RATIONAL DOUBLE POINTS ON SUPERSINGULAR $K3$ SURFACES

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Abstract. We investigate configurations of rational double points with the total Milnor number 21 on supersingular $K3$ surfaces. The complete list of possible configurations is given. As an application, we also give the complete list of extremal (quasi-)elliptic fibrations on supersingular $K3$ surfaces.

1. Introduction

Let Y be a normal projective surface defined over an algebraically closed field k, and let $f : X \to Y$ be the minimal resolution. Suppose that X is a $K3$ surface. Then the normal surface Y has only rational double points as its singularities. (See [2, 3, 5] for the definition of rational double points.) The *total Milnor number* $\mu(Y)$ of Y is, by definition, the number of (-2)-curves that are contracted by f. It is obvious that $\mu(Y)$ is less than the Picard number of X. In particular, $\mu(Y)$ cannot exceed 19 in characteristic 0. In positive characteristics, however, there exist *supersingular* $K3$ surfaces (in the sense of Shioda [26]), and we have singular $K3$ surfaces Y with $\mu(Y) \geq 20$.

For example, Dolgachev and Kondo [8] showed that a supersingular $K3$ surface X in characteristic 2 with the Artin invariant 1 is birational to the quartic surface in $\mathbb{P}^3$ defined by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 + x_0x_1x_2(x_0 + x_1 + x_2) = 0.$$

This quartic surface has seven rational double points of type A_3 so that its total Milnor number is 21. They also showed that X is birational to the purely inseparable double cover of $\mathbb{P}^2$ defined by

$$w^2 = x_0x_1x_2(x_0^3 + x_1^3 + x_2^3),$$

which has 21 rational double points of type A_1.

We say that Y is a *supersingular $K3$ surface with maximal rational double points* if $\mu(Y)$ attains the possible maximum 21. It is quite plausible that $K3$ surfaces with this property have many interesting features that are peculiar to algebraic varieties in positive characteristics. The purpose of this paper is to investigate the combinatorial aspects of such supersingular $K3$ surfaces by means of the lattice theory.

An ADE-type is a finite formal sum

$$R = \sum a_l A_l + \sum d_m D_m + \sum e_n E_n$$

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of symbols A_l ($l \geq 1$), D_m ($m \geq 4$) and E_n ($n = 6, 7, 8$) with non-negative integer coefficients, and its rank is defined by

$$\operatorname{rank}(R) := \sum a_l l + \sum d_m m + \sum e_n n.$$

For an ADE-type R, we denote by $Q(R)$ the *negative* definite integer lattice whose intersection matrix is the Cartan matrix of type R multiplied by -1. The rank of $Q(R)$ is therefore equal to $\operatorname{rank}(R)$.

Let Y be a supersingular $K3$ surface with maximal rational double points, and $f : X \to Y$ its minimal resolution. We denote by R_Y the ADE-type of rational double points on Y. In the Picard lattice S_X of X, we have a negative definite sublattice T_Y generated by the classes of (-2)-curves contracted by f. By definition, we have $\operatorname{rank}(R_Y) = \mu(Y) = 21$, and T_Y is isomorphic to $Q(R_Y)$. The orthogonal complement of T_Y in S_X is therefore generated by a vector $h_Y \in S_X$, and its norm

$$n_Y := h_Y^2$$

is uniquely determined. Note that n_Y is a positive even integer. Since X is a supersingular $K3$ surface, it is known from [4] that the discriminant of S_X is equal to $-p^{2\sigma_X}$, where p is the characteristic of the base field k, and σ_X is a positive integer ≤ 10, which is called the *Artin invariant* of X. Thus we obtain a triple

$$(R_Y, n_Y, \sigma_X).$$

For simplicity, we define an *RDP-triple* to be a triple (R, n, σ) consisting of an ADE-type R with $\operatorname{rank}(R) = 21$, a positive even integer n, and a positive integer $\sigma \leq 10$. We say that an RDP-triple (R, n, σ) is *geometrically realizable in characteristic* p if it satisfies the following conditions, which are equivalent to each other:

(i) There exists a supersingular $K3$ surface Y with maximal rational double points in characteristic p such that $(R_Y, n_Y, \sigma_X) = (R, n, \sigma)$.
(ii) Every (smooth) supersingular $K3$ surface X with the Artin invariant σ in characteristic p admits a birational morphism $f : X \to Y$ to a supersingular $K3$ surface Y with maximal rational double points such that $R_Y = R$ and $n_Y = n$.

See §2 for the equivalence of these conditions.

Our main results are as follows:

Theorem 1.1. *An RDP-triple (R, n, σ) is geometrically realizable in characteristic p if and only if (R, n, σ) is a member of the list given in Table* RDP *at the end of this paper.*

Corollary 1.2. *There exists a supersingular $K3$ surface with maximal rational double points in characteristic p if and only if $p \leq 19$.* □

The appearance of $(21A_1, 2, \sigma)$ with $\sigma = 1, \dots, 10$ in the list of RDP-triples geometrically realizable in characteristic 2 implies that every supersingular $K3$ surface in characteristic 2 is birational to a projective surface with 21 ordinary nodes. In fact, we will prove the following:

Proposition 1.3. *Every supersingular $K3$ surface in characteristic 2 is birational to a purely inseparable double cover of $\mathbb{P}^2$ that has 21 ordinary nodes.*

This proposition gives us another proof of the unirationality of supersingular $K3$ surfaces in characteristic 2, which was first proved by Rudakov-Šafarevič in [20].

As an application, we give the complete list of extremal (quasi-)elliptic fibrations on supersingular $K3$ surfaces.

Let X be a $K3$ surface. A (*quasi-*)*elliptic fibration* on X is, by definition, a surjective morphism $\phi : X \to \mathbb{P}^1$ such that the general fiber F of ϕ is a reduced irreducible curve of arithmetic genus 1, and that there exists a distinguished section $O : \mathbb{P}^1 \to X$ of ϕ. We say that $\phi : X \to \mathbb{P}^1$ is *elliptic* if F is smooth, and ϕ is *quasi-elliptic* if F is singular. A quasi-elliptic fibration exists only in characteristic 2 or 3 ([29, 23]).

Let $\phi : X \to \mathbb{P}^1$ be a (quasi-)elliptic fibration. The generic fiber E of ϕ is a curve defined over the rational function field $K := k(\mathbb{P}^1)$ of the base curve, and the set $E(K)$ of K-rational points of E is endowed with a structure of the abelian group such that the point $o \in E(K)$ corresponding to the section $O : \mathbb{P}^1 \to X$ is the zero element. We call $E(K)$ the *Mordell-Weil group* of $\phi : X \to \mathbb{P}^1$, and denote it by MW_ϕ. Let T_ϕ be the sublattice of S_X spanned by the classes of the general fiber of ϕ, the zero section O and the irreducible components of reducible fibers of ϕ that are disjoint from O. Then we have the following famous formula ([28], [12]):

$$MW_\phi \;\cong\; S_X/T_\phi. \tag{1.1}$$

Let $\mathcal{R}_\phi$ be the set of points $v \in \mathbb{P}^1$ of the base curve such that the fiber $\phi^{-1}(v)$ is reducible. It is well-known that, for each $v \in \mathcal{R}_\phi$, the classes of irreducible components of $\phi^{-1}(v)$ disjoint from the zero section span an indecomposable negative-definite ADE-lattice in S_X. Let R_v be the ADE-type of this sublattice, and put

$$R_\phi := \sum_{v \in \mathcal{R}_\phi} R_v.$$

Since $\operatorname{rank} T_\phi = \operatorname{rank}(R_\phi) + 2$, we have $\operatorname{rank}(R_\phi) \leq 20$. We say that ϕ is *extremal* if $\operatorname{rank}(R_\phi)$ attains the possible maximum 20. It follows that, if ϕ is an extremal (quasi-)elliptic fibration, then X is supersingular and MW_ϕ is finite. On the other hand, if ϕ is quasi-elliptic, then ϕ is necessarily extremal ([21]).

A triple $\langle R, \sigma, MW \rangle$ consisting of an ADE-type R of rank 20, an integer σ with $1 \leq \sigma \leq 10$, and a finite abelian group MW, is called an *elliptic triple.* An elliptic triple $\langle R, \sigma, MW \rangle$ is called a *triple of extremal elliptic* (*resp. quasi-elliptic*) $K3$ *surfaces in characteristic* p if it satisfies the following conditions, which are equivalent to each other:

(i) There exists a $K3$ surface X with the Artin invariant σ in characteristic p that has an extremal elliptic (resp. quasi-elliptic) fibration $\phi : X \to \mathbb{P}^1$ such that $R_\phi = R$ and $MW_\phi \cong MW$.
(ii) Every supersingular $K3$ surface X with the Artin invariant σ in characteristic p admits an extremal elliptic (resp. quasi-elliptic) fibration $\phi : X \to \mathbb{P}^1$ such that $R_\phi = R$ and $MW_\phi \cong MW$.

Theorem 1.4. *The complete lists of triples of extremal quasi-elliptic and elliptic* $K3$ *surfaces are given by Tables* QE *and* E.

The complete lists of triples of extremal elliptic $K3$ surfaces and of extremal quasi-elliptic $K3$ surfaces in characteristic 3 have been already obtained by Ito [12, 14, 15]. His method is completely different from ours, and he gave explicit defining

Table QE: The complete list of extremal quasi-elliptic $K3$ surfaces

$p = 2,\ MW = (\mathbb{Z}/2\mathbb{Z})^{\oplus r}$.

R	σ	r	R	σ	r
$2E_8 + D_4$	1	0	$D_{12} + D_8$	1, 2	$2-\sigma$
$E_8 + E_7 + 5A_1$	2	1	$D_{12} + 2D_4$	2, 3	$3-\sigma$
$E_8 + D_{12}$	1	0	$D_{12} + 8A_1$	3, 4	$5-\sigma$
$E_8 + D_8 + D_4$	2	0	$D_{10} + D_6 + 4A_1$	2, 3	$4-\sigma$
$E_8 + D_6 + 6A_1$	3	1	$D_{10} + D_4 + 6A_1$	3, 4	$5-\sigma$
$E_8 + 3D_4$	3	0	$D_{10} + 10A_1$	4, 5	$6-\sigma$
$E_8 + D_4 + 8A_1$	4	1	$2D_8 + D_4$	1, 2, 3	$3-\sigma$
$E_8 + 12A_1$	5	1	$D_8 + D_6 + 6A_1$	2, 3, 4	$5-\sigma$
$2E_7 + D_6$	1	1	$D_8 + 3D_4$	2, 3, 4	$4-\sigma$
$2E_7 + D_4 + 2A_1$	2	1	$D_8 + D_4 + 8A_1$	3, 4, 5	$6-\sigma$
$2E_7 + 6A_1$	3	1	$D_8 + 12A_1$	4, 5, 6	$7-\sigma$
$E_7 + D_{10} + 3A_1$	1, 2	$3-\sigma$	$3D_6 + 2A_1$	1, 2, 3	$4-\sigma$
$E_7 + D_8 + 5A_1$	2, 3	$4-\sigma$	$2D_6 + D_4 + 4A_1$	2, 3, 4	$5-\sigma$
$E_7 + 2D_6 + A_1$	2	1	$2D_6 + 8A_1$	3, 4, 5	$6-\sigma$
$E_7 + D_6 + D_4 + 3A_1$	2, 3	$4-\sigma$	$D_6 + 2D_4 + 6A_1$	2, 3, 4, 5	$6-\sigma$
$E_7 + D_6 + 7A_1$	3, 4	$5-\sigma$	$D_6 + D_4 + 10A_1$	3, 4, 5, 6	$7-\sigma$
$E_7 + 2D_4 + 5A_1$	3, 4	$5-\sigma$	$D_6 + 14A_1$	3, 4, 5, 6, 7	$8-\sigma$
$E_7 + D_4 + 9A_1$	3, 4, 5	$6-\sigma$	$5D_4$	1, 2, 3, 4, 5	$5-\sigma$
$E_7 + 13A_1$	4, 5, 6	$7-\sigma$	$3D_4 + 8A_1$	2, 3, 4, 5, 6	$7-\sigma$
D_{20}	1	0	$2D_4 + 12A_1$	3, 4, 5, 6, 7	$8-\sigma$
$D_{16} + D_4$	1, 2	$2-\sigma$	$D_4 + 16A_1$	2, 3, 4, 5, 6, 7, 8	$9-\sigma$
$D_{14} + 6A_1$	2, 3	$4-\sigma$	$20A_1$	3, 4, 5, 6, 7, 8, 9	$10-\sigma$

$p = 3,\ MW = (\mathbb{Z}/3\mathbb{Z})^{\oplus r}$.

R	σ	r	R	σ	r
$2E_8 + 2A_2$	1	0	$3E_6 + A_2$	1, 2	$2-\sigma$
$E_8 + 2E_6$	1	0	$2E_6 + 4A_2$	1, 2, 3	$3-\sigma$
$E_8 + E_6 + 3A_2$	2	0	$E_6 + 7A_2$	2, 3, 4	$4-\sigma$
$E_8 + 6A_2$	2, 3	$3-\sigma$	$10A_2$	1, 2, 3, 4, 5	$5-\sigma$

Table E: The complete list of extremal elliptic $K3$ surfaces

$[a] = \mathbb{Z}/a\mathbb{Z}, \quad [a,b] = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$.

p	R	σ	MW	p	R	σ	MW
11	$2A_{10}$	1	0	3	$D_7 + A_{11} + A_2$	1	[4]
7	$A_{13} + A_6 + A_1$	1	[2]	2	$A_{17} + 3A_1$	1	[6]
7	$E_8 + 2A_6$	1	0	2	$4A_5$	1	[3, 6]
5	$E_7 + A_9 + A_4$	1	[2]	2	$2A_9 + 2A_1$	1	[10]
5	$A_{14} + A_4 + A_2$	1	[3]	2	$E_6 + A_{11} + A_3$	1	[6]
3	$D_{16} + 2A_2$	1	[2]	2	$D_5 + A_{15}$	1	[4]
3	$D_{10} + 2A_5$	1	[2, 2]				

equations of these extremal (quasi-)elliptic $K3$ surfaces. The complete list of triples of extremal quasi-elliptic $K3$ surfaces in characteristic 2 seems to be new.

In [31] and [32], Yang classified all configurations of rational double points on complex sextic double planes and complex quartic surfaces. He used the ideas of Urabe [30], and reduced the problem of listing up all rational double points on these complex $K3$ surfaces to lattice theoretic calculations via Torelli theorem. By a similar method, the complete list of configurations of singular fibers on complex elliptic $K3$ surfaces has been obtained in [25] and [24]. In [9], the maximal configurations of ordinary nodes on rational surfaces in characteristic $\neq 2$ are investigated.

The plan of this paper is as follows. In §2, we review some known facts in the theory of $K3$ surfaces and the lattice theory. In §2.1, we give lattice theoretic conditions for a $K3$ surface to have a given configuration of (-2)-curves and to have a (quasi-)elliptic fibration with a given ADE-type of reducible fibers. In §2.2, we briefly review the theory of discriminant forms due to Nikulin [17]. In §2.3, we quote from Artin [4], Rudakov-Šafarevič [20, 21] and Shioda [27] some fundamental facts about the Picard lattices of supersingular $K3$ surfaces. These facts play, in positive characteristics, the same role as the one Torelli theorem played for complex $K3$ surfaces in [24, 25], [30] and [31, 32]. The algorithms for obtaining the lists of geometrically realizable RDP-triples and of triples of extremal (quasi-)elliptic $K3$ surfaces are presented in §3 and §4, respectively. In §5, we investigate the geometry of supersingular $K3$ surfaces in characteristic 2 with 21 ordinary nodes, and prove Proposition 1.3.

The lists in Tables RDP, QE and E are also available from the author's homepage:

`http://www.math.sci.hokudai.ac.jp/~shimada/ssK3.html`

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2. Preliminaries

2.1. Rational double points and (quasi-)elliptic fibrations on a $K3$ surface.

An integer lattice Λ is said to be *even* if $v^2 \in 2\mathbb{Z}$ for any $v \in \Lambda$. A vector v of an integer lattice Λ is said to be *primitive* if the intersection of $\mathbb{Q} \cdot v$ and Λ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated by v.

Let T be a *negative* definite even integer lattice. A vector $v \in T$ is called a *root* if $v^2 = -2$. Let $\mathrm{Roots}(T)$ be the set of roots of T, and let T_{roots} be the sublattice of T generated by $\mathrm{Roots}(T)$. We denote by $\Sigma(T)$ the ADE-type of the root lattice T_{roots}; that is, $\Sigma(T)$ is the ADE-type such that T_{roots} is isomorphic to $Q(\Sigma(T))$. (See [6], [10].)

Let X be a smooth $K3$ surface defined over an algebraically closed field of arbitrary characteristic. The Picard lattice S_X of X is an even integer lattice of signature $(1, \rho_X - 1)$, where ρ_X is the Picard number of X.

The purpose of this sub-section is to establish the following propositions:

Proposition 2.1. *Let R be an ADE-type, and X a smooth $K3$ surface. There exists a birational morphism $f : X \to Y$ with the singularity of Y being rational double points of ADE-type R if and only if there exists a vector $h \in S_X$ such that $h^2 > 0$ and $\Sigma(h^\perp) = R$, where $h^\perp$ is the orthogonal complement of h in S_X.*

Proposition 2.2. *Let R be an ADE-type, MW an abelian group* (*not necessarily finite*), *and X a smooth $K3$ surface. There exists a* (*quasi-*)*elliptic fibration $\phi : X \to \mathbb{P}^1$ such that $R_\phi = R$ and $MW_\phi \cong MW$ if and only if there exists an indefinite unimodular sublattice $U \subset S_X$ of rank 2 such that $\Sigma(U^\perp) = R$ and $MW \cong U^\perp/(U^\perp)_{\mathrm{roots}}$, where $U^\perp$ is the orthogonal complement of U in S_X.*

For the proof of these propositions, we need several lemmas. A line bundle L on X is said to be *numerically effective* (*nef*) if $L.C \geq 0$ holds for any curve C on X.

For a line bundle L and a divisor D on X, we denote by $[L] \in S_X$ and $[D] \in S_X$ the corresponding vectors in S_X.

Lemma 2.3. *Let L be a nef line bundle on X with $L^2 > 0$.*

(1) *The complete linear system $|2L|$ is fixed-component free and base-point free.*

(2) *Let $\mathcal{R}_L$ be the set of reduced irreducible curves C_i on X such that $L.C_i = 0$. Then $\mathcal{R}_L$ is an ADE-configuration of (-2)-curves on X whose ADE-type is equal to $\Sigma([L]^\perp)$.*

Proof. From Nikulin's proposition [18, Proposition 0.1], it follows that, if the complete linear system $|M|$ of a nef line bundle M on X with $M^2 > 0$ has a fixed component, then there exists a divisor E on X such that $E^2 = 0$ and $E.M = 1$. Therefore $|2L|$ is fixed component free. Then, by Saint-Donat's result ([22, Corollary 3.2]), $|2L|$ is base-point free.

Since $[L]^\perp$ is negative definite and C^2 is ≥ -2 for any reduced irreducible curve C on X, every $C_i \in \mathcal{R}_L$ satisfies $C_i^2 = -2$ and hence is a (-2)-curve. We put

$$R_L := \{\ [C_i]\ \mid\ C_i \in \mathcal{R}_L\ \}.$$

We will show that $[L]^\perp_{\mathrm{roots}}$ is generated by R_L. Let v be an arbitrary vector in $\mathrm{Roots}([L]^\perp)$. Since v or $-v$ is effective, we can and will assume that v is the class of an effective divisor $D = \sum \gamma_i C'_i$, where C'_i's are reduced irreducible curves and γ_i's are positive integers. Since L is nef and $L.D = 0$, it follows that $L.C'_i = 0$ for each C'_i, and hence $[C'_i]$ is contained in R_L.

The set R_L is contained in the set of positive roots

$$R^+ := \{\ r \in \mathrm{Roots}([L]^\perp)\ \mid\ r.[H] > 0\ \}$$

in $[L]^\perp$ with respect to the class $[H] \in S_X$ of a hyperplane section H of X. Since every $C_i \in \mathcal{R}_L$ is irreducible, each $[C_i] \in R_L$ is indecomposable in R^+. Therefore R_L is a fundamental root system of $[L]^\perp_{\mathrm{roots}}$. (See [6], [10].) The ADE-type $\Sigma([L]^\perp)$ of the root lattice $[L]^\perp_{\mathrm{roots}}$ and that of the configuration $\mathcal{R}_L$ of (-2)-curves are thus identical. □

Lemma 2.4. *Let R be an ADE-type. A smooth $K3$ surface X has a contraction $f : X \to Y$ of an ADE-configuration of (-2)-curves of type R if and only if there exists a nef line bundle L on X such that $L^2 > 0$ and $\Sigma([L]^\perp) = R$.*

Proof. Suppose that X admits a contraction $f : X \to Y$ of an ADE-configuration of (-2)-curves of type R. Let L be the line bundle $\mathcal{O}_X(f^*H)$, where H is a hyperplane section of Y. Then L is nef and $L^2 > 0$. Since the set of reduced irreducible curves on X contracted by f is equal to $\mathcal{R}_L$, $\Sigma([L]^\perp) = R$ follows from Lemma 2.3 (2).

Conversely, suppose that there exists a nef line bundle L on X with $L^2 > 0$ and $\Sigma([L]^\perp) = R$. By Lemma 2.3 (1), we have a morphism

$$\Phi_{|2L|}\ :\ X\ \to\ Z' := \Phi_{|2L|}(X)\ \subset\ \mathbb{P}^N$$

defined by $|2L|$. Since $L^2 > 0$, we have $\dim Z' = 2$. Let Z be the normalization of Z', and let

$$X\ \xrightarrow{f}\ Y\ \xrightarrow{g}\ Z$$

be the Stein factorization of the morphism $X \to Z$ induced from $\Phi_{|2L|}$. Since g is finite, the set $\mathcal{R}_L$ coincides with the set of reduced irreducible curves on X contracted by f. By Lemma 2.3 (2), f is a contraction of an ADE-configuration of (-2)-curves whose type is equal to $\Sigma([L]^\perp) = R$. □

Lemma 2.5. (1) *For any vector $v \in S_X$ with $v^2 > 0$, there exists an isometry γ of S_X such that $\gamma(v)$ is the class $[L]$ of a nef line bundle L.*

(2) *For any primitive vector $v \in S_X$ with $v^2 = 0$, there exists an isometry γ of S_X such that $\gamma(v)$ is the class $[F]$ of a reduced irreducible curve F of arithmetic genus* 1.

Proof. Using the scalar multiplication by -1 if necessary, we can assume that v is the class of an effective divisor. Then Lemma 2.5 follows from [21, Proposition 3 in Section 3]. □

Let $\phi : X \to \mathbb{P}^1$ be a (quasi-)elliptic fibration on a $K3$ surface X. We denote by U_ϕ the sublattice of S_X generated by the classes of the general fiber F of ϕ and the zero section. Then U_ϕ is an indefinite unimodular lattice of rank 2. Since U_ϕ is unimodular, we have an orthogonal decomposition

$$S_X = U_\phi \oplus U_\phi^\perp. \tag{2.1}$$

It is easy to prove $T_\phi = U_\phi \oplus (U_\phi^\perp)_{\mathrm{roots}}$ by the same argument as in the proof of Lemma 2.3. (See also [19, Lemma 6.1].) From the definition and (1.1), we have

$$R_\phi = \Sigma(U_\phi^\perp) \quad \text{and} \quad MW_\phi \cong U_\phi^\perp/(U_\phi^\perp)_{\mathrm{roots}}. \tag{2.2}$$

Consider the module $[F]^\perp/\mathbb{Z}[F]$, where F is the general fiber of ϕ, $\mathbb{Z}[F]$ is the sublattice of S_X generated by $[F]$, and $[F]^\perp$ is the orthogonal complement of $[F]$ in S_X. Since $F^2 = 0$, we can naturally regard $[F]^\perp/\mathbb{Z}[F]$ as an integer lattice. Since the orthogonal complement of $[F] \in U_\phi$ in U_ϕ is generated by $[F]$, we obtain from (2.1) the isomorphism of lattices

$$[F]^\perp/\mathbb{Z}[F] \cong U_\phi^\perp. \tag{2.3}$$

Lemma 2.6 ([16] Lemma 2.1)**.** *Suppose that S_X has an indefinite unimodular sublattice $U \subset S_X$ of rank* 2. *Then X has a* (*quasi-*)*elliptic fibration $\phi : X \to \mathbb{P}^1$ such that $[F]^\perp/\mathbb{Z}[F] \cong U^\perp$, where $U^\perp$ is the orthogonal complement of U in S_X.* □

Note that the proof in [16] is valid also in positive characteristics with the only exception that $\phi : X \to \mathbb{P}^1$ may be quasi-elliptic.

Proof of Propositions 2.1 and 2.2. Proposition 2.1 follows from Lemmas 2.4 and 2.5. Proposition 2.2 follows from the isomorphisms (2.2), (2.3) and Lemma 2.6. □

2.2. **The discriminant form of an even integer lattice.** Let Λ be a non-degenerate even integer lattice. We denote by $\Lambda^\vee$ the dual lattice of Λ, which is the free $\mathbb{Z}$-module $\mathrm{Hom}(\Lambda, \mathbb{Z})$ equipped with the natural symmetric bilinear form

$$\psi_\Lambda : \Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}.$$

There exists a natural inclusion $\Lambda \hookrightarrow \Lambda^\vee$. A submodule Λ' of $\Lambda^\vee$ containing Λ is said to be an *overlattice* of Λ if ψ_Λ restricted to $\Lambda' \times \Lambda'$ takes values in $\mathbb{Z}$.

The *discriminant group* G_Λ of Λ is defined to be $\Lambda^\vee/\Lambda$. Note that the order of G_Λ is equal to $|\operatorname{disc} \Lambda|$. There is a quadratic form

$$q_\Lambda : G_\Lambda \to \mathbb{Q}/2\mathbb{Z}$$

on G_Λ defined by $q_\Lambda(\bar{v}) := \psi_\Lambda(v, v) \bmod 2\mathbb{Z}$, where $\bar{v} := v + \Lambda \in G_\Lambda$. The pair (G_Λ, q_Λ) is called the *discriminant form* of Λ.

More generally, let (G, q) be a pair of a finite abelian group G and a quadratic form $q : G \to \mathbb{Q}/2\mathbb{Z}$. We define a symmetric bilinear form

$$b : G \times G \to \mathbb{Q}/\mathbb{Z}$$

by $b(x, y) := (q(x + y) - q(x) - q(y))/2$. For a subgroup S of G, we define its orthogonal complement $S^\perp$ by

$$S^\perp := \{ x \in G \mid b(x, y) = 0 \text{ for all } y \in S \}.$$

Proposition 2.7 ([17] Proposition 1.4.1)**.** *Let* $\mathrm{pr}_\Lambda : \Lambda^\vee \to G_\Lambda$ *be the natural projection. Then the correspondence* $S \mapsto \mathrm{pr}_\Lambda^{-1}(S)$ *yields a bijection from the set of subgroups of* (G_Λ, q_Λ) *to the set of even overlattices of* Λ*. If* S *is an isotropic subgroup of* (G_Λ, q_Λ)*, then the discriminant group of the even integer lattice* $\mathrm{pr}_\Lambda^{-1}(S)$ *is isomorphic to* $S^\perp/S$*.* □

2.3. **The Picard lattice of a supersingular $K3$ surface.** Let p be a prime integer. A non-degenerate even integer lattice Λ is called *p-elementary* if its discriminant group G_Λ is p-elementary; that is, $p \cdot \Lambda^\vee \subseteq \Lambda$ holds. A 2-elementary lattice Λ is said to be *of type* I if $\psi_\Lambda(v, v) \in \mathbb{Z}$ holds for any $v \in \Lambda^\vee$, where ψ_Λ is the natural symmetric bilinear form on $\Lambda^\vee$.

Let M_Λ be the intersection matrix of Λ with respect to a certain basis of Λ. Then Λ is p-elementary if and only if all the coefficients of $p \cdot M_\Lambda^{-1}$ are integers. Suppose that Λ is 2-elementary. Then Λ is of type I if and only if all the diagonal coefficients of M_Λ^{-1} are integers.

Theorem 2.8 ([4], [20])**.** *Let* X *be a supersingular* $K3$ *surface in characteristic* p*. Then the Picard lattice* S_X *of* X *is a* p*-elementary lattice of signature* $(1, 21)$*, and its discriminant is equal to* $-p^{2\sigma}$*, where* σ *is an integer satisfying* $1 \le \sigma \le 10$*.*

When $p = 2$*,* S_X *is of type* I*.* □

Theorem 2.9 ([20], [27])**.** *For each pair* (p, σ) *of a prime integer* p *and an integer* σ *satisfying* $1 \le \sigma \le 10$*, there exists a supersingular* $K3$ *surface with the Artin invariant* σ *in characteristic* p*.* □

For a pair (p, σ) of a prime integer p and an integer σ with $1 \le \sigma \le 10$, let $\Lambda_{p,\sigma}$ be a lattice of rank 22 satisfying the following conditions:

(1) $\Lambda_{p,\sigma}$ is a non-degenerate even integer lattice of signature $(1, 21)$,
(2) $\operatorname{disc} \Lambda_{p,\sigma} = -p^{2\sigma}$, and
(3) $\Lambda_{p,\sigma}$ is p-elementary.

When $p = 2$, we further impose on $\Lambda_{p,\sigma}$ the following condition:

(4) $\Lambda_{p,\sigma}$ is of type I.

Theorem 2.10 ([21], [7] Chapter 15)**.** *These conditions determine the lattice* $\Lambda_{p,\sigma}$ *uniquely up to isomorphisms.* □

Corollary 2.11. *If* X *is a supersingular* $K3$ *surface in characteristic* p *with the Artin invariant* σ*, then the Picard lattice* S_X *of* X *is isomorphic to* $\Lambda_{p,\sigma}$*.* □

By Propositions 2.1, 2.2 and Corollary 2.11, we see that the two conditions in the definition of geometric realizability of RDP-triples and elliptic triples given in Introduction are equivalent to each other.

Corollary 2.12. *An RDP-triple* (R, n, σ) *is geometrically realizable in characteristic* p *if and only if there exists a primitive vector* $h \in \Lambda_{p,\sigma}$ *such that* $h^2 = n$ *and* $\Sigma(h^\perp) = R$*, where* $h^\perp$ *is the orthogonal complement of* h *in* $\Lambda_{p,\sigma}$. □

Corollary 2.13. *An elliptic triple* $\langle R, \sigma, MW\rangle$ *is a triple of extremal (quasi-)elliptic* $K3$ *surface in characteristic* p *if and only if there exists an indefinite unimodular sublattice* $U \subset \Lambda_{p,\sigma}$ *of rank* 2 *such that* $\Sigma(U^\perp) = R$ *and* $MW \cong U^\perp/(U^\perp)_{\text{roots}}$*, where* $U^\perp$ *is the orthogonal complement of* U *in* $\Lambda_{p,\sigma}$. □

Proposition 2.14. *An elliptic triple* $\langle R, \sigma, MW\rangle$ *is a triple of extremal (quasi-) elliptic* $K3$ *surface in characteristic* p *if and only if there exist vectors* h *and* z *in* $\Lambda_{p,\sigma}$ *satisfying the following conditions:*

- (i) $h^2 = 2$, $\Sigma(h^\perp) = R + A_1$,
- (ii) $z \in \text{Roots}(h^\perp)$,
- (iii) $rz = 0$ *for any* $r \in \text{Roots}(h^\perp) \setminus \{\pm z\}$ *(so that* $\text{Roots}(h^\perp) \setminus \{\pm z\}$ *is a root system of type* R*),*
- (iv) $h - z$ *is divisible by* 2 *in* $\Lambda_{p,\sigma}$*, and*
- (v) *the sublattice* U *of* $\Lambda_{p,\sigma}$ *generated by* $f := (h - z)/2$ *and* z *satisfies* $MW \cong U^\perp/(U^\perp)_{\text{roots}}$.

In particular, if an elliptic triple $\langle R, \sigma, MW\rangle$ *is a triple of extremal (quasi-)elliptic* $K3$ *surface in characteristic* p*, then the RDP-triple* $(R + A_1, 2, \sigma)$ *is geometrically realizable in characteristic* p.

Proof. Suppose that there exist vectors h and z with the properties (i)-(v). Since $f^2 = 0, fz = 1$ and $z^2 = -2$, the sublattice U is indefinite and unimodular. From (iii), we have $\Sigma(U^\perp) = R$. Thus the condition in Corollary 2.13 is satisfied.

Conversely, suppose that there exists an indefinite unimodular sublattice $U \subset \Lambda_{p,\sigma}$ of rank 2 such that $\Sigma(U^\perp) = R$ and $MW \cong U^\perp/(U^\perp)_{\text{roots}}$. We can find vectors f and z in U that generate U and satisfy $f^2 = 0, fz = 1$ and $z^2 = -2$, because U is given by the intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to a certain basis of U. The vectors $h := 2f + z$ and z obviously satisfy $h^2 = 2$, and the conditions (ii), (iv) and (v). Since U is unimodular, we have an orthogonal decomposition $\Lambda_{p,\sigma} = U \oplus U^\perp$. Therefore we have an orthogonal decomposition $h^\perp = U^\perp \oplus \langle z\rangle$, and hence the condition (iii) is fulfilled and $\Sigma(h^\perp) = R + A_1$ holds.

The last assertion follows from Corollary 2.12, because a vector h with $h^2 = 2$ is necessarily primitive. □

In order to determine whether a (quasi-)elliptic fibration $\phi : X \to \mathbb{P}^1$ is elliptic or quasi-elliptic, we use the following:

Theorem 2.15 ([21] Theorem in Section 4)**.** *Let* $\phi : X \to \mathbb{P}^1$ *be a (quasi-)elliptic fibration on a* $K3$ *surface* X *in characteristic* p*, where* $p = 2$ *or* 3*. Then* ϕ *is quasi-elliptic if and only if* ϕ *is extremal and* $Q(R_\phi)$ *is* p*-elementary.* □

3. The algorithm for the list of RDP-triples

Let (R, n, σ) be an RDP-triple. We denote by $I(n)$ the lattice of rank 1 generated by a vector e_n with $e_n^2 = n$, and by $Q(R, n)$ the lattice $Q(R) \oplus I(n)$ of rank 22 with signature $(1, 21)$. By Theorem 2.10, we can rephrase Corollary 2.12 to the following:

Corollary 3.1. *An RDP-triple (R, n, σ) is geometrically realizable in characteristic p if and only if there exists an even overlattice Λ of $Q(R, n)$ with the following properties;*

(1) $\operatorname{disc} \Lambda = -p^{2\sigma}$,
(2) *Λ is p-elementary, and, if $p = 2$, Λ is of type* I,
(3) *the vector $\tilde{e}_n := (0, e_n) \in Q(R, n)$ remains primitive in Λ, and*
(4) $\operatorname{Roots}(\tilde{e}_n^{\perp}) = \operatorname{Roots}(Q(R))$*, where $\tilde{e}_n^{\perp}$ is the orthogonal complement of $\tilde{e}_n$ in Λ, and $Q(R)$ is regarded as a sublattice of Λ.* □

Using Proposition 2.7, we can further rephrase Corollary 3.1 as follows. For simplicity, we denote by (G_R, q_R), (G_n, q_n) and $(G_{R,n}, q_{R,n})$ the discriminant forms of $Q(R)$, $I(n)$ and $Q(R, n)$, respectively. There exists a natural isomorphism

$$(G_{R,n}, q_{R,n}) \cong (G_R, q_R) \oplus (G_n, q_n).$$

Note that G_n is a cyclic group of order n generated by

$$\varepsilon_n := e_n^{\vee} + I(n),$$

and we have $q_n(\varepsilon_n) = 1/n \bmod 2\mathbb{Z}$. See [24, §6] for the structure of (G_R, q_R). Let S be an isotropic subgroup of $(G_{R,n}, q_{R,n})$. We denote by Λ_S the even overlattice of $Q(R, n)$ corresponding to S via the bijection given in Proposition 2.7. We regard $Q(R)$ as a sublattice of Λ_S. Let $(\tilde{e}_n)^{\perp}_S$ denote the orthogonal complement of $\tilde{e}_n = (0, e_n) \in \Lambda_S$ in Λ_S, which is a negative definite even integer lattice. Since $Q(R)$ is contained in $(\tilde{e}_n)^{\perp}_S$, we have $\operatorname{Roots}(Q(R)) \subseteq \operatorname{Roots}((\tilde{e}_n)^{\perp}_S)$. We put

$$\varrho_R := |\operatorname{Roots}(Q(R))|, \quad \text{and} \quad \varrho_{R,n}(S) := |\operatorname{Roots}((\tilde{e}_n)^{\perp}_S)|.$$

See [25] for ϱ_R and for the method of calculating $\varrho_{R,n}(S)$.

Corollary 3.2. *An RDP-triple (R, n, σ) is geometrically realizable in characteristic p if and only if there exists an isotropic subgroup S of $(G_{R,n}, q_{R,n})$ with the following properties;*

(1) *$S^{\perp}/S$ is a p-elementary group of order $p^{2\sigma}$,*
(2) *$S \cap G_n$ is trivial,*
(3) *$\varrho_{R,n}(S) = \varrho_R$, and*
(4) *if $p = 2$, then Λ_S is of type* I. □

3.1. **Finiteness of the triples $[R, n, p]$.** In this sub-section, we show that there exist only a finite number of triples $[R, n, p]$ such that (R, n, σ) is geometrically realizable in characteristic p for some σ.

Let R be an ADE-type with $\operatorname{rank}(R) = 21$, and n a positive even integer. We denote by N_R the minimal positive integer such that

$$N_R \cdot q_R(x) = 0 \quad \text{in} \quad \mathbb{Q}/2\mathbb{Z} \quad \text{for all} \quad x \in G_R.$$

Then N_R is the least common multiple of N_X, where X runs through the set of indecomposable ADE-types that appear in R. From the intersection matrix of q_X given in [24, §6, Table 6.1], we obtain Table 3.1. We denote by P_R the set of prime factors of $|G_R| = |\operatorname{disc} Q(R)|$.

Lemma 3.3. *Let p be a prime integer. Suppose that $(G_{R,n}, q_{R,n})$ contains an isotropic subgroup S such that $S^{\perp}/S \cong (\mathbb{Z}/(p))^{\oplus 2\sigma}$ for some $\sigma > 0$, and that $S \cap G_n$ is trivial. Then (n, p) is contained in the following finite set $NP(R)$;*

$$\{\ (n, p) \ \mid\ p \in P_R,\ \ 2n \mid (N_R \cdot p^2),\ \ p^2 \mid (n \cdot |G_R|) \ \ \textit{and} \ \ \sqrt{n \cdot |G_R|} \in \mathbb{Z}\ \}.$$

TABLE 3.1. N_X for indecomposable ADE-type X

X	A_l		D_m			E_n		
	l : even	l : odd	$m = 4k$	$m = 4k + 2$	m : odd	$n = 6$	$n = 7$	$n = 8$
N_X	$l + 1$	$2(l + 1)$	2	4	8	3	4	1
$\lvert G_X \rvert$	$l + 1$		4			3	2	1

Proof. First we fix some notation. For an abelian group A and a prime integer l, we denote by A_l the maximal subgroup of A whose order is a power of l. For a discriminant form (G, q), we denote by q_l the restriction of q to G_l. Then we have a natural orthogonal decomposition

$$(G, q) = \bigoplus_l (G_l, q_l),$$

where l runs through the set of prime factors of $|G|$. Let S be an isotropic subgroup of (G, q). Then the l-part $(S^\perp)_l$ of $S^\perp$ coincides with the orthogonal complement $(S_l)^\perp$ of S_l in (G_l, q_l), and $(S^\perp/S)_l$ coincides with $(S_l)^\perp/S_l$.

Let l be a prime factor of n, and suppose that $l \notin P_R$. Then $(G_R)_l$ is trivial. Since $S \cap G_n$ is trivial, S_l is also trivial, and we have

$$(S^\perp/S)_l = (S_l)^\perp = (G_{R,n})_l = (G_n)_l,$$

which is a non-trivial. Therefore we have $l = p$. Then we get a contradiction with $(S^\perp/S)_p = (\mathbb{Z}/(p))^{\oplus 2\sigma}$, because $(G_n)_l$ is cyclic. Thus we have proved that every prime factor of n is contained in P_R. It then follows that p is contained in P_R, because p is a prime factor of $|G_{R,n}| = n \cdot |G_R|$.

We put

$$\begin{aligned} T &:= \{\, t \in G_n \mid N_R \cdot q_n(t) = 0 \,\}, \quad \text{and} \\ k &:= \min\{\, \nu \in \mathbb{Z}_{>0} \mid N_R \cdot \nu^2 = 0 \bmod 2n \,\}. \end{aligned}$$

Note that T is a cyclic group of order n/k generated by $k\varepsilon_n$. Let

$$\mathrm{pr}_n : G_{R,n} \to G_n$$

be the projection onto the second factor. Then $\mathrm{pr}_n(S)$ is contained in T. Indeed, since S is isotropic, we have $q_{R,n}(x, y) = 0$ for any $(x, y) \in S$ $(x \in G_R,\ y \in G_n)$. By the definition of N_R, we have

$$0 = N_R \cdot q_{R,n}(x, y) = N_R \cdot q_R(x) + N_R \cdot q_n(y) = N_R \cdot q_n(y)$$

for all $(x, y) \in S$. Let $i_n : G_n \hookrightarrow G_{R,n}$ be the inclusion given by $y \mapsto (0, y)$. Then $i_n^{-1}(S^\perp)$ coincides with the orthogonal complement of $\mathrm{pr}_n(S)$ in G_n, and hence $i_n^{-1}(S^\perp)$ contains the orthogonal complement $T^\perp$ of T in G_n, which is of order k generated by $(n/k) \cdot \varepsilon_n$. On the other hand, since $i_n^{-1}(S)$ is trivial by the assumed property $S \cap G_n = \{0\}$ of S, the composite

$$i_n^{-1}(S^\perp) \xrightarrow{i_n} S^\perp \longrightarrow S^\perp/S \cong (\mathbb{Z}/(p))^{\oplus 2\sigma}$$

is injective. Since $i_n^{-1}(S^\perp) \subset G_n$ is cyclic, it follows that $|i_n^{-1}(S^\perp)| = 1$ or $|i_n^{-1}(S^\perp)| = p$. Therefore $|T^\perp| = k$ is either 1 or p. If $k = 1$, then $N_R = 0 \bmod 2n$ holds. If $k = p$, then then $N_R \cdot p^2 = 0 \bmod 2n$ holds.

Finally, since $|S^{\perp}/S| = p^{2\sigma}$ is equal to $n|G_R|/|S|^2$, it follows that $n|G_R|$ is a square integer divisible by p^2. □

Lemma 3.3 implies, in particular, that if (R, n, σ) is geometrically realizable in characteristic p, then p divides $|G_R|$. The 'only if' part of Corollary 1.2 now follows. This assertion is also derived from a theorem of Goto [11, Theorem 3.7].

It is easy to list up all ADE-types R with rank 21. For each R, we calculate the set $NP(R)$, and make the list

$$\mathcal{R} := \{ \ [R, n, p] \ \mid \ \mathrm{rank}(R) = 21, \ \ (n, p) \in NP(R) \ \},$$

which consists of 20169 triples.

3.2. **Algorithm I.** We make the list

$$\overline{\mathcal{R}} := \{ \ [R, n] \ \mid \ [R, n, p] \in \mathcal{R} \quad \text{for some } p \ \},$$

which consists of 14487 pairs. For each pair $[R, n] \in \overline{\mathcal{R}}$, we do the following calculations.

Let $\mathcal{S}_{R,n}$ be the set of isotropic subgroups of $(G_{R,n}, q_{R,n})$, and let $\Gamma_{R,n}$ be the image of the natural homomorphism

$$\mathrm{Aut}(Q(R)) \times \mathrm{Aut}(I(n)) \ \to \ \mathrm{Aut}(G_{R,n}, q_{R,n}).$$

See [24] for the structure of the subgroup $\Gamma_{R,n}$ of $\mathrm{Aut}(G_{R,n}, q_{R,n})$. The group $\Gamma_{R,n}$ acts on $\mathcal{S}_{R,n}$. We find a subset $\mathcal{S}'_{R,n}$ of $\mathcal{S}_{R,n}$ such that the map

$$\mathcal{S}'_{R,n} \ \hookrightarrow \ \mathcal{S}_{R,n} \ \to \ \Gamma_{R,n} \backslash \mathcal{S}_{R,n}$$

is surjective. For each $S \in \mathcal{S}'_{R,n}$, we check the conditions in Corollary 3.2. Note that these conditions are invariant under the action of $\Gamma_{R,n}$. If all the conditions in Corollary 3.2 are satisfied, we put (R, n, σ) in the list of RDP-triples geometrically realizable in characteristic p.

3.3. **Algorithm II.** Algorithm I takes impractically long time when the coefficient of A_1 in R is large. We improve Algorithm I as follows. Observe the following trivial facts. Let S and T be two isotropic subgroups of $(G_{R,n}, q_{R,n})$ such that $T \subset S$.

(a) If $T \cap G_n$ is non-trivial, then $S \cap G_n$ is non-trivial.
(b) If $\varrho_{R,n}(T) > \varrho_R$, then $\varrho_{R,n}(S) > \varrho_R$.

We do the following calculations for each $[R, n, p] \in \mathcal{R}$.

Step 1. We decompose the set P_R of prime factors of $|G_R|$ into the disjoint union

$$P_R = A \bigsqcup B$$

of certain subsets A and B such that $p \in A$. Note that every prime factor of $|G_{R,n}|$ is contained in P_R by the definition of $NP(R)$. We then decompose $(G, q) := (G_{R,n}, q_{R,n})$ into the orthogonal direct sum

$$(G, q) = (G_A, q_A) \oplus (G_B, q_B),$$

where (G_A, q_A) and (G_B, q_B) are the orthogonal direct sum of (G_l, q_l) $(l \in A)$ and (G_l, q_l) $(l \in B)$, respectively.

Let $\Gamma_A \subset \mathrm{Aut}(G_A, q_A)$ and $\Gamma_B \subset \mathrm{Aut}(G_B, q_B)$ be the intersection of $\Gamma_{R,n} \subset \mathrm{Aut}(G, q)$ with the subgroups $\mathrm{Aut}(G_A, q_A)$ and $\mathrm{Aut}(G_B, q_B)$ of

$$\mathrm{Aut}(G, q) \ \cong \ \mathrm{Aut}(G_A, q_A) \times \mathrm{Aut}(G_B, q_B),$$

respectively.

Step 2. Let $\mathcal{S}_B$ be the set of the isotropic subgroups of (G_B, q_B), on which Γ_B acts. We find a subset $\mathcal{S}'_B$ of $\mathcal{S}_B$ such that

$$\mathcal{S}'_B \hookrightarrow \mathcal{S}_B \to \Gamma_B \backslash \mathcal{S}_B$$

is surjective. Then we make the subset $\mathcal{S}''_B$ of $\mathcal{S}'_B$ consisting of all $S_B \in \mathcal{S}'_B$ with the following properties;

- $|S_B|^2 = |G_B|$,
- $S_B \cap G_n$ is trivial, where $S_B \subset G_B$ is regarded as a subgroup of G, and
- $\varrho_{R,n}(S_B) = \varrho_R$.

Note that these properties are invariant under the action of Γ_B.

If $\mathcal{S}''_B \neq \emptyset$, then we go to the next step.

Step 3. Let $\mathcal{S}_A$ be the set of the isotropic subgroups of (G_A, q_A), on which Γ_A acts. We find a subset $\mathcal{S}'_A$ of $\mathcal{S}_A$ such that

$$\mathcal{S}'_A \hookrightarrow \mathcal{S}_A \to \Gamma_A \backslash \mathcal{S}_A$$

is surjective. Then, for each positive integer $\sigma \leq 10$, we make the subset $\mathcal{S}''_A(\sigma)$ of $\mathcal{S}'_A$ consisting of all $S_A \in \mathcal{S}'_A$ with the following properties;

- $S_A^{\perp}/S_A$ is a p-elementary group of order $p^{2\sigma}$, where $S_A^{\perp}$ is the orthogonal complement of S_A in (G_A, q_A),
- $S_A \cap G_n$ is trivial, where S_A is regarded as a subgroup of G, and
- $\varrho_{R,n}(S_A) = \varrho_R$.

Note again that these properties are invariant under the action of Γ_A.

If $\mathcal{S}''_A(\sigma) \neq \emptyset$, then we go to the next step.

Step 4. For each pair $(S_A, S_B) \in \mathcal{S}''_A(\sigma) \times \mathcal{S}''_B$, we make an isotropic subgroup

$$S := S_A \times S_B$$

of (G, q). Note that $S \cap G_n$ is still trivial. We check the condition

$$\varrho_{R,n}(S) = \varrho_R.$$

When $p = 2$, we further check the condition that the 2-elementary lattice Λ_S be of type I. If we find a pair (S_A, S_B) satisfying these conditions, then we put (R, n, σ) in the list of RDP-triples geometrically realizable in characteristic p.

In fact, Algorithm I is a special case of Algorithm II where B is taken to be an empty set.

3.4. **Remarks.** For many $[R, n, p] \in \mathcal{R}$, the set

$$\mathcal{S}''_A := \bigcup_{\sigma=1}^{10} \mathcal{S}''_A(\sigma)$$

or the set $\mathcal{S}''_B$ is empty, so that (R, n, σ) is not geometrically realizable in characteristic p for any σ.

Let $[R, n, p]$ be an element of $\mathcal{R}$. For a positive integer k, we denote by $\mathrm{ord}_p(k)$ the maximal integer ν such that $p^\nu \mid k$.

Lemma 3.4. *Let μ be the minimal non-negative integer such that $p^\mu x = 0$ holds for any x in the p-part $(G_R)_p$ of G_R. If $\mathrm{ord}_p(n) \geq \mu + 2$, then $\mathcal{S}''_A$ is empty.*

Proof. We put $\nu := \mathrm{ord}_p(n)$. Then the p-part $(G_n)_p$ of G_n is a cyclic group of order p^ν. Let η be a generator of $(G_n)_p$. Suppose that $\mathcal{S}''_A(\sigma) \neq \emptyset$. Then $\mathcal{S}''_{\{p\}}(\sigma)$ is not empty, because $p \in A$. Let $S_p^\perp$ be the orthogonal complement of S_p in (G_p, q_p). Since $S_p \cap (G_n)_p$ is trivial, we have $p^\mu y = 0$ in $(G_n)_p$ for any $(x, y) \in S_p$, and hence

$$b_{R,n}((x,y),(0,p^{\nu-2}\eta)) = b_n(p^{\nu-2}y,\eta) = 0 \bmod \mathbb{Z}$$

holds for any $(x, y) \in S_p$ by the assumption $\nu \geq \mu + 2$. Consequently, the cyclic group $\langle p^{\nu-2}\eta\rangle \subset (G_n)_p$ of order p^2 is contained in $S_p^\perp$. Since $S_p \cap (G_n)_p$ is trivial, we obtain an element of order p^2 in $S_p^\perp/S_p$. This contradicts the condition that $S_p^\perp/S_p$ be p-elementary. □

Lemma 3.5. *Suppose that $p \neq 3$ and $3 \in B$. Let a_2 and a_5 be the coefficients of A_2 and A_5 in R, respectively. Then $\mathcal{S}''_B$ is empty in the following cases:*

(i) $\mathrm{ord}_3(|G_R|) = 2$, $\mathrm{ord}_3(n) = 0$, *and* $a_2 = 2$;
(ii) $\mathrm{ord}_3(|G_R|) = 2$, $\mathrm{ord}_3(n) = 0$, *and* $a_5 = 2$;
(iii) $\mathrm{ord}_3(|G_R|) = 1$, $\mathrm{ord}_3(n) = 1$, $n/3 = 4 \bmod 6$ *and* $a_2 = 1$;
(iv) $\mathrm{ord}_3(|G_R|) = 1$, $\mathrm{ord}_3(n) = 1$, $n/3 = 2 \bmod 6$ *and* $a_5 = 1$.

Proof. We decompose (G_B, q_B) into the orthogonal direct sum of its 3-part (G_3, q_3) and the part prime to 3. In the cases above, G_3 is isomorphic to $(\mathbb{Z}/(3))^{\oplus 2}$, and q_3 is given by the matrix

$$\pm\begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix},$$

because $(q_{A_2})_3$ is given by the matrix $[-2/3]$, $(q_{A_5})_3$ is given by $[-2^2 \cdot 5/6] = [2/3]$, and $(q_n)_3$ is given by $[m^2/n] = [m/3]$, where $m := n/3$. Therefore (G_3, q_3) does not contain any non-zero isotropic vector. □

By the same argument, we can prove the following:

Lemma 3.6. *Suppose that $p \neq 7$, $\mathrm{ord}_7(|G_R|) = 2$ and $\mathrm{ord}_7(n) = 0$. Suppose also that $a_6 = 2$. If $7 \in B$, then $\mathcal{S}''_B$ is empty.* □

By these lemmas, we can remove 9247 triples from $\mathcal{R}$ and 7722 pairs from $\overline{\mathcal{R}}$, before we start the calculations.

3.5. **An example.** Let (G, q) be the discriminant form of the lattice $Q(21A_1, 2)$. Then (G, q) is naturally isomorphic to the vector space $\mathbb{F}_2^{21} \oplus \mathbb{F}_2$ with the quadratic form

$$(x_1, \dots, x_{21}, y) \;\mapsto\; \frac{1}{2}\Big(-\sum_{i=1}^{21} x_i^2 + y^2\Big) \;\in\; \mathbb{Q}/2\mathbb{Z}.$$

In Table 3.2, we present, for each σ, an example of an isotropic subgroup S_σ of (G, q) that yields an even overlattice of $Q(21A_1, 2)$ isomorphic to $\Lambda_{2,\sigma}$. A vector

$$(x_1, \dots, x_{21}, y) \in G, \quad \text{where } x_i \text{ and } y \text{ are 0 or 1},$$

is expressed by an integer

$$2^{21}x_1 + 2^{20}x_2 + \cdots + 2^2 x_{20} + 2x_{21} + y$$

in Table 3.2.

TABLE 3.2. Codes for supersingular $K3$ surfaces with 21 ordinary nodes

σ	Generators of $S_\sigma \subset G$
1	2097406, 1050398, 527206, 265642, 134866, 65657, 34069, 18113, 10633, 6693
2	2097183, 1048803, 525093, 263497, 133521, 69123, 37457, 21637, 14377
3	2097406, 1048607, 525091, 263493, 132745, 67985, 37457, 25649
4	2097183, 1048803, 525093, 263497, 137257, 75845, 51459
5	2097406, 1048607, 525091, 265253, 143401, 114737
6	2097183, 1048803, 526083, 276483, 245763
7	2097406, 1050398, 538654, 507905
8	2097406, 1081088, 1015809
9	2101246, 2093057
10	4194303

4. The algorithm for the list of elliptic triples

We use Proposition 2.14. First we make the list $\mathcal{E}$ of ADE-types R of rank 20 such that the RDP-triple $(R + A_1, 2, \sigma)$ is geometrically realizable for some σ and in some p. This list consists of 95 elements. For each $R \in \mathcal{E}$, we make the following calculations.

In the dual lattice

$$Q(R + A_1, 2)^\vee = Q(R)^\vee \oplus Q(A_1)^\vee \oplus I(2)^\vee,$$

we fix two vectors

$$h := (0, 0, 2) \quad \text{and} \quad z := (0, 2, 0),$$

both of which are in $Q(R + A_1, 2) \subset Q(R + A_1, 2)^\vee$. Let $\mathcal{S}_{R+A_1,2}$ be the set of isotropic subgroups of $(G_{R+A_1,2}, q_{R+A_1,2})$, and let $\Gamma_{R+A_1,2}$ be the image of the natural homomorphism

$$\mathrm{Aut}(Q(R)) \times \{\mathrm{Id}_{Q(A_1)}\} \times \{\mathrm{Id}_{I(2)}\} \;\to\; \mathrm{Aut}(G_{R+A_1,2}, q_{R+A_1,2}).$$

We find a subset $\mathcal{S}'_{R+A_1,2}$ of $\mathcal{S}_{R+A_1,2}$ such that the map

$$\mathcal{S}'_{R+A_1,2} \;\hookrightarrow\; \mathcal{S}_{R+A_1,2} \;\to\; \Gamma_{R+A_1,2}\backslash \mathcal{S}_{R+A_1,2}$$

is surjective. For each $S \in \mathcal{S}'_{R+A_1,2}$, we check the conditions in Corollary 3.2. If these conditions are satisfied, we then check the condition that $h - z$ be divisible by 2 in the overlattice Λ_S. Suppose that $h - z$ is divisible by 2 in Λ_S. We denote by U_S the indefinite unimodular sublattice of Λ_S spanned by $f := (h - z)/2$ and z, and calculate

$$MW := U_S^\perp / (U_S^\perp)_{\mathrm{roots}}.$$

If $p = 2$ or $p = 3$, we determine the quasi-ellipticity by Theorem 2.15; that is, we see whether $Q(R)$ is p-elementary or not. Then we put $\langle R, MW, \sigma\rangle$ in the list.

Remark 4.1. Let $\phi : X \to \mathbb{P}^1$ be an *elliptic* fibration on a $K3$ surface. Then the ADE-types and the Kodaira types of reducible fibers are corresponding in the following way:

$$A_1 \leftrightarrow \mathrm{I}_2 \text{ or III}, \quad A_2 \leftrightarrow \mathrm{I}_3 \text{ or IV}, \quad A_l\ (l>2) \leftrightarrow \mathrm{I}_{l+1},$$
$$D_m \leftrightarrow \mathrm{I}^*_{m-4}, \quad E_6 \leftrightarrow \mathrm{IV}^*, \quad E_7 \leftrightarrow \mathrm{III}^*, \quad E_8 \leftrightarrow \mathrm{II}^*.$$

When $\phi : X \to \mathbb{P}^1$ is a *quasi-elliptic* fibration in characteristic p, the correspondence becomes one-to-one:

$$\begin{array}{ll} A_1 \leftrightarrow \mathrm{III}, \quad D_{2m} \leftrightarrow \mathrm{I}^*_{2m-4}, \quad E_7 \leftrightarrow \mathrm{III}^*, \quad E_8 \leftrightarrow \mathrm{II}^* & \text{in characteristic } 2; \\ A_2 \leftrightarrow \mathrm{IV}, \quad E_6 \leftrightarrow \mathrm{IV}^*, \quad E_8 \leftrightarrow \mathrm{II}^* & \text{in characteristic } 3. \end{array}$$

Moreover, the Mordell-Weil group MW_ϕ is necessarily p-elementary, and the torsion rank $r := \dim_{\mathbb{F}_p} MW_\phi$ is related to the Artin invariant σ of X by the following formula, which is easily derived from the isomorphism (1.1):

$$2(\sigma + r) = \begin{cases} 2\sum \nu(\mathrm{I}^*_{2m}) + \nu(\mathrm{III}) + \nu(\mathrm{III}^*) & \text{in characteristic } 2, \\ \nu(\mathrm{IV}) + \nu(\mathrm{IV}^*) & \text{in characteristic } 3, \end{cases}$$

where $\nu(\tau)$ is the number of singular fibers of type τ. See [12, 13] for the detail.

Remark 4.2. In [20], it was shown that every supersingular $K3$ surface X in characteristic 2 has a quasi-elliptic pencil. Table QE shows that, if the Artin invariant of X is 10, then any quasi-elliptic pencil on X does not have a zero section.

Remark 4.3. In [24], it was shown that, over the complex number field, the torsion part of the Mordell-Weil group of an elliptic $K3$ surface is isomorphic to one of the following abelian groups:

$$0, [2], [3], [4], [5], [6], [7], [8], [2, 2], [4, 2], [6, 2], [3, 3], [4, 4],$$

where $[a] = \mathbb{Z}/(a)$ and $[a, b] = \mathbb{Z}/(a) \times \mathbb{Z}/(b)$. Therefore the appearance of $\langle 4A_5, 1, [3, 6]\rangle$ and $\langle 2A_9 + 2A_1, 1, [10]\rangle$ in Table E is a so-called pathological phenomenon in characteristic 2.

5. Supersingular $K3$ surfaces with 21 ordinary nodes

We prove Proposition 1.3. First note the following proposition, which holds in every characteristic.

Proposition 5.1 ([30] Proposition 1.7). *Let L be a nef line bundle with $L^2 = 2$ on a smooth $K3$ surface X. Then the complete linear system $|L|$ defines a surjective morphism $X \to \mathbb{P}^2$ if and only if there does* not *exist a divisor E such that $E.L = 1$ and $E^2 = 0$.* □

The proof in [30] is valid in positive characteristics if we replace Kawamata-Viehweg vanishing theorem by Nikulin's proposition [18, Proposition 0.1].

From now on, we will assume that the base field k is of characteristic 2.

For each $\sigma = 1, \dots, 10$, we explicitly construct from Table 3.2 a pair (Λ, h) of a lattice Λ isomorphic to $\Lambda_{2,\sigma}$ and a vector $h \in \Lambda$ with $h^2 = 2$ such that $\Sigma(h^\perp) = 21A_1$. It can be checked by direct calculations that the set of $u \in \Lambda$ satisfying $u^2 = 0$ and $uh = 1$ is empty. By Corollary 2.12 and Proposition 5.1, it follows that every supersingular $K3$ surface X has a nef line bundle L with $L^2 = 2$ such that $|L|$ defines a surjective morphism

$$\Phi : X \to \mathbb{P}^2$$

that decomposes into a composite

$$X \xrightarrow{f} Y \xrightarrow{\pi} \mathbb{P}^2 \tag{5.1}$$

of a contraction f of mutually disjoint twenty-one (-2)-curves and a finite morphism π of degree 2.

It remains to show that π is purely inseparable. Since $h^0(X, L^{\otimes m}) = m^2 + 2$ for each $m > 0$, the graded ring $\oplus_{m \geq 0} H^0(X, L^{\otimes m})$ is generated by elements

$$x_0, x_1, x_2 \in H^0(X, L), \quad w \in H^0(X, L^{\otimes 3}),$$

and the relations are generated by a relation

$$w^2 + C(x_0, x_1, x_2)w + G(x_0, x_1, x_2) = 0 \tag{5.2}$$

in degree 6. It is enough to show that the cubic homogeneous polynomial C is in fact zero.

We will assume that C is non-zero, and derive a contradiction. Let $\Gamma \subset \mathbb{P}^2$ be the divisor defined by the cubic equation

$$C(x_0, x_1, x_2) = 0.$$

We write Γ as $\sum \gamma_i \Gamma_i$, where Γ_i's are reduced irreducible curves distinct to each other, and γ_i's are positive integers. Let Y' be the surface defined by (5.2) in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, and let

$$X \xrightarrow{f'} Y' \xrightarrow{\pi'} \mathbb{P}^2 \tag{5.3}$$

be the natural morphisms. We have $\pi' \circ f' = \pi \circ f = \Phi$. The double cover π' is étale over $\mathbb{P}^2 \setminus \Gamma$. In particular, the singular locus $\operatorname{Sing}(Y')$ of Y' is contained in $\pi'^{-1}(\Gamma)$. We will show that Y' is normal, and hence the decomposition (5.3) of Φ coincides with the Stein factorization (5.1) of Φ. It suffices to show that $\dim \operatorname{Sing}(Y') = 0$ by [1, Corollary (3.15) in Chapter VII].

Suppose that $\dim \operatorname{Sing}(Y') = 1$. There exists a reduced irreducible component Γ_j of Γ such that $\pi'^{-1}(\Gamma_j)$ is contained in $\operatorname{Sing}(Y')$. Let ℓ be a general line on $\mathbb{P}^2$. The divisor $\tilde{\ell} := \Phi^*(\ell)$ on X is reduced and irreducible ([22, Proposition 2.6]), but may possibly be singular. The reduced irreducible curve $\tilde{\ell}' := \pi'^{-1}(\ell)$ on Y' is defined by a homogeneous equation

$$w^2 + C'(y_0, y_1)w + G'(y_0, y_1) = 0$$

of degree 6 in the weighted projective plane $\mathbb{P}(3, 1, 1)$. Hence its arithmetic genus $p_a(\tilde{\ell}')$ is 2. Let Q be an intersection point of ℓ and Γ_j. Since ℓ is general, we can find formal local parameters (ξ_0, ξ_1) of $\mathbb{P}^2$ at Q such that ℓ is defined by $\xi_0 = 0$ and Γ_j is defined by $\xi_1 = 0$. Since Y' is singular along $\pi'^{-1}(\Gamma_j)$, the surface Y' is defined over $k[[\xi_0, \xi_1]]$ by an equation of the form

$$\eta^2 \ + \ \xi_1^\nu \, a(\xi_0, \xi_1) \, \eta \ + \ \xi_1^\mu \, b(\xi_0, \xi_1) = 0 \qquad (\nu \geq 1, \ \ \mu \geq 2),$$

where $\eta = w + \beta(\xi_0, \xi_1)$ for some $\beta(\xi_0, \xi_1) \in k[[\xi_0, \xi_1]]$. Therefore the homomorphism $\mathcal{O}^\wedge_{Y', Q} \to (f'_* \mathcal{O}_X)^\wedge_{Y', Q}$ factors through

$$\mathcal{O}^\wedge_{Y', Q} \ \to \ \mathcal{O}^\wedge_{Y', Q}[\eta / \xi_1].$$

By local calculations, it follows that the cokernel of the homomorphism

$$\mathcal{O}_{\tilde{\ell}'} \ \to \ (f'|_{\tilde{\ell}})_* \mathcal{O}_{\tilde{\ell}}$$

has a non-trivial torsion subsheaf whose support is on $\pi'^{-1}(Q)$. Thus the arithmetic genus $p_a(\tilde{\ell})$ of $\tilde{\ell}$ is smaller that $p_a(\tilde{\ell}') = 2$, which contradicts $p_a(L) = 2$. Therefore Y' is normal.

Let $E_1, \dots, E_{21}$ be the (-2)-curves that are contracted by $f : X \to Y$. We denote by $\widetilde{\Gamma}_i$ the strict transform of Γ_i by $\Phi : X \to \mathbb{P}^2$, and put

$$\widetilde{\Gamma} := \sum \gamma_i \widetilde{\Gamma}_i.$$

Since Φ maps each E_i to a point of Γ, we have

$$\Phi^*(\Gamma) = \widetilde{\Gamma} + \sum \alpha_\nu E_\nu \ \in \ |3L|$$

with $\alpha_\nu \geq 1$ for each $\nu = 1, \dots, 21$, and hence

$$\widetilde{\Gamma}^2 = (3L)^2 + \sum \alpha_\nu^2 E_\nu^2 \leq -24. \tag{5.4}$$

Note that each $\widetilde{\Gamma}_i$ is irreducible. Let Δ_i be the reduced part of $\widetilde{\Gamma}_i$, and put $\widetilde{\Gamma}_i = \delta_i \Delta_i$, where $\delta_i = 1$ or 2. Since $\Delta_i^2 \geq -2$ for each i, we have

$$\widetilde{\Gamma}^2 \geq -2 \sum (\gamma_i \delta_i)^2. \tag{5.5}$$

From (5.4) and (5.5), it follows that only the following cases can occur:

Case I:	$\Gamma = 3\Gamma_1$,	$\widetilde{\Gamma} = 6\Delta_1$,
Case II-1:	$\Gamma = 2\Gamma_1 + \Gamma_2$,	$\widetilde{\Gamma} = 4\Delta_1 + \Delta_2$,
Case II-2:	$\Gamma = 2\Gamma_1 + \Gamma_2$,	$\widetilde{\Gamma} = 4\Delta_1 + 2\Delta_2$,
Case III:	$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$,	$\widetilde{\Gamma} = 2\Delta_1 + 2\Delta_2 + 2\Delta_3$,

where, in each cases, the components Γ_i are mutually distinct lines. Suppose that $\delta_j = 2$ for some Γ_j. We choose affine coordinates (x, y) on $\mathbb{P}^2$ such that Γ_j is defined by $x = 0$, and let

$$w^2 \ + \ b(x,y)\, xw \ + \ g(x,y) \ = \ 0$$

be the defining equation of the affine part of Y. Since $\delta_j = 2$ for some Γ_j, the curve defined by

$$w^2 + g(0, y) = 0$$

is non-reduced, and hence $g(x, y)$ is of the form $\gamma(y)^2 + h(x, y)\, x$. Replacing w by $w + \gamma(y)$, we can assume that $g(x, y)$ is equal to $h'(x, y)\, x$, where $h'(x, y)$ is a polynomial of degree ≤ 5. The points in $\mathrm{Sing}(Y) \cap \pi^{-1}(\Gamma_j)$ are therefore mapped by π to the intersection points of the line Γ_j and the curve $h' = 0$. Note that, since Y is normal, h' is not divisible by x. Hence $\mathrm{Sing}(Y) \cap \pi^{-1}(\Gamma_j)$ consists of at most 5 points. Therefore Cases I, II-2 and III cannot occur, because the cardinality of $\mathrm{Sing}(Y)$ is 21. In Case II-1, there are at least 16 points on $\mathrm{Sing}(Y) \cap \pi^{-1}(\Gamma_2)$. Hence

$$\Delta_2^2 = \widetilde{\Gamma}_2^2 \leq 2 + 16 \cdot (-2) = -30,$$

which is obviously impossible. □

$p=19$		
R	n	σ
$A_{18}+A_3$	76	1

$p=19$		
R	n	σ
$A_{18}+A_2+A_1$	114	1

$p=17$		
R	n	σ
$A_{16}+A_5$	102	1

$p=17$		
R	n	σ
$A_{16}+A_4+A_1$	170	1
$A_{16}+A_3+A_2$	204	1

$p=13$		
R	n	σ
$E_8+A_{12}+A_1$	26	1
$E_7+A_{12}+A_2$	78	1
$A_{12}+A_8+A_1$	234	1

$p=13$		
R	n	σ
$A_{12}+A_7+A_2$	312	1
$A_{12}+A_6+A_3$	364	1
$A_{12}+A_4+A_3+A_2$	780	1

$p=11$		
R	n	σ
$E_8+A_{10}+A_3$	44	1
$D_{11}+A_{10}$	44	1
$D_9+A_{10}+A_2$	132	1
$D_7+A_{10}+A_4$	220	1
A_{21}	22	1

$p=11$		
R	n	σ
$A_{11}+A_{10}$	132	1
$2A_{10}+A_1$	2	1
$A_{10}+A_8+A_3$	396	1
$A_{10}+A_6+A_5$	462	1
$A_{10}+A_6+A_4+A_1$	770	1

$p=7$		
R	n	σ
$E_8+E_7+A_6$	14	1
$E_8+D_7+A_6$	28	1
E_8+A_{13}	14	1
$E_8+A_7+A_6$	56	1
$E_8+2A_6+A_1$	2	1
$E_7+A_{13}+A_1$	14	1
$E_7+A_8+A_6$	126	1
$E_7+2A_6+A_2$	6	1
$E_6+A_9+A_6$	210	1
$E_6+2A_6+A_3$	12	1
$D_{15}+A_6$	28	1
$D_{14}+A_6+A_1$	14	1
$D_{12}+A_6+A_3$	28	1
D_9+2A_6	4	1
$D_9+A_6+A_4+A_2$	420	1
$D_8+A_7+A_6$	56	1
$D_7+A_{13}+A_1$	28	1
$D_5+A_{10}+A_6$	308	1

$p=7$		
R	n	σ
$D_5+2A_6+A_4$	20	1
$A_{20}+A_1$	42	1
$A_{15}+A_6$	112	1
$A_{15}+A_6$	28	1
$A_{14}+A_6+A_1$	210	1
$A_{13}+A_8$	126	1
$A_{13}+A_7+A_1$	56	1
$A_{13}+A_6+A_2$	6	1
$A_{13}+A_6+2A_1$	2	1
$A_{12}+A_6+A_2+A_1$	546	1
$A_{11}+A_6+A_4$	420	1
A_9+2A_6	10	1
$A_9+A_6+A_5+A_1$	210	1
$A_8+A_7+A_6$	504	1
$A_8+2A_6+A_1$	18	1
$A_8+A_6+A_5+A_2$	126	1
$3A_6+A_2+A_1$	42	$1,2$
$2A_6+A_5+A_4$	30	1

$p=5$			$p=5$		
R	n	σ	R	n	σ
$2E_8+A_4+A_1$	10	1	$A_{19}+A_2$	60	1
$E_8+D_7+A_4+A_2$	60	1	$A_{17}+A_4$	90	1
$E_8+A_9+A_4$	2	1	$A_{17}+A_4$	10	1
$E_8+A_9+A_3+A_1$	20	1	$A_{15}+A_4+A_2$	240	1
$E_8+A_6+A_4+A_3$	140	1	$A_{15}+A_4+A_2$	60	1
$E_8+3A_4+A_1$	10	$1,2$	$A_{14}+A_7$	120	1
$E_7+D_{10}+A_4$	10	1	$A_{14}+A_5+A_2$	30	1
$E_7+D_5+A_9$	20	1	$A_{14}+A_4+A_3$	12	1
E_7+A_{14}	30	1	$A_{14}+A_4+A_2+A_1$	2	1
$E_7+A_{10}+A_4$	110	1	$A_{13}+A_4+A_3+A_1$	140	1
$E_7+A_9+A_5$	30	1	$A_{12}+A_9$	130	1
$E_7+A_9+A_4+A_1$	2	1	$A_{12}+A_5+A_4$	390	1
$E_6+D_6+A_9$	30	1	$A_{11}+A_4+2A_3$	60	1
$D_{16}+A_4+A_1$	10	1	$A_{10}+A_9+A_2$	330	1
$D_{15}+A_4+A_2$	60	1	$A_{10}+A_7+A_4$	440	1
$D_{12}+A_4+A_3+A_2$	60	1	$A_9+A_8+A_4$	18	1
$D_{11}+A_9+A_1$	20	1	$A_9+A_8+A_3+A_1$	180	1
$D_{11}+A_6+A_4$	140	1	$A_9+A_7+A_5$	120	1
$D_7+A_9+A_4+A_1$	4	1	$A_9+A_6+A_4+A_2$	42	1
$D_7+A_7+A_4+A_3$	40	1	$A_9+A_5+A_4+A_3$	12	1
$D_7+3A_4+A_2$	60	$1,2$	A_9+3A_4	2	$1,2$
$D_6+A_{11}+A_4$	60	1	$A_9+2A_4+A_3+A_1$	20	$1,2$
$D_6+A_9+A_6$	70	1	$2A_8+A_4+A_1$	90	1
$D_6+A_9+A_4+A_2$	6	1	$A_8+A_6+A_4+A_3$	1260	1
$D_5+A_{14}+A_2$	20	1	$A_6+3A_4+A_3$	140	$1,2$
			$5A_4+A_1$	10	$1,2,3$

$p=3$			$p=3$		
R	n	σ	R	n	σ
$2E_8+A_5$	6	1	$E_8+A_7+A_5+A_1$	24	1
$2E_8+A_3+A_2$	12	1	$E_8+A_7+A_4+A_2$	120	1
$2E_8+2A_2+A_1$	2	1	$E_8+A_6+3A_2+A_1$	42	$1,2$
$E_8+E_7+E_6$	6	1	$E_8+2A_5+A_3$	4	1
$E_8+E_7+A_4+A_2$	30	1	$E_8+A_5+A_4+2A_2$	30	$1,2$
$E_8+E_7+3A_2$	6	$1,2$	$E_8+A_5+4A_2$	6	$1,2,3$
$E_8+2E_6+A_1$	2	1	$E_8+A_3+5A_2$	12	$2,3$
$E_8+E_6+A_6+A_1$	42	1	$E_8+6A_2+A_1$	2	$2,3$
$E_8+E_6+A_5+A_2$	6	$1,2$	$2E_7+D_5+A_2$	12	1
$E_8+E_6+A_3+2A_2$	12	$1,2$	$2E_7+A_3+2A_2$	4	1
$E_8+E_6+3A_2+A_1$	2	2	$E_7+2E_6+A_2$	6	$1,2$
$E_8+D_{11}+A_2$	12	1	$E_7+E_6+A_7+A_1$	24	1
$E_8+D_9+2A_2$	4	1	$E_7+E_6+A_5+A_3$	4	1
$E_8+D_5+A_6+A_2$	84	1	$E_7+E_6+A_4+2A_2$	30	$1,2$
$E_8+D_5+A_4+2A_2$	20	1	$E_7+E_6+4A_2$	6	$1,2,3$
$E_8+A_{11}+A_2$	4	1	$E_7+D_{10}+2A_2$	2	1
$E_8+A_{11}+2A_1$	12	1	$E_7+D_8+A_5+A_1$	6	1
$E_8+A_{10}+A_2+A_1$	66	1	$E_7+D_5+A_5+2A_2$	12	$1,2$
$E_8+A_9+2A_2$	10	1	$E_7+D_4+2A_5$	2	1
$E_8+A_8+2A_2+A_1$	18	$1,2$	$E_7+A_{10}+2A_2$	22	1

$p=3$		
R	n	σ
$E_7+A_9+2A_2+A_1$	10	1
$E_7+A_7+3A_2+A_1$	24	1, 2
$E_7+A_6+A_4+2A_2$	70	1
$E_7+A_5+A_3+3A_2$	4	2
$E_7+A_4+5A_2$	30	2, 3
E_7+7A_2	6	2, 3, 4
$3E_6+A_3$	12	1, 2
$3E_6+A_2+A_1$	2	1, 2
$2E_6+D_9$	4	1
$2E_6+D_5+A_4$	20	1
$2E_6+A_9$	10	1
$2E_6+A_8+A_1$	18	1, 2
$2E_6+A_6+A_2+A_1$	42	1, 2
$2E_6+A_5+A_4$	30	1, 2
$2E_6+A_5+2A_2$	6	1, 2, 3
$2E_6+A_3+3A_2$	12	1, 2, 3
$2E_6+4A_2+A_1$	2	1, 2, 3
$E_6+D_{14}+A_1$	6	1
$E_6+D_{11}+2A_2$	12	1, 2
$E_6+D_{10}+A_5$	2	1
$E_6+D_9+3A_2$	4	2
$E_6+D_5+A_9+A_1$	60	1
$E_6+D_5+A_6+2A_2$	84	1, 2
$E_6+D_5+2A_5$	12	1, 2
$E_6+D_5+A_4+3A_2$	20	2
$E_6+A_{14}+A_1$	10	1
$E_6+A_{13}+2A_1$	42	1
$E_6+A_{11}+2A_2$	4	1, 2
$E_6+A_{11}+A_2+2A_1$	12	1, 2
$E_6+A_{10}+A_5$	22	1
$E_6+A_{10}+A_4+A_1$	330	1
$E_6+A_{10}+2A_2+A_1$	66	1, 2
$E_6+A_9+A_5+A_1$	10	1
$E_6+A_9+3A_2$	10	2
$E_6+A_8+3A_2+A_1$	18	1, 2, 3
$E_6+A_7+A_5+A_2+A_1$	24	1, 2
$E_6+A_7+A_4+2A_2$	120	1, 2
$E_6+A_6+A_5+A_4$	70	1
$E_6+A_6+4A_2+A_1$	42	1, 2, 3
$E_6+2A_5+A_3+A_2$	4	1, 2
$E_6+A_5+A_4+3A_2$	30	1, 2, 3
$E_6+A_5+5A_2$	6	1, 2, 3, 4
$E_6+A_3+6A_2$	12	1, 2, 3, 4
$E_6+7A_2+A_1$	2	2, 3, 4
$D_{19}+A_2$	12	1
$D_{17}+2A_2$	4	1
$D_{16}+A_5$	6	1
$D_{16}+A_3+A_2$	12	1
$D_{16}+2A_2+A_1$	2	1
$D_{14}+A_4+A_2+A_1$	30	1

$p=3$		
R	n	σ
$D_{14}+3A_2+A_1$	6	1, 2
$D_{13}+A_6+A_2$	84	1
$D_{13}+A_4+2A_2$	20	1
$D_{12}+D_7+A_2$	12	1
$D_{12}+D_5+2A_2$	4	1
$D_{11}+2A_5$	4	1
$D_{11}+5A_2$	12	2, 3
$D_{10}+D_6+A_5$	6	1
$D_{10}+A_{11}$	12	1
$D_{10}+A_6+A_5$	42	1
$D_{10}+2A_5+A_1$	2	1
$D_{10}+A_5+3A_2$	2	2
D_9+6A_2	4	2, 3
$2D_8+A_3+A_2$	12	1
$D_8+A_{11}+A_2$	4	1
$D_8+A_7+A_5+A_1$	24	1
$D_8+A_7+A_4+A_2$	120	1
$D_8+2A_5+A_2+A_1$	6	1, 2
D_7+A_{14}	60	1
$D_7+A_{12}+A_2$	156	1
$D_7+A_{11}+A_3$	12	1
$D_7+A_{11}+A_2+A_1$	2	1
$D_7+A_9+A_5$	60	1
$D_7+2A_5+A_4$	20	1
$D_6+D_5+2A_5$	4	1
$2D_5+A_7+2A_2$	8	1
$D_5+A_{13}+A_2+A_1$	84	1
$D_5+A_{12}+2A_2$	52	1
$D_5+A_{11}+A_4+A_1$	30	1
$D_5+A_{11}+A_3+A_2$	4	1
$D_5+A_{11}+2A_2+A_1$	6	1, 2
$D_5+A_9+3A_2+A_1$	60	1, 2
$D_5+A_8+A_4+2A_2$	180	1, 2
$D_5+2A_7+A_2$	12	1
$D_5+A_6+5A_2$	84	2, 3
$D_5+2A_5+3A_2$	12	1, 2, 3
$D_5+A_4+6A_2$	20	2, 3
$D_4+A_{15}+A_2$	12	1
$D_4+A_{11}+A_6$	84	1
$D_4+A_{11}+A_4+A_2$	20	1
$D_4+3A_5+A_2$	2	1, 2
$A_{17}+2A_2$	2	1
$A_{16}+2A_2+A_1$	34	1
$A_{15}+2A_2+2A_1$	4	1
$A_{14}+3A_2+A_1$	10	1, 2
$A_{13}+A_4+2A_2$	70	1
$A_{13}+3A_2+2A_1$	42	1, 2
$A_{12}+A_4+2A_2+A_1$	130	1
$A_{11}+5A_2$	4	2, 3
$A_{11}+4A_2+2A_1$	12	1, 2, 3

$p=3$			$p=3$		
R	n	σ	R	n	σ
$A_{10}+A_5+3A_2$	22	2	$A_7+A_4+5A_2$	120	2, 3
$A_{10}+A_4+3A_2+A_1$	330	1, 2	$A_6+A_5+A_4+3A_2$	70	2
$A_{10}+5A_2+A_1$	66	2, 3	$A_6+7A_2+A_1$	42	2, 3, 4
$A_9+A_5+3A_2+A_1$	10	2	$2A_5+A_3+4A_2$	4	1, 2, 3
$A_9+2A_4+2A_2$	10	1	$A_5+A_4+6A_2$	30	1, 2, 3, 4
A_9+6A_2	10	2, 3	A_5+8A_2	6	1, 2, 3, 4, 5
$A_8+6A_2+A_1$	18	1, 2, 3, 4	A_3+9A_2	12	1, 2, 3, 4, 5
$A_7+A_5+4A_2+A_1$	24	1, 2, 3	$10A_2+A_1$	2	1, 2, 3, 4, 5

$p=2$		
R	n	σ
$2E_8+D_5$	4	1
$2E_8+D_4+A_1$	2	1
$2E_8+A_2+3A_1$	6	1
$2E_8+5A_1$	2	2
$E_8+E_7+D_6$	2	1
$E_8+E_7+D_4+A_2$	6	1
$E_8+E_7+D_4+2A_1$	2	2
$E_8+E_7+A_3+3A_1$	4	$1,2$
$E_8+E_7+A_2+4A_1$	6	2
$E_8+E_7+6A_1$	2	$2,3$
$E_8+E_6+D_7$	12	1
$E_8+E_6+D_4+A_3$	12	$1,2$
$E_8+E_6+A_4+3A_1$	30	1
E_8+D_{13}	4	1
$E_8+D_{12}+A_1$	2	1
$E_8+D_{10}+A_2+A_1$	6	1
$E_8+D_{10}+3A_1$	2	$1,2$
$E_8+D_9+D_4$	4	$1,2$
$E_8+D_9+A_4$	20	1
$E_8+D_8+D_5$	4	$1,2$
$E_8+D_8+D_4+A_1$	2	2
$E_8+D_8+A_2+3A_1$	6	2
$E_8+D_8+5A_1$	2	$2,3$
$E_8+D_7+6A_1$	4	$2,3$
$E_8+2D_6+A_1$	2	2
$E_8+D_6+D_4+A_2+A_1$	6	2
$E_8+D_6+D_4+3A_1$	2	$2,3$
$E_8+D_6+A_3+4A_1$	4	$2,3$
$E_8+D_6+A_2+5A_1$	6	$2,3$
$E_8+D_6+7A_1$	2	$3,4$
$E_8+D_5+2D_4$	4	$2,3$
$E_8+D_5+D_4+A_4$	20	$1,2$
$E_8+D_5+A_5+3A_1$	12	$1,2$
$E_8+D_5+8A_1$	4	$3,4$
$E_8+3D_4+A_1$	2	3
$E_8+2D_4+A_2+3A_1$	6	3
$E_8+2D_4+5A_1$	2	$3,4$
$E_8+D_4+A_9$	10	1

$p=2$		
R	n	σ
$E_8+D_4+A_8+A_1$	18	1
$E_8+D_4+A_6+A_2+A_1$	42	1
$E_8+D_4+A_5+A_4$	30	1
$E_8+D_4+A_3+6A_1$	4	$3,4$
$E_8+D_4+A_2+7A_1$	6	$3,4$
$E_8+D_4+9A_1$	2	$3,4,5$
$E_8+A_{10}+3A_1$	22	1
$E_8+A_9+4A_1$	10	$1,2$
$E_8+A_8+5A_1$	18	2
$E_8+A_7+A_2+4A_1$	24	$1,2$
$E_8+A_6+A_4+3A_1$	70	1
$E_8+A_6+A_2+5A_1$	42	2
$E_8+A_5+A_4+4A_1$	30	2
$E_8+A_5+A_3+5A_1$	12	$2,3$
$E_8+A_4+A_3+6A_1$	20	$2,3$
$E_8+A_3+10A_1$	4	$4,5$
$E_8+A_2+11A_1$	6	$4,5$
E_8+13A_1	2	$4,5,6$
$3E_7$	2	1
$2E_7+D_7$	4	1
$2E_7+D_6+A_1$	2	$1,2$
$2E_7+D_5+2A_1$	4	$1,2$
$2E_7+D_4+A_3$	4	$1,2$
$2E_7+D_4+A_2+A_1$	6	$1,2$
$2E_7+D_4+3A_1$	2	$1,2,3$
$2E_7+A_4+A_3$	20	1
$2E_7+A_3+4A_1$	4	$2,3$
$2E_7+A_2+5A_1$	6	$2,3$
$2E_7+7A_1$	2	$2,3,4$
$E_7+E_6+D_5+3A_1$	12	$1,2$
$E_7+E_6+D_4+A_4$	30	1
$E_7+E_6+A_4+4A_1$	30	2
$E_7+E_6+A_3+5A_1$	12	$2,3$
E_7+D_{14}	2	1
$E_7+D_{12}+A_2$	6	1
$E_7+D_{12}+2A_1$	2	$1,2$
$E_7+D_{11}+3A_1$	4	$1,2$
$E_7+D_{10}+D_4$	2	$1,2$

$p=2$		
R	n	σ
$E_7+D_{10}+A_3+A_1$	4	$1,2$
$E_7+D_{10}+A_2+2A_1$	6	$1,2$
$E_7+D_{10}+4A_1$	2	$1,2,3$
$E_7+D_9+A_5$	12	1
$E_7+D_9+5A_1$	4	$2,3$
$E_7+D_8+D_6$	2	$1,2$
$E_7+D_8+D_4+A_2$	6	2
$E_7+D_8+D_4+2A_1$	2	$1,2,3$
$E_7+D_8+A_3+3A_1$	4	$1,2,3$
$E_7+D_8+A_2+4A_1$	6	$1,2,3$
$E_7+D_8+6A_1$	2	$2,3,4$
$E_7+D_7+D_6+A_1$	4	$1,2$
$E_7+D_7+D_4+3A_1$	4	$2,3$
$E_7+D_7+A_5+2A_1$	12	$1,2$
$E_7+D_7+A_4+3A_1$	20	$1,2$
$E_7+D_7+7A_1$	4	$3,4$
$E_7+2D_6+A_2$	6	$1,2$
$E_7+2D_6+2A_1$	2	$1,2,3$
$E_7+D_6+D_5+3A_1$	4	$1,2,3$
$E_7+D_6+2D_4$	2	$2,3$
$E_7+D_6+D_4+A_3+A_1$	4	$1,2,3$
$E_7+D_6+D_4+A_2+2A_1$	6	$2,3$
$E_7+D_6+D_4+4A_1$	2	$2,3,4$
$E_7+D_6+A_8$	18	1
$E_7+D_6+A_6+A_2$	42	1
$E_7+D_6+A_5+A_3$	12	$1,2$
$E_7+D_6+A_4+A_3+A_1$	20	$1,2$
$E_7+D_6+A_3+5A_1$	4	$2,3,4$
$E_7+D_6+A_2+6A_1$	6	$2,3,4$
$E_7+D_6+8A_1$	2	$2,3,4,5$
$E_7+D_5+D_4+A_5$	12	$1,2$
$E_7+D_5+D_4+5A_1$	4	$2,3,4$
$E_7+D_5+A_5+4A_1$	12	$2,3$
$E_7+D_5+A_4+5A_1$	20	$2,3$
$E_7+D_5+9A_1$	4	$3,4,5$
$E_7+3D_4+A_2$	6	3
$E_7+3D_4+2A_1$	2	$2,3,4$
$E_7+2D_4+A_3+3A_1$	4	$2,3,4$
$E_7+2D_4+A_2+4A_1$	6	$2,3,4$
$E_7+2D_4+6A_1$	2	$2,3,4,5$
$E_7+D_4+A_{10}$	22	1
$E_7+D_4+A_9+A_1$	10	$1,2$
$E_7+D_4+A_8+2A_1$	18	2
$E_7+D_4+A_7+A_2+A_1$	24	$1,2$
$E_7+D_4+A_6+A_4$	70	1
$E_7+D_4+A_6+A_2+2A_1$	42	2
$E_7+D_4+A_5+A_4+A_1$	30	$1,2$
$E_7+D_4+A_5+A_3+2A_1$	12	$1,2,3$
$E_7+D_4+A_4+A_3+3A_1$	20	$2,3$
$E_7+D_4+A_3+7A_1$	4	$2,3,4,5$

$p=2$		
R	n	σ
$E_7+D_4+A_2+8A_1$	6	$3,4,5$
$E_7+D_4+10A_1$	2	$3,4,5,6$
$E_7+A_{11}+A_3$	6	1
$E_7+A_{10}+4A_1$	22	2
$E_7+A_9+A_3+A_2$	60	1
$E_7+A_9+5A_1$	10	$2,3$
$E_7+A_8+6A_1$	18	$2,3$
$E_7+A_7+2A_3+A_1$	8	$1,2$
$E_7+A_7+A_2+5A_1$	24	$2,3$
$E_7+A_6+A_5+A_3$	84	1
$E_7+A_6+A_4+4A_1$	70	2
$E_7+A_6+A_2+6A_1$	42	$2,3$
$E_7+A_5+A_4+5A_1$	30	$2,3$
$E_7+A_5+A_3+6A_1$	12	$2,3,4$
$E_7+A_4+A_3+7A_1$	20	$3,4$
$E_7+A_3+11A_1$	4	$3,4,5,6$
$E_7+A_2+12A_1$	6	$3,4,5,6$
E_7+14A_1	2	$3,4,5,6,7$
$3E_6+3A_1$	6	1
$2E_6+D_4+A_5$	6	1
$2E_6+A_5+4A_1$	6	2
E_6+D_{15}	12	1
$E_6+D_{12}+A_3$	12	$1,2$
$E_6+D_{11}+D_4$	12	$1,2$
$E_6+D_{10}+A_4+A_1$	30	1
$E_6+D_9+A_6$	84	1
$E_6+D_9+6A_1$	12	$2,3$
$E_6+D_8+D_7$	12	$1,2$
$E_6+D_8+D_4+A_3$	12	$1,2,3$
$E_6+D_8+A_4+3A_1$	30	2
$E_6+D_7+2D_4$	12	$2,3$
$E_6+D_7+8A_1$	12	$3,4$
$E_6+D_6+D_5+4A_1$	12	$2,3$
$E_6+D_6+D_4+A_4+A_1$	30	2
$E_6+D_6+A_4+5A_1$	30	$2,3$
$E_6+D_6+A_3+6A_1$	12	$2,3,4$
$E_6+D_5+D_4+A_6$	84	$1,2$
$E_6+D_5+D_4+6A_1$	12	$3,4$
$E_6+D_5+10A_1$	12	$4,5$
$E_6+3D_4+A_3$	12	$2,3,4$
$E_6+2D_4+A_4+3A_1$	30	3
$E_6+D_4+A_{11}$	4	$1,2$
$E_6+D_4+A_{10}+A_1$	66	1
$E_6+D_4+A_8+A_2+A_1$	18	1
$E_6+D_4+A_7+A_4$	120	$1,2$
$E_6+D_4+2A_5+A_1$	6	$1,2$
$E_6+D_4+A_4+7A_1$	30	$3,4$
$E_6+D_4+A_3+8A_1$	12	$3,4,5$
E_6+A_{15}	12	1
$E_6+A_{12}+A_3$	156	1

$p=2$		
R	n	σ
$E_6+A_{11}+A_4$	20	1
$E_6+A_{11}+A_3+A_1$	2	1
$E_6+A_{10}+5A_1$	66	2
$E_6+A_8+A_2+5A_1$	18	2
$E_6+A_6+A_3+6A_1$	84	$2,3$
E_6+3A_5	2	1
$E_6+2A_5+5A_1$	6	$2,3$
$E_6+A_4+11A_1$	30	$4,5$
$E_6+A_3+12A_1$	12	$4,5,6$
D_{21}	4	1
$D_{20}+A_1$	2	1
$D_{18}+A_2+A_1$	6	1
$D_{18}+3A_1$	2	$1,2$
$D_{17}+D_4$	4	$1,2$
$D_{17}+A_4$	20	1
$D_{16}+D_5$	4	$1,2$
$D_{16}+D_4+A_1$	2	$1,2$
$D_{16}+A_2+3A_1$	6	$1,2$
$D_{16}+5A_1$	2	$2,3$
$D_{15}+6A_1$	4	$2,3$
$D_{14}+D_6+A_1$	2	$1,2$
$D_{14}+D_4+A_2+A_1$	6	$1,2$
$D_{14}+D_4+3A_1$	2	$2,3$
$D_{14}+A_3+4A_1$	4	$1,2,3$
$D_{14}+A_2+5A_1$	6	$2,3$
$D_{14}+7A_1$	2	$2,3,4$
$D_{13}+D_8$	4	$1,2$
$D_{13}+2D_4$	4	$2,3$
$D_{13}+D_4+A_4$	20	$1,2$
$D_{13}+A_5+3A_1$	12	$1,2$
$D_{13}+8A_1$	4	$3,4$
$D_{12}+D_9$	4	$1,2$
$D_{12}+D_8+A_1$	2	$1,2$
$D_{12}+D_6+A_2+A_1$	6	$1,2$
$D_{12}+D_6+3A_1$	2	$1,2,3$
$D_{12}+D_5+D_4$	4	$1,2,3$
$D_{12}+D_5+A_4$	20	$1,2$
$D_{12}+2D_4+A_1$	2	$2,3$
$D_{12}+D_4+A_2+3A_1$	6	$2,3$
$D_{12}+D_4+5A_1$	2	$2,3,4$
$D_{12}+A_9$	10	1
$D_{12}+A_8+A_1$	18	1
$D_{12}+A_6+A_2+A_1$	42	1
$D_{12}+A_5+A_4$	30	1
$D_{12}+A_3+6A_1$	4	$2,3,4$
$D_{12}+A_2+7A_1$	6	$2,3,4$
$D_{12}+9A_1$	2	$3,4,5$
$D_{11}+D_6+4A_1$	4	$2,3$
$D_{11}+D_4+6A_1$	4	$3,4$
$D_{11}+A_5+5A_1$	12	$2,3$

$p=2$		
R	n	σ
$D_{11}+A_4+6A_1$	20	$2,3$
$D_{11}+10A_1$	4	$4,5$
$2D_{10}+A_1$	2	$1,2$
$D_{10}+D_8+A_2+A_1$	6	$1,2$
$D_{10}+D_8+3A_1$	2	$1,2,3$
$D_{10}+D_7+4A_1$	4	$1,2,3$
$D_{10}+D_6+D_4+A_1$	2	$1,2,3$
$D_{10}+D_6+A_3+2A_1$	4	$1,2,3$
$D_{10}+D_6+A_2+3A_1$	6	$1,2,3$
$D_{10}+D_6+5A_1$	2	$2,3,4$
$D_{10}+D_5+A_5+A_1$	12	$1,2$
$D_{10}+D_5+6A_1$	4	$2,3,4$
$D_{10}+2D_4+A_2+A_1$	6	$2,3$
$D_{10}+2D_4+3A_1$	2	$2,3,4$
$D_{10}+D_4+A_3+4A_1$	4	$2,3,4$
$D_{10}+D_4+A_2+5A_1$	6	$2,3,4$
$D_{10}+D_4+7A_1$	2	$2,3,4,5$
$D_{10}+A_{10}+A_1$	22	1
$D_{10}+A_9+2A_1$	10	$1,2$
$D_{10}+A_8+3A_1$	18	$1,2$
$D_{10}+A_7+A_2+2A_1$	24	$1,2$
$D_{10}+A_6+A_4+A_1$	70	1
$D_{10}+A_6+A_2+3A_1$	42	$1,2$
$D_{10}+A_5+A_4+2A_1$	30	$1,2$
$D_{10}+A_5+A_3+3A_1$	12	$1,2,3$
$D_{10}+A_4+A_3+4A_1$	20	$1,2,3$
$D_{10}+A_3+8A_1$	4	$3,4,5$
$D_{10}+A_2+9A_1$	6	$3,4,5$
$D_{10}+11A_1$	2	$3,4,5,6$
$D_9+D_8+D_4$	4	$1,2,3$
$D_9+D_8+A_4$	20	$1,2$
$D_9+D_6+A_5+A_1$	12	$1,2$
$D_9+D_6+6A_1$	4	$2,3,4$
$D_9+D_5+A_7$	8	$1,2$
D_9+3D_4	4	$2,3,4$
$D_9+2D_4+A_4$	20	$2,3$
$D_9+D_4+A_5+3A_1$	12	$2,3$
$D_9+D_4+8A_1$	4	$3,4,5$
D_9+A_{12}	52	1
$D_9+A_{11}+A_1$	6	1
$D_9+A_9+A_2+A_1$	60	1
$D_9+A_8+A_4$	180	1
$D_9+A_5+7A_1$	12	$3,4$
$D_9+A_4+8A_1$	20	$3,4$
D_9+12A_1	4	$4,5,6$
$2D_8+D_5$	4	$1,2,3$
$2D_8+D_4+A_1$	2	$1,2,3$
$2D_8+A_2+3A_1$	6	$1,2,3$
$2D_8+5A_1$	2	$1,2,3,4$
$D_8+D_7+6A_1$	4	$2,3,4$

$p=2$		
R	n	σ
$D_8+2D_6+A_1$	2	$1,2,3$
$D_8+D_6+D_4+A_2+A_1$	6	$1,2,3$
$D_8+D_6+D_4+3A_1$	2	$1,2,3,4$
$D_8+D_6+A_3+4A_1$	4	$1,2,3,4$
$D_8+D_6+A_2+5A_1$	6	$2,3,4$
$D_8+D_6+7A_1$	2	$2,3,4,5$
$D_8+D_5+2D_4$	4	$1,2,3,4$
$D_8+D_5+D_4+A_4$	20	$1,2,3$
$D_8+D_5+A_5+3A_1$	12	$1,2,3$
$D_8+D_5+8A_1$	4	$2,3,4,5$
$D_8+3D_4+A_1$	2	$2,3,4$
$D_8+2D_4+A_2+3A_1$	6	$2,3,4$
$D_8+2D_4+5A_1$	2	$2,3,4,5$
$D_8+D_4+A_9$	10	2
$D_8+D_4+A_8+A_1$	18	2
$D_8+D_4+A_6+A_2+A_1$	42	2
$D_8+D_4+A_5+A_4$	30	2
$D_8+D_4+A_3+6A_1$	4	$2,3,4,5$
$D_8+D_4+A_2+7A_1$	6	$2,3,4,5$
$D_8+D_4+9A_1$	2	$2,3,4,5,6$
$D_8+A_{10}+3A_1$	22	2
$D_8+A_9+4A_1$	10	$1,2,3$
$D_8+A_8+5A_1$	18	$2,3$
$D_8+A_7+A_2+4A_1$	24	$1,2,3$
$D_8+A_6+A_4+3A_1$	70	2
$D_8+A_6+A_2+5A_1$	42	$2,3$
$D_8+A_5+A_4+4A_1$	30	$1,2,3$
$D_8+A_5+A_3+5A_1$	12	$1,2,3,4$
$D_8+A_4+A_3+6A_1$	20	$2,3,4$
$D_8+A_3+10A_1$	4	$3,4,5,6$
$D_8+A_2+11A_1$	6	$3,4,5,6$
D_8+13A_1	2	$3,4,5,6,7$
$D_7+2D_6+2A_1$	4	$1,2,3$
$D_7+D_6+D_4+4A_1$	4	$2,3,4$
$D_7+D_6+A_5+3A_1$	12	$1,2,3$
$D_7+D_6+A_4+4A_1$	20	$2,3$
$D_7+D_6+8A_1$	4	$3,4,5$
$D_7+2D_4+6A_1$	4	$2,3,4,5$
$D_7+D_4+A_5+5A_1$	12	$2,3,4$
$D_7+D_4+A_4+6A_1$	20	$3,4$
$D_7+D_4+10A_1$	4	$3,4,5,6$
$D_7+A_{11}+3A_1$	6	$1,2$
$D_7+A_9+A_2+3A_1$	60	$1,2$
$D_7+A_7+A_3+4A_1$	8	$1,2,3$
$D_7+A_6+A_5+3A_1$	84	$1,2$
$D_7+A_5+9A_1$	12	$3,4,5$
$D_7+A_4+10A_1$	20	$4,5$
D_7+14A_1	4	$3,4,5,6,7$
$3D_6+A_3$	4	$1,2,3$
$3D_6+A_2+A_1$	6	$1,2,3$

$p=2$		
R	n	σ
$3D_6+3A_1$	2	$1,2,3,4$
$2D_6+D_5+4A_1$	4	$1,2,3,4$
$2D_6+2D_4+A_1$	2	$1,2,3,4$
$2D_6+D_4+A_3+2A_1$	4	$1,2,3,4$
$2D_6+D_4+A_2+3A_1$	6	$1,2,3,4$
$2D_6+D_4+5A_1$	2	$1,2,3,4,5$
$2D_6+A_9$	10	$1,2$
$2D_6+A_8+A_1$	18	2
$2D_6+A_7+A_2$	24	$1,2$
$2D_6+A_6+A_2+A_1$	42	2
$2D_6+A_5+A_4$	30	$1,2$
$2D_6+A_5+A_3+A_1$	12	$1,2,3$
$2D_6+A_4+A_3+2A_1$	20	$1,2,3$
$2D_6+A_3+6A_1$	4	$2,3,4,5$
$2D_6+A_2+7A_1$	6	$2,3,4,5$
$2D_6+9A_1$	2	$2,3,4,5,6$
$D_6+D_5+D_4+A_5+A_1$	12	$1,2,3$
$D_6+D_5+D_4+6A_1$	4	$2,3,4,5$
$D_6+D_5+A_5+5A_1$	12	$2,3,4$
$D_6+D_5+A_4+6A_1$	20	$2,3,4$
$D_6+D_5+10A_1$	4	$3,4,5,6$
$D_6+3D_4+A_2+A_1$	6	$2,3,4$
$D_6+3D_4+3A_1$	2	$1,2,3,4,5$
$D_6+2D_4+A_3+4A_1$	4	$1,2,3,4,5$
$D_6+2D_4+A_2+5A_1$	6	$2,3,4,5$
$D_6+2D_4+7A_1$	2	$2,3,4,5,6$
$D_6+D_4+A_{10}+A_1$	22	2
$D_6+D_4+A_9+2A_1$	10	$2,3$
$D_6+D_4+A_8+3A_1$	18	$2,3$
$D_6+D_4+A_7+A_2+2A_1$	24	$2,3$
$D_6+D_4+A_6+A_4+A_1$	70	2
$D_6+D_4+A_6+A_2+3A_1$	42	$2,3$
$D_6+D_4+A_5+A_4+2A_1$	30	$2,3$
$D_6+D_4+A_5+A_3+3A_1$	12	$1,2,3,4$
$D_6+D_4+A_4+A_3+4A_1$	20	$2,3,4$
$D_6+D_4+A_3+8A_1$	4	$2,3,4,5,6$
$D_6+D_4+A_2+9A_1$	6	$2,3,4,5,6$
$D_6+D_4+11A_1$	2	$2,3,4,5,6,$ 7
D_6+A_{15}	4	1
$D_6+A_{13}+A_2$	42	1
$D_6+A_{11}+A_3+A_1$	6	$1,2$
$D_6+A_{11}+2A_2$	12	1
$D_6+A_{10}+A_5$	66	1
$D_6+A_{10}+5A_1$	22	$2,3$
$D_6+A_9+A_3+A_2+A_1$	60	$1,2$
$D_6+A_9+6A_1$	10	$2,3,4$
$D_6+A_8+A_5+A_2$	18	1
$D_6+A_8+7A_1$	18	$3,4$
$D_6+A_7+2A_3+2A_1$	8	$1,2,3$

$p=2$		
R	n	σ
$D_6+A_7+A_2+6A_1$	24	$2,3,4$
$D_6+A_6+A_5+A_3+A_1$	84	$1,2$
$D_6+A_6+A_4+5A_1$	70	$2,3$
$D_6+A_6+A_2+7A_1$	42	$3,4$
D_6+3A_5	6	$1,2$
$D_6+A_5+A_4+6A_1$	30	$2,3,4$
$D_6+A_5+A_3+7A_1$	12	$2,3,4,5$
$D_6+A_4+A_3+8A_1$	20	$3,4,5$
$D_6+A_3+12A_1$	4	$2,3,4,5,6,7$
$D_6+A_2+13A_1$	6	$3,4,5,6,7$
D_6+15A_1	2	$2,3,4,5,6,7,8$
$2D_5+D_4+A_7$	8	$1,2,3$
D_5+4D_4	4	$1,2,3,4,5$
$D_5+3D_4+A_4$	20	$2,3,4$
$D_5+2D_4+A_5+3A_1$	12	$2,3,4$
$D_5+2D_4+8A_1$	4	$2,3,4,5,6$
$D_5+D_4+A_{12}$	52	$1,2$
$D_5+D_4+A_{11}+A_1$	6	$1,2$
$D_5+D_4+A_9+A_2+A_1$	60	$1,2$
$D_5+D_4+A_8+A_4$	180	$1,2$
$D_5+D_4+A_5+7A_1$	12	$2,3,4,5$
$D_5+D_4+A_4+8A_1$	20	$3,4,5$
$D_5+D_4+12A_1$	4	$3,4,5,6,7$
D_5+A_{16}	68	1
$D_5+A_{15}+A_1$	2	1
$D_5+A_{11}+A_5$	2	1
$D_5+A_{11}+5A_1$	6	$2,3$
$D_5+A_9+A_6+A_1$	140	1
$D_5+A_9+A_2+5A_1$	60	$2,3$
D_5+2A_8	36	1
$D_5+2A_7+2A_1$	4	$1,2$
$D_5+A_7+A_3+6A_1$	8	$2,3,4$
$D_5+A_6+A_5+5A_1$	84	$2,3$
$D_5+A_5+11A_1$	12	$3,4,5,6$
$D_5+A_4+12A_1$	20	$4,5,6$
D_5+16A_1	4	$2,3,4,5,6,7,8$
$5D_4+A_1$	2	$1,2,3,4,5$
$4D_4+A_2+3A_1$	6	$1,2,3,4,5$
$4D_4+5A_1$	2	$1,2,3,4,5,6$
$3D_4+A_9$	10	3
$3D_4+A_8+A_1$	18	3
$3D_4+A_6+A_2+A_1$	42	3
$3D_4+A_5+A_4$	30	3
$3D_4+A_3+6A_1$	4	$1,2,3,4,5,6$
$3D_4+A_2+7A_1$	6	$2,3,4,5,6$

$p=2$		
R	n	σ
$3D_4+9A_1$	2	$1,2,3,4,5,6,7$
$2D_4+A_{10}+3A_1$	22	3
$2D_4+A_9+4A_1$	10	$2,3,4$
$2D_4+A_8+5A_1$	18	$3,4$
$2D_4+A_7+A_2+4A_1$	24	$2,3,4$
$2D_4+A_6+A_4+3A_1$	70	3
$2D_4+A_6+A_2+5A_1$	42	$3,4$
$2D_4+A_5+A_4+4A_1$	30	$2,3,4$
$2D_4+A_5+A_3+5A_1$	12	$2,3,4,5$
$2D_4+A_4+A_3+6A_1$	20	$2,3,4,5$
$2D_4+A_3+10A_1$	4	$2,3,4,5,6,7$
$2D_4+A_2+11A_1$	6	$2,3,4,5,6,7$
$2D_4+13A_1$	2	$2,3,4,5,6,7,8$
D_4+A_{17}	2	1
$D_4+A_{16}+A_1$	34	1
$D_4+A_{15}+2A_1$	4	$1,2$
$D_4+A_{14}+A_2+A_1$	10	1
$D_4+A_{13}+A_4$	70	1
$D_4+A_{13}+A_2+2A_1$	42	$1,2$
$D_4+A_{12}+A_4+A_1$	130	1
$D_4+A_{11}+A_3+3A_1$	6	$2,3$
$D_4+A_{11}+2A_2+2A_1$	12	$1,2$
$D_4+A_{10}+A_5+2A_1$	66	2
$D_4+A_{10}+A_4+A_2+A_1$	330	1
$D_4+A_{10}+7A_1$	22	$3,4$
$D_4+A_9+2A_4$	10	1
$D_4+A_9+A_3+A_2+3A_1$	60	$2,3$
$D_4+A_9+8A_1$	10	$3,4,5$
$D_4+A_8+A_5+A_2+2A_1$	18	2
$D_4+A_8+9A_1$	18	$3,4,5$
$D_4+A_7+2A_3+4A_1$	8	$2,3,4$
$D_4+A_7+A_2+8A_1$	24	$3,4,5$
$D_4+A_6+A_5+A_3+3A_1$	84	$2,3$
$D_4+A_6+A_4+7A_1$	70	$3,4$
$D_4+A_6+A_2+9A_1$	42	$3,4,5$
$D_4+3A_5+2A_1$	6	$1,2,3$
$D_4+A_5+A_4+8A_1$	30	$3,4,5$
$D_4+A_5+A_3+9A_1$	12	$2,3,4,5,6$
$D_4+A_4+A_3+10A_1$	20	$3,4,5,6$
$D_4+A_3+14A_1$	4	$2,3,4,5,6,7,8$
$D_4+A_2+15A_1$	6	$2,3,4,5,6,7,8$
D_4+17A_1	2	$2,3,4,5,6,7,8,9$
$A_{19}+2A_1$	20	1

$p = 2$		
R	n	σ
$A_{18} + 3A_1$	38	1
$A_{17} + A_3 + A_1$	36	1
$A_{17} + A_3 + A_1$	4	1
$A_{17} + A_2 + 2A_1$	6	1
$A_{17} + 4A_1$	2	$1, 2$
$A_{16} + 5A_1$	34	2
$A_{15} + A_4 + 2A_1$	20	1
$A_{15} + A_3 + A_2 + A_1$	6	1
$A_{15} + A_3 + 3A_1$	2	$1, 2$
$A_{15} + 6A_1$	4	$2, 3$
$A_{14} + A_3 + 2A_2$	60	1
$A_{14} + A_2 + 5A_1$	10	2
$A_{13} + A_5 + A_3$	84	1
$A_{13} + A_4 + 4A_1$	70	$1, 2$
$A_{13} + A_2 + 6A_1$	42	$2, 3$
$A_{12} + A_6 + 3A_1$	182	1
$A_{12} + A_4 + 5A_1$	130	2
$A_{12} + A_3 + 6A_1$	52	$2, 3$
$A_{11} + A_9 + A_1$	60	1
$A_{11} + A_6 + A_3 + A_1$	42	1
$A_{11} + A_6 + 2A_2$	84	1
$A_{11} + 2A_5$	12	1
$A_{11} + A_5 + A_3 + A_2$	6	1
$A_{11} + A_5 + A_3 + 2A_1$	2	$1, 2$
$A_{11} + A_5 + 5A_1$	4	$1, 2, 3$
$A_{11} + 2A_3 + 2A_2$	12	$1, 2$
$A_{11} + A_3 + 7A_1$	6	$2, 3, 4$
$A_{11} + 2A_2 + 6A_1$	12	$2, 3$
$A_{10} + A_9 + 2A_1$	110	1
$A_{10} + A_7 + 4A_1$	88	$1, 2$
$A_{10} + A_6 + A_3 + A_2$	924	1
$A_{10} + A_5 + 6A_1$	66	$2, 3$
$A_{10} + A_4 + A_2 + 5A_1$	330	2
$A_{10} + 11A_1$	22	$4, 5$
$2A_9 + A_3$	4	1

$p = 2$		
R	n	σ
$2A_9 + A_2 + A_1$	6	1
$2A_9 + 3A_1$	2	$1, 2$
$A_9 + A_6 + A_3 + 3A_1$	140	$1, 2$
$A_9 + 2A_4 + 4A_1$	10	$1, 2$
$A_9 + A_3 + A_2 + 7A_1$	60	$2, 3, 4$
$A_9 + 12A_1$	10	$3, 4, 5, 6$
$A_8 + A_5 + A_2 + 6A_1$	18	$2, 3$
$A_8 + A_4 + A_3 + 6A_1$	180	$2, 3$
$A_8 + 13A_1$	18	$4, 5, 6$
$2A_7 + 2A_3 + A_1$	2	$1, 2$
$2A_7 + A_3 + 4A_1$	4	$1, 2, 3$
$A_7 + A_6 + A_4 + 4A_1$	280	$1, 2$
$A_7 + A_6 + 2A_3 + A_2$	168	$1, 2$
$A_7 + A_5 + A_4 + 5A_1$	120	$1, 2, 3$
$A_7 + 4A_3 + A_2$	24	$1, 2, 3$
$A_7 + 2A_3 + 8A_1$	8	$3, 4, 5$
$A_7 + A_2 + 12A_1$	24	$3, 4, 5, 6$
$3A_6 + A_3$	28	1
$3A_6 + 3A_1$	14	1
$A_6 + 3A_5$	42	1
$A_6 + A_5 + A_3 + 7A_1$	84	$3, 4$
$A_6 + A_4 + 11A_1$	70	$4, 5$
$A_6 + A_2 + 13A_1$	42	$4, 5, 6$
$4A_5 + A_1$	2	$1, 2$
$3A_5 + 6A_1$	6	$1, 2, 3, 4$
$A_5 + A_4 + 12A_1$	30	$3, 4, 5, 6$
$A_5 + A_3 + 13A_1$	12	$3, 4, 5, 6, 7$
$A_4 + A_3 + 14A_1$	20	$3, 4, 5, 6, 7$
$7A_3$	4	$1, 2, 3, 4$
$A_3 + 18A_1$	4	$2, 3, 4, 5, 6, 7, 8, 9$
$A_2 + 19A_1$	6	$2, 3, 4, 5, 6, 7, 8, 9$
$21A_1$	2	$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

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