THE EXPLANATION OF THE COMPUTATIONAL DATA ON THE HOLES OF THE LEECH LATTICE

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In [4], we present the following data of holes of the Leech lattice Λ used in the author's preprint [3]. These data are in `GAP` format [2].

- `ADEades` is the list

  ```
  [ "A1", "A2", ..., "A24",
    "D4", "D5", ..., "D24", "E6", "E7", "E8",
    "a1", "a2", ..., "a24", "a25",
    "d4", "d5", ..., "d24", "d25", "e6", "e7", "e8"]
  ```

  of names of indecomposable Coxeter–Dynkin diagrams.
- `GramLeech` is the Gram matrix of Λ with respect to the fixed basis of Λ; that is, the basis given in Figure 4.12 of [1].
- `CartanMatrices` is the record of the Cartan matrices of the indecomposable Coxeter–Dynkin diagrams in `ADEades`. For example, we have

  $$\texttt{CartanMatrices.A3} = \begin{aligned}&[[2,-1,0,-1],\\ &[-1,2,-1,0],\\ &[0,-1,2,-1],\\ &[-1,0,-1,2]].\end{aligned}$$

- `LeechHoleRecords` is the list whose ith member is the record `LHrec` that describes the following data of the ith equivalence class $[\mathbf{c}_i]$ of holes:
  - `LHrec.number` is the number i of the equivalence class, which ranges from 1 to $23 + 284 = 307$.
  - `LHrec.depth` is `"deep"` (when $i \leq 23$) or `"shallow"` (when $i \geq 24$).
  - `LHrec.type` is the list of indecomposable Coxeter–Dynkin types that indicates $\tau(\mathbf{c}_i)$. For example, when $i = 18$, we have

    ```
    LHrec.type=["D4", "A5", "A5", "A5", "A5"],
    ```

    which means that $\tau(\mathbf{c}_{18}) = D_4A_5^4$.
  - `LHrec.center` is a representative hole $\mathbf{c}_i$ of the equivalence class $[\mathbf{c}_i]$ written as a row vector with respect to the fixed basis of Λ.
  - `LHrec.vertices` is the list of vertices $\boldsymbol{\lambda}_j$ of the convex polytope $\overline{P}_{\mathbf{c}_i}$, each of which is written as a row vector with respect to the fixed basis of Λ. Suppose that `LHrec.type` $= [\mathtt{X}_1, \dots, \mathtt{X}_\mathtt{k}]$. Then the vertices of $\overline{P}_{\mathbf{c}_i}$ are sorted in the list `LHrec.vertices` $= [\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_n]$ in such a way that the $n \times n$ matrix

    $$[\ \|\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j\|^2\ ]$$

is equal to the matrix obtained from

$$\begin{bmatrix} \mathtt{CartanMatrices.}(\mathtt{X_1}) & & \\ & \cdots & \\ & & \mathtt{CartanMatrices.}(\mathtt{X_k}) \end{bmatrix}$$

by replacing the entries as follows: $2 \mapsto 0,\ 0 \mapsto 4,\ -1 \mapsto 6,\ -2 \mapsto 8$.

- `LHrec.s` is $s(\mathbf{c}_i)$.
- `LHrec.m` is $m(\mathbf{c}_i)$.
- `LHrec.N` is $N(\mathbf{c}_i)$.
- `LHrec.thetasquare` is $\theta(\mathbf{c}_i)^2$.
- `LHrec.svol` is the scaled volume $24! \cdot \mathrm{vol}(\overline{P}_{\mathbf{c}_i})$ of $\overline{P}_{\mathbf{c}_i}$.
- `LHrec.g` is the order of the group $\mathrm{Aut}(P_{\mathbf{c}_i}, \Lambda)$.

For the shallow holes except for the ones with numbers 293, 299, 303, 304, 305, 306, 307, we also record the following data:

- `LHrec.aut` is the structure of the group $\mathrm{Aut}(P_{\mathbf{c}_i}, \Lambda)$ calculated by `GAP`'s `StructureDescription`.
- `LHrec.generators` is a list of generators of $\mathrm{Aut}(P_{\mathbf{c}_i}, \Lambda)$ regarded as a permutation group of `LHrec.vertices`. This list of generators was calculated by `GAP`'s `GeneratorsSmallest`.

For the shallow holes with numbers 293, 299, 303, 304, 305, 306, 307, see the note in [4].

Example 0.1. Consider the shallow hole $\mathbf{c} = \mathbf{c}_{302}$ of type $a_3^8 a_1$. Let `LHrec` be the 302nd record in `LeechHoleRecords`:

$$\mathtt{LHrec} := \mathtt{LeechHoleRecords}[302].$$

The center `LHrec.center` is

$$\begin{aligned} \mathbf{c} \ = \ & [-1/3,\ 2/9,\ 2/9,\ 2/9,\ 1/3,\ 0,\ 2/9,\ 0,\ 1/9,\ -1/9,\ 0,\ 1/9, \\ & \quad 0,\ 1/9,\ -2/9,\ 1/9,\ 0,\ 1/9,\ -1/9,\ 0,\ -1/9,\ 1/9,\ 2/9,\ 2/9]. \end{aligned}$$

The list of vertices of $\overline{P}_{\mathbf{c}}$ is given in Table 0.1. The automorphism group $\mathrm{Aut}(P_{\mathbf{c}}, \Lambda)$ is of order 2688, and is isomorphic to

$$(C_2 \times C_2 \times C_2 \times C_2) : \mathrm{PSL}(3,2).$$

As a permutation group of the list `LHrec.vertices`, this group is generated by the six permutations in the following list:

$$\begin{aligned} \mathtt{LHrec.generators} := & \\ & [(7,9)(10,24)(11,23)(12,22)(13,15)(16,19)(17,20)(18,21), \\ & (7,10,16)(8,11,17)(9,12,18)(13,22,19)(14,23,20)(15,24,21), \\ & (4,6)(10,21)(11,20)(12,19)(13,15)(16,22)(17,23)(18,24), \\ & (4,7)(5,8)(6,9)(10,16)(11,17)(12,18)(19,21)(22,24), \\ & (1,3)(10,16)(11,17)(12,18)(13,15)(19,24)(20,23)(21,22), \\ & (1,4)(2,5)(3,6)(10,12)(16,19)(17,20)(18,21)(22,24)]. \end{aligned}$$

References

[1] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, third edition, 1999.

$$
\begin{aligned}
&[[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],\\
&[1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1],\\
&[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0],\\
&[0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0],\\
&[1,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1],\\
&[0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0],\\
&[0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0],\\
&[1,0,0,0,0,0,0,0,0,0,-1,0,0,0,0,0,0,0,0,0,0,0,0,1],\\
&[-1,0,0,0,1,0,0,1,1,0,1,0,0,0,-1,0,0,0,-1,0,0,0,1,0],\\
&[0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0],\\
&[2,0,0,0,-1,0,1,-1,-1,-1,0,0,-1,1,0,0,-1,1,0,0,1,0,0,0],\\
&[-6,2,2,2,2,1,1,-1,1,1,1,-1,0,0,-1,1,1,0,-1,0,-1,0,1,0],\\
&[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0],\\
&[1,0,-1,-1,1,0,1,0,0,-1,-1,2,0,0,0,0,1,0,0,-2,0,0,0,1],\\
&[-3,0,2,2,1,0,0,0,1,0,1,-1,0,1,-1,0,0,0,-1,1,-1,0,1,0],\\
&[-1,0,0,0,1,0,0,1,1,1,0,0,0,-1,0,0,0,-1,0,0,0,1,0,0],\\
&[-3,1,1,1,2,1,1,-1,0,-1,0,1,1,0,-1,0,0,1,-1,0,-1,0,1,0],\\
&[0,0,0,0,0,-1,-1,2,1,0,0,0,0,0,0,0,1,-1,0,0,-1,1,0,0],\\
&[-2,0,1,0,1,0,1,0,1,0,1,0,1,0,-2,0,0,0,0,0,0,0,0,1],\\
&[3,0,-2,0,0,0,0,-1,0,-1,-1,1,0,0,1,-1,0,1,0,-1,1,-1,-1,2],\\
&[-5,2,3,2,0,1,0,0,-1,1,1,-2,-1,0,0,2,-1,0,0,2,-2,2,2,-3],\\
&[-3,1,1,1,1,0,1,0,1,0,0,0,1,0,-1,0,0,0,0,0,-1,0,1,0],\\
&[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0],\\
&[5,-1,-1,-1,-1,-1,0,0,-2,-1,-1,1,-1,1,0,0,-1,0,1,0,1,0,0,0],\\
&[-3,2,2,0,1,0,0,0,1,1,0,-1,0,-1,0,1,0,-1,1,0,-1,1,0,0]].
\end{aligned}
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TABLE 0.1. `LeechHoleRecords[302].vertices`

[2] The GAP Group. GAP - Groups, Algorithms, and Programming. Version 4.7.9; 2015 (http://www.gap-system.org).

[3] Ichiro Shimada. Holes of the Leech lattice and the projective models of $K3$ surfaces. Preprint, 2015. arXiv:1502.02099.

[4] Ichiro Shimada. The list of holes of the Leech lattice, 2016. http://www.math.sci.hiroshima-u.ac.jp/∼shimada/Leech.html.

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