TOPOLOGY OF A COMPLEX DOUBLE PLANE BRANCHING ALONG A REAL LINE ARRANGEMENT

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Abstract. We investigate the topology of a double cover of a complex affine plane branching along a nodal real line arrangement. We define certain topological 2-cycles in the double plane using the real structure of the arrangement, and calculate their intersection numbers.

1. Introduction

Explicit description of topological cycles in complex algebraic varieties is an important task. It is essential, for example, for the study of periods and related differential equations in a family of algebraic varieties. For the recent development of *numerical algebraic geometry* (see, for example, [4, 5]), we need explicit descriptions of topological cycles suited for the numerical computation of periods by multiple integrals. For example, in [2], the arithmetic of certain Calabi-Yau threefolds obtained as the minimal desingularizations of double covers of $\mathbb{P}^3$ is studied by numerical integration over topological 3-cycles.

The first major general theory on topological cycles in complex algebraic varieties is the theory of vanishing cycles due to Lefschetz [7]. (See also [6] for the modern accounts of this theory.) In this paper, we investigate the topology of a smooth algebraic surface X obtained as the minimal desingularization of a double cover of the complex affine plane branching along a nodal *real* line arrangement. Using the real structure of the arrangement, we construct certain topological 2-cycles in X, which resemble, in some way, the vanishing cycles of Lefschetz.

To understand the topology of an algebraic surface, it is important to calculate the intersection form on the middle homology group. The main purpose of this paper is to calculate the intersection numbers of our topological 2-cycles in X. Since some pairs of these cycles intersect in loci of real dimension ≥ 1, we have to construct small displacements of the cycles.

Let $\mathbb{A}^2(\mathbb{R})$ be a real affine plane. An arrangement of N real lines

$$\mathcal{A} := \{\ell_1(\mathbb{R}), \dots, \ell_N(\mathbb{R})\}$$

on $\mathbb{A}^2(\mathbb{R})$ is called *nodal* if no three lines of $\mathcal{A}$ are concurrent. Suppose that $\mathcal{A}$ is a nodal real line arrangement. Let

$$\mathcal{A}_{\mathbb{C}} := \{\ell_1(\mathbb{C}), \dots, \ell_N(\mathbb{C})\}$$

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be the arrangement of complex affine lines in the complex affine plane $\mathbb{A}^2(\mathbb{C})$ obtained by complexifying the lines in $\mathcal{A}$. We put

$$B(\mathbb{R}) := \bigcup_{i=1}^{N} \ell_i(\mathbb{R}), \quad B(\mathbb{C}) := \bigcup_{i=1}^{N} \ell_i(\mathbb{C}).$$

We consider the morphisms

$$X \xrightarrow{\rho} W \xrightarrow{\pi} \mathbb{A}^2(\mathbb{C}), \tag{1.1}$$

where $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ is the double covering whose branch locus is equal to $B(\mathbb{C})$, and $\rho\colon X \to W$ is the minimal resolution of singularities. The purpose of this paper is to investigate the intersection form

$$\langle\quad\rangle \;:\; H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Z}$$

on the second homology group $H_2(X,\mathbb{Z})$ of X, where X is oriented as a smooth complex surface.

For simplicity, we put

$$\mathcal{P} := \operatorname{Sing} B(\mathbb{C}).$$

Let $\beta\colon Y(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ be the blowing-up at the points in $\mathcal{P}$. For a subset S of $\mathbb{A}^2(\mathbb{C})$, we put

$$S^\bullet := S \setminus (S \cap \mathcal{P}), \quad \beta^\sharp S := \text{the closure of } \beta^{-1}(S^\bullet) \text{ in } Y(\mathbb{C}),$$

and call $\beta^\sharp S$ the *strict transform* of S. We then put

$$Y(\mathbb{R}) := \beta^\sharp \mathbb{A}^2(\mathbb{R}).$$

Note that $\beta^\sharp B(\mathbb{C})$ is the disjoint union of smooth rational curves $\beta^\sharp \ell_i(\mathbb{C})$ on $Y(\mathbb{C})$. Then X fits in the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y(\mathbb{C}) & \underset{\beta}{\longrightarrow} & \mathbb{A}^2(\mathbb{C}), \end{array}$$

where $\phi\colon X \to Y(\mathbb{C})$ is the double covering whose branch locus is equal to $\beta^\sharp B(\mathbb{C})$. For $P \in \mathcal{P}$, let E_P denote the exceptional (-1)-curve of β over P, and we put

$$D_P := \phi^{-1}(E_P) = (\pi \circ \rho)^{-1}(P),$$

which is a smooth rational curve on X with self-intersection number -2. Note that D_P is the exceptional curve of the minimal desingularization $\rho\colon X \to W$ over the ordinary node of W that is mapped to P by π. If P is the intersection point of the lines $\ell_a(\mathbb{R})$ and $\ell_b(\mathbb{R})$ in $\mathcal{A}$, then $\phi|D_P\colon D_P \to E_P$ is the double covering branching at the intersection points of E_P and the strict transforms $\beta^\sharp \ell_a(\mathbb{C})$ and $\beta^\sharp \ell_b(\mathbb{C})$.

A *chamber* is the closure in $\mathbb{A}^2(\mathbb{R})$ of a connected component of the complement $\mathbb{A}^2(\mathbb{R}) \setminus B(\mathbb{R})$ of $B(\mathbb{R})$. We denote by $\boldsymbol{Ch}_{\mathrm{b}}$ the set of *bounded* chambers. Let C be a bounded chamber. We put

$$\operatorname{Vert}(C) := C \cap \mathcal{P}.$$

A point of $\operatorname{Vert}(C)$ is called a *vertex* of C. Let P be a vertex of C. Then $Y(\mathbb{R}) \cap E_P$ is a circle on the Riemann sphere E_P, on which the two branch points of the double covering $\phi|D_P\colon D_P \to E_P$ locate, and

$$J_{C,P} := \beta^\sharp C \cap E_P$$

is a part of the circle $Y(\mathbb{R}) \cap E_P$ connecting these two branch points. Therefore

$$S_{C,P} := \phi^{-1}(\beta^\sharp C) \cap D_P = \phi^{-1}(J_{C,P})$$

is a circle on the Riemann sphere D_P. The space $\phi^{-1}(\beta^\sharp C)$ is homeomorphic to a 2-sphere minus a union of disjoint open discs, and we have

$$\partial\, \phi^{-1}(\beta^\sharp C) = \bigsqcup_{P \in \mathrm{Vert}(C)} S_{C,P}.$$

Let γ_C be an orientation of $\phi^{-1}(\beta^\sharp C)$. We denote by $\Delta(C, \gamma_C)$ the topological 2-chain $\phi^{-1}(\beta^\sharp C)$ oriented by γ_C. Each boundary component $S_{C,P}$ of $\Delta(C, \gamma_C)$ is oriented by γ_C. Note that $S_{C,P}$ divides the 2-sphere D_P into the union of two closed hemispheres, and that the two hemispheres with their complex structures induce orientations on $S_{C,P}$ that are opposite to each other.

Definition 1.1. The *capping hemisphere* for γ_C at P is the closed hemisphere $H_{C,\gamma_C,P}$ on D_P with $\partial\, H_{C,\gamma_C,P} = S_{C,P}$ such that the orientation on $S_{C,P}$ induced by the complex structure of $H_{C,\gamma_C,P}$ is opposite to the orientation induced by the orientation γ_C of $\Delta(C, \gamma_C)$.

Let $H_{C,\gamma_C,P}$ be the capping hemisphere for γ_C at P. Then

$$\Sigma(C, \gamma_C) := \Delta(C, \gamma_C) \;\; \cup \bigsqcup_{P \in \mathrm{Vert}(C)} H_{C,\gamma_C,P}$$

with the orientations coming from the complex structure on each $H_{C,\gamma_C,P}$ is a topological 2-cycle homeomorphic to a 2-sphere. Figure 1.1 illustrates $\Sigma(C, \gamma_C)$ when C is a triangle.

In this paper, we show that this topological 2-cycle $\Sigma(C, \gamma_C)$ resembles, in many aspects, the vanishing cycle for an ordinary node of a complex surface. Hence we make the following:

Definition 1.2. The 2-cycle $\Sigma(C, \gamma_C)$ is called a *vanishing cycle* over the bounded chamber C. Its homology class $[\Sigma(C, \gamma_C)] \in H_2(X, \mathbb{Z})$ is also called a vanishing cycle over C.

By definition, the homology class $[\Sigma(C, \gamma_C)]$ depends on γ_C as follows:

$$[\Sigma(C, \gamma_C)] + [\Sigma(C, -\gamma_C)] = \sum_{P \in \mathrm{Vert}(C)} [D_P], \tag{1.2}$$

where $[D_P] \in H_2(X, \mathbb{Z})$ is the homology class of the smooth rational curve D_P.

In order to state our results precisely, we define *standard orientations* σ_C for bounded chambers C.

Definition 1.3. A *defining polynomial* of the arrangement $\mathcal{A}$ is the product

$$f := \prod_{i=1}^{n} \lambda_i,$$

where λ_i is an affine linear function $\lambda_i \colon \mathbb{A}^2(\mathbb{R}) \to \mathbb{R}$ such that $\ell_i(\mathbb{R}) = \lambda_i^{-1}(0)$.

A defining polynomial of $\mathcal{A}$ is unique up to real multiplicative constant. We can regard a defining polynomial $f \colon \mathbb{A}^2(\mathbb{R}) \to \mathbb{R}$ of $\mathcal{A}$ as a complex-valued polynomial function $f \colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{C}$.

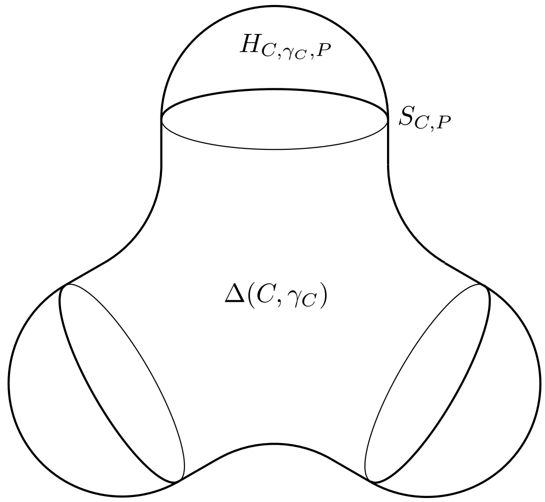

FIGURE 1.1. Vanishing cycle $\Sigma(C,\gamma_C)$

We fix an orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$ and a defining polynomial f of $\mathcal{A}$. Then the double covering $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ is given by the projection to the second factor from

$$W := \{\, (\omega, Q) \in \mathbb{C} \times \mathbb{A}^2(\mathbb{C}) \mid \omega^2 = f(Q) \,\}, \tag{1.3}$$

where ω is an affine coordinate of $\mathbb{C}$. Let C be a bounded chamber, and let C° be the interior of C in $\mathbb{A}^2(\mathbb{R})$. The pull-back $\pi^{-1}(C^\circ)$ has two connected components, which we call *sheets*. We can consider $\pi^{-1}(C^\circ)$ as an open subset of $\phi^{-1}(\beta^\sharp C)$ via $\rho\colon X \to W$. Let γ_C be an orientation of $\phi^{-1}(\beta^\sharp C)$. We let the two sheets be oriented by γ_C. Then $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ restricted to one sheet is an orientation-preserving homeomorphism to C° oriented by $\sigma_{\mathbb{A}}$, whereas π restricted to the other sheet is orientation-reversing.

Definition 1.4. The sheet on which π is orientation-preserving is called the *plus-sheet* with respect to $\sigma_{\mathbb{A}}$ and γ_C, and the other sheet is called the *minus-sheet*.

The orientation γ_C of $\phi^{-1}(\beta^\sharp C)$ is specified by indicating which sheet is plus. Note that we have either $f(C^\circ) \subset \mathbb{R}_{>0}$ or $f(C^\circ) \subset \mathbb{R}_{<0}$. In the former case, the two sheets are distinguished by the sign of the function $\omega = \pm\sqrt{f} \in \mathbb{R}$ on W, and in the latter case, the two are distinguished by the sign of $\omega/\sqrt{-1} = \pm\sqrt{-f} \in \mathbb{R}$.

Definition 1.5. The *standard orientation* σ_C for a bounded chamber C with respect to $\sigma_{\mathbb{A}}$ and f is the orientation of $\phi^{-1}(\beta^\sharp C)$ such that

- when $f(C^\circ) \subset \mathbb{R}_{>0}$, the sheet with $\omega > 0$ is plus, and
- when $f(C^\circ) \subset \mathbb{R}_{<0}$, the sheet with $\omega/\sqrt{-1} > 0$ is plus.

When f and $\sigma_{\mathbb{A}}$ are fixed and we use the standard orientation σ_C, we omit σ_C from $H_{C,\sigma_C,P}$, $\Delta(C,\sigma_C)$, and $\Sigma(C,\sigma_C)$:

$$H_{C,P} := H_{C,\sigma_C,P}, \quad \Delta(C) := \Delta(C,\sigma_C), \quad \Sigma(C) := \Sigma(C,\sigma_C).$$

Our main results are as follows. Let $\mathcal{A}$ be a nodal real line arrangement. We fix an orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$ and a defining polynomial f of $\mathcal{A}$, so that the standard orientation σ_C is defined for each bounded chamber $C \in \boldsymbol{Ch}_{\mathrm{b}}$.

Theorem 1.6. *The $\mathbb{Z}$-module $H_2(X,\mathbb{Z})$ is free. The homology classes $[\Sigma(C)]$, where C runs through $\boldsymbol{Ch}_{\mathrm{b}}$, and the homology classes $[D_P]$, where P runs through $\mathcal{P}$, form a basis of $H_2(X,\mathbb{Z})$.*

Definition 1.7. Let C be a chamber. We say that $\ell_i(\mathbb{R}) \in \mathcal{A}$ defines an *edge* $C \cap \ell_i(\mathbb{R})$ of C if $C \cap \ell_i(\mathbb{R})$ contains a non-empty open subset of $\ell_i(\mathbb{R})$.

Theorem 1.8. *Let C and C' be distinct bounded chambers.*

(1) *For $P \in \mathcal{P}$, we have*

$$\langle [\Sigma(C)], [D_P] \rangle = \begin{cases} -1 & \text{if } P \in \mathrm{Vert}(C), \\ 0 & \text{if } P \notin \mathrm{Vert}(C). \end{cases}$$

(2) *If C and C' are disjoint, then $\langle [\Sigma(C)], [\Sigma(C')] \rangle = 0$.*
(3) *If $C \cap C'$ consists of a single point, then $\langle [\Sigma(C)], [\Sigma(C')] \rangle = 0$.*
(4) *If C and C' share a common edge, then $\langle [\Sigma(C)], [\Sigma(C')] \rangle = -1$.*
(5) *The self-intersection number $\langle [\Sigma(C)], [\Sigma(C)] \rangle$ of $[\Sigma(C)]$ is equal to -2.*

In fact, using (1.2), we see that, in Theorem 1.8, the orientation σ_C of $\Sigma(C)$ matters only in assertion (3).

Corollary 1.9. *Let C and C' be distinct bounded chambers, and let γ_C and $\gamma_{C'}$ be orientations of $\phi^{-1}(\beta^\sharp C)$ and of $\phi^{-1}(\beta^\sharp C')$, respectively.*

- *For $P \in \mathrm{Vert}(C)$, we have $\langle [\Sigma(C,\gamma_C)], [D_P] \rangle = -1$.*
- *If C and C' share a common edge, then $\langle [\Sigma(C,\gamma_C)], [\Sigma(C',\gamma_{C'})] \rangle = -1$.*
- *$\langle [\Sigma(C,\gamma_C)], [\Sigma(C,\gamma_C)] \rangle = -2$.* □

These intersection numbers are checked extensively by generating random examples of nodal real line arrangements by computer. See Section 10.

Our construction of vanishing cycles over bounded chambers is similar to the construction of topological cycles by Pham [9] in hypersurfaces defined by equations of Fermat type. Pham's construction has many applications. For example, it was applied in [3] to the study of integral Hodge conjecture for Fermat varieties. It was also applied to the calculation of Picard lattices of quartic surfaces in [5].

A totally different algorithm to calculate the intersection form on the middle homology group of an open complex surface, which we called a *Zariski-van Kampen method*, was presented in [1] and [12]. It was applied to the construction of arithmetic Zariski pairs in [1] and [10].

This paper is organized as follows. In Section 2, we introduce the notion of simply branched coverings of topological spaces, and study its basic properties. In Section 3, we fix various terminologies about real line arrangements. In Section 4, we prove Theorem 1.6 by means of Lemma 2.4. In Section 5, we calculate the capping hemispheres explicitly. In Section 6, we introduce the notion of displacements of vanishing cycles. In Sections 7, 8, 9, we prove Theorem 1.8. In Section 10, we calculate some concrete examples.

2. Simply branched coverings

In this section, we define a notion of *simply branched coverings*, and show Lemma 2.4 about the liftability of strong deformation retractions.

Definition 2.1. Let d be a positive integer. Let S and T be topological spaces, and $\varpi\colon S \to T$ a continuous map. Let V be a closed subspace of T. We say that

$\varpi\colon S\to T$ is a *simply branched covering of degree d with branch locus V* if the following conditions hold:

(a) Every fiber of ϖ is discrete.
(b) We put
$$S_V := \varpi^{-1}(V), \quad S_{T\setminus V} := \varpi^{-1}(T\setminus V).$$
Then the restriction $\varpi|S_V\colon S_V\to V$ to S_V is a homeomorphism, and the restriction $\varpi|S_{T\setminus V}\colon S_{T\setminus V}\to T\setminus V$ to $S_{T\setminus V}$ is a covering map of degree d. (We do *not* assume that $S_{T\setminus V}$ is connected.)
(c) For any $P\in S_V$ and any open neighborhood $U_{P,S}$ of P in S, there exists an open neighborhood $U_{\varpi(P),T}$ of $\varpi(P)$ in T such that $\varpi^{-1}(U_{\varpi(P),T})$ is contained in $U_{P,S}$.

Example 2.2. Let T be a Hausdorff space, and $f\colon T\to\mathbb{C}$ a continuous map. We put
$$S := \{\,(s,Q)\in\mathbb{C}\times T \mid s^d = f(Q)\,\}.$$
Then the projection $\varpi\colon S\to T$ given by $\varpi(s,Q)=Q$ is a simply branched covering of degree d with branch locus $V:=\{f=0\}$. Indeed, every fiber of ϖ is finite and hence is discrete in S because S is Hausdorff. Condition (b) is obviously satisfied. Suppose that $P=(0,Q)$ is a point of S_V, and let $U_{P,S}$ be an open neighborhood of P in S. We choose a sufficiently small open disc $\varDelta_\varepsilon\subset\mathbb{C}$ of radius ε with the center 0 and an open neighborhood $U'_{Q,T}$ of $Q:=\varpi(P)$ in T such that
$$S\cap(\varDelta_\varepsilon\times U'_{Q,T}) \quad\subset\quad U_{P,S}.$$
We put
$$U_{Q,T} := U'_{Q,T}\cap\{\,Q'\in T \mid |f(Q')|<\varepsilon^d\,\},$$
which is an open neighborhood of Q in T. Then we have $\varpi^{-1}(U_{Q,T})\subset U_{P,S}$.

Example 2.3. Let $\varpi\colon S\to T$ be a simply branched covering of degree d with branch locus V. Let T' be a closed subset of T. Then
$$\varpi|\varpi^{-1}(T')\colon \varpi^{-1}(T')\to T'$$
is a simply topological branched covering of degree d with branch locus $V\cap T'$.

We denote by I the closed interval $[0,1]\subset\mathbb{R}$.

Lemma 2.4. *Let $\varpi\colon S\to T$ be a simply branched covering of degree d with the branch locus V. Let R be a closed subspace of T, and let*
$$F\colon T\times I\to T$$
be a strong deformation retraction of T onto R. Suppose the following:

(2.1) *For any $Q\in T$, if there exists a value $u_0\in I$ such that $F(Q,u_0)\in V$, then $F(Q,u)\in V$ holds for any $u\in[u_0,1]$.*

Then there exists a strong deformation retraction
$$\widetilde{F}\colon S\times I\to S$$
of S onto $\varpi^{-1}(R)$ that makes the following diagram commutative:

$$\begin{array}{ccc} S\times I & \xrightarrow{\widetilde{F}} & S \\ {\scriptstyle \varpi\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\varpi} \\ T\times I & \xrightarrow[F]{} & T. \end{array} \tag{2.2}$$

Proof. Let $S_V = \varpi^{-1}(V)$ and $S_{T\setminus V} = \varpi^{-1}(T \setminus V)$ be as in Definition 2.1. We put

$$\begin{aligned}(T\times I)_V &:= F^{-1}(V),\\ F_V &:= F|(T\times I)_V\colon (T\times I)_V \to V,\\ (S\times I)_V &:= (\varpi\times \mathrm{id})^{-1}((T\times I)_V),\end{aligned}$$

and

$$\begin{aligned}(T\times I)_{T\setminus V} &:= F^{-1}(T\setminus V),\\ F_{T\setminus V} &:= F|(T\times I)_{T\setminus V}\colon (T\times I)_{T\setminus V} \to T\setminus V,\\ (S\times I)_{T\setminus V} &:= (\varpi\times \mathrm{id})^{-1}((T\times I)_{T\setminus V}).\end{aligned}$$

Since the restriction $F|T\times\{0\}$ of F to $T\times\{0\}$ is the identity map of T, we have

$$\begin{aligned}(T\times I)_V\cap(T\times\{0\}) &= V\times\{0\},\\ (S\times I)_V\cap(S\times\{0\}) &= S_V\times\{0\},\\ (T\times I)_{T\setminus V}\cap(T\times\{0\}) &= (T\setminus V)\times\{0\},\\ (S\times I)_{T\setminus V}\cap(S\times\{0\}) &= S_{T\setminus V}\times\{0\}.\end{aligned}$$

The strategy of the proof is as follows. First we construct continuous maps

$$\widetilde{F}_V\colon (S\times I)_V \to S_V,\quad \widetilde{F}_{T\setminus V}\colon (S\times I)_{T\setminus V}\to S_{T\setminus V}$$

that form the commutative diagrams

$$\begin{array}{ccc}(S\times I)_V & \xrightarrow{\widetilde{F}_V} & S_V\\ \downarrow & & \downarrow\\ (T\times I)_V & \xrightarrow[F_V]{} & V\end{array},\quad \begin{array}{ccc}(S\times I)_{T\setminus V} & \xrightarrow{\widetilde{F}_{T\setminus V}} & S_{T\setminus V}\\ \downarrow & & \downarrow\\ (T\times I)_{T\setminus V} & \xrightarrow[F_{T\setminus V}]{} & T\setminus V\end{array}, \tag{2.3}$$

respectively, and that satisfy

$$\widetilde{F}_V(P,0) = P \text{ for any } P\in S_V,\quad \widetilde{F}_{T\setminus V}(Q,0) = Q \text{ for any } Q\in S_{T\setminus V}. \tag{2.4}$$

Here the vertical arrows in the diagrams in (2.3) are $\varpi\times\mathrm{id}$ and ϖ. Next we define $\widetilde{F}\colon S\times I\to S$ by (2.8) below, and show that $\widetilde{F}$ is continuous.

First, we construct $\widetilde{F}_V$. Since $\varpi|S_V\colon S_V\to V$ is a homeomorphism, we have its inverse $(\varpi|S_V)^{-1}\colon V\to S_V$. We define $\widetilde{F}_V$ to be the composite

$$(\varpi|S_V)^{-1}\ \circ\ F_V\ \circ\ ((\varpi\times\mathrm{id})|(S\times I)_V).$$

It is obvious that the left diagram of (2.3) is commutative. Suppose that $P\in S_V$. Since $F(\varpi(P),0) = \varpi(P)\in V$, we see that P is the unique point of S over $F(\varpi(P),0)$. Hence $\widetilde{F}_V(P,0) = P$ follows.

Next, we construct $\widetilde{F}_{T\setminus V}$. For any $Q\in T$, assumption (2.1) implies that, if $F(Q,u)\in T\setminus V$ for some $u\in I$, then $F(Q,0)\in T\setminus V$ and hence $Q = F(Q,0)\in T\setminus V$. Therefore $F^{-1}(T\setminus V)$ is contained in $(T\setminus V)\times I$. Since $\varpi\times\mathrm{id}$ is a covering map of degree d over $(T\setminus V)\times I$, we see that

$$(\varpi\times\mathrm{id})|(S\times I)_{T\setminus V}\colon (S\times I)_{T\setminus V}\to(T\times I)_{T\setminus V}$$

is a covering map of degree d. By assumption (2.1) again, we have a strong deformation retraction

$$H\colon (T\times I)_{T\setminus V}\times I\to (T\times I)_{T\setminus V} \tag{2.5}$$

of $(T\times I)_{T\setminus V}$ onto $(T\setminus V)\times\{0\}$ defined by

$$((Q,u),v)\mapsto (Q,u(1-v)). \tag{2.6}$$

By the homotopy lifting property of the covering map $(\varpi\times \mathrm{id})|(S\times I)_{T\setminus V}$, we obtain a lift

$$\widetilde{H}\colon (S\times I)_{T\setminus V}\times I\to (S\times I)_{T\setminus V}$$

of H that is the identity on $(S\times I)_{T\setminus V}\times\{0\}$:

$$\begin{array}{ccccc}
(S\times I)_{T\setminus V}\times\{0\} & & \xrightarrow{\mathrm{id}} & & (S\times I)_{T\setminus V}\\
\downarrow & & {}^{\widetilde{H}}\nearrow & & \downarrow \varpi\times\mathrm{id}\\
(S\times I)_{T\setminus V}\times I & \underset{(\varpi\times\mathrm{id})\times\mathrm{id}}{\longrightarrow} & (T\times I)_{T\setminus V}\times I & \underset{H}{\longrightarrow} & (T\times I)_{T\setminus V}.
\end{array}$$

We show that $\widetilde{H}$ is a strong deformation retraction of $(S\times I)_{T\setminus V}$ onto the subspace $S_{T\setminus V}\times\{0\}$, that is, the image of $(S\times I)_{T\setminus V}\times\{1\}$ by $\widetilde{H}$ is contained in $S_{T\setminus V}\times\{0\}$, and,

$$\text{if } (P,0)\in S_{T\setminus V}\times\{0\}, \text{ then } \widetilde{H}((P,0),v)=(P,0) \text{ for any } v\in I. \tag{2.7}$$

The first assertion follows from the fact that the image of $(T\times I)_{T\setminus V}\times\{1\}$ by H is contained in $(T\setminus V)\times\{0\}$. Let P be a point of $S_{T\setminus V}$ so that $(\varpi(P),0)$ belongs to $(T\setminus V)\times\{0\}$. We have

$$H((\varpi(P),0),v)=(\varpi(P),0)$$

for any $v\in I$ by the definition (2.6) of H. Therefore we have

$$\widetilde{H}((P,0),v)\in\varpi^{-1}(\varpi(P))\times\{0\}$$

for any $v\in I$. Since $\varpi^{-1}(\varpi(P))$ is discrete and $\widetilde{H}((P,0),0)=(P,0)$, we obtain (2.7).

We put

$$\widetilde{H}_1:=\widetilde{H}|(S\times I)_{T\setminus V}\times\{1\}\colon (S\times I)_{T\setminus V}\times\{1\}\to S_{T\setminus V}\times\{0\}=S_{T\setminus V}.$$

Note that $\widetilde{H}_1$ is the identity map on $(S_{T\setminus V}\times\{0\})\times\{1\}$ by (2.7) with $v=1$. Remark that, since $F|T\times\{0\}=\mathrm{id}_T$, the following diagram is commutative:

$$\begin{array}{ccccc}
(S\times I)_{T\setminus V} & \hookleftarrow & S_{T\setminus V}\times\{0\} & = & S_{T\setminus V}\\
\downarrow & & & & \downarrow\\
(T\times I)_{T\setminus V} & & \underset{F}{\longrightarrow} & & T\setminus V.
\end{array}$$

Therefore $\varpi\circ\widetilde{H}_1$ is equal to the restriction to $(S\times I)_{T\setminus V}\times\{1\}$ of the map

$$\mathcal{F}\colon (S\times I)_{T\setminus V}\times I\xrightarrow{\widetilde{H}}(S\times I)_{T\setminus V}\xrightarrow{(\varpi\times\mathrm{id})}(T\times I)_{T\setminus V}\xrightarrow{F} T\setminus V.$$

By the homotopy lifting property of the covering map $\varpi|S_{T\setminus V}\colon S_{T\setminus V}\to T\setminus V$, we obtain an extension

$$\widetilde{\mathcal{F}}\colon (S\times I)_{T\setminus V}\times I\to S_{T\setminus V}$$

of $\widetilde{H}_1$ that covers $\mathcal{F}$:

$$\begin{array}{ccc} (S\times I)_{T\setminus V}\times\{1\} & \xrightarrow{\widetilde{H}_1} & S_{T\setminus V} \\ \downarrow & \overset{\widetilde{\mathcal{F}}}{\nearrow} & \downarrow\varpi \\ (S\times I)_{T\setminus V}\times I & \underset{\mathcal{F}}{\longrightarrow} & T\setminus V. \end{array}$$

We then define $\widetilde{F}_{T\setminus V}$ to be the restriction of $\widetilde{\mathcal{F}}$ to $(S\times I)_{T\setminus V}\times\{0\}$:

$$\widetilde{F}_{T\setminus V} := \widetilde{\mathcal{F}}|(S\times I)_{T\setminus V}\times\{0\}\colon (S\times I)_{T\setminus V}\to S_{T\setminus V}.$$

Since $\widetilde{H}$ is the identity map on $(S\times I)_{T\setminus V}\times\{0\}$, we see that $\widetilde{F}_{T\setminus V}$ is a lift of F restricted to $(T\times I)_{T\setminus V}$, namely, for $(P,u)\in(S\times I)_{T\setminus V}$, we have

$$\begin{aligned}\varpi(\widetilde{F}_{T\setminus V}(P,u)) = \varpi(\widetilde{\mathcal{F}}((P,u),0)) = \\ = \mathcal{F}((P,u),0) = (F\circ(\varpi\times\mathrm{id}))(\widetilde{H}((P,u),0)) = F(\varpi(P),u).\end{aligned}$$

Thus the right diagram of (2.3) is commutative. We show that $\widetilde{F}_{T\setminus V}$ satisfies the second condition of (2.4), that is, $\widetilde{F}_{T\setminus V}$ restricted to $S_{T\setminus V}\times\{0\}$ is the identity map of $S_{T\setminus V}$. Suppose that $P\in S_{T\setminus V}$. It is enough to show that the point $\widetilde{\mathcal{F}}((P,0),v)$ does not depend on v, because we will then have

$$\widetilde{F}_{T\setminus V}(P,0) = \widetilde{\mathcal{F}}((P,0),0) = \widetilde{\mathcal{F}}((P,0),1) = \widetilde{H}_1(P,0) = P$$

by (2.7) with $v=1$. Because $\widetilde{H}$ is a strong deformation retraction of the space $(S\times I)_{T\setminus V}$ onto $S_{T\setminus V}\times\{0\}$, it follows that $\widetilde{H}((P,0),v)=(P,0)$, and hence

$$\mathcal{F}((P,0),v) = (F\circ(\varpi\times\mathrm{id}))(\widetilde{H}((P,0),v)) = F(\varpi(P),0) = \varpi(P)$$

holds for any $v\in I$. Since $\varpi^{-1}(\varpi(P))$ is discrete and the function $\widetilde{\mathcal{F}}((P,0),v)$ is continuous on v, the point $\widetilde{\mathcal{F}}((P,0),v)$ does not depend on v. Thus we have constructed $\widetilde{F}_{T\setminus V}$ with the desired properties.

We then define $\widetilde{F}\colon S\times I\to S$ by

$$\widetilde{F}(P,u) := \begin{cases}\widetilde{F}_V(P,u) & \text{if } (P,u)\in(S\times I)_V,\\ \widetilde{F}_{T\setminus V}(P,u) & \text{if } (P,u)\in(S\times I)_{T\setminus V}.\end{cases} \tag{2.8}$$

By (2.3), we see that the diagram (2.2) is commutative, that is, the mapping $\widetilde{F}$ is a lift of F. We show that $\widetilde{F}$ is continuous. It is obvious that $\widetilde{F}$ is continuous at every point of the open subset $(S\times I)_{T\setminus V}$ of $S\times I$. Let (P,u) be a point of $(S\times I)_V$. We put

$$P' := \widetilde{F}(P,u) = \widetilde{F}_V(P,u),$$

and let $U_{P',S}$ be an open neighborhood of P' in S. We show that there exist an open neighborhood $U_{P,S}$ of P in S and an open neighborhood J of u in I such that

$$U_{P,S}\times J\subset\widetilde{F}^{-1}(U_{P',S}).$$

Since $\varpi(P')=F(\varpi(P),u)\in V$, by requirement (c) in Definition 2.1, we have an open neighborhood $U_{\varpi(P'),T}$ of $\varpi(P')$ in T such that $\varpi^{-1}(U_{\varpi(P'),T})$ is contained in $U_{P',S}$. Note that

$$(\varpi\times\mathrm{id})^{-1}(F^{-1}(U_{\varpi(P'),T})) = \widetilde{F}^{-1}(\varpi^{-1}(U_{\varpi(P'),T}))\subset\widetilde{F}^{-1}(U_{P',S}).$$

Since F is continuous, we have an open neighborhood $U_{\varpi(P),T}$ of $\varpi(P)$ in T and an open neighborhood J of u in I such that

$$U_{\varpi(P),T} \times J \subset F^{-1}(U_{\varpi(P'),T}).$$

Then we have

$$\begin{aligned}\varpi^{-1}(U_{\varpi(P),T}) \times J &= (\varpi \times \mathrm{id})^{-1}(U_{\varpi(P),T} \times J)\\ &\subset (\varpi \times \mathrm{id})^{-1}(F^{-1}(U_{\varpi(P'),T})) \subset \widetilde{F}^{-1}(U_{P',S}).\end{aligned}$$

Thus $U_{P,S} := \varpi^{-1}(U_{\varpi(P),T})$ and J have the required property.

Finally, we show that $\widetilde{F}$ is a strong deformation retraction of S onto $\varpi^{-1}(R)$. The fact that $\widetilde{F}$ restricted to $S \times \{0\}$ is the identity map of S follows from (2.4). The fact that $\widetilde{F}(S \times \{1\}) \subset \varpi^{-1}(R)$ follows from $F(T \times \{1\}) \subset R$. Suppose that $P \in \varpi^{-1}(R)$. We have $F(\varpi(P), u) = \varpi(P)$ for any $u \in I$. Then we have $\widetilde{F}(P, 0) = P$ and

$$\widetilde{F}(P, u) \in \varpi^{-1}(F(\varpi(P), u)) = \varpi^{-1}(\varpi(P)).$$

Since $\widetilde{F}(P, u)$ is continuous with respect to u and $\varpi^{-1}(\varpi(P))$ is discrete, we see that $\widetilde{F}(P, u) = P$ for all $u \in I$. □

3. Terminologies

From now on to the end of the paper, we fix an orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$.

3.1. **The vector space of translations.** Let $T(\mathbb{A}^2(\mathbb{C}))$ denote the $\mathbb{C}$-vector space of translations of $\mathbb{A}^2(\mathbb{C})$, and let $T(\mathbb{A}^2(\mathbb{R}))$ denote the $\mathbb{R}$-vector space of translations of $\mathbb{A}^2(\mathbb{R})$. For a vector $\tau \in T(\mathbb{A}^2(\mathbb{C}))$ and $P \in \mathbb{A}^2(\mathbb{C})$, we denote by $P + \tau$ the image of P by the translation $\tau\colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$. For $Q, Q' \in \mathbb{A}^2(\mathbb{C})$, let $\tau_{Q,Q'} \in T(\mathbb{A}^2(\mathbb{C}))$ denote the unique translation such that $Q + \tau_{Q,Q'} = Q'$. We have

$$T(\mathbb{A}^2(\mathbb{C})) = T(\mathbb{A}^2(\mathbb{R})) \otimes \mathbb{C} = T(\mathbb{A}^2(\mathbb{R})) \oplus \sqrt{-1}\, T(\mathbb{A}^2(\mathbb{R})).$$

Let $\overline{Q}$ denote the complex conjugate of the point $Q \in \mathbb{A}^2(\mathbb{C})$. Then the mapping

$$Q \mapsto Q + (1/2)\tau_{Q,\overline{Q}} = (Q + \overline{Q})/2$$

that gives the real part of $Q \in \mathbb{A}^2(\mathbb{C})$ yields a projection

$$\mathrm{pr}_{\mathbb{R}}\colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{R}).$$

Then we have a natural identification

$$\mathrm{pr}_{\mathbb{R}}^{-1}(Q) \cong \sqrt{-1}\, T(\mathbb{A}^2(\mathbb{R})) \tag{3.1}$$

of the fiber $\mathrm{pr}_{\mathbb{R}}^{-1}(Q)$ over $Q \in \mathbb{A}^2(\mathbb{R})$ with $\sqrt{-1}\, T(\mathbb{A}^2(\mathbb{R}))$ by $Q' \mapsto \tau_{Q,Q'}$. For a real affine line $\lambda(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$, we put

$$T[\lambda(\mathbb{R})] := \{\, \tau \in T(\mathbb{A}^2(\mathbb{R})) \mid \lambda(\mathbb{R}) + \tau = \lambda(\mathbb{R}) \,\},$$

which is a 1-dimensional $\mathbb{R}$-vector subspace of $T(\mathbb{A}^2(\mathbb{R}))$. In the same way, we define the subspace of $T[\lambda(\mathbb{C})] \subset T(\mathbb{A}^2(\mathbb{C}))$ for a complex affine line $\lambda(\mathbb{C}) \subset \mathbb{A}^2(\mathbb{C})$. If $\lambda(\mathbb{C})$ is the complexification $\lambda(\mathbb{R}) \otimes \mathbb{C}$ of a real affine line $\lambda(\mathbb{R})$, then we have

$$T[\lambda(\mathbb{R}) \otimes \mathbb{C}] = T[\lambda(\mathbb{R})] \otimes \mathbb{C} = T[\lambda(\mathbb{R})] \oplus \sqrt{-1}\, T[\lambda(\mathbb{R})].$$

For affine coordinates (x, y) of $\mathbb{A}^2(\mathbb{R})$, we define a basis e_x, e_y of $T(\mathbb{A}^2(\mathbb{R}))$ by

$$e_x\colon (x, y) \to (x + 1, y), \quad e_y\colon (x, y) \to (x, y + 1). \tag{3.2}$$

Suppose that a real line $\lambda(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$ is defined by

$$ax + by + c = 0,$$

where $a, b, c \in \mathbb{R}$. Then the linear subspace $T[\lambda(\mathbb{R})]$ of $T(\mathbb{A}^2(\mathbb{R}))$ (and hence the linear subspace $T[\lambda(\mathbb{R}) \otimes \mathbb{C}]$ of $T(\mathbb{A}^2(\mathbb{C}))$) is generated by the vector

$$be_x - ae_y.$$

3.2. **Good coordinates.** Let (ξ, η) be affine coordinates of $\mathbb{A}^2(\mathbb{C})$. We put

$$\xi = x + \sqrt{-1}\, u, \quad \eta = y + \sqrt{-1}\, v,$$

where x, u, y, v are real-valued functions on $\mathbb{A}^2(\mathbb{C})$. We say that (ξ, η) is *compatible with the* $\mathbb{R}$*-structure* if the complex conjugation $Q \mapsto \overline{Q}$ is given by

$$(x + \sqrt{-1}\, u, y + \sqrt{-1}\, v) \;\mapsto\; (x - \sqrt{-1}\, u, y - \sqrt{-1}\, v).$$

Suppose that (ξ, η) is compatible with the $\mathbb{R}$-structure. Then $\mathbb{A}^2(\mathbb{R})$ is equal to $\{u = v = 0\}$, and the restriction of (x, y) to $\mathbb{A}^2(\mathbb{R})$ is an affine coordinate system of $\mathbb{A}^2(\mathbb{R})$. From now on, we regard (x, y) as affine coordinates of $\mathbb{A}^2(\mathbb{R})$ by restriction. Then $\mathrm{pr}_{\mathbb{R}} \colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{R})$ is given by

$$(\xi, \eta) \;\mapsto\; (x, y).$$

We say that the affine coordinates (ξ, η) compatible with the $\mathbb{R}$-structure are *good coordinates* if the ordered basis $(\partial/\partial x, \partial/\partial y)$ of tangent vectors of $\mathbb{A}^2(\mathbb{R})$ is positive with respect to the fixed orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$.

4. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. For a topological space T, we write $H_2(T)$ for $H_2(T, \mathbb{Z})$.

Lemma 4.1. *The homomorphism $\rho_* \colon H_2(X) \to H_2(W)$ is surjective, and its kernel is a free $\mathbb{Z}$-module with basis $[D_P]$, where P runs through $\mathcal{P}$.*

Proof. This follows from Lemma 4.4 of [4]. □

Next we prove the following:

Lemma 4.2. *The homology group $H_2(W)$ is a free $\mathbb{Z}$-module of which the classes $\rho_*([\Sigma(C)])$ form a basis, where C runs through $\boldsymbol{Ch}_{\mathrm{b}}$.*

Combining Lemmas 4.1 and 4.2, we will obtain a proof of Theorem 1.6.

Remark that Lemma 4.2 holds trivially when $\boldsymbol{Ch}_{\mathrm{b}} = \emptyset$. Indeed, if $\boldsymbol{Ch}_{\mathrm{b}}$ is empty, then either all lines in $\mathcal{A}$ are parallel, or all lines in $\mathcal{A}$ except one line are parallel. In these cases, we have $H_2(W) = 0$.

From now on, we assume $\boldsymbol{Ch}_{\mathrm{b}} \neq \emptyset$. We put

$$W_{\mathbb{R}} := \pi^{-1}(\mathbb{A}^2(\mathbb{R})), \;\; \mathscr{C} := \bigcup_{C \in \boldsymbol{Ch}_{\mathrm{b}}} C, \;\; W_{\mathscr{C}} := \pi^{-1}(\mathscr{C}) = \bigcup_{C \in \boldsymbol{Ch}_{\mathrm{b}}} \pi^{-1}(C).$$

Note that, for $C \in \boldsymbol{Ch}_{\mathrm{b}}$, the pull-back $\pi^{-1}(C)$ is homeomorphic to a 2-sphere, and is equal to the image $\rho(\Sigma(C))$ of the vanishing cycle $\Sigma(C)$ by the minimal desingularization $\rho \colon X \to W$. Hence Lemma 4.2 follows from the following two lemmas:

Lemma 4.3. *There exists a strong deformation retraction of W onto $W_{\mathscr{C}}$.*

Lemma 4.4. *The homology group $H_2(W_{\mathscr{C}})$ is a free $\mathbb{Z}$-module with basis $[\pi^{-1}(C)]$, where C runs through $\boldsymbol{Ch}_{\mathrm{b}}$, and each 2-sphere $\pi^{-1}(C)$ is equipped with an orientation.*

Proof of Lemma 4.3. First, we construct a strong deformation retraction of W onto $W_{\mathbb{R}}$. Since $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ is a simply branched covering of degree 2 with branch locus $B(\mathbb{C})$ in the sense of Definition 2.1, it is enough to construct a strong deformation retraction

$$F\colon \mathbb{A}^2(\mathbb{C}) \times I \to \mathbb{A}^2(\mathbb{C})$$

of $\mathbb{A}^2(\mathbb{C})$ onto $\mathbb{A}^2(\mathbb{R})$ such that, for $Q \in \mathbb{A}^2(\mathbb{C})$, we have

$$F(Q, t_0) \in B(\mathbb{C}) \text{ for } t_0 \in I \implies F(Q, t) \in B(\mathbb{C}) \text{ for all } t \in [t_0, 1]. \tag{4.1}$$

For $Q_0 \in \mathbb{A}^2(\mathbb{R})$, under the natural identification (3.1) of $\mathrm{pr}_{\mathbb{R}}^{-1}(Q_0)$ with $\sqrt{-1}\, T(\mathbb{A}^2(\mathbb{R}))$, we have

$$\mathrm{pr}_{\mathbb{R}}^{-1}(Q_0) \cap B(\mathbb{C}) = \begin{cases} \emptyset & \text{if } Q_0 \notin B(\mathbb{R}), \\ \sqrt{-1}\, T[\ell_i(\mathbb{R})] & \text{if } Q_0 \in \ell_i(\mathbb{R}) \setminus \mathcal{P}, \\ \sqrt{-1}\ \left(T[\ell_i(\mathbb{R})] \cup T[\ell_j(\mathbb{R})] \right) & \text{if } \{Q_0\} = \ell_i(\mathbb{R}) \cap \ell_j(\mathbb{R}). \end{cases}$$

Hence, for $Q \in \mathbb{A}^2(\mathbb{C})$, the line segment in $\mathrm{pr}_{\mathbb{R}}^{-1}(Q_0)$ connecting Q and its real part $Q_0 := \mathrm{pr}_{\mathbb{R}}(Q) \in \mathbb{A}^2(\mathbb{R})$ is either disjoint from $B(\mathbb{C})$ or entirely contained in $B(\mathbb{C})$ or intersecting $B(\mathbb{C})$ only at Q_0. Therefore the strong deformation retraction of $\mathbb{A}^2(\mathbb{C})$ onto $\mathbb{A}^2(\mathbb{R})$ defined by

$$F(Q, t) := (1-t) \cdot Q + t \cdot Q_0 = Q + t \cdot \tau_{Q, Q_0}$$

satisfies (4.1).

Next we construct a strong deformation retraction of $W_{\mathbb{R}}$ onto $W_{\mathscr{C}}$. Since $\pi | W_{\mathbb{R}}\colon W_{\mathbb{R}} \to \mathbb{A}^2(\mathbb{R})$ is a simply branched covering of degree 2 with branch locus $B(\mathbb{R})$, it is enough to construct a strong deformation retraction

$$G\colon \mathbb{A}^2(\mathbb{R}) \times I \to \mathbb{A}^2(\mathbb{R})$$

of $\mathbb{A}^2(\mathbb{R})$ onto $\mathscr{C}$ that satisfies

$$G(Q, t_0) \in B(\mathbb{R}) \text{ for } t_0 \in I \implies G(Q, t) \in B(\mathbb{R}) \text{ for all } t \in [t_0, 1]. \tag{4.2}$$

For $Q \in \mathbb{A}^2(\mathbb{R})$, we define $Q' \in \mathscr{C}$ by the following. (In the argument below, we use the assumption $\boldsymbol{Ch}_{\mathrm{b}} \neq \emptyset$ frequently.) If $Q \in \mathscr{C}$, then we put $Q' := Q$. Suppose that $Q \notin \mathscr{C}$. Then there exists a non-bounded chamber C' containing Q. Note that $C' \cap \mathscr{C}$ is non-empty, because of the assumption $\boldsymbol{Ch}_{\mathrm{b}} \neq \emptyset$.

- Suppose that Q is in the interior of C'. Since C' is non-bounded, C' has exactly two non-bounded edges. Let $\ell_i(\mathbb{R})$ and $\ell_j(\mathbb{R})$ be the lines defining these non-bounded edges.

  Suppose that $\ell_i(\mathbb{R})$ and $\ell_j(\mathbb{R})$ are not parallel, and let K be the intersection point of $\ell_i(\mathbb{R})$ and $\ell_j(\mathbb{R})$. Then the line segment QK intersect $C' \cap \mathscr{C}$ at a single point. We define Q' to be this point.

  Suppose that $\ell_i(\mathbb{R})$ and $\ell_j(\mathbb{R})$ are parallel. Let $\lambda(\mathbb{R})$ be the affine real line passing through Q parallel to $\ell_i(\mathbb{R})$. We define Q' to be the point of $\lambda(\mathbb{R}) \cap \mathscr{C}$ nearest to Q.
- Suppose that Q is not in the interior of C'. Then Q is located on some line $\ell_i(\mathbb{R}) \in \mathcal{A}$. We define Q' to be the point of $\ell_i(\mathbb{R}) \cap \mathscr{C}$ nearest to Q.

The mapping $Q \mapsto Q'$ is continuous. For $Q \in \mathbb{A}^2(\mathbb{R})$, the line segment connecting Q and Q' is either disjoint from $B(\mathbb{R})$, or entirely contained in $B(\mathbb{R})$, or intersect $B(\mathbb{R})$ only at Q'. Therefore the strong deformation retraction of $\mathbb{A}^2(\mathbb{R})$ onto $\mathscr{C}$ defined by

$$G(Q,t) := (1-t)\cdot Q + t\cdot Q' = Q + t\cdot \tau_{Q,Q'}$$

satisfies (4.2).

Thus Lemma 4.3 is proved. □

Proof of Lemma 4.4. We put $m := |\boldsymbol{Ch}_{\mathrm{b}}|$. It is enough to show that there exists a numbering $C_1, \dots, C_m$ of elements of $\boldsymbol{Ch}_{\mathrm{b}}$ such that, putting

$$\mathscr{C}_k := C_1 \cup \dots \cup C_k, \quad K_k := C_k \cap \mathscr{C}_{k-1} \subset \partial C_k,$$

we have $K_k \neq \partial C_k$ for $k = 2, \dots, m$. Indeed, if such a sequence $C_1, \dots, C_m$ exists, then we have $H_1(\pi^{-1}(K_k)) = H_2(\pi^{-1}(K_k)) = 0$ for $k = 2, \dots, m$, and hence we obtain

$$H_2(\pi^{-1}(\mathscr{C}_k)) = H_2(\pi^{-1}(\mathscr{C}_{k-1})) \oplus H_2(\pi^{-1}(C_k))$$

by the Mayer–Vietoris sequence, and thus

$$H_2(W_{\mathscr{C}}) = H_2(\pi^{-1}(\mathscr{C}_m)) = H_2(\pi^{-1}(C_1)) \oplus \dots \oplus H_2(\pi^{-1}(C_m))$$

is proved by induction on k. We construct the *reversed* sequence $C_m, \dots, C_1$ by the following procedure:

Set $k := m$ and $\boldsymbol{C} := \boldsymbol{Ch}_{\mathrm{b}}$
while $k > 0$ **do**
  Let $\mathscr{C}_k$ be the union of chambers in $\boldsymbol{C}$.
  Note that $\mathscr{C}_k$ is bounded.
  We choose $C \in \boldsymbol{C}$ such that an edge e_k of C is a part of $\partial \mathscr{C}_k$.
  We set $C_k := C$, remove C from $\boldsymbol{C}$, and decrement k by 1.
end while

Since $K_k = C_k \cap \mathscr{C}_{k-1}$ is contained in ∂C_k minus the interior of the edge e_k, the obtained sequence satisfies $K_k \neq \partial C_k$ for $k = 2, \dots, m$. □

5. Capping hemispheres

Throughout this section, we fix a defining polynomial $f\colon \mathbb{A}^2(\mathbb{R}) \to \mathbb{R}$ of $\mathcal{A}$, so that, together with the fixed orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$, a standard orientation σ_C is defined for each $C \in \boldsymbol{Ch}_{\mathrm{b}}$. We calculate the capping hemispheres $H_{C,P} := H_{C,\sigma_C,P}$ explicitly.

5.1. **Good systems of local coordinates.** Recall from Section 3.2 that an affine coordinate system (ξ, η) of $\mathbb{A}^2(\mathbb{C})$ is *good* if it is compatible with the $\mathbb{R}$-structure of $\mathbb{A}^2(\mathbb{C})$ and the affine coordinates $(x, y) := (\operatorname{Re}\xi, \operatorname{Re}\eta)|\mathbb{A}^2(\mathbb{R})$ of $\mathbb{A}^2(\mathbb{R})$ are compatible with the fixed orientation $\sigma_{\mathbb{A}}$. We extend this notion to the notion of *good local coordinate systems* on various (real and complex) surfaces.

Let P be a point in $\mathcal{P}$. We say that good coordinates (ξ, η) of $\mathbb{A}^2(\mathbb{C})$ are *good at* P if P is the origin $(0,0)$ and the two lines in $\mathcal{A}$ passing through P are defined in $\mathbb{A}^2(\mathbb{R})$ by

$$\ell_a(\mathbb{R})\colon y = ax \quad \text{and} \quad \ell_b(\mathbb{R})\colon y = bx$$

by some real numbers a, b satisfying $a < b$.

Remark 5.1. For any pair (a', b') of real numbers satisfying $a' < b'$, we have good coordinates (ξ, η) at P such that the pair (a, b) defined above is equal to the given pair (a', b').

Next we define local coordinates $(\tilde{\xi}, \mu)$ of $Y(\mathbb{C})$ and local coordinates $(\tilde{x}, m)$ of $Y(\mathbb{R}) = \beta^\sharp \mathbb{A}^2(\mathbb{R})$. Let (ξ, η) be affine coordinates of $\mathbb{A}^2(\mathbb{C})$ good at $P \in \mathcal{P}$ as above. Let r be a positive real number such that the open subset

$$U_r := \{\, (\xi, \eta) \mid |\xi| < r, \ \ |\eta| < r \,\}$$

of $\mathbb{A}^2(\mathbb{C})$ intersects no lines of $\mathcal{A}_{\mathbb{C}}$ other than $\ell_a(\mathbb{C})$ and $\ell_b(\mathbb{C})$:

$$\text{for } \ell(\mathbb{R}) \in \mathcal{A}, \text{ if } \ell(\mathbb{C}) \cap U_r \neq \emptyset, \text{ then we have } \ell(\mathbb{R}) \in \{\ell_a(\mathbb{R}), \ell_b(\mathbb{R})\}. \tag{5.1}$$

Then there exists a unique coordinate system $(\tilde{\xi}, \mu)$ on $\beta^{-1}(U_r) \subset Y(\mathbb{C})$ such that $\beta\colon Y(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ is given by

$$(\tilde{\xi}, \mu) \mapsto (\xi, \eta) = (\tilde{\xi}, \tilde{\xi}\mu). \tag{5.2}$$

In the chart $\beta^{-1}(U_r)$ of $(\tilde{\xi}, \mu)$, the exceptional curve E_P is defined by $\tilde{\xi} = 0$, and the strict transforms $\beta^\sharp \ell_a(\mathbb{C})$ and $\beta^\sharp \ell_b(\mathbb{C})$ of $\ell_a(\mathbb{C})$ and $\ell_b(\mathbb{C})$ are defined by $\mu = a$ and $\mu = b$, respectively. We put

$$\tilde{\xi} = \tilde{x} + \sqrt{-1}\, \tilde{u}, \quad \mu = m + \sqrt{-1}\, s,$$

where $\tilde{x}$, $\tilde{u}$, m, s are real-valued functions on $\beta^{-1}(U_r)$. Then we have

$$Y(\mathbb{R}) = \{\tilde{u} = s = 0\}$$

in $\beta^{-1}(U_r)$, and we can regard the restrictions of $(\tilde{x}, m)$ to $Y(\mathbb{R})$ as local coordinates of the real surface $Y(\mathbb{R})$. From now on, we consider $(\tilde{x}, m)$ as local coordinates of $Y(\mathbb{R})$ by restriction.

Definition 5.2. The local coordinate systems (ξ, η) on $\mathbb{A}^2(\mathbb{C})$, $(\tilde{\xi}, \mu)$ on $Y(\mathbb{C})$, (x, y) on $\mathbb{A}^2(\mathbb{R})$, and $(\tilde{x}, m)$ on $Y(\mathbb{R})$, are called *good local coordinate systems at* P.

5.2. **The orientation of** $Y(\mathbb{R})$**.** Let $\beta_{\mathbb{R}}\colon Y(\mathbb{R}) \to \mathbb{A}^2(\mathbb{R})$ denote the restriction of β to $Y(\mathbb{R})$, which is a local isomorphism at every point of

$$Y(\mathbb{R})^\circ := \beta^{-1}(\mathbb{A}^2(\mathbb{R}) \setminus \mathcal{P}).$$

Hence the fixed orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$ induces an orientation on $Y(\mathbb{R})^\circ$ via $\beta_{\mathbb{R}}$, which we denote by $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$.

Lemma 5.3. *Let Q be a point of $\beta^{-1}(U_r) \cap Y(\mathbb{R})^\circ$. If $\tilde{x} > 0$ at Q, then the ordered pair $(\partial/\partial\tilde{x}, \partial/\partial m)$ of tangent vectors of $Y(\mathbb{R})$ at Q is positive with respect to $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$, whereas if $\tilde{x} < 0$ at Q, then $(\partial/\partial\tilde{x}, \partial/\partial m)$ is negative with respect to $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$.*

Proof. The map $\beta_{\mathbb{R}}\colon Y(\mathbb{R}) \to \mathbb{A}^2(\mathbb{R})$ is given by $(\tilde{x}, m) \mapsto (x, y) = (\tilde{x}, \tilde{x}m)$. We can calculate $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$ by this formula. □

The restriction of the function μ to E_P is an affine coordinate of E_P. Let $Q_\infty \in E_P$ be the point $\mu = \infty$ of E_P. We consider the m-axis

$$(E_P \setminus \{Q_\infty\}) \cap Y(\mathbb{R}) = \{\tilde{x} = 0\}$$

in the chart $\beta^{-1}(U_r) \cap Y(\mathbb{R})$ of $(\tilde{x}, m)$.

Corollary 5.4. *The orientation $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$ on $Y(\mathbb{R})^\circ$ induces the downward orientation on the m-axis $(E_P \setminus \{Q_\infty\}) \cap Y(\mathbb{R})$, that is, the direction that m decreases is positive with respect to the orientation on the m-axis induced by $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$.*

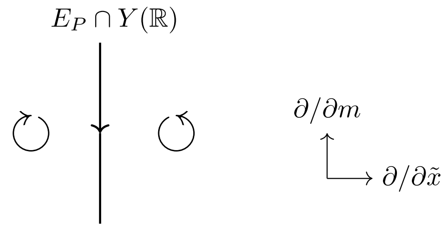

FIGURE 5.1. Orientation $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$

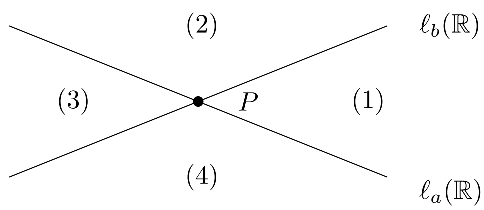

FIGURE 5.2. Location of chambers

Proof. See Figure 5.1 of $Y(\mathbb{R})$, in which $\partial/\partial \tilde{x}$ is rightward and $\partial/\partial m$ is upward. The orientation $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$ on $Y(\mathbb{R})^\circ$ is counter-clockwise $\circlearrowleft$ in the region $\tilde{x} > 0$, and is clockwise $\circlearrowright$ in the region $\tilde{x} < 0$. Both of them induce a downward orientation on the m-axis. □

5.3. **Capping hemispheres.** Let C be a bounded chamber such that $P \in \mathrm{Vert}(C)$. We use the good local coordinate systems at P given in the previous section. We calculate the capping hemisphere $H_{C,P} = H_{C,\sigma_C,P}$ explicitly. We have the following four cases: locally around P in $\mathbb{A}^2(\mathbb{R})$, the chamber C is equal to the region

$$\begin{aligned} &\text{Case (1):} \quad ax \le y \le bx, \\ &\text{Case (2):} \quad ax \le y \text{ and } bx \le y, \\ &\text{Case (3):} \quad ax \ge y \ge bx, \\ &\text{Case (4):} \quad ax \ge y \text{ and } bx \ge y. \end{aligned} \tag{5.3}$$

See Figure 5.2.

Remark 5.5. We can choose, without loss of generality, good coordinates (ξ, η) of $\mathbb{A}^2(\mathbb{C})$ at P such that Case (1) occurs.

In terms of the local coordinates $(\tilde{x}, m)$ of $Y(\mathbb{R})$, the closed region $\beta^\sharp C$ is given in $Y(\mathbb{R})$ as follows:

$$\begin{aligned} &\text{Case (1):} \quad \beta^\sharp C = \{\, (\tilde{x}, m) \mid \tilde{x} \ge 0, \ \ a \le m \le b \,\}, \\ &\text{Case (2):} \quad \beta^\sharp C = \{\, (\tilde{x}, m) \mid (\tilde{x} \ge 0 \text{ and } b \le m) \text{ or } (\tilde{x} \le 0 \text{ and } m \le a) \,\}, \\ &\text{Case (3):} \quad \beta^\sharp C = \{\, (\tilde{x}, m) \mid \tilde{x} \le 0, \ \ a \le m \le b \,\}, \\ &\text{Case (4):} \quad \beta^\sharp C = \{\, (\tilde{x}, m) \mid (\tilde{x} \le 0 \text{ and } b \le m) \text{ or } (\tilde{x} \ge 0 \text{ and } m \le a) \,\}. \end{aligned}$$

See Figure 5.3. Recall that the open subset U_r of $\mathbb{A}^2(\mathbb{C})$ is defined in such a way

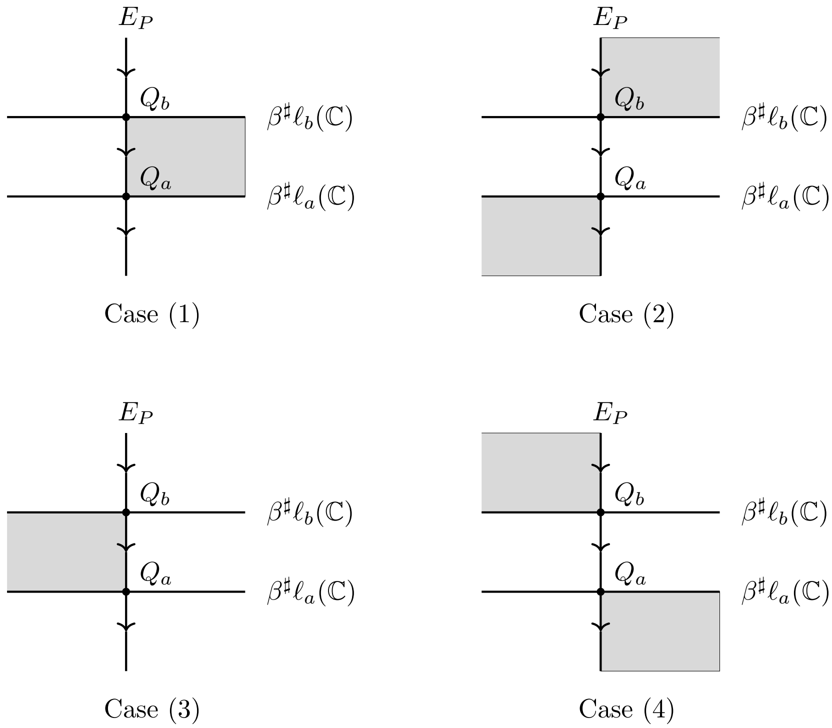

FIGURE 5.3. $\beta^\sharp C$ on $Y(\mathbb{R})$

that (5.1) holds. Hence the pullback $\beta^* f$ of the defining polynomial $f\colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{C}$ of $\mathcal{A}_\mathbb{C}$ by β is written as

$$\beta^* f = u_P \cdot \tilde{\xi}^2 \cdot (\mu - a) \cdot (\mu - b)$$

in the chart $\beta^{-1}(U_r)$ of $(\tilde{\xi}, \mu)$, where u_P is a complex-valued continuous function on $\beta^{-1}(U_r)$ that has *no zeros*, takes values in $\mathbb{R}$ on $\beta^{-1}(U_r) \cap Y(\mathbb{R})$, and is constant on $\beta^{-1}(U_r) \cap E_P$. We denote by

$$c_P \in \mathbb{R} \setminus \{0\}$$

the value of u_P on E_P. Note that $\beta^{-1}(U_r)$ is simply connected. Hence we can define a complex-valued continuous function v_P on $\beta^{-1}(U_r)$ such that $v_P^2 = u_P$ and that

$$v_P | (\beta^{-1}(U_r) \cap Y(\mathbb{R})) \text{ takes values in } \begin{cases} \mathbb{R}_{>0} & \text{if } c_P \in \mathbb{R}_{>0}, \\ \sqrt{-1}\,\mathbb{R}_{>0} & \text{if } c_P \in \mathbb{R}_{<0}. \end{cases} \tag{5.4}$$

Recall that ω is the function on W such that the covering $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ is given by $\omega^2 = f$. We regard ω as a function on X by $\rho\colon X \to W$. We also regard v_P, $\tilde{\xi}$, μ as functions on $\phi^{-1}(\beta^{-1}(U_r)) \subset X$ by $\phi\colon X \to Y(\mathbb{C})$. We then put

$$\zeta := \frac{\omega}{v_P \cdot \tilde{\xi} \cdot (\mu - b)}, \tag{5.5}$$

C	$f(C^\circ)$	$H_{C,P}$
Case (1)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Re} z \le 0$
Case (1)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Re} z \ge 0$
Case (2)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Im} z \le 0$
Case (2)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Im} z \le 0$
Case (3)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Re} z \ge 0$
Case (3)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Re} z \le 0$
Case (4)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Im} z \ge 0$
Case (4)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Im} z \ge 0$

TABLE 5.1. Capping hemispheres

which is a meromorphic function on $\phi^{-1}(\beta^{-1}(U_r)) \subset X$. Then we have

$$\zeta^2 = \frac{\mu - a}{\mu - b}, \tag{5.6}$$

which gives the double covering $\phi\colon X \to Y(\mathbb{C})$. We put

$$z := \zeta | D_P.$$

Let Q_a (resp. Q_b) be the intersection point of E_P and $\beta^\sharp \ell_a(\mathbb{C})$ (resp. $\beta^\sharp \ell_b(\mathbb{C})$). The restriction $\phi|D_P\colon D_P \to E_P$ of ϕ to $D_P \subset X$ is a double covering whose branch points are Q_a and Q_b. Let $\widetilde{Q}_a \in D_P$ (resp. $\widetilde{Q}_b \in D_P$) be the point of D_P lying over Q_a (resp. Q_b). Then z is an affine coordinate of the Riemann sphere D_P such that $\widetilde{Q}_a$ and $\widetilde{Q}_b$ are given by $z = 0$ and $z = \infty$, respectively. The main result of this section is as follows:

Proposition 5.6. *The capping hemisphere $H_{C,P}$ is given as in Table 5.1.*

Proof. To ease the notation, we put

$$\mathrm{P} := \mathbb{R}_{\ge 0} \cup \{\infty\}, \quad i\mathrm{P} := \sqrt{-1}\,\mathbb{R}_{\ge 0} \cup \{\infty\}.$$

Recall that $\phi^{-1}(\beta^{-1}(C^\circ)) = \rho^{-1}(\pi^{-1}(C^\circ))$ has two connected components, which we call sheets. We denote by $(\phi^{-1}\beta^{-1}C^\circ)_+$ the sheet on which we have

$$\begin{cases} \omega \in \mathrm{P} \text{ holds} & \text{if } f \in P \text{ on } C, \\ \omega \in i\mathrm{P} \text{ holds} & \text{if } f \in -P \text{ on } C. \end{cases} \tag{5.7}$$

Recall Definition 1.5 of the standard orientation σ_C. The sheet $(\phi^{-1}\beta^{-1}C^\circ)_+$ is the plus-sheet, that is, the restriction of ϕ to $(\phi^{-1}\beta^{-1}C^\circ)_+$ is an orientation-preserving isomorphism from $(\phi^{-1}\beta^{-1}C^\circ)_+$ with orientation σ_C to the open subset $\beta^{-1}(C^\circ)$ of $Y(\mathbb{R})$ with orientation $\beta_{\mathbb{R}}^*\sigma_{\mathbb{A}}$.

By the locations of C and the signs of $f(C)$, we have eight cases, which are given by the rows of Table 5.1. For each case, we calculate values at points of

$(\phi^{-1}\beta^{-1}C^{\circ})_+$ of the functions

$$\begin{aligned} g &:= (m-a)(m-b), \\ u_P &= \beta^* f/(\tilde{x}^2 g), \\ v_P & \quad \text{as defined by (5.4)}, \\ h &:= \tilde{x}\cdot(m-b), \\ \omega & \quad \text{as defined by (5.7)}, \\ \zeta &= \omega/(v_P h). \end{aligned}$$

The results are given in Table 5.2. The intersection

$$K_{C,P} := D_P \cap \overline{(\phi^{-1}\beta^{-1}C^{\circ})_+}$$

of D_P and the closure of $(\phi^{-1}\beta^{-1}C^{\circ})_+$ in X is an arc on D_P connecting the ramification points $\widetilde{Q}_a$ and $\widetilde{Q}_b$ of $\phi|D_P\colon D_P \to E_P$. In terms of the parameter z on D_P, the arc $K_{C,P}$ is the closed arc connecting $z=0$ and $z=\infty$ along the closure of the range of the function ζ on $(\phi^{-1}\beta^{-1}C^{\circ})_+$. Since we have calculated this range above, we can describe $K_{C,P}$ in terms of z. See the column $K_{C,P}$ of Table 5.2. Since the deck transformation of the double covering $\phi|D_P\colon D_P \to E_P$ is given by $z \mapsto -z$, we have

$$S_{C,P} = D_P \cap \Delta(C) = K_{C,P} \cup (-K_{C,P}). \tag{5.8}$$

The covering $\phi|D_P$ maps $K_{C,P}$ to $J_{C,P} = E_P \cap \beta^{\sharp}C$, which is a segment of the m-axis with Q_∞ being added. The orientation σ_C on $(\phi^{-1}\beta^{-1}C^{\circ})_+$ induces an orientation on $K_{C,P}$, and hence on $\phi(K_{C,P}) = J_{C,P}$. Since ϕ induces an orientation-preserving isomorphism from $(\phi^{-1}\beta^{-1}C^{\circ})_+$ with orientation σ_C to $\beta^{-1}(C^{\circ})$ with orientation $\beta^*_{\mathbb{R}}\sigma_{\mathbb{A}}$, Corollary 5.4 implies that this orientation on $J_{C,P}$ is downward, that is,

$$\begin{aligned} Q_b \searrow Q_a & \quad \text{in Cases (1) and (3)}, \\ Q_a \searrow Q_\infty \searrow Q_b & \quad \text{in Cases (2) and (4)}. \end{aligned}$$

See Figure 5.3. In the column ori_K of Table 5.2, this orientation is expressed in terms of the parameter z. Combining the computations of $K_{C,P}$ and ori_K, we obtain the orientation ori_C on the circle $S_{C,P}$ induced by the orientation σ_C of $\Delta(C)$ by (5.8). In Table 5.2, the circle $S_{C,P}$ and the orientation ori_C are given as follows:

$$\begin{aligned} \downarrow & \quad \text{means } S_{C,P} = \{\mathrm{Re}\, z = 0\} \text{ and the orientation is downward}, \\ \uparrow & \quad \text{means } S_{C,P} = \{\mathrm{Re}\, z = 0\} \text{ and the orientation is upward}, \\ \rightarrow & \quad \text{means } S_{C,P} = \{\mathrm{Im}\, z = 0\} \text{ and the orientation is rightward}, \\ \leftarrow & \quad \text{means } S_{C,P} = \{\mathrm{Im}\, z = 0\} \text{ and the orientation is leftward}. \end{aligned}$$

The orientation ori_H on $S_{C,P}$ given by the complex structure of the capping hemisphere $H_{C,P}$ is the opposite of ori_C. Thus we obtain $H_{C,P}$ as in the last column of Table 5.2. □

6. Displacement

We construct two types of displacements of vanishing cycles $\Sigma(C)$ in X. One is called *C-displacements*, and the other is called *E-displacements*.

C	$\beta^* f$	g	u_P	v_P	h	ω	ζ	$K_{C,P}$	ori_K	ori_C	ori_H	$H_{C,P}$
(1)	$+$	$-$	$-$	iP	$-$	P	iP	iP	$\infty \to 0$	$\downarrow$	$\uparrow$	$\mathrm{Re}\, z \leq 0$
(1)	$-$	$-$	$+$	P	$-$	iP	$-i$P	$-i$P	$\infty \to 0$	$\uparrow$	$\downarrow$	$\mathrm{Re}\, z \geq 0$
(2)	$+$	$+$	$+$	P	$+$	P	P	P	$0 \to \infty$	$\rightarrow$	$\leftarrow$	$\mathrm{Im}\, z \leq 0$
(2)	$-$	$+$	$-$	iP	$+$	iP	P	P	$0 \to \infty$	$\rightarrow$	$\leftarrow$	$\mathrm{Im}\, z \leq 0$
(3)	$+$	$-$	$-$	iP	$+$	P	$-i$P	$-i$P	$\infty \to 0$	$\uparrow$	$\downarrow$	$\mathrm{Re}\, z \geq 0$
(3)	$-$	$-$	$+$	P	$+$	iP	iP	iP	$\infty \to 0$	$\downarrow$	$\uparrow$	$\mathrm{Re}\, z \leq 0$
(4)	$+$	$+$	$+$	P	$-$	P	$-$P	$-$P	$0 \to \infty$	$\leftarrow$	$\rightarrow$	$\mathrm{Im}\, z \geq 0$
(4)	$-$	$+$	$-$	iP	$-$	iP	$-$P	$-$P	$0 \to \infty$	$\leftarrow$	$\rightarrow$	$\mathrm{Im}\, z \geq 0$

TABLE 5.2. Functions on $(\phi^{-1}\beta^{-1}C^\circ)_+$

6.1. **Notation and terminologies about displacements.** For a small positive real number ε, let I_ε denote the closed interval $[0, \varepsilon] \subset \mathbb{R}$.

Definition 6.1. A *displacement* of a subspace S of a topological space T is a continuous map

$$d \colon S \times I_\varepsilon \to T$$

such that

- $d(P, 0) = P$ for any $P \in S$, and
- the restriction $d_t := d|S \times \{t\}$ of d to $S \times \{t\}$ is a homeomorphism from S to its image $S_t := d(S \times \{t\})$ for any $t \in I_\varepsilon$.

We sometimes write

$$\{\, S_t \mid t \in I_\varepsilon \,\} \quad \text{or} \quad \{\, d_t \mid t \in I_\varepsilon \,\} \quad \text{or} \quad d_t \colon S \to T \ \ (t \in I_\varepsilon)$$

to denote the displacement $d \colon S \times I_\varepsilon \to T$. By further abuse of notation, we often say that *S_ε is a displacement of S in T.*

If $S = S_0$ is equipped with an orientation σ_0, then a displacement S_ε of S is also oriented by σ_0 via the homeomorphism $d_\varepsilon \colon S \to S_\varepsilon$. If S is a topological cycle, then S_ε is also a topological cycle, and their classes in the homology group of T are the same.

We say that a displacement $d_t \colon S \to T$ *preserves a subspace R of T* if $d_t(Q) \in R$ holds for any $Q \in S \cap R$ and any $t \in I_\varepsilon$. We say that d is *stationary* on a subspace $S' \subset S$ if $d(Q, t) = Q$ holds for any $(Q, t) \in S' \times I_\varepsilon$. A *trivial displacement* is a displacement that is stationary on the whole space S.

6.2. **C-displacement.** Recall from Section 3.1 that $T(\mathbb{A}^2(\mathbb{R}))$ is the $\mathbb{R}$-vector space of translations of $\mathbb{A}^2(\mathbb{R})$, and that $T[\lambda(\mathbb{R})]$ is the subspace of translations preserving a real affine line $\lambda(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$. Let C be a bounded chamber, and let

$$\delta \colon C \to T(\mathbb{A}^2(\mathbb{R}))$$

be a continuous function.

Definition 6.2. Suppose that $\ell_i(\mathbb{R}) \in \mathcal{A}$ defines an edge $C \cap \ell_i(\mathbb{R})$ of C. We say that δ satisfies the *e-condition* for the edge $C \cap \ell_i(\mathbb{R})$ if $\delta(Q) \in T[\ell_i(\mathbb{R})]$ holds for any $Q \in C \cap \ell_i(\mathbb{R})$.

Remark 6.3. If δ satisfies the e-condition for every edge of C, then we have $\delta(P)=0$ for every $P \in \mathrm{Vert}(C)$, because $T[\ell_i(\mathbb{R})] \cap T[\ell_j(\mathbb{R})] = \{0\}$ holds if $\ell_i(\mathbb{R}) \cap \ell_j(\mathbb{R}) = \{P\}$.

Suppose that $\delta\colon C \to T(\mathbb{A}^2(\mathbb{R}))$ satisfies the e-condition for every edge of C. We define $d\colon C \times I_\varepsilon \to \mathbb{A}^2(\mathbb{C})$ by

$$d(Q,t) := Q + \sqrt{-1}\,\delta(Q)t. \tag{6.1}$$

Note that we have

$$\mathrm{pr}_{\mathbb{R}}(d(Q,t)) = Q \ \text{ for any } \ (Q,t) \in C \times I_\varepsilon, \tag{6.2}$$

where $\mathrm{pr}_{\mathbb{R}}\colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{R})$ is the projection that takes the real part of points of $\mathbb{A}^2(\mathbb{C})$ (see Section 3.1). This implies that $d_t := d|C \times \{t\}$ is a homeomorphism from C to

$$C_t := d_t(C)$$

for any $t \in I_\varepsilon$, and hence d is a displacement of C in $\mathbb{A}^2(\mathbb{C})$. By the e-condition, the displacement d preserves $\ell_i(\mathbb{C})$ for each $\ell_i(\mathbb{R}) \in \mathcal{A}$ defining edges of C. Since $\mathcal{P} \subset \mathbb{A}^2(\mathbb{R})$ and $\mathrm{pr}_{\mathbb{R}}(B(\mathbb{C})) = B(\mathbb{R})$, we see from (6.2) that

$$Q \in C^\bullet = C \setminus \mathrm{Vert}(C) \implies d(Q,t) \notin \mathcal{P}, \tag{6.3}$$

$$Q \in C^\circ = C \setminus (C \cap B(\mathbb{R})) \implies d(Q,t) \notin B(\mathbb{C}). \tag{6.4}$$

Note that $\beta\colon B(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ induces an isomorphism from $\beta^{-1}(C^\bullet)$ to $C^\bullet$. By (6.3), there exists a unique continuous map

$$d^\bullet\colon \beta^{-1}(C^\bullet) \times I_\varepsilon \to Y(\mathbb{C})$$

that fits in the commutative diagram

$$\begin{array}{ccc} \beta^{-1}(C^\bullet) \times I_\varepsilon & \xrightarrow{d^\bullet} & Y(\mathbb{C}) \\ {\scriptstyle \beta\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \beta} \\ C \times I_\varepsilon & \xrightarrow[d]{} & \mathbb{A}^2(\mathbb{C}). \end{array} \tag{6.5}$$

It is obvious that $d^\bullet$ is a displacement of $\beta^{-1}(C^\bullet)$ in $Y(\mathbb{C})$. We define a map

$$d^\sharp\colon \beta^\sharp C \times I_\varepsilon \to Y(\mathbb{C})$$

by the following:

$$d^\sharp(Q,t) := \begin{cases} d^\bullet(Q,t) & \text{if } Q \in \beta^{-1}(C^\bullet), \\ Q & \text{if } Q \in \beta^\sharp C \setminus \beta^{-1}(C^\bullet) = \bigcup_{P \in \mathrm{Vert}(C)} J_{C,P}. \end{cases}$$

Definition 6.4. We say that $\delta\colon C \to T(\mathbb{A}^2(\mathbb{R}))$ is *regular at* $P \in \mathrm{Vert}(C)$ if $d^\sharp$ is continuous at (Q,t) for any $Q \in J_{C,P}$ and any $t \in I_\varepsilon$. We simply say that δ is *regular* if δ is regular at every vertex P of C, that is, if $d^\sharp$ is continuous.

If δ is regular, then $d^\sharp$ is a displacement of $\beta^\sharp C$ in $Y(\mathbb{C})$ that is stationary on each $J_{C,P}$, and that is preserving all $\beta^\sharp \ell_i(\mathbb{C}) \subset \beta^\sharp Y(\mathbb{C})$, where $\ell_i(\mathbb{R}) \in \mathcal{A}$ defines an edge of C. In particular, the displacement $d^\sharp$ preserves the intersection $\beta^\sharp C \cap \beta^\sharp B(\mathbb{C})$. Since the branch locus of $\phi\colon X \to Y(\mathbb{C})$ is $\beta^\sharp B(\mathbb{C})$, it follows that $d^\sharp$ lifts to a displacement

$$d^\Delta\colon \Delta(C) \times I_\varepsilon \to X$$

of $\Delta(C)$ in X that is stationary on each $S_{C,P} \subset \partial\,\Delta(C)$. We have

$$\Delta(C)_t = \phi^{-1}((\beta^\sharp C)_t)$$

with the orientation given by the standard orientation σ_C of $\phi^{-1}(\beta^\sharp C)$. Gluing d^Δ with the trivial displacements of the capping hemispheres $H_{C,P}$, we obtain a displacement

$$d^\Sigma \colon \Sigma(C) \times I_\varepsilon \to X$$

of $\Sigma(C)$ in X that is stationary on each $H_{C,P}$.

Definition 6.5. These displacements d, $d^\sharp$, d^Δ, and d^Σ are called the *C-displacements associated with a regular continuous map $\delta\colon C \to T(\mathbb{A}^2(\mathbb{R}))$* satisfying the e-condition for every edge of C.

We introduce an operation of *q_ρ-modification*, which is useful in obtaining a regular continuous map $\delta\colon C \to T(\mathbb{A}^2(\mathbb{R}))$. We use the good local coordinate systems (ξ, η) on $\mathbb{A}^2(\mathbb{C})$, $(\tilde{\xi}, \mu)$ on $Y(\mathbb{C})$, (x, y) on $\mathbb{A}^2(\mathbb{R})$, and $(\tilde{x}, m)$ on $Y(\mathbb{R})$, that are defined in Section 5.1. We can assume, without loss of generality, that we are in Case (1) in (5.3), so that there exist real numbers a, b with $a < b$ such that C is given by $ax \le y \le bx$ locally around $P = (0, 0)$ in $\mathbb{A}^2(\mathbb{R})$. We define real-valued functions $\delta_x(x, y)$ and $\delta_y(x, y)$ defined in a small neighborhood of the origin P in $\{\, (x, y) \mid ax \le y \le bx \,\}$ by

$$\delta(Q) = \delta_x(x(Q), y(Q)) \cdot e_x + \delta_y(x(Q), y(Q)) \cdot e_y, \tag{6.6}$$

where $(x(Q), y(Q))$ is the coordinates of $Q \in C$, and e_x, e_y are the basis of $T(\mathbb{A}^2(\mathbb{R}))$ given by (3.2). Because β restricted to $Y(\mathbb{R})$ is written as

$$(\tilde{x}, m) \mapsto (x, y) = (\tilde{x}, m\tilde{x}),$$

the displacement $d^\bullet \colon \beta^{-1}(C^\bullet) \times I_\varepsilon \to Y(\mathbb{C})$ defined by the diagram (6.5) is written in terms of the coordinate systems $(\tilde{x}, m)$ and $(\tilde{\xi}, \mu)$ as

$$\begin{aligned} &((\tilde{x}, m), t) \mapsto \\ &(\tilde{\xi}, \mu) = \left(\tilde{x} + \sqrt{-1}\, \delta_x(\tilde{x}, m\tilde{x}) \cdot t,\ \ \frac{m\tilde{x} + \sqrt{-1}\, \delta_y(\tilde{x}, m\tilde{x}) \cdot t}{\tilde{x} + \sqrt{-1}\, \delta_x(\tilde{x}, m\tilde{x}) \cdot t} \right). \end{aligned} \tag{6.7}$$

We introduce an inner-product on the $\mathbb{R}$-vector space $T(\mathbb{A}^2(\mathbb{R}))$. Then we have a distance on $\mathbb{A}^2(\mathbb{R})$. For points $Q, Q' \in \mathbb{A}^2(\mathbb{R})$, we denote by $|QQ'|$ the distance between Q and Q'. Let ρ be a small positive real number such that $|PP'| > 2\rho$ holds for any pair of distinct vertexes P, P' of C. For each vertex $P \in \mathrm{Vert}(C)$, we put

$$U_{P,\rho} := \{\, Q \in C \mid |PQ| \le \rho \,\}. \tag{6.8}$$

Then each point of C belongs at most one of $U_{P,\rho}$. For $Q \in U_{P,\rho}$, let $q_{P,\rho}(Q)$ denote the unique point on the line segment PQ such that the distance $|Pq_{P,\rho}(Q)|$ of P and $q_{P,\rho}(Q)$ satisfies

$$|Pq_{P,\rho}(Q)| = |PQ|^2/\rho.$$

Since $u \mapsto u^2/\rho$ is a bijection from the closed interval $[0, \rho] \subset \mathbb{R}$ to itself, we see that $q_{P,\rho}$ is a self-homeomorphism of $U_{P,\rho} \subset C$. We construct $q_\rho \colon C \to C$ by

$$q_\rho(Q) := \begin{cases} q_{P,\rho}(Q) & \text{if } Q \in U_{P,\rho} \text{ for some } P \in \mathrm{Vert}(C), \\ Q & \text{otherwise.} \end{cases}$$

Since $q_{P,\rho}(Q) = Q$ for $Q \in U_{P,\rho}$ with $|PQ| = \rho$, we see that q_ρ is continuous on the whole C, and hence is a self-homeomorphism of C.

Definition 6.6. Let $\delta\colon C\to T(\mathbb{A}^2(\mathbb{R}))$ be a continuous map. We call the composite $\delta\circ q_\rho\colon C\to T(\mathbb{A}^2(\mathbb{R}))$ of q_ρ and δ the *q_ρ-modification* of δ.

Note that $q_\rho\colon C\to C$ preserves each edge of C. Hence, if δ satisfies the e-condition for every edge of C, then so does its q_ρ-modification $\delta\circ q_\rho$.

Lemma 6.7. *Let $\delta\colon C\to T(\mathbb{A}^2(\mathbb{R}))$ be a continuous map that satisfies the e-condition for every edge of C. Suppose that δ is affine-linear in a small neighborhood of each vertex P of C. Then the q_ρ-modification $\delta\circ q_\rho$ is regular.*

Proof. We define continuous functions $\delta_x\circ q_\rho(x,y)$ and $\delta_y\circ q_\rho(x,y)$ on $U_{P,\rho}$ by

$$\delta\circ q_\rho(Q)=(\delta_x\circ q_\rho)(x(Q),y(Q))\cdot e_x+(\delta_y\circ q_\rho)(x(Q),y(Q))\cdot e_y.$$

Note that the function m in the coordinates $(\tilde{x},m)$ of $Y(\mathbb{R})$ is bounded in a neighborhood of $J_{C,P}$ in $\beta^\sharp C$. By the assumption that δ is affine-linear locally around P, it follows that

$$|\delta_x(\tilde{x},m\tilde{x})|/|\tilde{x}| \quad\text{and}\quad |\delta_y(\tilde{x},m\tilde{x})|/|\tilde{x}|$$

are bounded in a neighborhood of $J_{C,P}$ in $\beta^\sharp C$. On the other hand, since $|u^2/\rho|$ tends to 0 faster than $|u|$ does as $u\to 0$, we have

$$|(\delta_x\circ q_\rho)(\tilde{x},m\tilde{x})|/|\tilde{x}|\ \to\ 0,\quad |(\delta_y\circ q_\rho)(\tilde{x},m\tilde{x})|/|\tilde{x}|\ \to\ 0,$$

as $\tilde{x}\to 0$. Hence we have

$$\frac{m\tilde{x}+\sqrt{-1}\,(\delta_y\circ q_\rho)(\tilde{x},m\tilde{x})\cdot t}{\tilde{x}+\sqrt{-1}\,(\delta_x\circ q_\rho)(\tilde{x},m\tilde{x})\cdot t}\quad\to\quad m$$

as $\tilde{x}\to 0$. Therefore by (6.7) with δ_x and δ_y replaced by $\delta_x\circ q_\rho$ and $\delta_y\circ q_\rho$ respectively, we see that, if δ is q_ρ-modified, then $d^\bullet(Q,t)$ tends to Q as $Q\in\beta^{-1}(C^\bullet)$ approaches to a point of $J_{C,P}\subset\beta^\sharp C$, and hence $\delta\circ q_\rho$ is regular at P. □

6.3. ***E*-displacement.** Next we define *E-displacements* of various subspaces in $Y(\mathbb{C})$ and X. Let C be a bounded chamber, and P a vertex of C.

Definition 6.8. For a point $Q\in E_P$, we denote by $\lambda_Q(\mathbb{C})$ the complex affine line in $\mathbb{A}^2(\mathbb{C})$ passing through P such that its strict transform $\beta^\sharp\lambda_Q(\mathbb{C})$ in $Y(\mathbb{C})$ intersects E_P at Q. A *tubular neighborhood* of E_P is an open neighborhood $\mathcal{N}_P\subset Y(\mathbb{C})$ of E_P in $Y(\mathbb{C})$ equipped with a continuous map

$$p_{\mathcal{N}}\colon\mathcal{N}_P\to E_P$$

such that, for any $Q\in E_P$, the fiber $p_{\mathcal{N}}^{-1}(Q)$ is an open disk of $\beta^\sharp\lambda_Q(\mathbb{C})$ with the center Q. We call $p_{\mathcal{N}}$ the *projection* of the tubular neighborhood $\mathcal{N}_P$.

Let $p_{\mathcal{N}}\colon\mathcal{N}_P\to E_P$ be a tubular neighborhood of E_P. We consider the inclusion $E_P\hookrightarrow\mathcal{N}_P$ as a section of $p_{\mathcal{N}}$, which we call the *zero section*. For $Q'\in p_{\mathcal{N}}^{-1}(Q)$ and $t\in I:=[0,1]$, let tQ' denote the unique point on $p_{\mathcal{N}}^{-1}(Q)\subset\beta^\sharp\lambda_Q(\mathbb{C})$ such that

$$\beta(tQ')=P+t\cdot\tau_{P,\beta(Q')},$$

where $\tau_{P,\beta(Q')}\in T(\mathbb{A}^2(\mathbb{C}))$ is the translation that maps P to $\beta(Q')$.

We describe a tubular neighborhood in terms of local coordinates. Let (ξ,η), $(\tilde{\xi},\mu)$, (x,y), $(\tilde{x},m)$ be good local coordinate systems of $\mathbb{A}^2(\mathbb{C})$, $Y(\mathbb{C})$, $\mathbb{A}^2(\mathbb{R})$, $Y(\mathbb{R})$, respectively, given in Section 5.1. We can assume, without loss of generality, that the location of C is Case (1) in (5.3), so that there exist real numbers a,b with $a<b$ such that $\beta^\sharp C$ is defined by $\tilde{x}\geq 0$ and $a\leq m\leq b$ in the chart of $(\tilde{x},m)$ on

$Y(\mathbb{R})$. By definition, in terms of the coordinates $(\tilde{\xi}, \mu)$ of $Y(\mathbb{C})$, the projection $p_{\mathcal{N}}$ is given by

$$\mathcal{N}_P \ni (\tilde{\xi}, \mu) \quad \mapsto \quad (0, \mu) \in E_P,$$

and the map $Q' \mapsto tQ'$ is given by

$$(\tilde{\xi}, \mu) \quad \mapsto \quad (t\tilde{\xi}, \mu).$$

Definition 6.9. A continuous section

$$s\colon E_P \to \mathcal{N}_P$$

of $p_{\mathcal{N}}$ is said to be *admissible with respect to* C if it satisfies the following.

(s1) There exists one and only one point $Q_0 \in E_P$ such that $s(Q_0) \in E_P$.
(s2) The point $Q_0 \in E_P$ is not on the arc $J_{C,P} = E_P \cap \beta^\sharp C$.
(s3) If $Q \in J_{C,P}$, then $s(Q) \in \beta^\sharp C$.

Suppose that $s\colon E_P \to \mathcal{N}_P$ is a section admissible with respect to C. We can write s as

$$\mu \mapsto (\tilde{\xi}, \mu) = (\sigma(\mu), \mu)$$

in a neighborhood of $J_{C,P}$ in E_P, where σ is a continuous function such that, if m is a real number in the closed interval $[a, b]$, then $\sigma(m) \in \mathbb{R}_{>0}$.

Example 6.10. Let c be a sufficiently small positive real number. Then we have a section $s\colon E_P \to \mathcal{N}_P$ admissible with respect to C such that the function $\sigma(\mu)$ is constantly equal to c in a small neighborhood of $J_{C,P}$ in E_P.

Note that the image $s(E_P)$ of s and E_P intersect only at Q_0, and the local intersection number is -1. For $t \in I$, we denote by

$$ts\colon E_P \to \mathcal{N}_P \subset Y(\mathbb{C})$$

the continuous map $Q \mapsto t \cdot s(Q)$. Let ε' be a sufficiently small positive real number. For $t \in I_{\varepsilon'}$, let $(E_P)_t \subset \mathcal{N}_P$ be the image of E_P by ts, and let $(J_{C,P})_t$ be the image of $J_{C,P}$ by ts. Note that we have

$$(J_{C,P})_t = (E_P)_t \cap \beta^\sharp C.$$

Let Q_a (resp. Q_b) be the intersection point of E_P and $\beta^\sharp \ell_a(\mathbb{C})$ (resp. $\beta^\sharp \ell_b(\mathbb{C})$). Then $(J_{C,P})_t$ is an arc on $(E_P)_t$ connecting $ts(Q_a)$ and $ts(Q_b)$. We put

$$(D_P)_t := \phi^{-1}((E_P)_t).$$

Then $\phi|(D_P)_t\colon (D_P)_t \to (E_P)_t$ is a double covering whose branch points are the end points $ts(Q_a)$ and $ts(Q_b)$ of the arc $(J_{C,P})_t$. Therefore

$$(S_{C,P})_t := \phi^{-1}((J_{C,P})_t)$$

is a circle on $(D_P)_t$. Note that $(S_{C,P})_t$ decomposes $(D_P)_t$ into the union of two closed hemispheres. Let $(H_{C,P})_t$ denote the hemisphere obtained from $(H_{C,P})_0 = H_{C,P}$ by continuity. Thus, from the section $s\colon E_P \to \mathcal{N}_P$ admissible with respect to C, we obtain displacements

$$\begin{aligned}
&\{\,(J_{C,P})_t \mid t\in I_{\varepsilon'}\,\} \text{ of } J_{C,P} \text{ in } \beta^\sharp C,\\
&\{\,(E_P)_t \mid t\in I_{\varepsilon'}\,\} \text{ of } E_P \text{ in } Y(\mathbb{C}),\\
&\{\,(D_P)_t \mid t\in I_{\varepsilon'}\,\} \text{ of } D_P \text{ in } X,\\
&\{\,(S_{C,P})_t \mid t\in I_{\varepsilon'}\,\} \text{ of } S_{C,P} \text{ in } \phi^{-1}(\beta^\sharp C), \text{ and}\\
&\{\,(H_{C,P})_t \mid t\in I_{\varepsilon'}\,\} \text{ of } H_{C,P} \text{ in } X,
\end{aligned}$$

which we call *E-displacements* associated with $s\colon E_P\to\mathcal{N}_P$.

For positive real numbers α,β with $\alpha<\beta$, let $g_{\alpha,\beta}\colon \mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ denote the continuous function defined by

$$g_{\alpha,\beta}(u) := \begin{cases} \alpha+(\beta-\alpha)u/\beta & \text{if } 0\le u\le\beta,\\ u & \text{if } u\ge\beta.\end{cases}$$

Then $g_{\alpha,\beta}$ is a homeomorphism from $\mathbb{R}_{\ge 0}$ to $\mathbb{R}_{\ge\alpha}$ that is the identity on $\mathbb{R}_{\ge\beta}$. We choose ε' so small that the point $(\tilde{x},m)=(2\varepsilon'\sigma(m),m)$ is in the chart for the coordinate system $(\tilde{\xi},\mu)$ for all $m\in[a,b]$. Let U denote the chart. We then define a displacement

$$\beta^\sharp C\times I_{\varepsilon'}\to\beta^\sharp C \tag{6.9}$$

of $\beta^\sharp C$ in $\beta^\sharp C$ by

$$(Q',t)\mapsto\begin{cases}(g_{t\sigma(m),2t\sigma(m)}(\tilde{x}),m) & \text{if } Q'=(\tilde{x},m)\in U\cap\beta^\sharp C,\\ Q' & \text{otherwise.}\end{cases}$$

Note that this map is continuous because we have $g_{t\sigma(m),2t\sigma(m)}(\tilde{x})=\tilde{x}$ for $\tilde{x}\ge 2t\sigma(m)$. Let $(\beta^\sharp C)_t$ be the closed subset of $\beta^\sharp C$ obtained by removing

$$\{\,(\tilde{x},m)\mid a\le m\le b,\ \ 0\le\tilde{x}<t\sigma(m)\,\}$$

from $\beta^\sharp C$. Then the displacement (6.9) is a family of maps that shrink $\beta^\sharp C$ to $(\beta^\sharp C)_t\subset\beta^\sharp C$ homeomorphically. Note that the displacement (6.9) preserves the subspaces $\beta^\sharp C\cap\beta^\sharp\ell_a(\mathbb{C})$ and $\beta^\sharp C\cap\beta^\sharp\ell_b(\mathbb{C})$. Putting

$$\Delta(C)_t := \phi^{-1}((\beta^\sharp C)_t)$$

with the orientation σ_C, we obtain a displacement $\{\,\Delta(C)_t\mid t\in I_{\varepsilon'}\,\}$ of $\Delta(C)$ in X. We have

$$\partial\,\Delta(C)_t=(S_{C,P})_t\ \sqcup \bigsqcup_{P'\in\mathrm{Vert}(C)\setminus\{P\}} S_{C,P'}.$$

Therefore we have a displacement

$$\Sigma(C)_t \ := \ \Delta(C)_t\ \cup\ \left((H_{C,P})_t\ \sqcup \bigsqcup_{P'\in\mathrm{Vert}(C)\setminus\{P\}} H_{C,P'}\right)$$

of the topological 2-cycle $\Sigma(C)_0=\Sigma(C)$ in X. These displacements

$$\begin{aligned}
&\{\,(\beta^\sharp C)_t\mid t\in I_{\varepsilon'}\,\} \quad \text{of } \beta^\sharp C \text{ in } \beta^\sharp C,\\
&\{\,\Delta(C)_t\mid t\in I_{\varepsilon'}\,\} \quad \text{of } \Delta(C) \text{ in } \Delta(C), \text{ and}\\
&\{\,\Sigma(C)_t\mid t\in I_{\varepsilon'}\,\} \quad \text{of } \Sigma(C) \text{ in } X
\end{aligned} \tag{6.10}$$

are also called the *E-displacements* associated with the section $s\colon E_P\to\mathcal{N}_P$.

We can easily extend the definition of E-displacements to the case where we are given sections at several vertexes. Note that the displacements in (6.10) are stationary outside small neighborhoods of $J_{C,P}$ in $\beta^\sharp C$, of $S_{C,P}$ in $\Delta(C)$, and of $H_{C,P}$ in $\Sigma(C)$, respectively. By choosing sufficiently small ε', we can make these neighborhoods arbitrarily small. Therefore, if we are given sections

$$s_i\colon E_{P_i} \to \mathcal{N}_{P_i} \ \ (i = 1, \dots, k)$$

admissible with respect to C for distinct vertexes $P_1, \dots, P_k$ of C, then, assuming ε' to be sufficiently small, we can glue the displacements associated with these s_i together. For example, let $\mathcal{U}(P_i)$ be a neighborhood of $H_{C,P}$ in $\Sigma(C)$ such that the E-displacement $d_{s_i}\colon \Sigma(C) \times I_{\varepsilon'} \to X$ associated with s_i is stationary outside of $\mathcal{U}(P_i)$. Then we can define a displacement

$$d := d_{s_1,\dots,s_k}\colon \Sigma(C) \times I_{\varepsilon'} \to X$$

by the following:

$$d(Q,t) := \begin{cases} d_{s_i}(Q,t) & \text{if } Q \in \mathcal{U}(P_i) \text{ for some } i, \\ Q & \text{if } Q \notin \mathcal{U}(P_i) \text{ for any } i. \end{cases}$$

Definition 6.11. These displacements of $\beta^\sharp C$ in $\beta^\sharp C$, of $\Delta(C)$ in $\Delta(C)$, and of $\Sigma(C)$ in X, are called the *E-displacements* associated with the continuous sections $s_i\colon E_{P_i} \to \mathcal{N}_{P_i}$.

By construction, the E-displacements associated with these sections $s_i\colon E_{P_i} \to \mathcal{N}_{P_i}$ $(i = 1, \dots, k)$ satisfy the following:

$$(\beta^\sharp C)_{\varepsilon'} \cap E_{P_i} = \emptyset \ \text{ and hence } \ \Delta(C)_{\varepsilon'} \cap D_{P_i} = \emptyset \quad \text{for } i = 1, \dots, k. \tag{6.11}$$

7. Proof of Theorem 1.8 (1), (2), (3)

We keep on using f, and prove assertions (1), (2), (3) of Theorem 1.8.

7.1. **Proof of assertion (1).** The case where $P \notin \mathrm{Vert}(C)$ is obvious. Suppose that $P \in \mathrm{Vert}(C)$. As in Section 6.3, we choose a section $s\colon E_P \to \mathcal{N}_P$ of a tubular neighborhood $\mathcal{N}_P \to E_P$ admissible with respect to C, and construct the E-displacement $\{\, \Sigma(C)_t \mid t \in I_{\varepsilon'} \,\}$ associated with s. Note that $\Delta(C)_{\varepsilon'}$ is disjoint from D_P. (See (6.11).) Recall that $(E_P)_{\varepsilon'}$ and E_P intersect only at one point Q_0, and the local intersection number at Q_0 is -1. Since $Q_0 \notin J_{C,P}$, we see that $(D_P)_{\varepsilon'}$ and D_P intersect only at the two points in $\phi^{-1}(Q_0)$ and that the local intersection number at each point is -1. Assuming ε' to be small enough, we have that one of the two points in $\phi^{-1}(Q_0)$ is on $(H_{C,P})_{\varepsilon'}$, whereas the other does not belong to $(H_{C,P})_{\varepsilon'}$. Hence $\Sigma(C)_{\varepsilon'}$ and D_P intersect only at one point, at which the local intersection number is -1. Thus assertion (1) is proved.

7.2. **Proof of assertion (2).** Obvious.

7.3. **Proof of assertion (3).** Suppose that $C \cap C'$ consists of a single point P. Let (ξ, η), $(\tilde{\xi}, \mu)$, (x, y), $(\tilde{x}, m)$ be good local coordinates on $\mathbb{A}^2(\mathbb{C})$, $Y(\mathbb{C})$, $\mathbb{A}^2(\mathbb{R})$, $Y(\mathbb{R})$, respectively, given in Section 5.1. Without loss of generality, we can assume that the location of C is Case (1) and the location of C' is Case (3) in (5.3). Note that we have

$$J_{C,P} = J_{C',P} \ \text{ on } E_P, \quad S_{C,P} = S_{C',P} \ \text{ on } D_P.$$

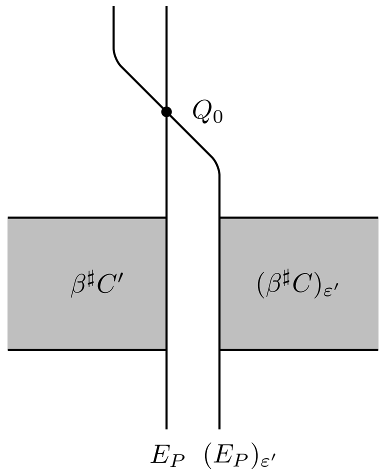

FIGURE 7.1. E-displacement for the proof of assertion (3)

Note also that we have either

$$(\ f(C^\circ) \subset \mathbb{R}_{>0} \text{ and } f(C'^\circ) \subset \mathbb{R}_{>0}\ ) \text{ or } (\ f(C^\circ) \subset \mathbb{R}_{<0} \text{ and } f(C'^\circ) \subset \mathbb{R}_{<0}\ ).$$

By Proposition 5.6, we see that $H_{C,P}$ and $H_{C',P}$ are the closures of *distinct* connected components of $D_P \setminus S_{C,P} = D_P \setminus S_{C',P}$.

As in Section 6.3, we choose a section $s\colon E_P \to \mathcal{N}_P$ of a tubular neighborhood $\mathcal{N}_P \to E_P$ admissible with respect to C, and construct the E-displacement $\Sigma(C)_{\varepsilon'}$ associated with s. We illustrate the E-displacements of $\beta^\sharp C$ and E_P in $Y(\mathbb{C})$ in Figure 7.1, where Q_0 is the zero of the section s. Since $\beta^\sharp C'$ and $(\beta^\sharp C)_{\varepsilon'}$ are disjoint, it follows that $\Delta(C')$ and $\Delta(C)_{\varepsilon'}$ are disjoint. Since E_P and $(\beta^\sharp C)_{\varepsilon'}$ are disjoint, we see that $H_{C',P}$ and $\Delta(C)_{\varepsilon'}$ are disjoint. Since $\beta^\sharp C'$ and $(E_P)_{\varepsilon'}$ are disjoint, it follows that $\Delta(C')$ and $(H_{C,P})_{\varepsilon'}$ are disjoint. Note that D_P and $(D_P)_{\varepsilon'}$ intersect only at the two points in $\phi^{-1}(Q_0)$. Since $H_{C,P}$ and $H_{C',P}$ are distinct hemispheres of D_P given by $S_{C,P} = S_{C',P}$, and ε' is sufficiently small, neither of the two points of $\phi^{-1}(Q_0)$ is on $H_{C',P} \cap (H_{C,P})_{\varepsilon'}$. Therefore $\Sigma(C')$ and $(\Sigma(C))_{\varepsilon'}$ are disjoint.

8. PROOF OF THEOREM 1.8 (4)

We first prove that the intersection number $\langle[\Sigma(C)], [\Sigma(C')]\rangle$ for distinct bounded chambers C and C' sharing a common edge does not depend on $\mathcal{A}$. See Lemma 8.1. We then compare the intersection number $\langle[\Sigma(C)], [\Sigma(C')]\rangle$ with the intersection number $\langle[\Sigma^-(C)], [\Sigma^-(C')]\rangle^-$ that is calculated by replacing f with $-f$. Using (1.2), we obtain a proof of assertion (4) of Theorem 1.8.

8.1. **A preliminary lemma.** We consider two nodal real line arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$. Let j be in $\{1, 2\}$. Let $\mathcal{A}_j$ be a nodal real line arrangement with a defining polynomial f_j. We consider the morphisms

$$X_j \xrightarrow{\rho_j} W_j \xrightarrow{\pi_j} \mathbb{A}^2(\mathbb{C}).$$

Here W_j is defined in $\mathbb{C} \times \mathbb{A}^2(\mathbb{C})$ by

$$\omega_j^2 = f_j,$$

where ω_j is an affine coordinate of $\mathbb{C}$, and π_j denotes the double covering $(\omega_j, Q) \mapsto Q$. The morphism $\rho_j \colon X_j \to W_j$ is the minimal desingularization. Then, for a bounded chamber C of $\mathcal{A}_j$, we have a vanishing cycle $\Sigma(C)$ in X_j with the standard orientation determined by $\sigma_{\mathbb{A}}$ and f_j. We denote by

$$\langle \quad \rangle_j \colon H_2(X_j, \mathbb{Z}) \times H_2(X_j, \mathbb{Z}) \to \mathbb{Z}$$

the intersection form on X_j.

Lemma 8.1. *Let C_j and C'_j be distinct bounded chambers of $\mathcal{A}_j$ that share a common edge. Then we have*

$$\langle [\Sigma(C_1)], [\Sigma(C'_1)] \rangle_1 = \langle [\Sigma(C_2)], [\Sigma(C'_2)] \rangle_2. \tag{8.1}$$

Proof. The idea of the proof is as follows. We construct a biholomorphic isomorphism from an open neighborhood $U_{X,1}$ of $\Sigma(C_1) \cap \Sigma(C'_1)$ in X_1 to an open neighborhood $U_{X,2}$ of $\Sigma(C_2) \cap \Sigma(C'_2)$ in X_2 that induces orientation-preserving isomorphisms

$$U_{X,1} \cap \Sigma(C_1) \;\cong\; U_{X,2} \cap \Sigma(C_2) \quad \text{and} \quad U_{X,1} \cap \Sigma(C'_1) \;\cong\; U_{X,2} \cap \Sigma(C'_2).$$

Since the intersection number of topological cycles is determined by the geometric situation in a neighborhood of the set-theoretic intersection of the topological cycles, this biholomorphic isomorphism proves (8.1).

Since C_j and C'_j are adjacent, the signs of f_j on C_j and on C'_j are opposite. Interchanging C_j and C'_j if necessary, we can assume without loss of generality that

$$f_j(C_j) \subset \mathbb{R}_{\ge 0}, \quad f_j(C'_j) \subset \mathbb{R}_{\le 0}.$$

Let $e_j := P_j P'_j$ be the common edge of C_j and C'_j, where we have

$$\mathrm{Vert}(C_j) \cap \mathrm{Vert}(C'_j) = \{P_j, P'_j\}.$$

Let $\ell_{ej}(\mathbb{R}) \in \mathcal{A}_j$ be the line defining the edge $P_j P'_j$. Let (ξ_j, η_j) be good affine coordinates of $\mathbb{A}^2(\mathbb{C})$ with $(x_j, y_j) = (\mathrm{Re}\,\xi_j, \mathrm{Re}\,\eta_j)$ regarded as affine coordinates of $\mathbb{A}^2(\mathbb{R})$ such that $\ell_{ej}(\mathbb{R})$ is defined by $y_j = 0$, and such that $\{P_j, P'_j\}$ is $\{(\pm 1, 0)\}$. See Section 3.2. Note that, if (ξ_j, η_j) satisfies these properties, then so does $(-\xi_j, -\eta_j)$. Replacing (ξ_j, η_j) with $(-\xi_j, -\eta_j)$ if necessary, we can assume that

$$C_j \subset \{y_j \ge 0\}, \quad C'_j \subset \{y_j \le 0\}.$$

Interchanging P_j and P'_j if necessary, we can further assume that

$$P_j = (-1, 0), \quad P'_j = (1, 0)$$

in terms of (x_j, y_j). Let $\ell_{Pj}(\mathbb{R})$ (resp. $\ell_{P'j}(\mathbb{R})$) be the line in $\mathcal{A}_j$ other than $\ell_{ej}(\mathbb{R})$ that passes through P_j (resp. P'_j). Then there exist real numbers g_j, g'_j such that

$$\ell_{Pj}(\mathbb{R}) \;:\; x_j = -1 + g_j y_j, \quad \ell_{P'j}(\mathbb{R}) \;:\; x_j = 1 + g'_j y_j.$$

See Figure 8.1.

We prepare another copy of $\mathbb{A}^2(\mathbb{C})$ with good affine coordinates (ξ_0, η_0), and we regard $(x_0, y_0) = (\mathrm{Re}\,\xi_0, \mathrm{Re}\,\eta_0)$ as affine coordinates of $\mathbb{A}^2(\mathbb{R})$. We put

$$P_0 := (-1, 0), \quad P'_0 := (1, 0), \quad e_0 := P_0 P'_0,$$

and define three lines $\ell_{e0}(\mathbb{R}), \ell_{P0}(\mathbb{R}), \ell_{P'0}(\mathbb{R})$ by

$$\ell_{e0}(\mathbb{R}) \;:\; y_0 = 0, \quad \ell_{P0}(\mathbb{R}) \;:\; x_0 = -1, \quad \ell_{P'0}(\mathbb{R}) \;:\; x_0 = 1.$$

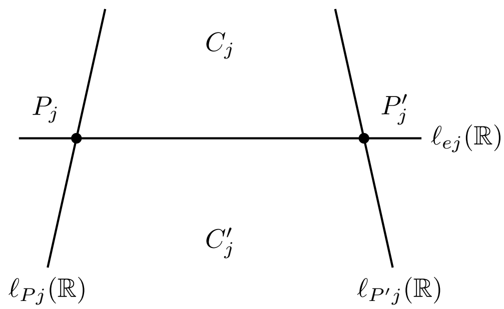

FIGURE 8.1. C_j and C'_j

We then put

$$C_0 := \{ (x_0, y_0) \mid -1 \le x_0 \le 1,\ 0 \le y_0 \},$$
$$C'_0 := \{ (x_0, y_0) \mid -1 \le x_0 \le 1,\ 0 \ge y_0 \}.$$

Finally, we put

$$f_0 := y_0(1 - x_0)(1 + x_0),$$

which is a defining polynomial of the nodal real line arrangement $\mathcal{A}_0$ consisting of three lines $\ell_{e0}(\mathbb{R}), \ell_{P0}(\mathbb{R}), \ell_{P'0}(\mathbb{R})$. Note that we have

$$f_0(C_0) \subset \mathbb{R}_{\ge 0}, \quad f_0(C'_0) \subset \mathbb{R}_{\le 0}.$$

We define a quadratic map $\psi_j \colon \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ by

$$(\xi_0, \eta_0) \ \mapsto \ (\xi_j, \eta_j) = \left(\xi_0 + \left(\frac{1 - \xi_0}{2} g_j + \frac{1 + \xi_0}{2} g'_j \right) \eta_0,\ \eta_0 \right).$$

It is easy to see that $\psi_j(P_0) = P_j$, $\psi_j(P'_0) = P'_j$, and that ψ_j maps the line segment e_0 to the line segment e_j isomorphically. Let U_0 be a sufficiently small open neighborhood of e_0 in $\mathbb{A}^2(\mathbb{C})$. Then there exists an open neighborhood U_j of e_j in $\mathbb{A}^2(\mathbb{C})$ such that ψ_j induces a biholomorphic isomorphism

$$\psi_{jU} \colon U_0 \xrightarrow{\sim} U_j.$$

The inverse of ψ_{jU} is given by

$$(\xi_j, \eta_j) \ \mapsto \ (\xi_0, \eta_0) = \left(\frac{2\xi_j - (g_j + g'_j)\eta_j}{2 - (g_j - g'_j)\eta_j},\ \eta_j \right).$$

We can easily check that ψ_{jU} maps $U_0 \cap \mathbb{A}^2(\mathbb{R})$ to $U_j \cap \mathbb{A}^2(\mathbb{R})$ isomorphically and orientation-preservingly, and that it also maps isomorphically

$$\begin{array}{lll} U_0 \cap C_0 & \text{to} & U_j \cap C_j, \\ U_0 \cap C'_0 & \text{to} & U_j \cap C'_j, \\ U_0 \cap \ell_{e0}(\mathbb{C}) & \text{to} & U_j \cap \ell_{ej}(\mathbb{C}), \\ U_0 \cap \ell_{P0}(\mathbb{C}) & \text{to} & U_j \cap \ell_{Pj}(\mathbb{C}), \\ U_0 \cap \ell_{P'0}(\mathbb{C}) & \text{to} & U_j \cap \ell_{P'j}(\mathbb{C}). \end{array}$$

We make U_0 so small that U_j does not intersect any element of the complexified line arrangement $\mathcal{A}_{j\mathbb{C}}$ other than $\ell_{ej}(\mathbb{C}), \ell_{Pj}(\mathbb{C}), \ell_{P'j}(\mathbb{C})$. Then

$$u_j := f_0 / \psi^*_{jU} f_j$$

is a holomorphic function on U_0 that does *not* have any zero. Moreover we have $u_j(Q) \in \mathbb{R}_{>0}$ for any $Q \in C_0 \cup C_0'$. We can assume that U_0 is simply-connected. Then we can define a holomorpic function v_j on U_0 such that $v_j^2 = u_j$ (in particular v_j does not have any zero) and that

$$v_j(Q) \in \mathbb{R}_{>0} \text{ for any } Q \in C_0 \cup C_0'. \tag{8.2}$$

Let W_0 be defined in $\mathbb{C} \times \mathbb{A}^2(\mathbb{C})$ by

$$\omega_0^2 = f_0,$$

where ω_0 is an affine coordinate of $\mathbb{C}$, and let $\pi_0\colon W_0 \to \mathbb{A}^2(\mathbb{C})$ be the double covering $(\omega_0, Q_0) \mapsto Q_0$. We put

$$U_{W,0} := \pi_0^{-1}(U_0), \quad U_{W,j} := \pi_j^{-1}(U_j).$$

We define an isomorphism

$$\psi_{jW}\colon U_{W,0} \xrightarrow{\sim} U_{W,j}$$

that covers the isomorphism $\psi_{jU}\colon U_0 \xrightarrow{\sim} U_j$ by

$$(\omega_0, Q_0) \mapsto (\omega_j, Q_j) = (\omega_0/v_j(Q_0), \psi_{jU}(Q_0)). \tag{8.3}$$

We put

$$\psi_W := \psi_{2W} \circ \psi_{1W}^{-1} \ :\ U_{W,1} \xrightarrow{\sim} U_{W,2},$$

which induces isomorphisms from $U_{W,1} \cap \pi_1^{-1}(C_1)$ to $U_{W,2} \cap \pi_2^{-1}(C_2)$ and from $U_{W,1} \cap \pi_1^{-1}(C_1')$ to $U_{W,2} \cap \pi_2^{-1}(C_2')$. Recall from Definition 1.5 that the plus-sheet and the minus-sheet are defined by the sign of $\omega \in \mathbb{R}$ or $\omega/\sqrt{-1} \in \mathbb{R}$. Therefore it follows from (8.2) and (8.3) that ψ_W maps the plus-sheet of $U_{W,1} \cap \pi_1^{-1}(C_1^\circ)$ to the plus-sheet of $U_{W,2} \cap \pi_2^{-1}(C_2^\circ)$, and the plus-sheet of $U_{W,1} \cap \pi_1^{-1}(C_1'^\circ)$ to the plus-sheet of $U_{W,2} \cap \pi_2^{-1}(C_2'^\circ)$. Hence the isomorphism by ψ_W from $U_{W,1} \cap \pi_1^{-1}(C_1)$ to $U_{W,2} \cap \pi_2^{-1}(C_2)$ is orientation-preserving, where orientations are the standard orientations defined by $\sigma_{\mathbb{A}}$ and f_j. We put

$$U_{X,j} := \rho_j^{-1}(U_{W,j}).$$

Since the minimal resolution of an ordinary node of an algebraic surface is unique, the isomorphism ψ_W induces an isomorphism

$$\psi_X \ :\ U_{X,1} \xrightarrow{\sim} U_{X,2}$$

that maps D_{P_1} to D_{P_2} and $D_{P_1'}$ to $D_{P_2'}$. Since ψ_X induces orientation-preserving isomorphisms

$$U_{X,1} \cap \Delta(C_1) \ \cong\ U_{X,2} \cap \Delta(C_2), \quad U_{X,1} \cap \Delta(C_1') \ \cong\ U_{X,2} \cap \Delta(C_2'),$$

it follows that ψ_X induces isomorphisms of the capping hemispheres

$$\begin{array}{ccccccc} H_{C_1,P_1} & \xrightarrow{\sim} & H_{C_2,P_2}, & \quad & H_{C_1,P_1'} & \xrightarrow{\sim} & H_{C_2,P_2'}, \\ H_{C_1',P_1} & \xrightarrow{\sim} & H_{C_2',P_2}, & \quad & H_{C_1',P_1'} & \xrightarrow{\sim} & H_{C_2',P_2'}. \end{array}$$

Therefore ψ_X induces isomorphisms from $U_{X,1} \cap \Sigma(C_1)$ to $U_{X,2} \cap \Sigma(C_2)$ and from $U_{X,1} \cap \Sigma(C_1')$ to $U_{X,2} \cap \Sigma(C_2')$, and these isomorphisms are orientation-preserving.

Note that $\Sigma(C_j) \cap \Sigma(C_j')$ is contained in $U_{X,j}$. The intersection number of $\Sigma(C_1)$ and $\Sigma(C_1')$ is calculated by making a small displacement $\Sigma(C_1')_\varepsilon$ of $\Sigma(C_1')$ that is stationary outside $U_{X,1}$ so that $\Sigma(C_1)$ and $\Sigma(C_1')_\varepsilon$ intersect transversely. Transplanting this procedure into $U_{X,2}$ by ψ_X, we see that $\Sigma(C_2)$ and a displacement

$\Sigma(C_2')_\varepsilon$ of $\Sigma(C_2')$ intersect in the same way as $\Sigma(C_1)$ and $\Sigma(C_1')_\varepsilon$. Therefore we obtain (8.1). □

8.2. **Proof of assertion (4) of Theorem 1.8.** Interchanging C and C' if necessary, we can assume that

$$f(C) \subset \mathbb{R}_{\ge 0}, \quad f(C') \subset \mathbb{R}_{\le 0}. \tag{8.4}$$

We construct morphisms (1.1) with f replaced by $-f$, and denote them by

$$X^- \xrightarrow{\rho^-} W^- \xrightarrow{\pi^-} \mathbb{A}^2(\mathbb{C}), \tag{8.5}$$

where W^- is defined in $\mathbb{C} \times \mathbb{A}^2(\mathbb{C})$ by

$$\omega^2 = -f$$

with $\pi^-(\omega, Q) = Q$, and $\rho^- \colon X^- \to W^-$ is the minimal desingularization. For a bounded chamber C'', let $\Sigma^-(C'')$ denote the vanishing cycle over C'' in X^- equipped with the standard orientation $\sigma^-_{C''}$ defined by $\sigma_{\mathbb{A}}$ and $-f$. Let

$$\langle \quad \rangle^- \colon H_2(X^-, \mathbb{Z}) \times H_2(X^-, \mathbb{Z}) \to \mathbb{Z}$$

denote the intersection form on X^-. By Lemma 8.1, we have

$$\langle [\Sigma(C)], [\Sigma(C')] \rangle = \langle [\Sigma^-(C)], [\Sigma^-(C')] \rangle^-. \tag{8.6}$$

Let $\Phi_W \colon W \xrightarrow{\sim} W^-$ be the isomorphism defined by

$$(\omega, Q) \mapsto (\sqrt{-1}\,\omega, Q).$$

Note that we have

$$\omega \in \mathbb{R}_{\ge 0} \Longrightarrow \sqrt{-1}\,\omega \in \sqrt{-1}\,\mathbb{R}_{\ge 0}, \quad \omega \in \sqrt{-1}\,\mathbb{R}_{\ge 0} \Longrightarrow \sqrt{-1}\,\omega \in -\mathbb{R}_{\ge 0}.$$

Therefore, by (8.4), we see that Φ_W maps the plus-sheet of $\pi^{-1}(C^\circ)$ to the plus-sheet of $(\pi^-)^{-1}(C^\circ)$, whereas it maps the plus-sheet of $\pi^{-1}(C'^\circ)$ to the *minus*-sheet of $(\pi^-)^{-1}(C'^\circ)$. Therefore the isomorphism $\Phi_X \colon X \xrightarrow{\sim} X^-$ induced by Φ_W induces an isomorphism

$$\Sigma(C, \sigma_C) \cong \Sigma^-(C, \sigma^-_C),$$

whereas it induces an isomorphism

$$\Sigma(C', \sigma_{C'}) \cong \Sigma^-(C', -\sigma^-_{C'}).$$

Hence the homomorphism $\Phi_{X*} \colon H_2(X, \mathbb{Z}) \xrightarrow{\sim} H_2(X^-, \mathbb{Z})$ satisfies

$$\Phi_{X*}([\Sigma(C)]) = [\Sigma^-(C)], \tag{8.7}$$

$$\Phi_{X*}([\Sigma(C')]) = \sum_{P'' \in \mathrm{Vert}(C')} [D_{P''}] - [\Sigma^-(C')], \tag{8.8}$$

where (8.8) is derived from (1.2). Since Φ_{X*} preserves the intersection form, we have

$$\langle [\Sigma(C)], [\Sigma(C')] \rangle = \langle \Phi_{X*}([\Sigma(C)]), \Phi_{X*}([\Sigma(C')]) \rangle^-. \tag{8.9}$$

Comparing (8.6), (8.7), (8.8), (8.9), we obtain

$$2\,\langle [\Sigma(C)], [\Sigma(C')] \rangle = \sum_{P'' \in \mathrm{Vert}(C')} \langle [\Sigma^-(C)], [D_{P''}] \rangle^- = -2,$$

where we use assertion (1) of Theorem 1.8 and $|\mathrm{Vert}(C) \cap \mathrm{Vert}(C')| = 2$ in the second equality. Thus Theorem 1.8 (4) is proved. □

9. Proof of Theorem 1.8 (5)

9.1. **Target displacements.** We construct an E-displacement

$$\Sigma(C)_{\varepsilon'} \;=\; \Delta(C)_{\varepsilon'} \cup \bigsqcup_{P\in \mathrm{Vert}(C)} (H_{C,P})_{\varepsilon'} \tag{9.1}$$

of $\Sigma(C)$ in X associated with sections $s_P \colon E_P \to \mathcal{N}_P$ for all $P \in \mathrm{Vert}(C)$, and a C-displacement

$$\Sigma(C)_{\varepsilon} \;=\; \Delta(C)_{\varepsilon} \cup \bigsqcup_{P\in \mathrm{Vert}(C)} H_{C,P} \tag{9.2}$$

of $\Sigma(C)$ in X associated with a continuous function $\delta \colon C \to T(\mathbb{A}^2(\mathbb{R}))$. By the definition of E-displacements (see (6.11)), for any $P \in \mathrm{Vert}(C)$, we have that $\Delta(C)_{\varepsilon'}$ and $H_{C,P}$ are disjoint, and that $(H_{C,P})_{\varepsilon'}$ and $H_{C,P}$ intersect only at one point, at which the local intersection number is -1. By choosing a sufficiently small ε', we have that $(H_{C,P})_{\varepsilon'}$ and $H_{C,P'}$ are disjoint for $P \neq P'$. We construct these displacements (9.1) and (9.2) in such a way that they have the following additional properties.

- (EC1) The two spaces $\Delta(C)_{\varepsilon'}$ and $\Delta(C)_{\varepsilon}$ intersect only at two points, each intersection point is in the interior of $\Delta(C)_{\varepsilon'}$ and of $\Delta(C)_{\varepsilon}$, and the local intersection number is -1 at each intersection point.
- (EC2) For any $P \in \mathrm{Vert}(C)$, we have that $(H_{C,P})_{\varepsilon'}$ and $\Delta(C)_{\varepsilon}$ intersect only at one point, that the intersection point is in the interior of $(H_{C,P})_{\varepsilon'}$ and of $\Delta(C)_{\varepsilon}$, and that the local intersection number is 1.

Then we have

$$\begin{aligned}\langle [\Sigma(C)], [\Sigma(C)] \rangle &= \langle [\Sigma(C)_{\varepsilon'}], [\Sigma(C)_{\varepsilon}] \rangle = \\ &= \sum_{P\in \mathrm{Vert}(C)} (-1) + \sum_{P\in \mathrm{Vert}(C)} 1 + (-2) = -2,\end{aligned}$$

where the first sum comes from the intersections of $(H_{C,P})_{\varepsilon'}$ and $H_{C,P}$, the second sum comes from the intersections of $(H_{C,P})_{\varepsilon'}$ and $\Delta(C)_{\varepsilon}$, and the third term (-2) comes from the intersection of $\Delta(C)_{\varepsilon'}$ and $\Delta(C)_{\varepsilon}$. Thus assertion (5) of Theorem 1.8 will be proved.

9.2. **Construction of the E-displacement.** For each $P \in \mathrm{Vert}(C)$, we fix good local coordinates (ξ_P, η_P), (x_P, y_P), $(\tilde{\xi}_P, \mu_P)$, and $(\tilde{x}_P, m_P)$ on $\mathbb{A}^2(\mathbb{C})$, $\mathbb{A}^2(\mathbb{R})$, $Y(\mathbb{C})$, and $Y(\mathbb{R})$, given in Section 5.1. We can assume, without loss of generality, that C is located as Case (1) in (5.3) so that there exist real numbers a_P, b_P with $a_P < b_P$ such that $\beta^{\sharp}C$ is give by $\tilde{x}_P \geq 0$ and $a_P \leq m_P \leq b_P$ in the chart of $(\tilde{x}_P, m_P)$ on $Y(\mathbb{R})$. To make the computation easy, we choose (ξ_P, η_P) in such a way that we have

$$a_P = -1, \quad b_P = 1. \tag{9.3}$$

See Remark 5.1. We choose a sufficiently small real number c, and choose a section $s_P \colon E_P \to \mathcal{N}_P$ of a tubular neighborhood $\mathcal{N}_P \to E_P$ such that, in a neighborhood of $J_{C,P}$ in E_P, the section s_P is written as

$$(0, \mu) \mapsto (c, \mu)$$

in terms of $(\tilde{\xi}_P, \mu_P)$. Then s_P is admissible with respect to C. See Example 6.10. Now we construct the E-displacement (9.1) of $\Sigma(C)$ in X associated with all these sections s_P.

We denote by $\mathit{\Gamma}_P$ the image by β of the region

$$\{\, (\tilde{x}_P, m_P) \mid 0 \le \tilde{x}_P \le c\varepsilon',\ -1 \le m_P \le 1 \,\}$$

in $\beta^\sharp C$. In terms of the coordinates (x_P, y_P) of $\mathbb{A}^2(\mathbb{R})$, we have

$$\mathit{\Gamma}_P = \{\, (x_P, y_P) \mid 0 \le x_P \le c\varepsilon',\ -x_P \le y_P \le x_P \,\} \ \subset\ C. \tag{9.4}$$

9.3. **Construction of the C-displacement.** As in Section 6.2, we introduce an inner-product on the $\mathbb{R}$-vector space $T(\mathbb{A}^2(\mathbb{R}))$. In particular, we have a distance and angles on $\mathbb{A}^2(\mathbb{R})$.

Let n be the size of $\mathrm{Vert}(C)$. We index the vertexes of C by elements ν of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ as

$$\mathrm{Vert}(C) = \{\, P_\nu \mid \nu \in \mathbb{Z}/n\mathbb{Z} \,\}$$

in such a way that, for each $\nu \in \mathbb{Z}/n\mathbb{Z}$, the line segment $P_\nu P_{\nu+1}$ is an edge of C, and that the cyclic sequence $P_0, P_1, \dots, P_{n-1}, P_n = P_0$ of vertexes goes around C in the positive-direction with respect to the fixed orientation $\sigma_{\mathbb{A}}$ of $\mathbb{A}^2(\mathbb{R})$. Let $\ell_\nu(\mathbb{R}) \in \mathcal{A}$ be the line defining the edge $P_\nu P_{\nu+1}$, and let $\overrightarrow{P_\nu P_{\nu+1}} \in T(\mathbb{A}^2(\mathbb{R}))$ be the translation such that

$$P_\nu + \overrightarrow{P_\nu P_{\nu+1}} = P_{\nu+1}.$$

Obviously we have $\overrightarrow{P_\nu P_{\nu+1}} \in T[\ell_\nu(\mathbb{R})]$. We define a unit vector $\tau_\nu \in T[\ell_\nu(\mathbb{R})]$ by

$$\tau_\nu := \overrightarrow{P_\nu P_{\nu+1}} \,/\, |P_\nu P_{\nu+1}|.$$

Let $o \in T(\mathbb{A}^2(\mathbb{R}))$ be the zero vector. Then $\tau_{\nu+1}$ is obtained by rotating τ_ν in a positive-direction by the angle

$$\theta_\nu := \angle \tau_\nu o \tau_{\nu+1} = \pi - \angle P_\nu P_{\nu+1} P_{\nu+2},$$

where the positive-direction is with respect to the orientation of $T(\mathbb{A}^2(\mathbb{R}))$ induced by $\sigma_{\mathbb{A}}$. We denote by Θ_ν the triangle $\tau_\nu o \tau_{\nu+1}$ in $T(\mathbb{A}^2(\mathbb{R}))$. Then the interiors of these n triangles are disjoint, and, since the sum of $\theta_0, \dots, \theta_{n-1}$ is equal to 2π, their union

$$\Theta := \bigcup_{\nu \in \mathbb{Z}/n\mathbb{Z}} \Theta_\nu$$

contains o in its interior. The cyclic sequence $\tau_0, \dots, \tau_{n-1}, \tau_n = \tau_0$ of the vertexes of the n-gon Θ goes in a positive-direction around Θ. See Figure 9.1.

We define a continuous function

$$\delta \colon C \to \Theta \subset T(\mathbb{A}^2(\mathbb{R}))$$

as follows. Let M_ν be the mid-point of the edge $P_\nu P_{\nu+1}$, and let M be the n-gon $M_0 M_1 \dots M_{n-1} M_0$. Then we have a homeomorphism

$$\delta_M \colon M \xrightarrow{\sim} \Theta$$

with the following properties.

- For each $\nu \in \mathbb{Z}/n\mathbb{Z}$, we have $\delta_M(M_\nu) = \tau_\nu$. Moreover, the restriction of δ_M to $M_\nu M_{\nu+1}$ is an affine-linear isomorphism of line segments from $M_\nu M_{\nu+1}$ to $\tau_\nu \tau_{\nu+1}$. In particular, δ_M is orientation-preserving.
- The inverse-image $\delta_M^{-1}(o)$ consists of a single point Q_0 in the interior of M, δ_M is differentiable at the point Q_0, and Q_0 is not a critical point of δ_M.

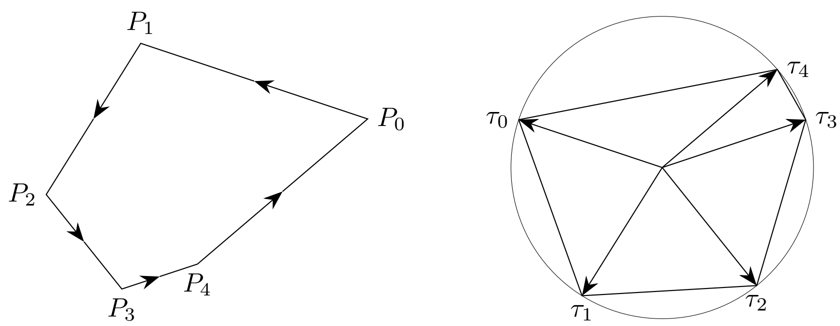

FIGURE 9.1. C and Θ

For each $\nu \in \mathbb{Z}/n\mathbb{Z}$, there exists a unique affine-linear isomorphism

$$\delta_\nu \colon \triangle M_\nu P_{\nu+1} M_{\nu+1} \xrightarrow{\sim} \Theta_\nu = \triangle \tau_\nu o \tau_{\nu+1}$$

that maps M_ν to τ_ν, $P_{\nu+1}$ to o, and $M_{\nu+1}$ to $\tau_{\nu+1}$, respectively. In particular, the map δ_ν is orientation-*reversing*. Then the restrictions to the line segment $M_\nu M_{\nu+1}$ of δ_M and of δ_ν coincide. Thus, gluing δ_M and δ_ν along $M_\nu M_{\nu+1}$ for $\nu \in \mathbb{Z}/n\mathbb{Z}$, we obtain a continuous map

$$\delta' \colon C \to \Theta \subset T(\mathbb{A}^2(\mathbb{R})).$$

Note that δ' maps the edge $P_\nu P_{\nu+1}$ to the line-segment $o\tau_\nu \subset T[\ell_\nu(\mathbb{R})]$. Hence δ' satisfies the e-condition for every edge of C.

We assume that we have chosen $c \in \mathbb{R}_{>0}$ and ε' in the construction of the E-displacement so small that, for each $P_{\nu+1} \in \operatorname{Vert}(C)$, the triangle $\Gamma_{P_{\nu+1}}$ defined by (9.4) is contained in $\triangle M_\nu P_{\nu+1} M_{\nu+1}$ and that the edge

$$\gamma_{\nu+1} := C \cap \{ x_{P_{\nu+1}} = c\varepsilon' \}$$

of $\Gamma_{P_{\nu+1}}$ is disjoint from the edge $M_\nu M_{\nu+1}$ of $\triangle M_\nu P_{\nu+1} M_{\nu+1}$. We choose a sufficiently small positive real number ρ so that, for each $P_{\nu+1} \in \operatorname{Vert}(C)$, the closed region $U_{P_{\nu+1},\rho}$ defined by (6.8) is contained in $\Gamma_{P_{\nu+1}}$ and disjoint from the edge $\gamma_{\nu+1} \subset \Gamma_{P_{\nu+1}}$. (See Figure 9.2.) We execute the q_ρ-modification to δ' at each $P_{\nu+1} \in \operatorname{Vert}(C)$, and obtain a continuous function

$$\delta \colon C \to \Theta \subset T(\mathbb{A}^2(\mathbb{R}))$$

that satisfies the e-condition for every edge of C and regular at each vertex of C. See Lemma 6.7. Note that, since this q_ρ-modification does not affect the restriction of δ' to a small neighborhood of the line segment $\gamma_{\nu+1}$ in C, the function δ restricted to a small neighborhood of $\gamma_{\nu+1}$ in C is equal to the unique affine-linear map that maps M_ν, $P_{\nu+1}$, $M_{\nu+1}$ to τ_ν, o, $\tau_{\nu+1}$, respectively.

9.4. **Intersection of $\Delta(C)_{\varepsilon'}$ and $\Delta(C)_\varepsilon$.** We prove that the E-displacement and the C-displacement thus constructed satisfy condition (EC1). Recall that

$$C_\varepsilon = \{ Q + \sqrt{-1}\, \delta(Q)\varepsilon \mid Q \in C \}.$$

Hence we have

$$C \cap C_\varepsilon := \{ Q \in C \mid \delta(Q) = 0 \} = \{Q_0\} \sqcup \operatorname{Vert}(C),$$

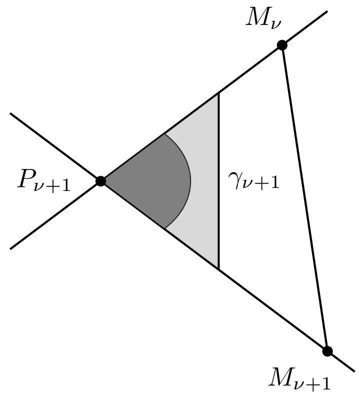

FIGURE 9.2. $\triangle M_\nu P_{\nu+1} M_{\nu+1}$, $\mathit{\Gamma}_{P_{\nu+1}}$, and $U_{P_{\nu+1},\rho}$

where $\{Q_0\} = \delta_M^{-1}(o)$. Let $Q'_0 \in Y(\mathbb{R})$ be the point such that $\beta(Q'_0) = Q_0$. We have

$$\beta^\sharp C \cap (\beta^\sharp C)_\varepsilon = \{Q'_0\} \quad \sqcup \bigsqcup_{P \in \mathrm{Vert}(C)} J_{C,P}.$$

Since $(\beta^\sharp C)_{\varepsilon'} \subset \beta^\sharp C$ and $(\beta^\sharp C)_{\varepsilon'} \cap J_{C,P} = \emptyset$, we obtain

$$(\beta^\sharp C)_{\varepsilon'} \cap (\beta^\sharp C)_\varepsilon = \{Q'_0\}.$$

Recall that $\Delta(C)_{\varepsilon'} = \phi^{-1}((\beta^\sharp C)_{\varepsilon'})$ and $\Delta(C)_\varepsilon = \phi^{-1}((\beta^\sharp C)_\varepsilon)$, each equipped with the standard orientation σ_C. We put

$$\phi_{\Delta,\varepsilon'} := \phi|\Delta(C)_{\varepsilon'} \colon \Delta(C)_{\varepsilon'} \to (\beta^\sharp C)_{\varepsilon'}, \quad \phi_{\Delta,\varepsilon} := \phi|\Delta(C)_\varepsilon \colon \Delta(C)_\varepsilon \to (\beta^\sharp C)_\varepsilon.$$

Let $Q'_{0,+}$ and $Q'_{0,-}$ be the points of X such that $\phi^{-1}(Q'_0) = \{Q'_{0,+}, Q'_{0,-}\}$. We then have

$$\Delta(C)_{\varepsilon'} \cap \Delta(C)_\varepsilon = \{Q'_{0,+}, Q'_{0,-}\}.$$

Interchanging $Q'_{0,+}$ and $Q'_{0,-}$ if necessary, we can assume that the restrictions $\phi_{\Delta,\varepsilon'}$ and $\phi_{\Delta,\varepsilon}$ of ϕ to $\Delta(C)_{\varepsilon'}$ and to $\Delta(C)_\varepsilon$ are both orientation-preserving at $Q'_{0,+}$, and that they are both orientation-reversing at $Q'_{0,-}$. In particular, the local intersection number $(\Delta(C)_{\varepsilon'}, \Delta(C)_\varepsilon)_{Q'_{0,+}}$ of $\Delta(C)_{\varepsilon'}$ and $\Delta(C)_\varepsilon$ at $Q'_{0,+}$ is equal to their local intersection number $(\Delta(C)_{\varepsilon'}, \Delta(C)_\varepsilon)_{Q'_{0,-}}$ at $Q'_{0,-}$. Since $\beta \circ \phi$ is a local isomorphism locally around each of the points $Q'_{0,\pm}$ of X, we see that

$$(\Delta(C)_{\varepsilon'}, \Delta(C)_\varepsilon)_{Q'_{0,+}} = (\Delta(C)_{\varepsilon'}, \Delta(C)_\varepsilon)_{Q'_{0,-}} = (C, C_\varepsilon)_{Q_0},$$

where $(C, C_\varepsilon)_{Q_0}$ is the local intersection number of C and C_ε at Q_0. We show that $(C, C_\varepsilon)_{Q_0} = -1$.

We choose a coordinate system (ξ, η) of $\mathbb{A}^2(\mathbb{C})$ that is good at Q_0. (See Section 5.1). We then put

$$(x, y) = (\operatorname{Re}\xi, \operatorname{Re}\eta), \quad (u, v) = (\operatorname{Im}\xi, \operatorname{Im}\eta).$$

Then an ordered basis of the real tangent space at Q_0 of $\mathbb{A}^2(\mathbb{C})$ oriented by the complex structure is

$$\partial/\partial x,\ \partial/\partial u,\ \partial/\partial y,\ \partial/\partial v. \tag{9.5}$$

We write tangent vectors at Q_0 in terms of this basis. The tangent space at Q_0 of C oriented by $\sigma_{\mathbb{A}}$ has the ordered basis

$$(1,0,0,0),\quad (0,0,1,0).$$

Recall that C_ε is given by the equations

$$u=\varepsilon\delta_x(x,y),\quad v=\varepsilon\delta_y(x,y),$$

where δ_x and δ_y are functions defined by (6.6), and that C_ε is oriented by $\sigma_{\mathbb{A}}$ via the homeomorphism $\mathrm{pr}_{\mathbb{R}}|C_\varepsilon\colon C_\varepsilon\cong C$. Therefore the tangent space at Q_0 of C_ε with this orientation has the ordered basis

$$\left(1,\ \varepsilon\frac{\partial\delta_x}{\partial x}(Q_0),\ 0,\ \varepsilon\frac{\partial\delta_y}{\partial x}(Q_0)\right),\quad \left(0,\ \varepsilon\frac{\partial\delta_x}{\partial y}(Q_0),\ 1,\ \varepsilon\frac{\partial\delta_y}{\partial y}(Q_0)\right).$$

Since δ is orientation-preserving at Q_0, its Jacobian at Q_0 is >0:

$$\frac{\partial(\delta_x,\delta_y)}{\partial(x,y)}(Q_0)\ >\ 0.$$

Thus we obtain $(C,C_\varepsilon)_{Q_0}=-1$, and hence condition (EC1) is satisfied.

9.5. **Intersection of** $(H_{C,P})_{\varepsilon'}$ **and** $\Delta(C)_\varepsilon$**.** Let P be a vertex of C. We investigate the intersection of $(H_{C,P})_{\varepsilon'}$ and $\Delta(C)_\varepsilon$, and show that condition (EC2) is satisfied. Recall that we have fixed good local coordinate systems (ξ_P,η_P), (x_P,y_P), $(\tilde{\xi}_P,\mu_P)$, $(\tilde{x}_P,m_P)$ at P (see Definition 5.2) such that $a_P=-1$ and $b_P=1$ hold (see (9.3)), and that the location of C is Case (1) of (5.3). For simplicity, we omit the subscript P from these coordinates. Let $\nu+1\in\mathbb{Z}/n\mathbb{Z}$ be the index of P. By (9.3), there exist positive real numbers r,r',r'',r''' such that

$$M_\nu=r(1,1),\ \ M_{\nu+1}=r'(1,-1),\ \ \tau_\nu=-r''(e_x+e_y),\ \ \tau_{\nu+1}=r'''(e_x-e_y).$$

in terms of (x,y), where e_x,e_y is the basis of $T(\mathbb{A}^2(\mathbb{R}))$ given by (3.2). Recall that, in a small neighborhood U_γ of the line-segment $\gamma_{\nu+1}=\{x=c\varepsilon'\}\cap C$ in C, the function δ is equal to the affine-linear map such that $\delta(P)=o$, $\delta(M_\nu)=\tau_\nu$, $\delta(M_{\nu+1})=\tau_{\nu+1}$. Hence there exist real numbers g,h satisfying

$$-h<g<h \tag{9.6}$$

such that, on U_γ, we have

$$\delta_x(x,y)=gx-hy,\quad \delta_y(x,y)=-hx+gy, \tag{9.7}$$

where δ_x and δ_y are defined by (6.6). Then, by the formula (6.7), the C-displacement $d_\varepsilon^\sharp\colon\beta^\sharp C\to(\beta^\sharp C)_\varepsilon$ associated with δ is given in a small neighborhood $\beta^{-1}(U_\gamma)$ of $\beta^{-1}(\gamma_{\nu+1})$ in $\beta^\sharp C$ by

$$(\tilde{x},m)\ \mapsto$$
$$(\tilde{\xi},\mu)\ =\ \left(\tilde{x}(1+\sqrt{-1}\,\varepsilon(g-hm)),\ \frac{m+\sqrt{-1}\,\varepsilon(-h+gm)}{1+\sqrt{-1}\,\varepsilon(g-hm)}\right)\ \in\ (\beta^\sharp C)_\varepsilon.$$

Recall that the E-displacement $(E_P)_{\varepsilon'}$ is given by $\tilde{\xi}=c\varepsilon'$ in a small neighborhood of $\beta^{-1}(\gamma_{\nu+1})$ in $Y(\mathbb{C})$. Since $c\varepsilon'\in\mathbb{R}_{>0}$, it follows that $(E_P)_{\varepsilon'}$ and $(\beta^\sharp C)_\varepsilon$ intersect only at one point, which we write as

$$R=d_\varepsilon^\sharp(R_0),$$

where R_0 is the point

$$(\tilde{x},m)=(c\varepsilon',g/h).$$

The coordinates of the point R in terms of $(\tilde{\xi}, \mu)$ is

$$(\tilde{\xi}_R, \mu_R) = \left(c\varepsilon', \ \frac{g}{h} + \sqrt{-1}\,\varepsilon \left(-h + \frac{g^2}{h} \right) \right).$$

By (9.6) and ε being very small, the value μ_R of μ at R satisfies

$$-1 < \operatorname{Re}\mu_R < 1, \quad \operatorname{Im}\mu_R < 0, \quad |\operatorname{Im}\mu_R| \ll 1. \tag{9.8}$$

More precisely, we can make $|\operatorname{Im}\mu_R|$ arbitrarily small by choosing a sufficiently small ε. Then $(D_P)_{\varepsilon'} = \phi^{-1}((E_P)_{\varepsilon'})$ and $\Delta(C)_\varepsilon$ intersect at two points in $\phi^{-1}(R)$. Let ζ be the function on X defined by (5.6). In the current situation, we have

$$\zeta^2 = \frac{\mu + 1}{\mu - 1}.$$

We calculate the value of ζ at the points of $\phi^{-1}(R)$. Note that, by (9.8), making ε smaller if necessary, we can assume that

$$\operatorname{Re}\frac{\mu_R + 1}{\mu_R - 1} < 0, \quad 0 < \operatorname{Im}\frac{\mu_R + 1}{\mu_R - 1} \ll 1.$$

Hence one solution ζ_R of the equation $\zeta^2 = (\mu_R + 1)/(\mu_R - 1)$ satisfies

$$0 < \operatorname{Re}\zeta_R \ll 1, \quad 0 < \operatorname{Im}\zeta_R, \tag{9.9}$$

and the other solution is $-\zeta_R$. Let $\widetilde{R}_+$ and $\widetilde{R}_-$ be the points of $\phi^{-1}(R)$ such that $\zeta(\widetilde{R}_+) = \zeta_R$ and $\zeta(\widetilde{R}_-) = -\zeta_R$.

Recall from Table 5.1 that, because we are in Case (1) of (5.3), the circle $S_{C,P}$ is given by $\operatorname{Re}\zeta = 0$ on $D_P = \{\phi^*\tilde{\xi} = 0\}$. Hence one and only one of $\widetilde{R}_+$ or $\widetilde{R}_-$ belongs to $(H_{C,P})_{\varepsilon'}$. In particular, the set-theoretic intersection of $(H_{C,P})_{\varepsilon'}$ and $(\Delta(C))_\varepsilon$ consists of a single point. We will show that the local intersection number at this point is 1.

Suppose that $\widetilde{R}_+ \in (H_{C,P})_{\varepsilon'}$. By (9.9), the hemisphere $H_{C,P}$ is given by $\operatorname{Re}\zeta | D_P \geq 0$, that is, we are in the second row of Table 5.2. The orientation ori_H on $S_{C,P} = \partial H_{C,P}$ given by the complex structure of $H_{C,P}$ is $\downarrow$. Hence the orientation ori_C on $S_{C,P} = \partial\,\Delta(C)$ given by the standard orientation on $\Delta(C)$ is $\uparrow$. Since ϕ is given by

$$\mu = \frac{\zeta^2 + 1}{\zeta^2 - 1},$$

and since $\operatorname{Im}\zeta_R > 0$ by (9.9), the image by ϕ of the orientation ori_C of $S_{C,P} = \partial\,\Delta(C)$ near $\widetilde{R}_+$ is the m-increasing direction on $J_{C,P}$. Indeed, when ζ moves from 0 to $\infty\sqrt{-1}$ along $\sqrt{-1}\,\mathbb{R}_{\geq 0}$, then μ moves from -1 to 1 along $\mathbb{R}$. Recall that the standard orientation of $\beta^\sharp C$ induces the m-decreasing orientation on $J_{C,P}$. (See Corollary 5.4 and Figure 5.3.) Therefore $\widetilde{R}_+$ is on the *minus*-sheet of $(\Delta(C))_\varepsilon$.

Suppose that $\widetilde{R}_- \in (H_{C,P})_{\varepsilon'}$. By (9.9), we see that $H_{C,P}$ is given by $\operatorname{Re}\zeta \leq 0$, and hence we are in the first row of Table 5.2. The orientation ori_H on $S_{C,P} = \partial\,H_{C,P}$ is $\uparrow$. Hence the orientation ori_C on $S_{C,P} = \partial\,\Delta(C)$ is $\downarrow$. Since $-\operatorname{Im}\zeta_R < 0$ by (9.9), the image by ϕ of the orientation ori_C of $S_{C,P} = \partial\,\Delta(C)$ near $\widetilde{R}_-$ is the m-increasing orientation on $J_{C,P}$. Therefore $\widetilde{R}_-$ is also on the *minus*-sheet of $(\Delta(C))_\varepsilon$.

In any case, the restriction of $\beta \circ \phi$ to a neighborhood in $\Delta(C)_\varepsilon$ of the intersection point $\widetilde{R}_+$ or $\widetilde{R}_-$ of $(H_{C,P})_{\varepsilon'}$ and $\Delta(C)_\varepsilon$ is an orientation-*reversing* local isomorphism to an open neighborhood of $\beta(R)$ in C_ε. Hence, to show that the local intersection number of $(H_{C,P})_{\varepsilon'}$ and $(\Delta(C))_\varepsilon$ is 1, it is enough to prove that, in $\mathbb{A}^2(\mathbb{C})$, the local intersection number at $\beta(R)$ of

$$\beta((E_P)_{\varepsilon'}) = \{\xi = c\varepsilon'\}$$

with the orientation coming from the complex structure and the space C_ε with the orientation induced by $\sigma_{\mathbb{A}}$ is -1. Note that C_ε is given locally around $\beta(R)$ by

$$\begin{aligned}\xi &= x + \sqrt{-1}\,\varepsilon\delta_x(x,y) = x + \sqrt{-1}\,\varepsilon(gx - hy),\\ \eta &= y + \sqrt{-1}\,\varepsilon\delta_y(x,y) = y + \sqrt{-1}\,\varepsilon(-hx + gy).\end{aligned}$$

We use the ordered basis (9.5) of the real tangent space of $\mathbb{A}^2(\mathbb{C})$. Then the tangent space of $\beta((E_P)_{\varepsilon'})$ oriented by the complex structure has an ordered basis

$$(0,0,1,0), \quad (0,0,0,1).$$

On the other hand, the oriented tangent space of C_ε has an ordered basis

$$\left(1, \varepsilon\frac{\partial\delta_x}{\partial x}, 0, \varepsilon\frac{\partial\delta_y}{\partial x}\right) = (1, \varepsilon g, 0, -\varepsilon h), \quad (0, \varepsilon\frac{\partial\delta_x}{\partial y}, 1, \varepsilon\frac{\partial\delta_y}{\partial y}) = (0, -\varepsilon h, 1, \varepsilon g).$$

Since $h > 0$ by (9.6), we see that the local intersection number is -1. Thus (EC2) holds at P.

10. Examples

Using a smooth projective completion $\widetilde{X}$ of X, we check certain consequences of Theorem 1.8 in several examples.

10.1. **Lattice.** A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a *non-degenerate* symmetric bilinear form $\langle\ \ \rangle\colon L \times L \to \mathbb{Z}$. Let L be a lattice. The *signature* of L is the pair (s_+, s_-) of numbers of positive and negative eigenvalues of the Gram matrix of L. A lattice L is embedded naturally into its dual $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$. The *discriminant group* of L is the finite abelian group

$$\mathrm{disc}(L) := L^\vee/L.$$

We say that L is *unimodular* if its discriminant group is trivial. By definition, we have the following:

Proposition 10.1. *Let L' be a sublattice of a lattice L with a finite index m. Then* $\mathrm{disc}(L)$ *is a sub-quotient of* $\mathrm{disc}(L')$*, and* $|\mathrm{disc}(L)|$ *is equal to* $|\mathrm{disc}(L')|/m^2$. □

The following is well-known. See, for example, [8].

Proposition 10.2. *Let L be a sublattice of a unimodular lattice M. Suppose that M/L is torsion-free. We put*

$$L^\perp := \{\, x \in M \mid \langle x, y\rangle = 0 \ \textit{for all}\ \ y \in L \,\}. \tag{10.1}$$

Then we have $\mathrm{disc}(L) \cong \mathrm{disc}(L^\perp)$. □

Combining Propositions 10.1 and 10.2, we obtain the following:

Corollary 10.3. *Let L be a sublattice of a unimodular lattice M. Then* $\mathrm{disc}(L^\perp)$ *is a sub-quotient of* $\mathrm{disc}(L)$. □

Let $\mathcal{A}$ be a nodal real line arrangement, and let $\langle\ \ \rangle$ be the intersection form on $H_2(X,\mathbb{Z})$. We put

$$\operatorname{Ker}\langle\ \ \rangle := \{\, x \in H_2(X,\mathbb{Z}) \mid \langle x, y\rangle = 0 \ \text{ for any } \ y \in H_2(X,\mathbb{Z}) \,\}. \tag{10.2}$$

Then $\langle\ \ \rangle$ yields a natural structure of the lattice on

$$\overline{H}(X) := H_2(X,\mathbb{Z})/\operatorname{Ker}\langle\ \ \rangle,$$

and we can calculate the Gram matrix of $\overline{H}(X)$ by Theorem 1.8. In particular, we can calculate the signature and the discriminant group of $\overline{H}(X)$ by Theorem 1.8.

10.2. **A projective completion of X.** Let $\mathbb{P}^2(\mathbb{C})$ be the complex projective plane containing $\mathbb{A}^2(\mathbb{C})$ as an affine part. We put

$$\tilde{\ell}_\infty(\mathbb{C}) := \mathbb{P}^2(\mathbb{C}) \setminus \mathbb{A}^2(\mathbb{C}).$$

For $\ell_i(\mathbb{C}) \in \mathcal{A}_\mathbb{C}$, we denote by $\tilde{\ell}_i(\mathbb{C})$ the projective completion of $\ell_i(\mathbb{C})$ in $\mathbb{P}^2(\mathbb{C})$, and put

$$\widetilde{\mathcal{A}}_\mathbb{C} := \begin{cases} \{\, \tilde{\ell}_i(\mathbb{C}) \mid \ell_i(\mathbb{C}) \in \mathcal{A}_\mathbb{C} \,\} \cup \{\tilde{\ell}_\infty(\mathbb{C})\} & \text{if } |\mathcal{A}| \text{ is odd,} \\ \{\, \tilde{\ell}_i(\mathbb{C}) \mid \ell_i(\mathbb{C}) \in \mathcal{A}_\mathbb{C} \,\} & \text{if } |\mathcal{A}| \text{ is even.} \end{cases}$$

Let $\widetilde{B}(\mathbb{C})$ be the union of the complex projective lines in $\widetilde{\mathcal{A}}_\mathbb{C}$. Then $\deg \widetilde{B}(\mathbb{C})$ is even. We consider the morphisms

$$\widetilde{X} \ \xrightarrow{\tilde{\rho}}\ \widetilde{W} \ \xrightarrow{\tilde{\pi}}\ \mathbb{P}^2(\mathbb{C}), \tag{10.3}$$

where $\tilde{\pi}\colon \widetilde{W} \to \mathbb{P}^2(\mathbb{C})$ is the double covering whose branch locus is equal to $\widetilde{B}(\mathbb{C})$, and $\tilde{\rho}\colon \widetilde{X} \to \widetilde{W}$ is the minimal desingularization. We put

$$\Lambda_\infty := \tilde{\rho}^{-1}(\tilde{\pi}^{-1}(\tilde{\ell}_\infty(\mathbb{C}))) \ \subset\ \widetilde{X}.$$

Then we have $X = \widetilde{X} \setminus \Lambda_\infty$, and the inclusion $\iota\colon X \hookrightarrow \widetilde{X}$ induces a natural homomorphism

$$\iota_* :\ H_2(X,\mathbb{Z}) \to H_2(\widetilde{X},\mathbb{Z}),$$

which preserves the intersection form. We assume that $H_2(\widetilde{X},\mathbb{Z})$ is torsion-free. Since $\widetilde{X}$ is smooth and compact, the intersection form makes $H_2(\widetilde{X},\mathbb{Z})$ a unimodular lattice. Let

$$H_\infty \subset H_2(\widetilde{X},\mathbb{Z})$$

be the submodule generated by the classes of irreducible components of Λ_∞. Suppose that H_∞ is a sublattice of $H_2(\widetilde{X},\mathbb{Z})$, that is, the intersection form restricted to H_∞ is non-degenerate. Then the image of ι_* is equal to the orthogonal complement $H_\infty^\perp$ of H_∞ in $H_2(\widetilde{X},\mathbb{Z})$ defined by (10.1). On the other hand, the kernel of ι_* is equal to $\operatorname{Ker}\langle\ \ \rangle$ defined by (10.2). Hence the two lattices $\overline{H}(X)$ and $H_\infty^\perp$ are isomorphic:

$$\overline{H}(X) \ \cong\ H_\infty^\perp. \tag{10.4}$$

10.3. **The lattices $H_2(\widetilde{X}, \mathbb{Z})$ and $H_\infty^\perp$.** In this section, we consider arrangements $\mathcal{A}$ with the following property:

$$\text{for any line } \lambda(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R}), \text{ at most two lines in } \mathcal{A} \text{ are parallel to } \lambda(\mathbb{R}). \tag{10.5}$$

We put $N := |\mathcal{A}|$, and assume $N \geq 3$. We then put

$$\widetilde{N} := |\widetilde{\mathcal{A}}_{\mathbb{C}}| = \begin{cases} N+1 & \text{if } N \text{ is odd,} \\ N & \text{if } N \text{ is even.} \end{cases}$$

Suppose that $\mathcal{A}$ has exactly p parallel pairs, where $2p \leq N$. Then we have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = \frac{(N-1)(N-2)}{2} - p, \quad |\mathcal{P}| = \frac{N(N-1)}{2} - p. \tag{10.6}$$

Proposition 10.4. *Suppose that (10.5) holds and that $N \geq 3$. Let p be the number of parallel pairs in $\mathcal{A}$.*

(1) *The homology group $H_2(\widetilde{X}, \mathbb{Z})$ is torsion-free, and the unimodular lattice $H_2(\widetilde{X}, \mathbb{Z})$ is of rank $\widetilde{N}^2 - 3\widetilde{N} + 4$ with signature*

$$\left(\frac{1}{4}\widetilde{N}^2 - \frac{3}{2}\widetilde{N} + 3,\ \frac{3}{4}\widetilde{N}^2 - \frac{3}{2}\widetilde{N} + 1\right).$$

(2) *The submodule H_∞ of $H_2(\widetilde{X}, \mathbb{Z})$ is a sublattice of rank*

$$r_\infty := \operatorname{rank} H_\infty = \begin{cases} 1 + N + 2p & \text{if } N \text{ is odd,} \\ 1 + p & \text{if } N \text{ is even and } N \neq 2p, \\ 2 + p & \text{if } N = 2p, \end{cases}$$

with the signature $(1, r_\infty - 1)$. The discriminant group is given by

$$\operatorname{disc}(H_\infty) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{N-1} & \text{if } N \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^{p+1} & \text{if } N \text{ is even and } N \neq 2p, \\ (\mathbb{Z}/2\mathbb{Z})^{p-1} \times (\mathbb{Z}/2(p-1)\mathbb{Z}) & \text{if } N = 2p. \end{cases}$$

Suppose that N is odd. Then we see that

$$\operatorname{rank} H_2(X, \mathbb{Z}) = |\boldsymbol{Ch}_{\mathrm{b}}| + |\mathcal{P}| = (N-1)^2 - 2p$$

is equal to

$$\operatorname{rank} H_2(\widetilde{X}, \mathbb{Z}) - \operatorname{rank} H_\infty = \widetilde{N}^2 - 3\widetilde{N} + 4 - (1 + N + 2p).$$

By (10.4), it follows that $\operatorname{Ker} \langle\ \ \rangle$ is 0. Therefore we obtain the following:

Corollary 10.5. *Suppose that (10.5) holds and that N is odd ≥ 3. Then the intersection form $\langle\ \ \rangle$ on $H_2(X, \mathbb{Z})$ is non-degenerate.* □

Proof of Proposition 10.4. Let $\mathcal{P}_\infty$ denote the set of singular points of $\widetilde{B}(\mathbb{C})$ located on $\widetilde{\ell}_\infty(\mathbb{C})$. By assumption (10.5), it follows that $\mathcal{P}_\infty$ consists of simple singular points of type a_1 (ordinary double points) and/or of type d_4 (ordinary triple points). Let

$$\mathcal{P}_\infty = \mathcal{P}_{\infty,\mathrm{a1}} \sqcup \mathcal{P}_{\infty,\mathrm{d4}}$$

be the decomposition of $\mathcal{P}_\infty$ according to the types. We have the following.

- If N is odd, then $|\mathcal{P}_{\infty,\mathrm{a1}}| = N - 2p$ and $|\mathcal{P}_{\infty,\mathrm{d4}}| = p$.
- If N is even, then $|\mathcal{P}_{\infty,\mathrm{a1}}| = p$ and $|\mathcal{P}_{\infty,\mathrm{d4}}| = 0$.

In particular, the branch curve $\widetilde{B}(\mathbb{C})$ of the double covering $\tilde{\pi}\colon \widetilde{W} \to \mathbb{P}^2(\mathbb{C})$ has only simple singularities, and hence $\widetilde{W}$ has only rational double points as its singularities. By the theory of simultaneous resolution of rational double points of algebraic surfaces, we see that $\widetilde{X}$ is diffeomorphic to a double cover $\widetilde{X}'$ of $\mathbb{P}^2(\mathbb{C})$ branching along a smooth projective plane curve of degree $\widetilde{N}$. We can calculate the second betti number b_2 and the geometric genus p_g of $\widetilde{X}'$ easily. Note that the signature of $H_2(\widetilde{X}', \mathbb{Z})$ is

$$(1 + 2p_g, b_2 - 1 - 2p_g).$$

Therefore we obtain a proof of assertion (1).

For $P \in \mathcal{P}_{\infty,\mathrm{a1}}$, let $a(P)$ denote the smooth rational curve on $\widetilde{X}$ that is contracted to P. We have $\langle a(P), a(P)\rangle = -2$. For $Q \in \mathcal{P}_{\infty,\mathrm{d4}}$, there exist four smooth rational curves

$$d_1(Q),\ \ d_2(Q),\ \ d_3(Q),\ \ d_4(Q)$$

on $\widetilde{X}$ that are contracted to Q. These curves have the self-intersection (-2), and form the Coxeter-Dynkin diagram of type D_4 as the dual graph. Let R_∞ denote the reduced part of the strict transform of $\tilde{\ell}_\infty(\mathbb{C})$ in $\widetilde{X}$. Note that R_∞ is irreducible if and only if $N \neq 2p$. Then H_∞ is generated by

(a) the classes of $a(P)$, where P runs though $\mathcal{P}_{\infty,\mathrm{a1}}$,
(d) the classes of $d_1(Q), \dots, d_4(Q)$, where Q runs though $\mathcal{P}_{\infty,\mathrm{d4}}$, and
(R) the classes of irreducible components of R_∞.

Let H'_∞ be the submodule of H_∞ generated by the classes in (a) and (d) above, and

(L) the class of the total transform $\mathcal{L}_\infty$ of $\tilde{\ell}_\infty(\mathbb{C})$ in $\widetilde{X}$.

Since we have $\langle \mathcal{L}_\infty, \mathcal{L}_\infty\rangle = 2$ and $\mathcal{L}_\infty$ is orthogonal to the classes of exceptional curves in (a) and (d) above, it follows that H'_∞ is a lattice with

$$\operatorname{rank} H'_\infty = 1 + |\mathcal{P}_{\infty,\mathrm{a1}}| + 4|\mathcal{P}_{\infty,\mathrm{d4}}|, \quad \operatorname{disc} H'_\infty \cong (\mathbb{Z}/2\mathbb{Z})^{1+|\mathcal{P}_{\infty,\mathrm{a1}}|+2|\mathcal{P}_{\infty,\mathrm{d4}}|}.$$

Suppose that N is odd. Then R_∞ is irreducible and the multiplicity of R_∞ in $\mathcal{L}_\infty$ is 2. Hence H'_∞ is a submodule of index 2 in H_∞. Therefore H_∞ is a lattice, and the rank and the discriminant group of H_∞ can be derived from those of H'_∞.

Suppose that N is even and $2p < N$. Then R_∞ is irreducible and the multiplicity of R_∞ in $\mathcal{L}_\infty$ is 1. Hence we have $H'_\infty = H_\infty$.

Suppose that $N = 2p$. Then R_∞ is a disjoint union of two smooth rational curves $R_{\infty,+}$ and $R_{\infty,-}$. For $P \in \mathcal{P}_{\infty,\mathrm{a1}}$, we have $\langle a(P), R_{\infty,+}\rangle = \langle a(P), R_{\infty,-}\rangle = 1$. Using

$$\mathcal{L}_\infty = R_{\infty,+} + R_{\infty,-} + \sum_{P \in \mathcal{P}_{\infty,\mathrm{a1}}} a(P)$$

and $\langle \mathcal{L}_\infty, \mathcal{L}_\infty\rangle = 2$, we obtain

$$\langle R_{\infty,+}, R_{\infty,+}\rangle = \langle R_{\infty,-}, R_{\infty,-}\rangle = 1 - p.$$

By these formulas, we see that H_∞ is a lattice of rank $2 + |\mathcal{P}_{\infty,\mathrm{a1}}|$. From the Gram matrix of H_∞, we can compute the discriminant group. The computation is left to the reader.

Since H_∞ is contained in the Hodge part $H^{1,1}(\widetilde{X})$ of $H^2(\widetilde{X}, \mathbb{C})$ and $\mathcal{L}_\infty \in H_\infty$ is an ample class, we see that the signature of H_∞ is $(1, r_\infty - 1)$. □

10.4. **Experiments.** For fixed $N \geq 3$ and $p \leq N/2$, we randomly generate a nodal real line arrangement $\mathcal{A}$ satisfying (10.5), and calculate the signature and the discriminant group of $\overline{H}(X)$ topologically by Theorem 1.8. We compare the result with the signature and the discriminant group of H_∞ computed by Proposition 10.4. We see that the equality of signatures holds for $\overline{H}(X)$ and $H_\infty^\perp$, as is expected from (10.4), and that $\mathrm{disc}(\overline{H}(X))$ is a sub-quotient of $\mathrm{disc}(H_\infty)$, as is expected by (10.4) and Corollary 10.3.

Remark 10.6. In fact, for all examples we computed, we obtain an isomorphism $\mathrm{disc}(\overline{H}(X)) \cong \mathrm{disc}(H_\infty)$, and hence, in these cases, the sublattice H_∞ is primitive in $H_2(\widetilde{X}, \mathbb{Z})$. In general, the primitivity of a sublattice of $H_2(\widetilde{X}, \mathbb{Z})$ generated by the classes of given curves is a subtle problem. See [11] and [12].

Example 10.7. Suppose that $N = 6$ and $p = 0$. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 10, \quad |\mathcal{P}| = 15,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 25. On the other hand, the unimodular lattice $H_2(\widetilde{X}, \mathbb{Z})$ is of rank 22 with signature $(3, 19)$, and the sublattice H_∞ is of rank 1 with signature $(1, 0)$ and $\mathrm{disc}(H_\infty) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $H_\infty^\perp$ is of rank 21 with signature $(2, 19)$.

For randomly generated arrangements satisfying (10.5) with $N = 6$ and $p = 0$, we checked that $\overline{H}(X)$ is of rank 21 with signature $(2, 19)$ and $\mathrm{disc}(\overline{H}(X)) \cong \mathbb{Z}/2\mathbb{Z}$. Remark that there exist several combinatorial structures of nodal arrangements of six real lines with no parallel pairs. For example, the numbers of n-gons in $\boldsymbol{Ch}_{\mathrm{b}}$ can vary as follows:

n	3	4	5	6
	4	4	2	0
	4	5	1	0
	4	5	0	1
	4	6	0	0
	5	3	2	0
	5	4	1	0

n	3	4	5	6
	5	4	0	1
	6	2	2	0
	6	3	1	0
	6	3	0	1
	7	0	3	0 .

Example 10.8. Suppose that $N = 6$ and $p = 3$. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 7, \quad |\mathcal{P}| = 12,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 19. On the other hand, the unimodular lattice $H_2(\widetilde{X}, \mathbb{Z})$ is of rank 22 with signature $(3, 19)$, and the lattice H_∞ is of rank 5 with signature $(1, 4)$ and $\mathrm{disc}(H_\infty) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$. Hence $H_\infty^\perp$ is of rank 17 with signature $(2, 15)$.

For randomly generated such arrangements, we checked that $\overline{H}(X)$ is of rank 17 with signature $(2, 15)$ and $\mathrm{disc}(\overline{H}(X)) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$, regardless of combinatorial structures of $\boldsymbol{Ch}_{\mathrm{b}}$.

Example 10.9. Suppose that $N = 24$ and $p = 10$. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 243, \quad |\mathcal{P}| = 266,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 509. On the other hand, the unimodular lattice $H_2(\widetilde{X}, \mathbb{Z})$ is of rank 508 with signature $(111, 397)$, and the sublattice H_∞ is of rank 11 with signature $(1, 10)$ and $\mathrm{disc}(H_\infty) \cong (\mathbb{Z}/2\mathbb{Z})^{11}$. Hence $H_\infty^\perp$ is of rank 497 with signature $(110, 387)$. For randomly generated such arrangements, we obtained expected signature $(110, 387)$, and $\mathrm{disc}(\overline{H}(X)) \cong (\mathbb{Z}/2\mathbb{Z})^{11}$.

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