A NOTE ON QUEBBEMANN'S EXTREMAL LATTICES OF RANK 64: COMPUTATION DATA

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This note is the explanation of the computation data that are used to obtain the main result of the paper

[Q] I. Shimada: A note on Quebbemann's extremal lattices of rank 64.

The data and the paper above are available from the author's webpage [1]. The data are made by `GAP` (see [2]).

We use the notions and notation of the paper [Q].

The matrix `GramE` is the Gram matrix of E with respect to the basis $e_1, \dots, e_8$.

The matrix `GramS` is the Gram matrix of $S = E^8$ with respect to the basis

$$e_1^{(1)}, \dots, e_8^{(1)}, e_1^{(2)}, \dots, e_8^{(2)}, \dots\dots\dots, e_1^{(8)}, \dots, e_8^{(8)}. \tag{0.1}$$

(See (3.2) of the paper [Q].)

The matrices `V0basis`, `WIbasis`, `WIIbasis` are the bases of the maximal isotropic subspaces V_0, W^{I}, W^{II} of $U = E/3E$ with respect to the basis $e_1, \dots, e_8$. (See Table 2.1 of the paper [Q].)

The other part of the data consists of records `Qrec`. Each record `Qrec` describes a Quebbemann lattice $Q = Q(\Delta, B)$ obtained by $\Delta \in \mathcal{D}^8$ and a ternary code $B \subset V_T$ satisfying p_2-condition. The record `Qrec` has the following components.

- `no` is the number of this example $Q = Q(\Delta, B)$. If `Qrec.no` ≤ 1000, then the component `Qrec.Aut` below is `"trivial"`, whereas if `Qrec.no` > 1000, then the component `Qrec.Aut` is `"order8"`.
- `Aut` is either `"trivial"` or `"order8"`. In the former case, we have $\mathrm{O}(Q) = \{\pm 1\}$, and in the latter case, we have $\mathrm{O}(Q) \cong \{\pm 1\} \times \mathbb{Z}/8\mathbb{Z}$.
- `Delta` indicates $\Delta \in \mathcal{D}^8$ that is used in the construction of $Q = Q(\Delta, B)$. `Delta` is a sequence $[i_1, \dots, i_8]$ of 8 indexes $i_j \in \{1, 2\}$, which means

  $$\Delta = ((V_0, W_1), \dots, (V_0, W_8)),$$

  where, for $j = 1, \dots, 8$, the second factor W_j is W^{I} (reps. W^{II}) if $i_j = 1$ (resp. $i_j = 2$). (The first factors are all V_0, and hence $V_T = V_0^8$.)
- `Bbasis` is an 8×32 matrix with $\mathbb{F}_3$-components whose row vectors form a basis of the ternary code $B \subset V_T = V_0^8$. Each row vector v is of the form $(v_1 | \dots | v_8)$, where $v_j \in V_0$ is the j^{th} component of $v \in B$ and is written with respect to the basis `V0basis` of V_0.

- `Bperpbasis` is a 24×32 matrix with $\mathbb{F}_3$-components whose row vectors form a basis of the ternary code $B^{\perp} \subset W_T = W_1 \oplus \cdots \oplus W_8$. Each row vector v is of the form $(v_1|\dots|v_8)$, where $v_j \in W_j$ is the j^{th} component of $v \in B^{\perp}$ and is written with respect to the basis `WIbasis` (resp. `WIIbasis`) of $W_j \cong W^{\mathrm{I}}$ (resp. $W_j \cong W^{\mathrm{II}}$).
- `Qbasis` is a 64×64 matrix with $\mathbb{Z}$-components whose row vectors form a basis of $Q = Q(\Delta, B) \subset S = E^8$. Each row vector $v \in Q$ is written with respect to the basis (0.1) of $S = E^8$.
- `GramQ` is the Gram matrix of Q with respect to the basis `Qbasis`, that is, `GramQ` is equal to $(1/3) \cdot$ `Qbasis` $\cdot$ `GramS` $\cdot\, {}^t$`Qbasis`, where ^{t}`Qbasis` is the transposed matrix of `Qbasis`.
- `minvects` is the list of minimal-norm vectors of Q modulo the action of $\{\pm 1\}$. Each vector is written with respect to the basis (0.1) of $S = E^8$ (*not* with respect to the basis `Qbasis` of Q). From each pair $\{v, -v\}$ of minimal-norm vectors, we choose the one whose left-most nonzero component is positive. The size of `minvects` is therefore 1305600.
- `intpatterns` is the list of intersection patterns of minimal-norm vectors. The i^{th} element of this list is the intersection pattern $a(v_i) = [a_1(v_i), a_2(v_i), a_3(v_i)]$ of the i^{th} element v_i of `minvects`.
- `distribution` describes the distribution A_Q of intersection patterns of Q by a list of $[a, A_Q(a)]$, where $a = [a_1, a_2, a_3]$ runs through the set of intersection patterns such that $A_Q(a) > 0$. The elements $[a, A_Q(a)]$ in this list are sorted according to the lexicographic order on the 1^{st} component $a = [a_1, a_2, a_3]$.
- `rigidifying` is a 64×64 matrix with $\mathbb{Z}$-components whose row vectors form a Γ-rigidifying basis, where $\Gamma = \{\pm 1\}$ when `Qrec.Aut` is `"trivial"`, and $\Gamma = \{\pm 1\} \times \langle \tilde{\gamma}_Q \rangle$ when `Qrec.Aut` is `"order8"`.

When `Qrec.Aut` is `"order8"`, the ternary code B is of the form $B(\gamma, v)$ and the record `Qrec` has the following additional components.

- `gamma` is the matrix representation of $\gamma \in \mathrm{O}(E)$ with respect to the basis $e_1, \dots, e_8$ of E.
- `gammatilde` is the matrix representation of $\tilde{\gamma} \in \mathrm{O}(S)$ with respect to the basis (0.1) of S.
- `generatorv` is the vector $v = (v_1|\dots|v_8) \in V_T = V_0^8$, where each $v_i \in V_0$ is written with respect to the basis `V0basis`. Then the k^{th} row vectors of `Bbasis` is $v^{(\tilde{\gamma}^k)}$.
- `orbits` is the list of indexes $\{k_1, \dots, k_8\}$ such that the vectors at ${k_j}^{\text{th}}$ positions $(j = 1, \dots, 8)$ in the list `Qrec.minvects` form an orbit of the action of $\mathrm{O}(Q)/\{\pm 1\} \cong \mathbb{Z}/8\mathbb{Z}$ on $\mathrm{Min}(Q)/\{\pm 1\}$.

Remark 0.1. We have produced $300 + 100$ records `Qrec`, 300 records with `Qrec.Aut` being `"trivial"` and 100 records with `Qrec.Aut` being `"order8"`. We put only

$10+10$ of them on the webpage, because of the restriction on the disk usage. Their names are `Qrec1 ...Qrec10` and `Qrec1001 ...Qrec1010`.

Remark 0.2. The $2+2$ examples explained in Section 4 of the paper [Q] is `Qrec1, Qrec2` and `Qrec1001, Qrec1002`.

References

[1] Ichiro Shimada. A note on Quebbemann's extremal lattices of rank 64: computation data. http://www.math.sci.hiroshima-u.ac.jp/shimada/lattice.html, 2021.

[2] The GAP Group. *GAP - Groups, Algorithms, and Programming.* Version 4.11.0; 2020 (http://www.gap-system.org).

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