigma\left(\mathbf{c}, \sqrt{2-\frac{d}{t_0^2}}\right) \ge \frac{2}{m(\mathbf{c})t_0}$$

if and only if $\psi_{\mathbf{c}}(t_0)$ is non-negative and

$$\sqrt{R(\mathbf{c})^2-\theta(\mathbf{c})^2} - \frac{2}{m(\mathbf{c})\,t_0} \ge 0$$

holds. We put

$$\Psi_{\mathbf{c}}(t) := t^2\,\psi_{\mathbf{c}}(t) = \left(\frac{4}{m(\mathbf{c})^2}+d\right) - \frac{4\sqrt{R(\mathbf{c})^2-\theta(\mathbf{c})^2}}{m(\mathbf{c})}t + (R(\mathbf{c})^2-2)\,t^2.$$

Note that $\Psi_{\mathbf{c}}$ is a strictly decreasing linear function of t having a positive root $\beta(\mathbf{c},d)$ if $\mathbf{c}$ is deep, whereas $\Psi_{\mathbf{c}}$ is an upward convex quadratic function of t having a negative root $\alpha(\mathbf{c},d)$ and a positive root $\beta(\mathbf{c},d)$ if $\mathbf{c}$ is shallow. Hence we have

$$\mathcal{S}_{\mathrm{III}}([\mathbf{c}],d) = \left\{ b \in \mathbb{Z}_{>0}\setminus\mathcal{S}_{\mathrm{II}}([\mathbf{c}],d) \;\middle|\; \frac{2}{m(\mathbf{c})\sqrt{R(\mathbf{c})^2-\theta(\mathbf{c})^2}} \le b \le \beta(\mathbf{c},d) \right\}.$$

In terms of the invariants s, m, and θ^2, the function $\beta(\mathbf{c},d)$ is given as follows:

$$\beta(\mathbf{c},d) = \frac{d\,m(\mathbf{c})^2+4}{4\,m(\mathbf{c})\sqrt{2-\theta(\mathbf{c})^2}} \tag{4.3}$$

when $\mathbf{c}$ is deep, whereas

$$\beta(\mathbf{c},d) = \frac{\sqrt{4\,s(\mathbf{c})^2\,(2-\theta(\mathbf{c})^2)+d\,s(\mathbf{c})\,m(\mathbf{c})^2} - \sqrt{4\,s(\mathbf{c})^2\,(2-\theta(\mathbf{c})^2)-4\,s(\mathbf{c})}}{m(\mathbf{c})}$$

when $\mathbf{c}$ is shallow.

Example 4.3. Let $\mathbf{c}_1$ be the deep hole with $\tau(\mathbf{c}_1) = D_{24}$ given in Example 4.2. Recall that we have $m(\mathbf{c}_1) = 46$ and $2 - \theta(\mathbf{c}_1)^2 = 1/4324$. By (4.3), we see that $\beta(\mathbf{c}_1, d)$ is equal to the function $\phi(d)$ given in the statement of Theorem 1.2. On the other hand, we have

$$\frac{2}{m(\mathbf{c}_1)\sqrt{R(\mathbf{c}_1)^2 - \theta(\mathbf{c}_1)^2}} = \frac{2}{23}\sqrt{1081} = 2.859\ldots.$$

Hence we have

$$\mathcal{S}_{\mathrm{II}}([\mathbf{c}_1], d) \cup \mathcal{S}_{\mathrm{III}}([\mathbf{c}_1], d) = \{\ b \in \mathbb{Z}_{>0} \ \mid\ b \le \phi(d)\ \}.$$

Finally, we put

$$\mathcal{S}(d) := \bigcup_{[\mathbf{c}]} \Big(\ \mathcal{S}_{\mathrm{I}}([\mathbf{c}], d) \cup \mathcal{S}_{\mathrm{II}}([\mathbf{c}], d) \cup \mathcal{S}_{\mathrm{III}}([\mathbf{c}], d)\ \Big),$$

where $[\mathbf{c}]$ ranges through the set of all equivalence classes of holes. Then Proposition 3.2 can be rephrased as follows:

Proposition 4.4. *Let $w \in \mathbf{L}$ be a Weyl vector, and let d be an even positive integer. Then, for any vector $v \in \mathcal{D}(w) \cap \mathbf{L}$ with $\langle v, v\rangle_{\mathbf{L}} = d$, we have $\langle v, w\rangle_{\mathbf{L}} \in \mathcal{S}(d)$.*

4.3. **Proof of Theorem 1.2.** We compare the sets $\mathcal{S}_{\mathrm{I}}([\mathbf{c}], d), \mathcal{S}_{\mathrm{II}}([\mathbf{c}], d), \mathcal{S}_{\mathrm{III}}([\mathbf{c}], d)$ and prove Theorem 1.2. After the comparison, it turns out that the the set $\mathcal{S}_{\mathrm{III}}([\mathbf{c}_1], d)$ given by the deep hole $\mathbf{c}_1$ of type D_{24} is the largest.

Theorem 1.2 follows from Proposition 4.4 by the following lemma.

Lemma 4.5. *The set $\mathcal{S}(d)$ coincides with $\{b \in \mathbb{Z}_{>0} \mid b \le \phi(d)\}$.*

Proof. The fact that $\mathcal{S}(d)$ includes $\{b \in \mathbb{Z}_{>0} \mid b \le \phi(d)\}$ follows from Example 4.3. In order to show the opposite inclusion, we prove the following claims.

Claim 4.6. If $b \in \mathcal{S}_{\mathrm{I}}([\mathbf{c}], d)$, then $b \le \phi(d)$.

We put

$$\mu_{\mathbf{c}} := \min(\, N(\mathbf{c}), \sqrt{s(\mathbf{c})}\,).$$

Then $\mathcal{S}_{\mathrm{I}}([\mathbf{c}], d)$ is included in $\{b \in \mathbb{Z}_{>0} \mid b \le \mu_{\mathbf{c}}\sqrt{d}\}$. Since $\sqrt{d} < d$ for any even positive integer d and $\phi(0) > 0$, Claim 4.6 follows from

$$\mu_{\mathbf{c}} < \frac{529\sqrt{1081}}{23} = 756.20\cdots,$$

which can be confirmed by numerical computation for each equivalence class $[\mathbf{c}]$.

Claim 4.7. If $b \in \mathcal{S}_{\mathrm{II}}([\mathbf{c}], d)$, then $b \le \beta(\mathbf{c}, d)$.

This claim follows from

$$\Psi_{\mathbf{c}}\left(\sqrt{\frac{d}{2 - \theta(\mathbf{c})^2}}\right) = \left(\sqrt{\frac{R(\mathbf{c})^2 - \theta(\mathbf{c})^2}{2 - \theta(\mathbf{c})^2}}\sqrt{d} - \frac{2}{m}\right)^2 \ge 0.$$

Claim 4.8. Suppose that $[\mathbf{c}] \ne [\mathbf{c}_1]$. Then $\beta(\mathbf{c}, d) \le \phi(d)$ holds for all even positive integers d.

i	$\operatorname{disc} T_i$	T_i			$\langle h_i, h_i \rangle_S$	References
		a	b	c		
1	3	2	1	2	78	[29]
2	4	2	0	2	55	[29]
3	7	2	1	4	28	[28]
4	8	2	0	4	61/2	[23]
5	12	2	0	6	18	[23]
6	12	4	2	4	16	[8]
7	15	2	1	8	12	[23], [25]
8	16	4	0	4	10	[8]
9	20	4	2	6	11	
10	24	2	0	12	15/2	[25]
11	36	6	0	6	5	[25]

TABLE 5.1. Singular $K3$ surfaces of simple Borcherds type

Suppose that $\mathbf{c}$ is deep. Then $\beta(\mathbf{c}, d)$ is a linear function of d, and hence we can write it as $f(\mathbf{c})\, d + g(\mathbf{c})$. We have $f(\mathbf{c}) > 0$. Hence the hoped-for inequality $\beta(\mathbf{c}, d) \le \beta(\mathbf{c}_1, d)$ follows from

$$f(\mathbf{c}) < f(\mathbf{c}_1) = \frac{529\sqrt{1081}}{23} \quad \text{and} \quad -\frac{g(\mathbf{c}) - g(\mathbf{c}_1)}{f(\mathbf{c}) - f(\mathbf{c}_1)} < 2,$$

which we can confirm by numerical computation again. Suppose that $\mathbf{c}$ is shallow. In order to prove $\beta(\mathbf{c}, d) \le \phi(d)$, it is enough to show that $\Psi_{\mathbf{c}}(\phi(d)) \le 0$. Since $\Psi_{\mathbf{c}}(\phi(d))$ is a quadratic polynomial in d, and its coefficient of d^2 is negative, we can prove $\Psi_{\mathbf{c}}(\phi(d)) \le 0$ for any even positive integer d by showing that the quadratic equation $\Psi_{\mathbf{c}}(\phi(x)) = 0$ in variable x has no roots larger than 2.

Combining these three claims, we complete the proof of Lemma 4.5 and hence that of Theorem 1.2. $\square$

5. EXAMPLES AND REMARKS

We continue the list of polarized $K3$ surfaces (X, h) of simple Borcherds type in Example 1.9.

A complex $K3$ surface X is said to be *singular* if S_X is of rank 20. For a singular $K3$ surface X, the orthogonal complement of S_X in $H_X = H^2(X, \mathbb{Z})$ is called the *transcendental lattice* of X. By [26], we see that, for each even positive-definite lattice T_i of rank 2 whose Gram matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

is given in Table 5.1, there exists a singular $K3$ surface X_i, unique up to isomorphism, such that the transcendental lattice of X_i is isomorphic to T_i. Then X_i possesses an ample class h_i such that (X_i, h_i) is of simple Borcherds type. The automorphism group $\operatorname{Aut}(X_i)$ of each X_i has been determined in the papers cited in Table 5.1.

In [6], it was shown that the generic quartic Hessian surface X possesses an ample class $h \in S_X \otimes \mathbb{Q}$ with $h^2 = 20$ such that (X, h) is of simple Borcherds type. In this case, we have $\operatorname{rank} S_X = 16$.

In [8], it was shown that the complex Kummer surface $\mathrm{Km}(E \times E)$, where E is a generic elliptic curve, possesses an ample class $h \in S_X \otimes \mathbb{Q}$ with $h^2 = 19$ such that (X, h) is of simple Borcherds type. In this case, we have $\operatorname{rank} S_X = 19$.

Remark 5.1. In [5], it was shown that the supersingular $K3$ surface X in characteristic 2 with Artin invariant 1 possesses an ample class $h \in S_X \otimes \mathbb{Q}$ with $h^2 = 14$ such that Corollary 1.8 holds for (X, h).

Remark 5.2. There exists a singular $K3$ surface X, unique up to isomorphism, such that its transcendental lattice is of discriminant 11. We showed in [23] that there exists a primitive embedding $S_X \hookrightarrow \mathbf{L}$ satisfying Assumption 1.3 and $\mathcal{P}(X) \subset \mathcal{P}_{\mathbf{L}}$ such that the number of G_X-congruence classes of induced chambers is 1098.

Remark 5.3. In all known examples of polarized $K3$ surfaces (X, h) of simple Borcherds type, the orthogonal complement R of S_X in $\mathbf{L}$ contains a sublattice of finite index generated by the set $\mathcal{R}_R$ of vectors of R with square norm -2. See [1, Lemma 5.1] and [23, Remark 6.7].

Remark 5.4. Let $S_X \hookrightarrow \mathbf{L}$ be a primitive embedding satisfying Assumption 1.3 and $\mathcal{P}(X) \subset \mathcal{P}_{\mathbf{L}}$, and let $a := \mathrm{pr}_S(w)$ be the image of a Weyl vector $w \in \mathbf{L}$ by the orthogonal projection $\mathrm{pr}_S : \mathbf{L} \to S_X^{\vee}$. We show that $\langle a, a \rangle_S > 0$. Since the orthogonal complement R of S_X in $\mathbf{L}$ is negative-definite, we have $\langle a, a \rangle_S \geq \langle w, w \rangle_{\mathbf{L}} = 0$, and the equality holds if and only if $a = w$. Therefore, if $\langle a, a \rangle_S = 0$, then we have $w \in S_X$, and hence $\langle w \rangle^{\perp} / \langle w \rangle \cong \Lambda^{-}$ contains R, which contradicts condition (b) in Assumption 1.3 .

References

[1] Richard Borcherds. Automorphism groups of Lorentzian lattices. *J. Algebra*, 111(1):133–153, 1987.

[2] Richard E. Borcherds. Coxeter groups, Lorentzian lattices, and $K3$ surfaces. *Internat. Math. Res. Notices*, 1998(19):1011–1031, 1998.

[3] J. H. Conway. The automorphism group of the 26-dimensional even unimodular Lorentzian lattice. *J. Algebra*, 80(1):159–163, 1983.

[4] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.

[5] I. Dolgachev and S. Kondō. A supersingular $K3$ surface in characteristic 2 and the Leech lattice. *Int. Math. Res. Not.*, 2003(1):1–23, 2003.

[6] Igor Dolgachev and Jonghae Keum. Birational automorphisms of quartic Hessian surfaces. *Trans. Amer. Math. Soc.*, 354(8):3031–3057 (electronic), 2002.

[7] The GAP Group. GAP - Groups, Algorithms, and Programming. Version 4.7.9; 2015 (http://www.gap-system.org).

[8] Jonghae Keum and Shigeyuki Kondō. The automorphism groups of Kummer surfaces associated with the product of two elliptic curves. *Trans. Amer. Math. Soc.*, 353(4):1469–1487 (electronic), 2001.

[9] Donald E. Knuth. Estimating the efficiency of backtrack programs. *Math. Comp.*, 29:122–136, 1975. Collection of articles dedicated to Derrick Henry Lehmer on the occasion of his seventieth birthday.

[10] Shigeyuki Kondō. The automorphism group of a generic Jacobian Kummer surface. *J. Algebraic Geom.*, 7(3):589–609, 1998.

[11] Shigeyuki Kondō and Ichiro Shimada. The automorphism group of a supersingular $K3$ surface with Artin invariant 1 in characteristic 3. *Int. Math. Res. Not. IMRN*, 2014(7):1885–1924, 2014.

[12] Abhinav Kumar. Elliptic fibrations on a generic Jacobian Kummer surface. *J. Algebraic Geom.*, 23(4):599–667, 2014.
[13] Masato Kuwata and Tetsuji Shioda. Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface. In *Algebraic geometry in East Asia—Hanoi 2005*, volume 50 of *Adv. Stud. Pure Math.*, pages 177–215. Math. Soc. Japan, Tokyo, 2008.
[14] Max Lieblich and Davesh Maulik. A note on the cone conjecture for $K3$ surfaces in positive characteristic, 2011. preprint, arXiv:1102.3377.
[15] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979. English translation: Math USSR-Izv. 14 (1979), no. 1, 103–167 (1980).
[16] Ken-ichi Nishiyama. The Jacobian fibrations on some $K3$ surfaces and their Mordell-Weil groups. *Japan. J. Math. (N.S.)*, 22(2):293–347, 1996.
[17] Keiji Oguiso. On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves. *J. Math. Soc. Japan*, 41(4):651–680, 1989.
[18] Arthur Ogus. Supersingular $K3$ crystals. In *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II*, volume 64 of *Astérisque*, pages 3–86. Soc. Math. France, Paris, 1979.
[19] Arthur Ogus. A crystalline Torelli theorem for supersingular $K3$ surfaces. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 361–394. Birkhäuser Boston, Boston, MA, 1983.
[20] I. I. Piatetski-Shapiro and I. R. Shafarevich. Torelli's theorem for algebraic surfaces of type K3. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:530–572, 1971. Reprinted in I. R. Shafarevich, Collected Mathematical Papers, Springer-Verlag, Berlin, 1989, pp. 516–557.
[21] Vera Pless. On the uniqueness of the Golay codes. *J. Combinatorial Theory*, 5:215–228, 1968.
[22] Tathagata Sengupta. *Supersingular K3 Surfaces*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–Brandeis University.
[23] Ichiro Shimada. An algorithm to compute automorphism groups of $K3$ surfaces and an application to singular $K3$ surfaces. *Int. Math. Res. Not. IMRN*, (22):11961–12014, 2015.
[24] Ichiro Shimada. The list of holes of the Leech lattice, 2016. http://www.math.sci.hiroshima-u.ac.jp/~shimada/Leech.html.
[25] Ichiro Shimada. The automorphism groups of certain singular $K3$ surfaces and an Enriques surface. In *$K3$ surfaces and their moduli*, volume 315 of *Progr. Math.*, pages 297–343. Birkhäuser/Springer, Basel, 2016.
[26] T. Shioda and H. Inose. On singular $K3$ surfaces. In *Complex analysis and algebraic geometry*, pages 119–136. Iwanami Shoten, Tokyo, 1977.
[27] Hans Sterk. Finiteness results for algebraic $K3$ surfaces. *Math. Z.*, 189(4):507–513, 1985.
[28] Masashi Ujikawa. The automorphism group of the singular $K3$ surface of discriminant 7. *Comment. Math. Univ. St. Pauli*, 62(1):11–29, 2013.
[29] È. B. Vinberg. The two most algebraic $K3$ surfaces. *Math. Ann.*, 265(1):1–21, 1983.

Appendix A. Reconfirmation of the enumeration of holes

This appendix is a detailed version of Remark 2.10. In the following, TABLE means Table 25.1 of [4, Chapter 25] calculated by Borcherds, Conway, and Queen. In TABLE, the equivalence classes of holes of the Leech lattice Λ are enumerated. The purpose of this appendix is to explain a method to reconfirm the correctness of TABLE.

The fact that there exist at least $23 + 284$ equivalence classes of holes can be established by giving explicitly the set $P_{\mathbf{c}}$ of vertices of the polytope $\overline{P}_{\mathbf{c}}$ for a representative $\mathbf{c}$ of each equivalence class $[\mathbf{c}]$. See Remark 3.1 and the computational data given in the author's web page [24]. (See also Appendix B.)

In order to see that there exist no other equivalence classes, Borcherds, Conway, and Queen used the volume formula (2.5). The volume $\mathrm{vol}(\overline{P}_{\mathbf{c}})$ of $\overline{P}_{\mathbf{c}}$ can be easily calculated from the set $P_{\mathbf{c}}$ of vertices, and the result coincides with the values given in the third column of TABLE. The equality (2.5) holds when $|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)|$ is replaced by the value $g = g(\mathbf{c})$ given in the second column of TABLE and the

no.	type	α	β	ν	$\lvert\mathrm{Aut}(\overline{P}_{\mathbf{c}})\rvert$	$g(\mathbf{c})$
293	$a_5 a_2^{10}$	a_5	a_2	10	$2 \cdot 2^{10} \cdot 10!$	720
299	$d_4 a_1^{21}$	d_4	a_1	21	$6 \cdot 21!$	120960
303	$a_3 a_2^{11}$	a_3	a_2	11	$2 \cdot 2^{11} \cdot 11!$	7920
304	$a_3 a_1^{22}$	a_3	a_1	22	$2 \cdot 22!$	887040
305	$a_1 a_2^{12}$	a_1	a_2	12	$2^{12} \cdot 12!$	190080
306	$a_2 a_1^{23}$	a_2	a_1	23	$2 \cdot 23!$	10200960
307	a_1^{25}	a_1	a_1	24	$25!$	244823040

TABLE A.1. Shallow holes with large $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$

summation is taken over the set of the equivalence classes of holes listed in TABLE. Therefore, in order to show the completeness of TABLE, it is enough to prove the inequality

$$|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)| \;\le\; g(\mathbf{c}) \tag{A.1}$$

for each hole $\mathbf{c}$ that appears in TABLE. The groups $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ for deep holes are studied in detail in [4, Chapters 23 and 24]. Hence we will prove the inequality (A.1) for shallow holes $\mathbf{c}$.

Let $\mathbf{c}$ be a shallow hole that appears in TABLE. Then $\overline{P}_{\mathbf{c}}$ is a 24-dimensional simplex, and $P_{\mathbf{c}}$ consists of 25 points of Λ. Recall that $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is the group of permutations g of $P_{\mathbf{c}}$ such that $\|p^g - q^g\| = \|p - q\|$ holds for any $p, q \in P_{\mathbf{c}}$. Each permutation $g \in \mathrm{Aut}(\overline{P}_{\mathbf{c}})$ induces an affine isometry $g_\Lambda \colon \Lambda \otimes \mathbb{Q} \xrightarrow{\sim} \Lambda \otimes \mathbb{Q}$, and we have

$$g \in \mathrm{Aut}(P_{\mathbf{c}}, \Lambda) \iff g_\Lambda \text{ preserves } \Lambda \subset \Lambda \otimes \mathbb{Q}. \tag{A.2}$$

When $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is not very large, we can make the list of elements of $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ by the criterion (A.2). We can also use the following trick to reduce the amount of the computation.

Example A.1. Consider the shallow hole $\mathbf{c}_{297}$ of type $d_4^4 a_1^9$. We have $|\mathrm{Aut}(\overline{P}_{\mathbf{c}})| = 6^4 \cdot 4! \cdot 9! = 11287019520$. We choose two vertices v_1 and v_2 that correspond to nodes of two a_1 in $d_4^4 a_1^9$, and consider the subgroup $\mathrm{Stab}(v_1, v_2)$ of $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ consisting of permutations that fix each of v_1 and v_2. Then the index of $\mathrm{Stab}(v_1, v_2)$ in $\mathrm{Aut}(\overline{P}_{\mathbf{c}})$ is at most 72. We see by the criterion (A.2) that $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda) \cap \mathrm{Stab}(v_1, v_2)$ is of order 6, and hence $|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)|$ is at most $72 \times 6 = 432 = g(\mathbf{c}_{297})$. In fact, $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is isomorphic to $(((C_3 \times C_3) : Q_8) : C_3) : C_2$, where C_n is the cyclic group of order n and Q_8 is the quaternion group.

This brute-force method works for shallow holes except for the seven cases listed in Table A.1.

A.1. **Golay codes and Mathieu groups.** The values $g(\mathbf{c})$ in Table A.1 suggest that the groups $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ are related to Mathieu groups. (See Table A.2.) For each shallow hole $\mathbf{c}$ in Table A.1, we construct a code that is related to a Golay code, and clarify the relation between $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ and the corresponding Mathieu group.

Remark A.2. In Remarks (ii) of [4, Chapter 25], it is stated that $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is isomorphic to the Mathieu group M_{24} for the shallow hole $\mathbf{c}_{307}$ of type a_1^{25}.

$$\begin{array}{rcrcl}
|M_{21}| & = & 20160 & = & g(\mathbf{c}_{299})/6 \\
|M_{22}| & = & 443520 & = & g(\mathbf{c}_{304})/2 \\
|M_{23}| & = & 10200960 & = & g(\mathbf{c}_{306}) \\
|M_{24}| & = & 244823040 & = & g(\mathbf{c}_{307}) \\
|M_{11}| & = & 7920 & = & g(\mathbf{c}_{303}) \\
|M_{12}| & = & 95040 & = & g(\mathbf{c}_{305})/2
\end{array}$$

TABLE A.2. Orders of Mathieu groups

We fix notions and notation about codes, and recall the definitions of Golay codes and Mathieu groups. Let $\mathbb{F}$ be either $\mathbb{F}_2$ or $\mathbb{F}_3$, and let l be a positive integer. A *code of length* l over $\mathbb{F}$ is a linear subspace of $\mathbb{F}^l$. Let C be a code of length l. When $\mathbb{F} = \mathbb{F}_2$, we say that C is *binary*, and when $\mathbb{F} = \mathbb{F}_3$, we say that C is *ternary*. When $\dim C = d$, we say that C is an (l, d)-code. Each element of C is called a *codeword*. The *weight* $\mathrm{wt}(x)$ of a codeword $x = (x_1, \dots, x_l)$ is defined to be the cardinality of $\{i \mid x_i \neq 0\}$. The *minimal weight* of C is the minimum of $\{\mathrm{wt}(x) \mid x \in C \setminus \{0\}\}$. The *weight distribution of a code* C is the expression

$$0^1\, w_1^{n_1}\, w_2^{n_2}\, \dots\, w_m^{n_m}$$

that indicates that C contains exactly n_i codewords of weight w_i for $i = 1, \dots, m$, where $0, w_1, \dots, w_m$ are distinct weights, and that $|C| = 1 + n_1 + \dots + n_m$ holds.

For a linear subspace V of $\mathbb{F}^l$, the intersection $C \cap V$ is also a code of length l. For a positive integer $k < l$, let $\mathrm{pr}_k \colon \mathbb{F}^l \to \mathbb{F}^k$ denote the projection

$$(x_1, \dots, x_l) \mapsto (x_1, \dots, x_k).$$

Then $\mathrm{pr}_k(C)$ is a code of length k.

Let $\mathcal{G}_l$ denote the subgroup of $GL_l(\mathbb{F})$ consisting of monomial transformations, that is, $\mathcal{G}_l$ is the group of linear automorphisms of $\mathbb{F}^l$ generated by permutations of coordinates and multiplications by a non-zero scalar on one coordinate. When $\mathbb{F} = \mathbb{F}_2$, we have $\mathcal{G}_l \cong \mathfrak{S}_l$, and when $\mathbb{F} = \mathbb{F}_3$, we have $\mathcal{G}_l \cong \{\pm 1\}^l \rtimes \mathfrak{S}_l$. The automorphism group of a code C of length l is defined to be

$$\mathrm{Aut}(C) := \{\, g \in \mathcal{G}_l \;\mid\; C^g = C \,\}.$$

Two codes C and C' of length l are said to be *equivalent* if there exists a monomial transformation $g \in \mathcal{G}_l$ such that $C' = C^g$. The weight distribution and the isomorphism class of the automorphism group depend only on the equivalence class of codes.

The *binary Golay code* $\mathcal{C}_{24}$ is the binary $(24, 12)$-code generated by the row vectors of the matrix in Table A.3. The *ternary Golay code* $\mathcal{C}_{12}$ is the ternary $(12, 6)$-code generated by the row vectors of the matrix in Table A.4. We have the following theorem, which will be used frequently in the next section.

Theorem A.3 (Pless [21]). (1) *Let C be a binary $(24, 12)$-code. Then the following conditions are equivalent:*

- *C is equivalent to the binary Golay code $\mathcal{C}_{24}$,*
- *the minimal weight of C is 8, and*
- *the weight distribution of C is $0^1\, 8^{759}\, 12^{2576}\, 16^{759}\, 24^1$.*

(2) *Let C be a ternary $(12, 6)$-code. Then the following conditions are equivalent:*

$$
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&1&1&0&0&0&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&1&0&1&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&1&0&1&1\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&1&0&0&0&1&1&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&1&0&0&1&0&1
\end{bmatrix}
$$

TABLE A.3. A basis of $\mathcal{C}_{24}$

$$
\begin{bmatrix}
1&0&0&0&0&0&0&1&1&1&1&1\\
0&1&0&0&0&0&2&0&1&2&2&1\\
0&0&1&0&0&0&2&1&0&1&2&2\\
0&0&0&1&0&0&2&2&1&0&1&2\\
0&0&0&0&1&0&2&2&2&1&0&1\\
0&0&0&0&0&1&2&1&2&2&1&0
\end{bmatrix}
$$

TABLE A.4. A basis of $\mathcal{C}_{12}$

- *C is equivalent to the ternary Golay code $\mathcal{C}_{12}$,*
- *the minimal weight of C is 6, and*
- *the weight distribution of C is $0^1\, 6^{264}\, 9^{440}\, 12^{24}$.*

Let $\mathbb{F}$ be $\mathbb{F}_2$. The automorphism group of $\mathcal{C}_{24}$ is the Mathieu group M_{24}. As a subgroup of the full symmetric group $\mathfrak{S}_{24}$ of the set $\{x_1, \dots, x_{24}\}$ of coordinate positions of $\mathbb{F}_2^{24}$, the Mathieu group M_{24} is 5-transitive. For a positive integer $k < 24$, let $\mathfrak{S}_k$ denote the subgroup of $\mathfrak{S}_{24}$ consisting of permutations that fix each of $x_{k+1}, \dots, x_{24}$. For $k = 21, 22, 23$, we define the Mathieu group M_k by

$$M_k := M_{24} \cap \mathfrak{S}_k.$$

Let $\mathbb{F}$ be $\mathbb{F}_3$. We have a natural homomorphism from $\mathcal{G}_{12}$ to the full symmetric group $\mathfrak{S}_{12}$ of the set $\{x_1, \dots, x_{12}\}$ of coordinate positions of $\mathbb{F}_3^{12}$. The image of $\mathrm{Aut}(\mathcal{C}_{12})$ by this homomorphism is the Mathieu group M_{12}. The kernel of the projection $\mathrm{Aut}(\mathcal{C}_{12}) \to M_{12}$ is of order 2 and is generated by the scalar multiplication by -1. The action of M_{12} on $\{x_1, \dots, x_{12}\}$ is 5-transitive. The stabilizer subgroup of x_{12} in M_{12} is the Mathieu group M_{11}.

A.2. **Construction of a code.** Let $[\mathbf{c}]$ be one of the equivalence classes listed in Table A.1. The hole type $\tau(\mathbf{c})$ is of the form $\alpha\beta^\nu$, where α, β, and ν are given in Table A.1. We put

$$
\begin{aligned}
&p = 2, \ \ \mathbb{F} = \mathbb{F}_2, \ \ \text{when } \beta = a_1, \quad \text{and} \\
&p = 3, \ \ \mathbb{F} = \mathbb{F}_3, \ \ \text{when } \beta = a_2.
\end{aligned}
$$

We consider the case $\mathbf{c} \neq \mathbf{c}_{307}$. (The case $\mathbf{c} = \mathbf{c}_{307}$ will be treated in Section A.4.) We decompose $P_{\mathbf{c}}$ to the disjoint union of A and B, where the vertices in A correspond to the nodes of α and the vertices in B correspond to the nodes of β^ν. Since $\alpha \neq \beta$, we have a direct product decomposition

$$\mathrm{Aut}(\overline{P}_{\mathbf{c}}) = \mathrm{Aut}(A) \times \mathrm{Aut}(B),$$

where $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ are the groups of symmetries of the Coxeter–Dynkin diagrams α of A and β^ν of B, respectively. Since $\mathrm{Aut}(A)$ is very small, we can easily calculate $\mathrm{Aut}(A) \cap \mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ by the criterion (A.2). It turns out that, in all cases, the group $\mathrm{Aut}(A) \cap \mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is trivial. Therefore the second projection $\mathrm{Aut}(\overline{P}_{\mathbf{c}}) \to \mathrm{Aut}(B)$ embeds $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ into $\mathrm{Aut}(B)$. We denote by

$$\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda) \subset \mathrm{Aut}(B)$$

the image of $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$. For the proof of the inequality (A.1), it is enough to show that the order of $\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)$ is at most $g(\mathbf{c})$.

Let $\langle A\rangle$ and $\langle B\rangle$ denote the minimal affine subspaces of $\Lambda_{\mathbb{R}}$ that contain A and B, respectively. We have

$$\dim\langle A\rangle = |A|-1, \quad \dim\langle B\rangle = |B|-1, \quad \dim\langle A\rangle + \dim\langle B\rangle = 23, \quad \langle A\rangle \cap \langle B\rangle = \emptyset.$$

Let $\Lambda_{\mathbb{R}}/\langle A\rangle$ be the quotient of $\Lambda_{\mathbb{R}}$ by the equivalence relation

$$x \sim y \iff a + x - y \in \langle A\rangle \text{ for one (and hence all) } a \in \langle A\rangle,$$

that is, we have $x \sim y$ if and only if $x - y$ is parallel to $\langle A\rangle$. We denote by

$$\rho:\ \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}/\langle A\rangle$$

the quotient map. Then $\Lambda_{\mathbb{R}}/\langle A\rangle$ has a natural structure of the linear space of dimension $|B|$ over $\mathbb{R}$ with $\rho(\langle A\rangle)$ being the origin, and

$$L := \rho(\Lambda)$$

is a discrete $\mathbb{Z}$-submodule of $\Lambda_{\mathbb{R}}/\langle A\rangle$ with full rank. Let M denote the $\mathbb{Z}$-submodule of $\Lambda_{\mathbb{R}}/\langle A\rangle$ generated by $\rho(B)$. Then M is also a discrete $\mathbb{Z}$-submodule with full rank, and is equipped with a canonical basis $\{\rho(b) \mid b \in B\}$. It is obvious that M is contained in L. Therefore we have

$$M \subset L \subset M \otimes \mathbb{Q}.$$

Note that $\mathrm{Aut}(B)$ acts on M naturally, and that each element of the subgroup $\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)$ of $\mathrm{Aut}(B)$ preserves $L \subset M \otimes \mathbb{Q}$.

Let n denote the least positive integer such that $nL \subset M$. Then we have a submodule nL/nM of $M/nM = (\mathbb{Z}/n\mathbb{Z})^B$. It turns out that n is divisible by p. We define a submodule F of M/nM as follows.

- When $\beta = a_1$, we put $\tilde{b} := (n/2)\, b$, and
$$F := \bigoplus_{b \in B} (\mathbb{Z}/n\mathbb{Z})\, \tilde{b}.$$
- Suppose that $\beta = a_2$. We label the elements of B as $b_1, b_1', \dots, b_\nu, b_\nu'$ in such a way that the nodes corresponding to b_i and b_i' are connected in the Coxeter–Dynkin diagram a_2^ν. We then put $\tilde{b}_i := (n/3)\, b_i + (2n/3)\, b_i'$, and
$$F := \bigoplus_{i=1}^{\nu} (\mathbb{Z}/n\mathbb{Z})\, \tilde{b}_i.$$

Note that F does not change even if we interchange b_i and b'_i, because we have $(n/3)\,(b_i+2b'_i) = -(n/3)\,(2b_i+b'_i)$ in M/nM.

Then we have $F = \mathbb{F}^\nu$. We define a code Γ of length ν over $\mathbb{F}$ by

$$\Gamma := (nL/nM) \cap F.$$

The group $\mathrm{Aut}(B)$ acts on F, and is identified with the group $\mathcal{G}_\nu$ of monomial transformations of $\mathbb{F}^\nu$. (When $\beta = \alpha_2$, the transposition of b_i and b'_i corresponds to the multiplication by -1 on the ith coordinate of $\mathbb{F}^\nu$.) Under this identification, we have

$$\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda) \subset \mathrm{Aut}(\Gamma).$$

In the next section, we describe this code Γ explicitly, and derive an upper bound of $|\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)| = |\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)|$ from $\mathrm{Aut}(\Gamma)$.

A.3. **Description of the code Γ.**

A.3.1. *The shallow hole* $\mathbf{c}_{293}$ *of type* $a_5 a_2^{10}$. In this case, we have $n = 15$. The ternary code Γ is a $(10, 5)$-code with weight distribution

$$0^1\, 4^{30}\, 6^{60}\, 7^{120}\, 9^{20}\, 10^{12}.$$

It turns out that Γ is equivalent to the code $\mathrm{pr}_{10}(\mathcal{C}_{12} \cap V)$, where V is the linear subspace of $\mathbb{F}_3^{12}$ defined by $x_{11} + x_{12} = 0$. We can calculate its automorphism group directly, and see that $\mathrm{Aut}(\Gamma)$ is of order 1440. Hence $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is contained in the group $\mathrm{Aut}(A) \times \mathrm{Aut}(\Gamma)$ of order 2880. We calculate $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ by applying the criterion (A.2) to these 2880 elements. Then we see that $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is isomorphic to the symmetric group of degree 6, and hence its order is $g(\mathbf{c}_{293}) = 720$.

A.3.2. *The shallow hole* $\mathbf{c}_{299}$ *of type* $d_4 a_1^{21}$. In this case, we have $n = 14$. The binary code Γ is a $(21, 11)$-code with weight distribution

$$0^1\, 6^{168}\, 8^{210}\, 10^{1008}\, 12^{280}\, 14^{360}\, 16^{21}.$$

We construct a linear embedding

$$\iota\colon \Gamma \hookrightarrow \mathbb{F}_2^{24}$$

such that $\mathrm{pr}_{21} \circ \iota$ is the identity map of Γ, and that every codeword of the image $\Gamma' := \iota(\Gamma)$ is of weight 0, 8, 12, or 16. Let $\beta_1, \dots, \beta_{11}$ be a basis of Γ. We define $\beta'_i \in \mathbb{F}_2^{24}$ as follows. When the weight of β_i is 6, 10, or 14, we put

$$\beta'_i := (\,\beta_i \,|\, 0, 1, 1\,), \quad \text{or} \quad \beta'_i := (\beta_i \,|\, 1, 0, 1\,), \quad \text{or} \quad \beta'_i := (\,\beta_i \,|\, 1, 1, 0\,). \tag{A.3}$$

When the weight of β_i is 8, 12, or 16, we put

$$\beta'_i := (\,\beta_i \,|\, 0, 0, 0\,).$$

We search for a combination of choices in (A.3) such that every element of the linear subspace of $\mathbb{F}_2^{24}$ generated by $\beta'_1, \dots, \beta'_{11}$ has weight 0, 8, 12, or 16. If $\beta'_1, \dots, \beta'_{11}$ satisfy this condition, then the linear embedding $\Gamma \hookrightarrow \mathbb{F}_2^{24}$ defined by $\beta_i \mapsto \beta'_i$ satisfies the properties required for ι. By this method, we find exactly six such embeddings. We fix one of them. The weight distribution of Γ' is

$$0^1\, 8^{378}\, 12^{1288}\, 16^{381}.$$

Then the code $\tilde{\Gamma}$ generated by Γ' and the vector $\varepsilon := (1, 1, \dots, 1) \in \mathbb{F}_2^{24}$ of weight 24 is equivalent to $\mathcal{C}_{24}$. This means that Γ is equivalent to the code $\mathrm{pr}_{21}(\mathcal{C}_{24} \cap V)$, where $V \subset \mathbb{F}_2^{24}$ is the linear subspace defined by $x_{22} + x_{23} + x_{24} = 0$.

Let $\mathfrak{S}'_3$ be the full symmetric group of the coordinate positions $\{x_{22}, x_{23}, x_{24}\}$. We have $\mathfrak{S}_{21} \times \mathfrak{S}'_3 \subset \mathfrak{S}_{24}$. We will construct an injective homomorphism

$$\mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\tilde{\Gamma}) \cap (\mathfrak{S}_{21} \times \mathfrak{S}'_3).$$

Since $\mathrm{Aut}(\tilde{\Gamma}) \cap \mathfrak{S}_{21}$ is isomorphic to M_{21}, the order of $\mathrm{Aut}(\tilde{\Gamma}) \cap (\mathfrak{S}_{21} \times \mathfrak{S}'_3)$ is at most $6 \times |M_{21}| = g(\mathbf{c}_{299})$. Since $\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda) \subset \mathrm{Aut}(\Gamma)$, the existence of such an injective homomorphism will imply the desired inequality $|\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)| \le g(\mathbf{c}_{299})$.

Let $\mathrm{pr}'_3 \colon \mathbb{F}_2^{24} \to \mathbb{F}_2^3$ denote the projection $(x_1, \dots, x_{24}) \mapsto (x_{22}, x_{23}, x_{24})$. Then $T := \mathrm{pr}'_3(\Gamma')$ is defined in $\mathbb{F}_2^3$ by $x_{22} + x_{23} + x_{24} = 0$, and hence we have a natural identification

$$GL(T) = \mathfrak{S}'_3. \tag{A.4}$$

Let $g \in \mathfrak{S}_{21}$ be an automorphism of Γ. Then, via $\iota \colon \Gamma \cong \Gamma'$, the automorphism g induces a linear automorphism g' of the linear space Γ'. Since the linear subspace $\iota^{-1}(\mathrm{Ker}\, \mathrm{pr}'_3|_{\Gamma'})$ of Γ consists exactly of codewords of weight 0, 8, 12, and 16, it is preserved by g, and hence g' induces a linear automorphism of T. By (A.4), there exists a unique permutation $g'' \in \mathfrak{S}'_3$ such that $(g, g'') \in \mathfrak{S}_{21} \times \mathfrak{S}'_3$ preserves Γ'. Since (g, g'') preserves $\varepsilon = (1, 1, \dots, 1)$, this pair (g, g'') is in fact an automorphism of $\tilde{\Gamma}$.

A.3.3. *The shallow hole* $\mathbf{c}_{303}$ *of type* $a_3 a_2^{11}$. In this case, we have $n = 18$. The ternary code Γ is an $(11, 5)$-code with weight distribution

$$0^1\, 6^{132}\, 9^{110}.$$

Let $\Gamma \hookrightarrow \mathbb{F}_3^{12}$ be the linear embedding given by $x \mapsto (\, x \,|\, 0\,)$, and let Γ' denote its image. We put

$$Y := \{\, y \in \mathbb{F}_3^{11} \;\mid\; \mathrm{wt}(y) = 11, \text{ and } \mathrm{wt}(x + y) \equiv 2 \bmod 3 \text{ for all } x \in \Gamma \,\}.$$

Then Y consists of 24 vectors. We choose an element $y_0 \in Y$, and let $\tilde{\Gamma}_1$ (resp. $\tilde{\Gamma}_2$) be the code of length 12 generated by Γ' and $(\, y_0 \,|\, 1\,)$ (resp. $(\, y_0 \,|\, 2\,)$). Then both of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are equivalent to $\mathcal{C}_{12}$. This means that Γ is equivalent to $\mathrm{pr}_{11}(\mathcal{C}_{12} \cap V)$, where V is the linear subspace of $\mathbb{F}_3^{12}$ defined by $x_{12} = 0$. Moreover, the two codes $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are distinct, and for each $y \in Y$, one and only one of the following holds:

$$(\, (\, y \,|\, 1\,) \in \tilde{\Gamma}_1 \text{ and } (\, y \,|\, 2\,) \in \tilde{\Gamma}_2 \,) \quad \text{or} \quad (\, (\, y \,|\, 1\,) \in \tilde{\Gamma}_2 \text{ and } (\, y \,|\, 2\,) \in \tilde{\Gamma}_1 \,).$$

Let $g \in \mathcal{G}_{11}$ be an automorphism of Γ. Since g preserves Y, one and only one of $(\, g \,|\, 1\,) \in \mathcal{G}_{12}$ or $(\, g \,|\, -1\,) \in \mathcal{G}_{12}$ is an automorphism of $\tilde{\Gamma}_1$. Hence $|\mathrm{Aut}(\Gamma)|$ is bounded by the order of $2.M_{11}$.

On the other hand, let $f_A \in \mathrm{Aut}(A)$ be the non-trivial element of $\mathrm{Aut}(A) \cong \mathbb{Z}/2\mathbb{Z}$, and let f_B be the element of $\mathrm{Aut}(B)$ which corresponds to the scalar multiplication by -1, that is, f_B is the product of transpositions of b_i and b'_i for $i = 1, \dots, 11$. Note that f_B belongs to $\mathrm{Aut}(\Gamma)$. By the criterion (A.2), we see that neither f_B nor $f_A f_B$ is in $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$. Hence $\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)$ is a proper subgroup of $\mathrm{Aut}(\Gamma)$. In particular, its order is at most $|M_{11}| = 7920 = g(\mathbf{c}_{303})$.

A.3.4. *The shallow hole* $\mathbf{c}_{304}$ *of type* $a_3 a_1^{22}$. In this case, we have $n = 16$. The binary code Γ is a $(22, 11)$-code with weight distribution

$$0^1\, 6^{77}\, 8^{330}\, 10^{616}\, 12^{616}\, 14^{330}\, 16^{77}\, 22^1.$$

Let $\beta_1, \dots, \beta_{11}$ be a basis of Γ. We define $\beta_i' \in \mathbb{F}_2^{24}$ by

$$\beta_i' := \begin{cases} (\,\beta_i \,|\, 0, 0\,) & \text{if } \mathrm{wt}(\beta_i) \text{ is 8, 12, or 16,} \\ (\,\beta_i \,|\, 1, 1\,) & \text{if } \mathrm{wt}(\beta_i) \text{ is 6, 10, 14, or 22.} \end{cases}$$

Then the image Γ' of the linear embedding $\Gamma \hookrightarrow \mathbb{F}_2^{24}$ defined by $\beta_i \mapsto \beta_i'$ is a binary $(24, 11)$-code with weight distribution

$$0^1\, 8^{407}\, 12^{1232}\, 16^{407}\, 24^1.$$

We enumerate the set

$$Y := \{\, y \in \mathbb{F}_2^{22} \;\;|\;\; \mathrm{wt}(y) = 7, \text{ and } \mathrm{wt}(x+y) \equiv 3 \bmod 4 \text{ for all } x \in \Gamma \,\}.$$

Then Y consists of 352 vectors. We choose $y_0 \in Y$, and define the code $\tilde{\Gamma}_{01}$ (resp. $\tilde{\Gamma}_{10}$) to be the code of length 24 generated by Γ' and $(\,y_0 \,|\, 0, 1\,)$ (resp. $(\,y_0 \,|\, 1, 0\,)$). Then both of $\tilde{\Gamma}_{01}$ and $\tilde{\Gamma}_{10}$ are equivalent to $\mathcal{C}_{24}$. This means that Γ is equivalent to the code $\mathrm{pr}_{22}(\mathcal{C}_{24} \cap V)$, where $V \subset \mathbb{F}_2^{24}$ is the linear subspace defined by $x_{23} + x_{24} = 0$. Moreover, the two codes $\tilde{\Gamma}_{01}$ and $\tilde{\Gamma}_{10}$ are distinct, and for each $y \in Y$, one and only one of the following holds:

$$\Big(\, (y \,|\, 0, 1) \in \tilde{\Gamma}_{01} \text{ and } (y \,|\, 1, 0) \in \tilde{\Gamma}_{10} \,\Big) \quad \text{or} \quad \Big(\, (y \,|\, 0, 1) \in \tilde{\Gamma}_{10} \text{ and } (y \,|\, 1, 0) \in \tilde{\Gamma}_{01} \,\Big).$$

Let $\sigma \in \mathfrak{S}_{24}$ denote the transposition of x_{23} and x_{24}, and let $\mathfrak{S}_2'$ be the subgroup $\{\mathrm{id}, \sigma\}$ of $\mathfrak{S}_{24}$. We have $\mathfrak{S}_{22} \times \mathfrak{S}_2' \subset \mathfrak{S}_{24}$. Since $\mathrm{Aut}(\tilde{\Gamma}_{01}) \cap \mathfrak{S}_{22}$ is isomorphic to M_{22} and $2 \times |M_{22}| = g(\mathbf{c}_{304})$, it is enough to construct an injective homomorphism

$$\mathrm{Aut}(\Gamma) \hookrightarrow \mathrm{Aut}(\tilde{\Gamma}_{01}) \cap (\mathfrak{S}_{22} \times \mathfrak{S}_2').$$

Note that σ interchanges $\tilde{\Gamma}_{01}$ and $\tilde{\Gamma}_{10}$. Let $g \in \mathfrak{S}_{22}$ be an automorphism of Γ. Since g preserves Y, one and only one of $(g, \mathrm{id}) \in \mathfrak{S}_{22} \times \mathfrak{S}_2'$ or $(g, \sigma) \in \mathfrak{S}_{22} \times \mathfrak{S}_2'$ induces an isomorphism of $\tilde{\Gamma}_{01}$. Hence the mapping

$$g \mapsto \begin{cases} (g, \mathrm{id}) & \text{if } (g, \mathrm{id}) \text{ maps } \tilde{\Gamma}_{01} \text{ to } \tilde{\Gamma}_{01}, \\ (g, \sigma) & \text{if } (g, \mathrm{id}) \text{ maps } \tilde{\Gamma}_{01} \text{ to } \tilde{\Gamma}_{10}, \end{cases}$$

gives the desired injective homomorphism.

A.3.5. *The shallow hole* $\mathbf{c}_{305}$ *of type* $a_1 a_2^{12}$. In this case, we have $n = 21$. The ternary code Γ is a $(12, 6)$-code of minimal weigh 6, and hence is equivalent to $\mathcal{C}_{12}$. Therefore $|\mathrm{Aut}_B(P_{\mathbf{c}}, \Lambda)|$ is at most $|2.M_{12}| = 2 \times 95040 = g(\mathbf{c}_{305})$.

A.3.6. *The shallow hole* $\mathbf{c}_{306}$ *of type* $a_2 a_1^{23}$. In this case, we have $n = 18$. The binary code Γ is a $(23, 11)$-code with weight distribution

$$0^1\, 8^{506}\, 12^{1288}\, 16^{253}.$$

Let $\Gamma \hookrightarrow \mathbb{F}_2^{24}$ be the linear embedding given by $x \mapsto (\,x \,|\, 0\,)$. Then the code $\tilde{\Gamma}$ in $\mathbb{F}_2^{24}$ generated by the image of this embedding and the vector $\varepsilon = (1, 1, \dots, 1) \in \mathbb{F}_2^{24}$ is equivalent to $\mathcal{C}_{24}$. This means that Γ is equivalent to the code $\mathrm{pr}_{23}(\mathcal{C}_{24} \cap V)$, where $V \subset \mathbb{F}_2^{24}$ is the linear subspace defined by $x_{24} = 0$. Hence we obtain an injective homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\tilde{\Gamma}) \cap \mathfrak{S}_{23} \cong M_{23}$.

A.4. **The shallow hole $\mathbf{c}_{307}$ of type a_1^{25}.** Let $\mathbf{c}$ be a shallow hole with $\tau(\mathbf{c}) = a_1^{25}$. Let $v_0, \dots, v_{24}$ be the vertices of $\overline{P}_{\mathbf{c}}$, and let c_i be the circumcenter of the 23-dimensional face of $\overline{P}_{\mathbf{c}}$ that does not contain v_i. Then there exists a unique vertex v_k such that $m(c_k) = 12$ and $m(c_j) = 24$ for $j \neq k$, where $m\colon \Lambda \otimes \mathbb{Q} \to \mathbb{Z}_{>0}$ is defined in Section 3. We put $A := \{v_k\}$ and $B := P_{\mathbf{c}} \setminus A$. Then $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is contained in $\mathrm{Aut}(B) \subset \mathrm{Aut}(\overline{P}_{\mathbf{c}})$. We construct a code Γ of length 24 by the method described in Section A.2. In this case, the quotient map $\rho\colon \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}/\langle A\rangle$ is just the translation $x \mapsto x - v_k$, and M is the sublattice of Λ generated by $v_j - v_k$ ($j \neq k$). We have $n = 10$, and the binary code $\Gamma := (10\Lambda \cap 5M)/10M$ of length 24 is equivalent to $\mathcal{C}_{24}$. Hence $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is embedded into M_{24}.

Appendix B. The explanation of the computational data

The part of the LaTeX source file of this preprint between `\end{appendix}` and `\end{document}` contains the following data of holes of the Leech lattice Λ in `GAP` format [7].

- `ADEades` is the list

```
[ "A1", "A2", ..., "A24",
  "D4", "D5", ..., "D24", "E6", "E7", "E8",
  "a1", "a2", ..., "a24", "a25",
  "d4", "d5", ..., "d24", "d25", "e6", "e7", "e8"]
```

  of names of indecomposable Coxeter–Dynkin diagrams.
- `GramLeech` is the Gram matrix of Λ with respect to the fixed basis of Λ; that is, the basis given in Figure 4.12 of [4].
- `CartanMatrices` is the record of the Cartan matrices of the indecomposable Coxeter–Dynkin diagrams in `ADEades`. For example, we have

$$\texttt{CartanMatrices.A3} \;=\; \begin{array}{l}[[2,-1,0,-1],\\ [-1,2,-1,0],\\ [0,-1,2,-1],\\ [-1,0,-1,2]].\end{array}$$

- `LeechHoleRecords` is the list whose ith member is the record `LHrec` that describes the following data of the ith equivalence class $[\mathbf{c}_i]$ of holes:
  - `LHrec.number` is the number i of the equivalence class, which ranges from 1 to $23 + 284 = 307$.
  - `LHrec.depth` is `"deep"` (when $i \leq 23$) or `"shallow"` (when $i \geq 24$).
  - `LHrec.type` is the list of indecomposable Coxeter–Dynkin types that indicates $\tau(\mathbf{c}_i)$. For example, when $i = 18$, we have

    `LHrec.type=["D4", "A5", "A5", "A5", "A5"],`

    which means that $\tau(\mathbf{c}_{18}) = D_4 A_5^4$.
  - `LHrec.center` is a representative hole $\mathbf{c}_i$ of the equivalence class $[\mathbf{c}_i]$ written as a row vector with respect to the fixed basis of Λ.
  - `LHrec.vertices` is the list of vertices $\boldsymbol{\lambda}_j$ of the convex polytope $\overline{P}_{\mathbf{c}_i}$, each of which is written as a row vector with respect to the fixed basis of Λ. Suppose that `LHrec.type` $= [\mathtt{X}_1, \dots, \mathtt{X}_k]$. Then the vertices of

$\overline{P}_{\mathbf{c}_i}$ are sorted in the list `LHrec.vertices` $= [\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_n]$ in such a way that the $n \times n$ matrix

$$[\ \|\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j\|^2\ ]$$

is equal to the matrix obtained from

$$\begin{bmatrix} \texttt{CartanMatrices.(X}_1\texttt{)} & & \\ & \dots & \\ & & \texttt{CartanMatrices.(X}_\texttt{k}\texttt{)} \end{bmatrix}$$

by replacing the entries as follows: $2 \mapsto 0,\ 0 \mapsto 4,\ -1 \mapsto 6,\ -2 \mapsto 8$.

- `LHrec.s` is $s(\mathbf{c}_i)$.
- `LHrec.m` is $m(\mathbf{c}_i)$.
- `LHrec.N` is $N(\mathbf{c}_i)$.
- `LHrec.thetasquare` is $\theta(\mathbf{c}_i)^2$.
- `LHrec.svol` is the scaled volume $24! \cdot \mathrm{vol}(\overline{P}_{\mathbf{c}_i})$ of $\overline{P}_{\mathbf{c}_i}$.
- `LHrec.g` is the order of the group $\mathrm{Aut}(P_{\mathbf{c}_i}, \Lambda)$.

For the shallow holes except for the ones with numbers 293, 299, 303, 304, 305, 306, 307, we also record the following data:

- `LHrec.aut` is the structure of the group $\mathrm{Aut}(P_{\mathbf{c}_i}, \Lambda)$ calculated by `GAP`'s `StructureDescription`.
- `LHrec.generators` is a list of generators of $\mathrm{Aut}(P_{\mathbf{c}_i}, \Lambda)$ regarded as a permutation group of `LHrec.vertices`. This list of generators was calculated by `GAP`'s `GeneratorsSmallest`.

For the shallow holes with numbers 293, 299, 303, 304, 305, 306, 307, see Appendix A.

Example B.1. Consider the shallow hole $\mathbf{c} = \mathbf{c}_{302}$ of type $a_3^8 a_1$. Let `LHrec` be the 302nd record in `LeechHoleRecords`:

$$\texttt{LHrec} := \texttt{LeechHoleRecords}[302].$$

The center `LHrec.center` is

$$\begin{aligned} \mathbf{c} \ = \ & [-1/3,\ 2/9,\ 2/9,\ 2/9,\ 1/3,\ 0,\ 2/9,\ 0,\ 1/9,\ -1/9,\ 0,\ 1/9, \\ & \quad 0,\ 1/9,\ -2/9,\ 1/9,\ 0,\ 1/9,\ -1/9,\ 0,\ -1/9,\ 1/9,\ 2/9,\ 2/9]. \end{aligned}$$

The list of vertices of $\overline{P}_{\mathbf{c}}$ is given in Table B.1. The automorphism group $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is of order 2688, and is isomorphic to

$$(C_2 \times C_2 \times C_2 \times C_2) : \mathrm{PSL}(3, 2).$$

As a permutation group of the list `LHrec.vertices`, this group is generated by the six permutations in the following list:

$$\begin{aligned} &\texttt{LHrec.generators} := \\ &\qquad [(7,9)(10,24)(11,23)(12,22)(13,15)(16,19)(17,20)(18,21), \\ &\qquad (7,10,16)(8,11,17)(9,12,18)(13,22,19)(14,23,20)(15,24,21), \\ &\qquad (4,6)(10,21)(11,20)(12,19)(13,15)(16,22)(17,23)(18,24), \\ &\qquad (4,7)(5,8)(6,9)(10,16)(11,17)(12,18)(19,21)(22,24), \\ &\qquad (1,3)(10,16)(11,17)(12,18)(13,15)(19,24)(20,23)(21,22), \\ &\qquad (1,4)(2,5)(3,6)(10,12)(16,19)(17,20)(18,21)(22,24)]. \end{aligned}$$

$[[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],$
$[1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1],$
$[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0],$
$[0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],$
$[1,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1],$
$[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0],$
$[0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0],$
$[1,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,1],$
$[-1,0,0,0,1,0,0,1,1,0,1,0,0,0,-1,0,0,0,-1,0,0,0,1,0],$
$[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0],$
$[2,0,0,0,-1,0,1,-1,-1,-1,0,0,-1,1,0,0,-1,1,0,0,1,0,0,0],$
$[-6,2,2,2,2,1,1,-1,1,1,1,-1,0,0,-1,1,1,0,-1,0,-1,0,1,0],$
$[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0],$
$[1,0,-1,-1,1,0,1,0,0,-1,-1,2,0,0,0,0,1,0,0,-2,0,0,0,1],$
$[-3,0,2,2,1,0,0,0,1,0,1,-1,0,1,-1,0,0,0,-1,1,-1,0,1,0],$
$[-1,0,0,0,1,0,0,1,1,1,0,0,0,-1,0,0,0,-1,0,0,0,1,0,0],$
$[-3,1,1,1,2,1,1,-1,0,-1,0,1,1,0,-1,0,0,1,-1,0,-1,0,1,0],$
$[0,0,0,0,0,-1,-1,2,1,0,0,0,0,0,0,0,1,-1,0,0,-1,1,0,0],$
$[-2,0,1,0,1,0,1,0,1,0,1,0,1,0,-2,0,0,0,0,0,0,0,0,1],$
$[3,0,-2,0,0,0,0,-1,0,-1,-1,1,0,0,1,-1,0,1,0,-1,1,-1,-1,2],$
$[-5,2,3,2,0,1,0,0,-1,1,1,-2,-1,0,0,2,-1,0,0,2,-2,2,2,-3],$
$[-3,1,1,1,1,0,1,0,1,0,0,0,1,0,-1,0,0,0,0,0,-1,0,1,0],$
$[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0],$
$[5,-1,-1,-1,-1,-1,0,0,-2,-1,-1,1,-1,1,0,0,-1,0,1,0,1,0,0,0],$
$[-3,2,2,0,1,0,0,0,1,1,0,-1,0,-1,0,1,0,-1,1,0,-1,1,0,0]].$

TABLE B.1. `LeechHoleRecords[302].vertices`

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