

# TOPOLOGY OF A COMPLEX DOUBLE PLANE BRANCHING ALONG A REAL LINE ARRANGEMENT

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ABSTRACT. We investigate the topology of the double cover of the complex affine plane branching along a nodal real line arrangement. We define certain topological 2-cycles in the double plane using the real structure of the arrangement, and calculate their intersection numbers.

## 1. INTRODUCTION

Explicit description of topological cycles in complex algebraic varieties is an important task. It is essential, for example, in the study of periods and related differential equations in a family of algebraic varieties. For the recent development of *numerical algebraic geometry* (see, for example, [4, 6]), we need explicit descriptions of topological cycles suited for the numerical computation of periods by multiple integrals. For example, in [2], the arithmetic of certain Calabi-Yau threefolds obtained as the minimal desingularizations of double covers of  $\mathbb{P}^3$  is studied by numerical integration over some topological 3-cycles.

The first major general theory on topological cycles in complex algebraic varieties is the theory of vanishing cycles due to Lefschetz [8]. (See also [7] for the modern accounts of this theory.) In the present work, we investigate the topology of the smooth algebraic surface  $X$  obtained as the minimal desingularization of the double cover of the complex affine plane branching along a nodal *real* line arrangement. Using the real structure of the arrangement, we construct certain topological 2-cycles in  $X$ , which resemble, in some way, the vanishing cycles of Lefschetz.

To understand the topology of an algebraic surface, it is important to calculate the intersection form on the middle homology group. The main purpose of this paper is to calculate the intersection numbers of our topological 2-cycles in  $X$ . Since some pairs of these cycles intersect in loci of real dimension  $\geq 1$ , we have to construct small displacements of these cycles.

Let  $\mathbb{A}^2(\mathbb{R})$  be a real affine plane. An arrangement of  $N$  real lines

$$\mathcal{A} := \{\ell_1(\mathbb{R}), \dots, \ell_N(\mathbb{R})\}$$

on  $\mathbb{A}^2(\mathbb{R})$  is called *nodal* if no three lines of  $\mathcal{A}$  are concurrent. Suppose that  $\mathcal{A}$  is a nodal real line arrangement. Let

$$\mathcal{A}_{\mathbb{C}} := \{\ell_1(\mathbb{C}), \dots, \ell_N(\mathbb{C})\}$$

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be the arrangement of complex affine lines in the complex affine plane  $\mathbb{A}^2(\mathbb{C})$  obtained by complexifying the lines in  $\mathcal{A}$ . We put

$$B(\mathbb{R}) := \bigcup_{i=1}^N \ell_i(\mathbb{R}), \quad B(\mathbb{C}) := \bigcup_{i=1}^N \ell_i(\mathbb{C}).$$

We consider the morphisms

$$(1.1) \quad X \xrightarrow{\rho} W \xrightarrow{\pi} \mathbb{A}^2(\mathbb{C}),$$

where  $\pi: W \rightarrow \mathbb{A}^2(\mathbb{C})$  is the double covering whose branch locus is equal to  $B(\mathbb{C})$ , and  $\rho: X \rightarrow W$  is the minimal resolution of singularities. The purpose of this paper is to investigate the intersection form

$$\langle \quad \rangle : H_2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$$

on the second homology group  $H_2(X; \mathbb{Z})$  of  $X$ , where  $X$  is oriented as a smooth complex surface.

For simplicity, we put

$$\mathcal{P} := \text{Sing } B(\mathbb{C}).$$

Let  $\beta: Y(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{C})$  be the blowing-up at the points in  $\mathcal{P}$ . For a subset  $S$  of  $\mathbb{A}^2(\mathbb{C})$ , we put

$$S^\bullet := S \setminus (S \cap \mathcal{P}), \quad \beta^\# S := \text{the closure of } \beta^{-1}(S^\bullet) \text{ in } Y(\mathbb{C}),$$

and call  $\beta^\# S$  the *strict transform* of  $S$ . We then put

$$Y(\mathbb{R}) := \beta^\# \mathbb{A}^2(\mathbb{R}).$$

Note that  $\beta^\# B(\mathbb{C})$  is the disjoint union of smooth rational curves  $\beta^\# \ell_i(\mathbb{C})$  on  $Y(\mathbb{C})$ . Then  $X$  fits in the following commutative diagram:

$$(1.2) \quad \begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y(\mathbb{C}) & \xrightarrow{\beta} & \mathbb{A}^2(\mathbb{C}), \end{array}$$

where  $\phi: X \rightarrow Y(\mathbb{C})$  is the double covering whose branch locus is equal to  $\beta^\# B(\mathbb{C})$ . For  $P \in \mathcal{P}$ , let  $E_P$  denote the exceptional  $(-1)$ -curve of  $\beta$  over  $P$ , and we put

$$D_P := \phi^{-1}(E_P) = (\pi \circ \rho)^{-1}(P),$$

which is a smooth rational curve on  $X$  with self-intersection number  $-2$ . Note that  $D_P$  is the exceptional curve of the minimal desingularization  $\rho: X \rightarrow W$  over the ordinary node of  $W$  that is mapped to  $P$  by  $\pi$ . If  $P$  is the intersection point of the lines  $\ell_a(\mathbb{R})$  and  $\ell_b(\mathbb{R})$  in  $\mathcal{A}$ , then  $\phi|D_P: D_P \rightarrow E_P$  is the double covering branching at the intersection points of  $E_P$  and the strict transforms  $\beta^\# \ell_a(\mathbb{C})$  and  $\beta^\# \ell_b(\mathbb{C})$ .

A *chamber* is the closure in  $\mathbb{A}^2(\mathbb{R})$  of a connected component of the complement  $\mathbb{A}^2(\mathbb{R}) \setminus B(\mathbb{R})$  of  $B(\mathbb{R})$ . We denote by  $\mathbf{Ch}_b$  the set of *bounded* chambers. Let  $C$  be a bounded chamber. We put

$$\text{Vert}(C) := C \cap \mathcal{P}.$$

A point of  $\text{Vert}(C)$  is called a *vertex* of  $C$ . Let  $P$  be a vertex of  $C$ . Then  $Y(\mathbb{R}) \cap E_P$  is a circle on the Riemann sphere  $E_P$ , on which the two branch points of the double covering  $\phi|D_P: D_P \rightarrow E_P$  locate, and

$$J_{C,P} := \beta^\# C \cap E_P$$

is a part of the circle  $Y(\mathbb{R}) \cap E_P$  connecting these two branch points. Therefore

$$S_{C,P} := \phi^{-1}(\beta^\sharp C) \cap D_P = \phi^{-1}(J_{C,P})$$

is a circle on the Riemann sphere  $D_P$ . The space  $\phi^{-1}(\beta^\sharp C)$  is homeomorphic to a 2-sphere minus a union of disjoint open discs, and we have

$$\partial \phi^{-1}(\beta^\sharp C) = \bigsqcup_{P \in \text{Vert}(C)} S_{C,P}.$$

Let  $\gamma_C$  be an orientation of  $\phi^{-1}(\beta^\sharp C)$ . We denote by  $\Delta(C, \gamma_C)$  the topological 2-chain  $\phi^{-1}(\beta^\sharp C)$  oriented by  $\gamma_C$ . Each boundary component  $S_{C,P}$  of  $\Delta(C, \gamma_C)$  is oriented by  $\gamma_C$ . Note that  $S_{C,P}$  divides the 2-sphere  $D_P$  into the union of two closed hemispheres, and that the two hemispheres with their complex structures induce orientations on  $S_{C,P}$  that are opposite to each other.

**Definition 1.1.** The *capping hemisphere* for  $\gamma_C$  at  $P$  is the closed hemisphere  $H_{C,\gamma_C,P}$  on  $D_P$  with  $\partial H_{C,\gamma_C,P} = S_{C,P}$  such that the orientation on  $S_{C,P}$  induced by the complex structure of  $H_{C,\gamma_C,P}$  is opposite to the orientation induced by the orientation  $\gamma_C$  of  $\Delta(C, \gamma_C)$ .

Let  $H_{C,\gamma_C,P}$  be the capping hemisphere for  $\gamma_C$  at  $P$ . Then

$$\Sigma(C, \gamma_C) := \Delta(C, \gamma_C) \cup \bigsqcup_{P \in \text{Vert}(C)} H_{C,\gamma_C,P}$$

with the orientations coming from the complex structure on each  $H_{C,\gamma_C,P}$  is a topological 2-cycle homeomorphic to a 2-sphere. Figure 1.1 illustrates  $\Sigma(C, \gamma_C)$  when  $C$  is a triangle.

In this paper, we show that this topological 2-cycle  $\Sigma(C, \gamma_C)$  resembles, in many aspects, the vanishing cycle for an ordinary node of a complex surface. Hence we make the following:

**Definition 1.2.** The 2-cycle  $\Sigma(C, \gamma_C)$  is called a *vanishing cycle* over the bounded chamber  $C$ . Its homology class  $[\Sigma(C, \gamma_C)] \in H_2(X; \mathbb{Z})$  is also called a vanishing cycle over  $C$ .

By definition, the homology class  $[\Sigma(C, \gamma_C)]$  depends on  $\gamma_C$  as follows:

$$(1.3) \quad [\Sigma(C, \gamma_C)] + [\Sigma(C, -\gamma_C)] = \sum_{P \in \text{Vert}(C)} [D_P],$$

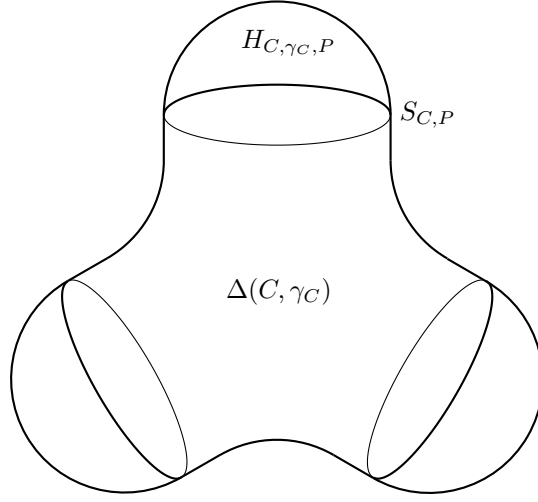
where  $[D_P] \in H_2(X; \mathbb{Z})$  is the homology class of the smooth rational curve  $D_P$ .

Our first main result is as follows.

**Theorem 1.3.** *The  $\mathbb{Z}$ -module  $H_2(X; \mathbb{Z})$  is free. We fix an orientation  $\gamma_C$  for each  $C \in \mathbf{Ch}_b$ . Then the homology classes  $[\Sigma(C, \gamma_C)]$ , where  $C$  runs through  $\mathbf{Ch}_b$ , and the homology classes  $[D_P]$ , where  $P$  runs through  $\mathcal{P}$ , form a basis of  $H_2(X; \mathbb{Z})$ .*

To state the second result, we need the following:

**Definition 1.4.** Let  $C$  and  $C'$  be bounded chambers such that  $C \cap C'$  consists of a single point  $P \in \mathcal{P}$ . Note that we have  $S_{C,P} = S_{C',P}$ . We say that the orientations  $\gamma_C$  and  $\gamma_{C'}$  are *coherent* if the capping hemispheres  $H_{C,\gamma_C,P}$  and  $H_{C',\gamma_{C'},P}$  are distinct. We say that a collection  $\{\gamma_C \mid C \in \mathbf{Ch}_b\}$  of orientations are *coherent* if  $\gamma_C$  and  $\gamma_{C'}$  are coherent whenever  $C \cap C'$  consists of a single point.

FIGURE 1.1. Vanishing cycle  $\Sigma(C, \gamma_C)$ 

We have the following:

**Proposition 1.5.** *A coherent collection of orientations exists.*

**Definition 1.6.** Let  $C$  be a chamber. We say that  $\ell_i(\mathbb{R}) \in \mathcal{A}$  defines an *edge*  $C \cap \ell_i(\mathbb{R})$  of  $C$  if  $C \cap \ell_i(\mathbb{R})$  contains a non-empty open subset of  $\ell_i(\mathbb{R})$ .

Our second main result is as follows.

**Theorem 1.7.** *For each bounded chamber  $C \in \mathbf{Ch}_b$ , we fix an orientation  $\gamma_C$ . Let  $C$  and  $C'$  be distinct bounded chambers.*

(1) *For  $P \in \mathcal{P}$ , we have*

$$\langle [\Sigma(C, \gamma_C)], [D_P] \rangle = \begin{cases} -1 & \text{if } P \in \text{Vert}(C), \\ 0 & \text{if } P \notin \text{Vert}(C). \end{cases}$$

(2) *If  $C$  and  $C'$  are disjoint, then  $\langle [\Sigma(C, \gamma_C)], [\Sigma(C', \gamma_{C'})] \rangle = 0$ .*

(3) *Suppose that  $C \cap C'$  consists of a single point. Then we have*

$$\langle [\Sigma(C, \gamma_C)], [\Sigma(C', \gamma_{C'})] \rangle = \begin{cases} 0 & \text{if } \gamma_C \text{ and } \gamma_{C'} \text{ are coherent,} \\ -1 & \text{otherwise.} \end{cases}$$

(4) *If  $C$  and  $C'$  share a common edge, then  $\langle [\Sigma(C, \gamma_C)], [\Sigma(C', \gamma_{C'})] \rangle = -1$ .*

(5) *The self-intersection number  $\langle [\Sigma(C, \gamma_C)], [\Sigma(C, \gamma_C)] \rangle$  is equal to  $-2$ .*

*Remark 1.8.* Using (1.3), we can easily check that, if Theorem 1.7 is valid for one choice of orientations, then Theorem 1.7 is true for any choice of orientations. Hence it is enough to prove Theorem 1.7 for a fixed collection of orientations. We will prove Theorem 1.7 for *standard orientations*, which will be defined in Section 2.1. The collection of standard orientations is in fact coherent.

Our construction of vanishing cycles over bounded chambers is similar to the construction of topological cycles by Pham [10] in hypersurfaces defined by equations of Fermat type. Pham's construction has many applications. For example, it

was applied in [3] to the study of integral Hodge conjecture for Fermat varieties. It was also applied to the calculation of Picard lattices of quartic surfaces in [6].

A totally different algorithm to calculate the intersection form on the middle homology group of an open complex surface, which we called a *Zariski-van Kampen method*, was presented in [1] and [12]. It was applied to the construction of arithmetic Zariski pairs in [1] and [11].

This paper is organized as follows. In Section 2, we fix terminology about real line arrangements. In particular, we define standard orientations  $\sigma_C$ . In Section 3, we prove Theorem 1.3. In Section 4, we calculate the capping hemispheres explicitly. In Section 5, we introduce the notion of displacements of vanishing cycles. In Sections 6, 7, 8, we prove Theorem 1.7. In Section 9, we calculate some concrete examples.

*Remark 1.9.* A referee for the first version of this paper suggests a simpler proof of Theorem 1.7 (5), which uses a normal vector field.

## 2. TERMINOLOGY

Let  $\mathcal{A} = \{\ell_1(\mathbb{R}), \dots, \ell_N(\mathbb{R})\}$  be a nodal real line arrangement.

**2.1. Standard orientations.** We define *standard orientations*  $\sigma_C$  for bounded chambers  $C$ .

**Definition 2.1.** A *defining polynomial* of the arrangement  $\mathcal{A}$  is the product

$$f := \prod_{i=1}^n \lambda_i,$$

where  $\lambda_i$  is an affine linear function  $\lambda_i: \mathbb{A}^2(\mathbb{R}) \rightarrow \mathbb{R}$  such that  $\ell_i(\mathbb{R}) = \lambda_i^{-1}(0)$ .

A defining polynomial of  $\mathcal{A}$  is unique up to real multiplicative constant. We can regard a defining polynomial  $f: \mathbb{A}^2(\mathbb{R}) \rightarrow \mathbb{R}$  of  $\mathcal{A}$  as a complex-valued polynomial function  $f: \mathbb{A}^2(\mathbb{C}) \rightarrow \mathbb{C}$ .

We fix an orientation  $\sigma_{\mathbb{A}}$  of  $\mathbb{A}^2(\mathbb{R})$  and a defining polynomial  $f$  of  $\mathcal{A}$ . Then the double covering  $\pi: W \rightarrow \mathbb{A}^2(\mathbb{C})$  in the diagram (1.2) is given by the projection to the second factor from

$$(2.1) \quad W := \{(\omega, Q) \in \mathbb{C} \times \mathbb{A}^2(\mathbb{C}) \mid \omega^2 = f(Q)\},$$

where  $\omega$  is an affine coordinate of  $\mathbb{C}$ . Let  $C$  be a bounded chamber of  $\mathcal{A}$ , and let  $C^\circ$  be the interior of  $C$  in  $\mathbb{A}^2(\mathbb{R})$ . The pull-back  $\pi^{-1}(C^\circ)$  has two connected components, (which we call *sheets*). We can consider  $\pi^{-1}(C^\circ)$  as an open subset of  $\phi^{-1}(\beta^\sharp C)$  via  $\rho: X \rightarrow W$ , where  $\phi$  and  $\rho$  are given (1.2). Let  $\gamma_C$  be an orientation of  $\phi^{-1}(\beta^\sharp C)$ . We let the two sheets be oriented by  $\gamma_C$ . Then  $\pi: W \rightarrow \mathbb{A}^2(\mathbb{C})$  restricted to one sheet is an orientation-preserving homeomorphism to  $C^\circ$  oriented by  $\sigma_{\mathbb{A}}$ , whereas  $\pi$  restricted to the other sheet is orientation-reversing.

**Definition 2.2.** The sheet on which  $\pi$  is orientation-preserving is called the *positive-sheet* with respect to  $\sigma_{\mathbb{A}}$  and  $\gamma_C$ , and the other sheet is called the *negative-sheet*.

The orientation  $\gamma_C$  of  $\phi^{-1}(\beta^\sharp C)$  is specified by indicating which sheet is positive. Note that we have either  $f(C^\circ) \subset \mathbb{R}_{>0}$  or  $f(C^\circ) \subset \mathbb{R}_{<0}$ . In the former case, the two sheets are distinguished by the sign on  $\phi^{-1}(\beta^\sharp C)$  of the function  $\omega = \pm\sqrt{f} \in \mathbb{R}$ , and in the latter case, the two are distinguished by the sign of  $\omega/\sqrt{-1} = \pm\sqrt{-f} \in \mathbb{R}$ .

**Definition 2.3.** The *standard orientation*  $\sigma_C$  for a bounded chamber  $C$  with respect to  $\sigma_{\mathbb{A}}$  and  $f$  is the orientation of  $\phi^{-1}(\beta^\sharp C)$  such that

- when  $f(C^\circ) \subset \mathbb{R}_{>0}$ , the sheet with  $\omega > 0$  is positive, and
- when  $f(C^\circ) \subset \mathbb{R}_{<0}$ , the sheet with  $\omega/\sqrt{-1} > 0$  is positive.

From now on to the end of this paper, we fix  $f$  and  $\sigma_{\mathbb{A}}$ , and use the standard orientation  $\sigma_C$ . We omit  $\sigma_C$  from the notation  $H_{C,\sigma_C,P}$ ,  $\Delta(C,\sigma_C)$ , and  $\Sigma(C,\sigma_C)$  introduced in Section 1:

$$(2.2) \quad H_{C,P} := H_{C,\sigma_C,P}, \quad \Delta(C) := \Delta(C,\sigma_C), \quad \Sigma(C) := \Sigma(C,\sigma_C).$$

In Section 4.3, we show that the collection  $\{\sigma_C \mid C \in \mathbf{Ch}_b\}$  of standard orientations is coherent in the sense of Definition 1.4. In Sections 6, 7, 8, we prove Theorem 1.7 for the standard orientations. See Remark 1.8.

**2.2. The vector space of translations.** Let  $T(\mathbb{A}^2(\mathbb{C}))$  denote the  $\mathbb{C}$ -vector space of translations of  $\mathbb{A}^2(\mathbb{C})$ , and let  $T(\mathbb{A}^2(\mathbb{R}))$  denote the  $\mathbb{R}$ -vector space of translations of  $\mathbb{A}^2(\mathbb{R})$ . For a vector  $\tau \in T(\mathbb{A}^2(\mathbb{C}))$  and  $P \in \mathbb{A}^2(\mathbb{C})$ , we denote by  $P + \tau$  the image of  $P$  by the translation  $\tau: \mathbb{A}^2(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{C})$ . For  $Q, Q' \in \mathbb{A}^2(\mathbb{C})$ , let  $\tau_{Q,Q'} \in T(\mathbb{A}^2(\mathbb{C}))$  denote the unique translation such that  $Q + \tau_{Q,Q'} = Q'$ . We have

$$T(\mathbb{A}^2(\mathbb{C})) = T(\mathbb{A}^2(\mathbb{R})) \otimes \mathbb{C} = T(\mathbb{A}^2(\mathbb{R})) \oplus \sqrt{-1}T(\mathbb{A}^2(\mathbb{R})).$$

Let  $\bar{Q}$  denote the complex conjugate of the point  $Q \in \mathbb{A}^2(\mathbb{C})$ . Then the mapping

$$Q \mapsto Q + (1/2)\tau_{Q,\bar{Q}} = (Q + \bar{Q})/2$$

that gives the real part of  $Q \in \mathbb{A}^2(\mathbb{C})$  yields a projection

$$\text{pr}_{\mathbb{R}}: \mathbb{A}^2(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{R}).$$

Then we have a natural identification

$$(2.3) \quad \text{pr}_{\mathbb{R}}^{-1}(Q) \cong \sqrt{-1}T(\mathbb{A}^2(\mathbb{R}))$$

of the fiber  $\text{pr}_{\mathbb{R}}^{-1}(Q)$  over  $Q \in \mathbb{A}^2(\mathbb{R})$  with  $\sqrt{-1}T(\mathbb{A}^2(\mathbb{R}))$  by  $Q' \mapsto \tau_{Q,Q'}$ . For a real affine line  $\lambda(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$ , we put

$$T[\lambda(\mathbb{R})] := \{\tau \in T(\mathbb{A}^2(\mathbb{R})) \mid \lambda(\mathbb{R}) + \tau = \lambda(\mathbb{R})\},$$

which is a 1-dimensional  $\mathbb{R}$ -vector subspace of  $T(\mathbb{A}^2(\mathbb{R}))$ . In the same way, we define the subspace of  $T[\lambda(\mathbb{C})] \subset T(\mathbb{A}^2(\mathbb{C}))$  for a complex affine line  $\lambda(\mathbb{C}) \subset \mathbb{A}^2(\mathbb{C})$ . If  $\lambda(\mathbb{C})$  is the complexification  $\lambda(\mathbb{R}) \otimes \mathbb{C}$  of a real affine line  $\lambda(\mathbb{R})$ , then we have

$$T[\lambda(\mathbb{R}) \otimes \mathbb{C}] = T[\lambda(\mathbb{R})] \otimes \mathbb{C} = T[\lambda(\mathbb{R})] \oplus \sqrt{-1}T[\lambda(\mathbb{R})].$$

For affine coordinates  $(x, y)$  of  $\mathbb{A}^2(\mathbb{R})$ , we define a basis  $e_x, e_y$  of  $T(\mathbb{A}^2(\mathbb{R}))$  by

$$(2.4) \quad e_x: (x, y) \mapsto (x + 1, y), \quad e_y: (x, y) \mapsto (x, y + 1).$$

Suppose that a real line  $\lambda(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$  is defined by

$$ax + by + c = 0,$$

where  $a, b, c \in \mathbb{R}$ . Then the linear subspace  $T[\lambda(\mathbb{R})]$  of  $T(\mathbb{A}^2(\mathbb{R}))$  (and hence the linear subspace  $T[\lambda(\mathbb{R}) \otimes \mathbb{C}]$  of  $T(\mathbb{A}^2(\mathbb{C}))$ ) is generated by the vector

$$be_x - ae_y.$$

**2.3. Good coordinates.** Let  $(\xi, \eta)$  be affine coordinates of  $\mathbb{A}^2(\mathbb{C})$ . We put

$$\xi = x + \sqrt{-1}u, \quad \eta = y + \sqrt{-1}v,$$

where  $x, u, y, v$  are real-valued functions on  $\mathbb{A}^2(\mathbb{C})$ . We say that  $(\xi, \eta)$  is *compatible with the  $\mathbb{R}$ -structure* if the complex conjugation  $Q \mapsto \overline{Q}$  is given by

$$(x + \sqrt{-1}u, y + \sqrt{-1}v) \mapsto (x - \sqrt{-1}u, y - \sqrt{-1}v).$$

Suppose that  $(\xi, \eta)$  is compatible with the  $\mathbb{R}$ -structure. Then  $\mathbb{A}^2(\mathbb{R})$  is equal to  $\{u = v = 0\}$ , and the restriction of  $(x, y)$  to  $\mathbb{A}^2(\mathbb{R})$  is an affine coordinate system of  $\mathbb{A}^2(\mathbb{R})$ . From now on, we regard  $(x, y)$  as affine coordinates of  $\mathbb{A}^2(\mathbb{R})$  by restriction. Then  $\text{pr}_{\mathbb{R}}: \mathbb{A}^2(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{R})$  is given by

$$(\xi, \eta) \mapsto (x, y).$$

We say that the affine coordinates  $(\xi, \eta)$  compatible with the  $\mathbb{R}$ -structure are *good coordinates* if the ordered basis  $(\partial/\partial x, \partial/\partial y)$  of tangent vectors of  $\mathbb{A}^2(\mathbb{R})$  is positive with respect to the orientation  $\sigma_{\mathbb{A}}$  of  $\mathbb{A}^2(\mathbb{R})$  fixed in Section 2.1.

### 3. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3. For a topological space  $T$ , we write  $H_2(T)$  for  $H_2(T; \mathbb{Z})$ . Recall the commutative diagram (1.2).

**Lemma 3.1.** *The homomorphism  $\rho_*: H_2(X) \rightarrow H_2(W)$  is surjective, and its kernel is a free  $\mathbb{Z}$ -module with basis  $[D_P]$ , where  $P$  runs through  $\mathcal{P} = \text{Sing } B(\mathbb{C})$ .*

*Proof.* This follows from Lemma 4.4 of [4]. □

Next we prove the following:

**Lemma 3.2.** *The homology group  $H_2(W)$  is a free  $\mathbb{Z}$ -module of which the classes  $\rho_*([\Sigma(C)])$  form a basis, where  $C$  runs through  $\mathbf{Ch}_b$ .*

Combining Lemmas 3.1 and 3.2, we will obtain a proof of Theorem 1.3.

Remark that Lemma 3.2 holds trivially when  $\mathbf{Ch}_b = \emptyset$ . Indeed, if  $\mathbf{Ch}_b$  is empty, then either all lines in  $\mathcal{A}$  are parallel, or all lines in  $\mathcal{A}$  except one line are parallel. In these cases, we have  $H_2(W) = 0$ , because there exists a deformation retraction of  $W$  onto a CW-complex of dimension 1.

From now on, we assume  $\mathbf{Ch}_b \neq \emptyset$ . We put

$$W_{\mathbb{R}} := \pi^{-1}(\mathbb{A}^2(\mathbb{R})), \quad \mathcal{C} := \bigcup_{C \in \mathbf{Ch}_b} C, \quad W_{\mathcal{C}} := \pi^{-1}(\mathcal{C}) = \bigcup_{C \in \mathbf{Ch}_b} \pi^{-1}(C).$$

Note that, for  $C \in \mathbf{Ch}_b$ , the pull-back  $\pi^{-1}(C)$  is homeomorphic to a 2-sphere, and is equal to the image  $\rho(\Sigma(C))$  of the vanishing cycle  $\Sigma(C)$  by the minimal desingularization  $\rho: X \rightarrow W$ . Hence Lemma 3.2 follows from the following two lemmas:

**Lemma 3.3.** *There exists a strong deformation retraction of  $W$  onto  $W_{\mathcal{C}}$ .*

**Lemma 3.4.** *The homology group  $H_2(W_{\mathcal{C}})$  is a free  $\mathbb{Z}$ -module with basis  $[\pi^{-1}(C)]$ , where  $C$  runs through  $\mathbf{Ch}_b$ . Here we understand that each 2-sphere  $\pi^{-1}(C)$  is equipped with an orientation.*

*Proof of Lemma 3.3.* First, we construct a strong deformation retraction of  $W$  onto  $W_{\mathbb{R}}$ . To begin with, we construct a strong deformation retraction

$$F: \mathbb{A}^2(\mathbb{C}) \times I \rightarrow \mathbb{A}^2(\mathbb{C})$$

of  $\mathbb{A}^2(\mathbb{C})$  onto  $\mathbb{A}^2(\mathbb{R})$  such that, for  $Q \in \mathbb{A}^2(\mathbb{C})$ , we have

$$(3.1) \quad F(Q, t_0) \in B(\mathbb{C}) \text{ for } t_0 \in I \implies F(Q, t) \in B(\mathbb{C}) \text{ for all } t \in [t_0, 1].$$

For  $Q_0 \in \mathbb{A}^2(\mathbb{R})$ , under the natural identification (2.3) of  $\text{pr}_{\mathbb{R}}^{-1}(Q_0)$  with  $\sqrt{-1}T(\mathbb{A}^2(\mathbb{R}))$ , we have

$$\text{pr}_{\mathbb{R}}^{-1}(Q_0) \cap B(\mathbb{C}) = \begin{cases} \emptyset & \text{if } Q_0 \notin B(\mathbb{R}), \\ \sqrt{-1}T[\ell_i(\mathbb{R})] & \text{if } Q_0 \in \ell_i(\mathbb{R}) \setminus \mathcal{P}, \\ \sqrt{-1}(T[\ell_i(\mathbb{R})] \cup T[\ell_j(\mathbb{R})]) & \text{if } \{Q_0\} = \ell_i(\mathbb{R}) \cap \ell_j(\mathbb{R}). \end{cases}$$

Hence, for  $Q \in \mathbb{A}^2(\mathbb{C})$ , the line segment in  $\text{pr}_{\mathbb{R}}^{-1}(Q_0)$  connecting  $Q$  and its real part  $Q_0 := \text{pr}_{\mathbb{R}}(Q) \in \mathbb{A}^2(\mathbb{R})$  is either disjoint from  $B(\mathbb{C})$  or entirely contained in  $B(\mathbb{C})$  or intersecting  $B(\mathbb{C})$  only at  $Q_0$ . Therefore the strong deformation retraction of  $\mathbb{A}^2(\mathbb{C})$  onto  $\mathbb{A}^2(\mathbb{R})$  defined by

$$F(Q, t) := (1-t) \cdot Q + t \cdot Q_0 = Q + t \cdot \tau_{Q, Q_0}$$

satisfies (3.1). (Recall that  $\tau_{Q, Q_0} \in T(\mathbb{A}^2(\mathbb{C}))$  is the vector  $Q_0 - Q$ .) Now, since the double covering  $\pi: W \rightarrow \mathbb{A}^2(\mathbb{C})$  is étale outside  $B(\mathbb{C})$ , and induces a homeomorphism from  $\pi^{-1}(B(\mathbb{C}))$  to  $B(\mathbb{C})$ , the property (3.1) enables us to construct a strong deformation retraction

$$\tilde{F}: W \times I \rightarrow W$$

of  $W$  onto  $W_{\mathbb{R}}$  as a lift of  $F$ . Indeed, for  $Q' \in W$ , we define  $\tilde{F}(Q', t)$  using the homotopy lifting property of  $\pi$  outside  $B(\mathbb{C})$  as long as  $F(\pi(Q'), t) \notin B(\mathbb{C})$ , while if  $F(\pi(Q'), t) \in B(\mathbb{C})$ , we define  $\tilde{F}(Q', t)$  to be the unique point of  $\pi^{-1}(B(\mathbb{C}))$  that is mapped to  $F(\pi(Q'), t)$ .

Next we construct a strong deformation retraction of  $W_{\mathbb{R}}$  onto  $W_{\mathcal{C}}$ . Since  $\pi|_{W_{\mathbb{R}}}: W_{\mathbb{R}} \rightarrow \mathbb{A}^2(\mathbb{R})$  is a local isomorphism outside of  $B(\mathbb{R})$ , and induces a homeomorphism from  $(\pi|_{W_{\mathbb{R}}})^{-1}(B(\mathbb{R}))$  to  $B(\mathbb{R})$ , the same argument as above shows that it is enough to construct a strong deformation retraction

$$G: \mathbb{A}^2(\mathbb{R}) \times I \rightarrow \mathbb{A}^2(\mathbb{R})$$

of  $\mathbb{A}^2(\mathbb{R})$  onto  $\mathcal{C}$  that satisfies

$$G(Q, t_0) \in B(\mathbb{R}) \text{ for } t_0 \in I \implies G(Q, t) \in B(\mathbb{R}) \text{ for all } t \in [t_0, 1].$$

First we make a strong deformation retraction of each unbounded chamber to its boundary, so that  $\mathbb{A}^2(\mathbb{R})$  is retracted onto the union  $\mathcal{C} \cup B(\mathbb{R})$ . Next we make a strong deformation retraction of  $\mathcal{C} \cup B(\mathbb{R})$  onto  $\mathcal{C}$ .

Thus Lemma 3.3 is proved.  $\square$

*Proof of Lemma 3.4.* We put  $m := |\mathbf{Ch}_b|$ . It is enough to show that there exists a numbering  $C_1, \dots, C_m$  of elements of  $\mathbf{Ch}_b$  such that, putting

$$\mathcal{C}_k := C_1 \cup \dots \cup C_k, \quad K_k := C_k \cap \mathcal{C}_{k-1} \subset \partial C_k,$$

we have  $K_k \neq \partial C_k$  for  $k = 2, \dots, m$ . Indeed, if such a sequence  $C_1, \dots, C_m$  exists, then we have  $H_1(\pi^{-1}(K_k)) = H_2(\pi^{-1}(K_k)) = 0$  for  $k = 2, \dots, m$ , and hence we obtain

$$H_2(\pi^{-1}(\mathcal{C}_k)) = H_2(\pi^{-1}(\mathcal{C}_{k-1})) \oplus H_2(\pi^{-1}(C_k))$$



by the Mayer–Vietoris sequence, and thus

$$H_2(W_{\mathcal{C}}) = H_2(\pi^{-1}(\mathcal{C}_m)) = H_2(\pi^{-1}(C_1)) \oplus \cdots \oplus H_2(\pi^{-1}(C_m))$$

is proved by induction on  $k$ . We construct the *reversed* sequence  $C_m, \dots, C_1$  by the following procedure:

Set  $k := m$  and  $\mathbf{C} := \mathbf{Ch}_b$   
**while**  $k > 0$  **do**  
 Let  $\mathcal{C}_k$  be the union of chambers in  $\mathbf{C}$ .  
 Note that  $\mathcal{C}_k$  is bounded.  
 We choose  $C \in \mathbf{C}$  such that an edge  $e_k$  of  $C$  is a part of  $\partial\mathcal{C}_k$ .  
 We set  $C_k := C$ , remove  $C$  from  $\mathbf{C}$ , and decrement  $k$  by 1.  
**end while**

Since  $K_k = C_k \cap \mathcal{C}_{k-1}$  is contained in  $\partial C_k$  minus the interior of the edge  $e_k$ , the obtained sequence satisfies  $K_k \neq \partial C_k$  for  $k = 2, \dots, m$ .  $\square$

#### 4. CAPPING HEMISPHERES

Recall that we have fixed a defining polynomial  $f: \mathbb{A}^2(\mathbb{R}) \rightarrow \mathbb{R}$  of  $\mathcal{A}$  and an orientation  $\sigma_{\mathbb{A}}$  of  $\mathbb{A}^2(\mathbb{R})$ , so that a standard orientation  $\sigma_C$  is defined for each  $C \in \mathbf{Ch}_b$ . We calculate the capping hemispheres  $H_{C,P} := H_{C,\sigma_C,P}$  explicitly.

**4.1. Good systems of local coordinates.** Recall from Section 2.3 that an affine coordinate system  $(\xi, \eta)$  of  $\mathbb{A}^2(\mathbb{C})$  is *good* if it is compatible with the  $\mathbb{R}$ -structure of  $\mathbb{A}^2(\mathbb{C})$  and the affine coordinates  $(x, y) := (\operatorname{Re} \xi, \operatorname{Re} \eta)|_{\mathbb{A}^2(\mathbb{R})}$  of  $\mathbb{A}^2(\mathbb{R})$  are compatible with the fixed orientation  $\sigma_{\mathbb{A}}$ . We refine this notion to the notion of *good local coordinate systems* on various (real and complex) surfaces.

Let  $P$  be a point in  $\mathcal{P} = \operatorname{Sing} B(\mathbb{C})$ . We say that good coordinates  $(\xi, \eta)$  of  $\mathbb{A}^2(\mathbb{C})$  are *good at  $P$*  if  $P$  is the origin  $(0, 0)$  and the two lines in  $\mathcal{A}$  passing through  $P$  are defined in  $\mathbb{A}^2(\mathbb{R})$  by

$$\ell_a(\mathbb{R}): y = ax \quad \text{and} \quad \ell_b(\mathbb{R}): y = bx$$

by some real numbers  $a, b$  satisfying  $a < b$ .

*Remark 4.1.* For any pair  $(a', b')$  of real numbers satisfying  $a' < b'$ , we have good coordinates  $(\xi, \eta)$  at  $P$  such that the pair  $(a, b)$  defined above is equal to the given pair  $(a', b')$ .

Next we define local coordinates  $(\tilde{\xi}, \mu)$  of  $Y(\mathbb{C})$  and local coordinates  $(\tilde{x}, m)$  of  $Y(\mathbb{R}) = \beta^{\sharp} \mathbb{A}^2(\mathbb{R})$ . Let  $(\xi, \eta)$  be affine coordinates of  $\mathbb{A}^2(\mathbb{C})$  good at  $P \in \mathcal{P}$  as above. Let  $r$  be a positive real number such that the open subset

$$U_r := \{ (\xi, \eta) \mid |\xi| < r, \quad |\eta| < r \}$$

of  $\mathbb{A}^2(\mathbb{C})$  intersects no lines of  $\mathcal{A}_{\mathbb{C}}$  other than  $\ell_a(\mathbb{C})$  and  $\ell_b(\mathbb{C})$ :

$$(4.1) \quad \text{for } \ell(\mathbb{R}) \in \mathcal{A}, \text{ if } \ell(\mathbb{C}) \cap U_r \neq \emptyset, \text{ then we have } \ell(\mathbb{R}) \in \{ \ell_a(\mathbb{R}), \ell_b(\mathbb{R}) \}.$$

Then there exists a unique coordinate system  $(\tilde{\xi}, \mu)$  on  $\beta^{-1}(U_r) \subset Y(\mathbb{C})$  such that  $\beta: Y(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{C})$  is given by

$$(4.2) \quad (\tilde{\xi}, \mu) \mapsto (\xi, \eta) = (\tilde{\xi}, \tilde{\xi}\mu).$$

In the chart  $\beta^{-1}(U_r)$  of  $(\tilde{\xi}, \mu)$ , the exceptional curve  $E_P$  is defined by  $\tilde{\xi} = 0$ , and the strict transforms  $\beta^\sharp \ell_a(\mathbb{C})$  and  $\beta^\sharp \ell_b(\mathbb{C})$  of  $\ell_a(\mathbb{C})$  and  $\ell_b(\mathbb{C})$  are defined by  $\mu = a$  and  $\mu = b$ , respectively. We put

$$\tilde{\xi} = \tilde{x} + \sqrt{-1} \tilde{u}, \quad \mu = m + \sqrt{-1} s,$$

where  $\tilde{x}$ ,  $\tilde{u}$ ,  $m$ ,  $s$  are real-valued functions on  $\beta^{-1}(U_r)$ . Then we have

$$Y(\mathbb{R}) = \{\tilde{u} = s = 0\}$$

in  $\beta^{-1}(U_r)$ , and we can regard the restrictions of  $(\tilde{x}, m)$  to  $Y(\mathbb{R})$  as local coordinates of the real surface  $Y(\mathbb{R})$ . From now on, we consider  $(\tilde{x}, m)$  as local coordinates of  $Y(\mathbb{R})$  by restriction.

**Definition 4.2.** The local coordinate systems  $(\xi, \eta)$  on  $\mathbb{A}^2(\mathbb{C})$ ,  $(\tilde{\xi}, \mu)$  on  $Y(\mathbb{C})$ ,  $(x, y)$  on  $\mathbb{A}^2(\mathbb{R})$ , and  $(\tilde{x}, m)$  on  $Y(\mathbb{R})$ , are called *good local coordinate systems at  $P$* .

**4.2. The orientation of  $Y(\mathbb{R})$ .** Let  $\beta_{\mathbb{R}}: Y(\mathbb{R}) \rightarrow \mathbb{A}^2(\mathbb{R})$  denote the restriction of  $\beta$  to  $Y(\mathbb{R})$ , which is a local isomorphism at every point of

$$Y(\mathbb{R})^\circ := \beta^{-1}(\mathbb{A}^2(\mathbb{R}) \setminus \mathcal{P}).$$

Hence the fixed orientation  $\sigma_{\mathbb{A}}$  of  $\mathbb{A}^2(\mathbb{R})$  induces an orientation on  $Y(\mathbb{R})^\circ$  via  $\beta_{\mathbb{R}}$ , which we denote by  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$ .

**Lemma 4.3.** *Let  $Q$  be a point of  $\beta^{-1}(U_r) \cap Y(\mathbb{R})^\circ$ . If  $\tilde{x} > 0$  at  $Q$ , then the ordered pair  $(\partial/\partial\tilde{x}, \partial/\partial m)$  of tangent vectors of  $Y(\mathbb{R})$  at  $Q$  is positive with respect to  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$ , whereas if  $\tilde{x} < 0$  at  $Q$ , then  $(\partial/\partial\tilde{x}, \partial/\partial m)$  is negative with respect to  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$ .*

*Proof.* The map  $\beta_{\mathbb{R}}: Y(\mathbb{R}) \rightarrow \mathbb{A}^2(\mathbb{R})$  is given by  $(\tilde{x}, m) \mapsto (x, y) = (\tilde{x}, \tilde{x}m)$ . We can calculate  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$  by this formula.  $\square$

The restriction of the function  $\mu$  to  $E_P$  is an affine coordinate of  $E_P$ . Let  $Q_\infty \in E_P$  be the point  $\mu = \infty$  of  $E_P$ . We consider the  $m$ -axis

$$(E_P \setminus \{Q_\infty\}) \cap Y(\mathbb{R}) = \{\tilde{x} = 0\}$$

in the chart  $\beta^{-1}(U_r) \cap Y(\mathbb{R})$  of  $(\tilde{x}, m)$ .

**Corollary 4.4.** *The orientation  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$  on  $Y(\mathbb{R})^\circ$  induces the downward orientation on the  $m$ -axis  $(E_P \setminus \{Q_\infty\}) \cap Y(\mathbb{R})$ , that is, the direction that  $m$  decreases is positive with respect to the orientation on the  $m$ -axis induced by  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$ .*

*Proof.* See Figure 4.1 of  $Y(\mathbb{R})$ , in which  $\partial/\partial\tilde{x}$  is rightward and  $\partial/\partial m$  is upward. The orientation  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$  on  $Y(\mathbb{R})^\circ$  is counter-clockwise  $\circlearrowleft$  in the region  $\tilde{x} > 0$ , and is clockwise  $\circlearrowright$  in the region  $\tilde{x} < 0$ . Both of them induce a downward orientation on the  $m$ -axis.  $\square$

**4.3. Capping hemispheres.** Let  $C$  be a bounded chamber such that  $P \in \text{Vert}(C)$ . We use the good local coordinate systems at  $P$  given in the previous section. We calculate the capping hemisphere  $H_{C,P} = H_{C,\sigma_C,P}$  explicitly. We have the following four cases: locally around  $P$  in  $\mathbb{A}^2(\mathbb{R})$ , the chamber  $C$  is equal to the region

$$(4.3) \quad \begin{aligned} \text{Case (1):} & \quad ax \leq y \leq bx, \\ \text{Case (2):} & \quad ax \leq y \text{ and } bx \leq y, \\ \text{Case (3):} & \quad ax \geq y \geq bx, \\ \text{Case (4):} & \quad ax \geq y \text{ and } bx \geq y. \end{aligned}$$

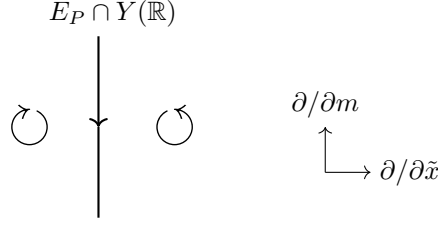


FIGURE 4.1. Orientation  $\beta_{\mathbb{R}}^* \sigma_{\mathbb{A}}$

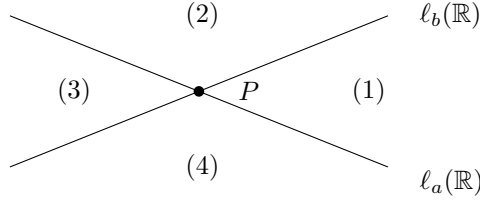


FIGURE 4.2. Location of chambers

See Figure 4.2.

*Remark 4.5.* We can choose, without loss of generality, good coordinates  $(\xi, \eta)$  of  $\mathbb{A}^2(\mathbb{C})$  at  $P$  such that Case (1) occurs.

In terms of the local coordinates  $(\tilde{x}, m)$  of  $Y(\mathbb{R})$ , the closed region  $\beta^{\sharp}C$  is given in  $Y(\mathbb{R})$  as follows:

- Case (1):  $\beta^{\sharp}C = \{ (\tilde{x}, m) \mid \tilde{x} \geq 0, a \leq m \leq b \}$ ,
- Case (2):  $\beta^{\sharp}C = \{ (\tilde{x}, m) \mid (\tilde{x} \geq 0 \text{ and } b \leq m) \text{ or } (\tilde{x} \leq 0 \text{ and } m \leq a) \}$ ,
- Case (3):  $\beta^{\sharp}C = \{ (\tilde{x}, m) \mid \tilde{x} \leq 0, a \leq m \leq b \}$ ,
- Case (4):  $\beta^{\sharp}C = \{ (\tilde{x}, m) \mid (\tilde{x} \leq 0 \text{ and } b \leq m) \text{ or } (\tilde{x} \geq 0 \text{ and } m \leq a) \}$ .

See Figure 4.3. Recall that the open subset  $U_r$  of  $\mathbb{A}^2(\mathbb{C})$  is defined in such a way that (4.1) holds. Hence the pullback  $\beta^*f$  of the defining polynomial  $f: \mathbb{A}^2(\mathbb{C}) \rightarrow \mathbb{C}$  of  $\mathcal{A}$  by  $\beta$  is written as

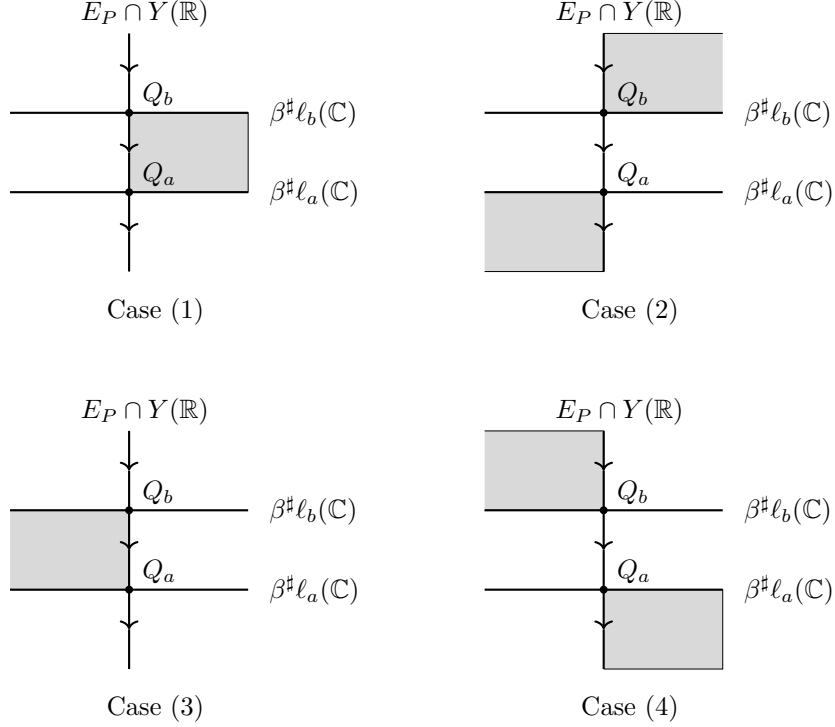
$$\beta^*f = u_P \cdot \tilde{\xi}^2 \cdot (\mu - a) \cdot (\mu - b)$$

in the chart  $\beta^{-1}(U_r)$  of  $(\tilde{\xi}, \mu)$ , where  $u_P$  is a complex-valued continuous function on  $\beta^{-1}(U_r)$  that has *no zeros*, takes values in  $\mathbb{R}$  on  $\beta^{-1}(U_r) \cap Y(\mathbb{R})$ , and is constant on  $\beta^{-1}(U_r) \cap E_P$ . We denote by

$$c_P \in \mathbb{R} \setminus \{0\}$$

the value of  $u_P$  on  $E_P$ . Note that  $\beta^{-1}(U_r)$  is simply connected. Hence we can define a complex-valued continuous function  $v_P$  on  $\beta^{-1}(U_r)$  such that  $v_P^2 = u_P$  and that

$$(4.4) \quad v_P|_{(\beta^{-1}(U_r) \cap Y(\mathbb{R}))} \text{ takes values in } \begin{cases} \mathbb{R}_{>0} & \text{if } c_P \in \mathbb{R}_{>0}, \\ \sqrt{-1} \mathbb{R}_{>0} & \text{if } c_P \in \mathbb{R}_{<0}. \end{cases}$$

FIGURE 4.3.  $\beta^\# C$  on  $Y(\mathbb{R})$ 

Recall from Section 2.1 that  $\omega$  is the function on  $W$  such that the double covering  $\pi: W \rightarrow \mathbb{A}^2(\mathbb{C})$  is given by  $\omega^2 = f$ . We regard  $\omega$  as a function on  $X$  by  $\rho: X \rightarrow W$ . We also regard  $v_P, \tilde{\xi}, \mu$  as functions on  $\phi^{-1}(\beta^{-1}(U_r)) \subset X$  by  $\phi: X \rightarrow Y(\mathbb{C})$ . We then put

$$(4.5) \quad \zeta := \frac{\omega}{v_P \cdot \tilde{\xi} \cdot (\mu - b)},$$

which is a meromorphic function on  $\phi^{-1}(\beta^{-1}(U_r)) \subset X$ . Then we have

$$(4.6) \quad \zeta^2 = \frac{\mu - a}{\mu - b},$$

which gives the double covering  $\phi: X \rightarrow Y(\mathbb{C})$ . We put

$$z := \zeta|_{D_P}.$$

Let  $Q_a$  (resp.  $Q_b$ ) be the intersection point of  $E_P$  and  $\beta^\# \ell_a(\mathbb{C})$  (resp.  $\beta^\# \ell_b(\mathbb{C})$ ). The restriction  $\phi|_{D_P}: D_P \rightarrow E_P$  of  $\phi$  to  $D_P \subset X$  is a double covering whose branch points are  $Q_a$  and  $Q_b$ . Let  $\tilde{Q}_a \in D_P$  (resp.  $\tilde{Q}_b \in D_P$ ) be the point of  $D_P$  lying over  $Q_a$  (resp.  $Q_b$ ). Then  $z$  is an affine coordinate of the Riemann sphere  $D_P$  such that  $\tilde{Q}_a$  and  $\tilde{Q}_b$  are given by  $z = 0$  and  $z = \infty$ , respectively. The main result of this section is as follows:

**Proposition 4.6.** *The capping hemisphere  $H_{C,P}$  is given as in Table 4.1.*

$C$	$f(C^\circ)$	$H_{C,P}$
Case (1)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Re} z \leq 0$
Case (1)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Re} z \geq 0$
Case (2)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Im} z \leq 0$
Case (2)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Im} z \leq 0$
Case (3)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Re} z \geq 0$
Case (3)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Re} z \leq 0$
Case (4)	$f(C^\circ) \subset \mathbb{R}_{>0}$	$\operatorname{Im} z \geq 0$
Case (4)	$f(C^\circ) \subset \mathbb{R}_{<0}$	$\operatorname{Im} z \geq 0$

TABLE 4.1. Capping hemispheres

*Proof.* To ease the notation, we put

$$P := \mathbb{R}_{\geq 0} \cup \{\infty\}, \quad iP := \sqrt{-1}\mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Recall that  $\phi^{-1}(\beta^{-1}(C^\circ)) = \rho^{-1}(\pi^{-1}(C^\circ))$  has two connected components, which we call sheets. We denote by  $(\phi^{-1}\beta^{-1}C^\circ)_+$  the sheet on which we have

$$(4.7) \quad \begin{cases} \omega \in P \text{ holds} & \text{if } f \in P \text{ on } C, \\ \omega \in iP \text{ holds} & \text{if } f \in -P \text{ on } C. \end{cases}$$

Recall Definition 2.3 of the standard orientation  $\sigma_C$ . The sheet  $(\phi^{-1}\beta^{-1}C^\circ)_+$  is the positive-sheet, that is, the restriction of  $\phi$  to  $(\phi^{-1}\beta^{-1}C^\circ)_+$  is an orientation-preserving isomorphism from  $(\phi^{-1}\beta^{-1}C^\circ)_+$  with orientation  $\sigma_C$  to the open subset  $\beta^{-1}(C^\circ)$  of  $Y(\mathbb{R})$  with orientation  $\beta_{\mathbb{R}}^*\sigma_{\mathbb{A}}$ .

By the locations of  $C$  and the signs of  $f(C)$ , we have eight cases, which are given by the rows of Table 4.1. For each case, we calculate values at points of  $(\phi^{-1}\beta^{-1}C^\circ)_+$  of the functions

$$\begin{aligned} g &:= (m-a)(m-b), \\ u_P &= \beta^*f/(\tilde{x}^2g), \\ v_P &\text{ as defined by (4.4),} \\ h &:= \tilde{x} \cdot (m-b), \\ \omega &\text{ as defined by (4.7),} \\ \zeta &= \omega/(v_P h). \end{aligned}$$

The results are given in Table 4.2. The intersection

$$K_{C,P} := D_P \cap \overline{(\phi^{-1}\beta^{-1}C^\circ)_+}$$

of  $D_P$  and the closure of  $(\phi^{-1}\beta^{-1}C^\circ)_+$  in  $X$  is an arc on  $D_P$  connecting the ramification points  $\tilde{Q}_a$  and  $\tilde{Q}_b$  of  $\phi|D_P: D_P \rightarrow E_P$ . In terms of the parameter  $z$  on  $D_P$ , the arc  $K_{C,P}$  is the closed arc connecting  $z = 0$  and  $z = \infty$  along the closure of the range of the function  $\zeta$  on  $(\phi^{-1}\beta^{-1}C^\circ)_+$ . Since we have calculated this range above, we can describe  $K_{C,P}$  in terms of  $z$ . See the column  $K_{C,P}$  of Table 4.2.

$C$	$\beta^*f$	$g$	$u_P$	$v_P$	$h$	$\omega$	$\zeta$	$K_{C,P}$	$\text{ori}_K$	$\text{ori}_C$	$\text{ori}_H$	$H_{C,P}$
(1)	+	-	-	$iP$	-	$P$	$iP$	$iP$	$\infty \rightarrow 0$	$\downarrow$	$\uparrow$	$\text{Re } z \leq 0$
(1)	-	-	+	$P$	-	$iP$	$-iP$	$-iP$	$\infty \rightarrow 0$	$\uparrow$	$\downarrow$	$\text{Re } z \geq 0$
(2)	+	+	+	$P$	+	$P$	$P$	$P$	$0 \rightarrow \infty$	$\rightarrow$	$\leftarrow$	$\text{Im } z \leq 0$
(2)	-	+	-	$iP$	+	$iP$	$P$	$P$	$0 \rightarrow \infty$	$\rightarrow$	$\leftarrow$	$\text{Im } z \leq 0$
(3)	+	-	-	$iP$	+	$P$	$-iP$	$-iP$	$\infty \rightarrow 0$	$\uparrow$	$\downarrow$	$\text{Re } z \geq 0$
(3)	-	-	+	$P$	+	$iP$	$iP$	$iP$	$\infty \rightarrow 0$	$\downarrow$	$\uparrow$	$\text{Re } z \leq 0$
(4)	+	+	+	$P$	-	$P$	$-P$	$-P$	$0 \rightarrow \infty$	$\leftarrow$	$\rightarrow$	$\text{Im } z \geq 0$
(4)	-	+	-	$iP$	-	$iP$	$-P$	$-P$	$0 \rightarrow \infty$	$\leftarrow$	$\rightarrow$	$\text{Im } z \geq 0$

TABLE 4.2. Functions on  $(\phi^{-1}\beta^{-1}C^\circ)_+$ 

Since the deck transformation of the double covering  $\phi|D_P: D_P \rightarrow E_P$  is given by  $z \mapsto -z$ , we have

$$(4.8) \quad S_{C,P} = D_P \cap \Delta(C) = K_{C,P} \cup (-K_{C,P}).$$

The covering  $\phi|D_P$  maps  $K_{C,P}$  to  $J_{C,P} = E_P \cap \beta^\sharp C$ , which is a segment of the  $m$ -axis with  $Q_\infty$  being added. The orientation  $\sigma_C$  on  $(\phi^{-1}\beta^{-1}C^\circ)_+$  induces an orientation on  $K_{C,P}$ , and hence on  $\phi(K_{C,P}) = J_{C,P}$ . Since  $\phi$  induces an orientation-preserving isomorphism from  $(\phi^{-1}\beta^{-1}C^\circ)_+$  with orientation  $\sigma_C$  to  $\beta^{-1}(C^\circ)$  with orientation  $\beta_{\mathbb{R}}^*\sigma_{\mathbb{A}}$ , Corollary 4.4 implies that this orientation on  $J_{C,P}$  is downward, that is,

$$\begin{aligned} Q_b \searrow Q_a & \text{ in Cases (1) and (3),} \\ Q_a \searrow Q_\infty \searrow Q_b & \text{ in Cases (2) and (4).} \end{aligned}$$

See Figure 4.3. In the column  $\text{ori}_K$  of Table 4.2, this orientation is expressed in terms of the parameter  $z$ . Combining the computations of  $K_{C,P}$  and  $\text{ori}_K$ , we obtain the orientation  $\text{ori}_C$  on the circle  $S_{C,P}$  induced by the orientation  $\sigma_C$  of  $\Delta(C)$  by (4.8). In Table 4.2, the circle  $S_{C,P}$  and the orientation  $\text{ori}_C$  are given as follows:

- $\downarrow$  means  $S_{C,P} = \{\text{Re } z = 0\}$  and the orientation is downward,
- $\uparrow$  means  $S_{C,P} = \{\text{Re } z = 0\}$  and the orientation is upward,
- $\rightarrow$  means  $S_{C,P} = \{\text{Im } z = 0\}$  and the orientation is rightward,
- $\leftarrow$  means  $S_{C,P} = \{\text{Im } z = 0\}$  and the orientation is leftward.

The orientation  $\text{ori}_H$  on  $S_{C,P}$  given by the complex structure of the capping hemisphere  $H_{C,P}$  is the opposite of  $\text{ori}_C$ . Thus we obtain  $H_{C,P}$  as in the last column of Table 4.2.  $\square$

**Corollary 4.7.** *The collection  $\{\sigma_C \mid C \in \mathbf{Ch}_b\}$  of standard orientations is coherent in the sense of Definition 1.4.*

*Proof.* Suppose that  $C \cap C'$  consists of a single point  $P$ . Then the signs of  $f|C$  and of  $f|C'$  are the same. Looking at Cases (1) and (3) (or Cases (2) and (4)) of Table 4.1, we see that  $H_{C,P}$  and  $H_{C',P}$  are distinct.  $\square$

Now Proposition 1.5 follows immediately from Corollary 4.7.  $\square$

5. DISPLACEMENT

We construct two types of displacements of vanishing cycles  $\Sigma(C)$  in  $X$ . One is called *C-displacements*, and the other is called *E-displacements*.

**5.1. Notation and terminology about displacements.** For a small positive real number  $\varepsilon$ , let  $I_\varepsilon$  denote the closed interval  $[0, \varepsilon] \subset \mathbb{R}$ .

**Definition 5.1.** A *displacement* of a subspace  $S$  of a topological space  $T$  is a continuous map

$$d: S \times I_\varepsilon \rightarrow T$$

such that

- $d(P, 0) = P$  for any  $P \in S$ , and
- the restriction  $d_t := d|_{(S \times \{t\})}$  of  $d$  to  $S \times \{t\}$  is a homeomorphism from  $S$  to its image  $S_t := d(S \times \{t\})$  for any  $t \in I_\varepsilon$ .

We sometimes write

$$\{S_t \mid t \in I_\varepsilon\} \quad \text{or} \quad \{d_t \mid t \in I_\varepsilon\} \quad \text{or} \quad d_t: S \rightarrow T \quad (t \in I_\varepsilon)$$

to denote the displacement  $d: S \times I_\varepsilon \rightarrow T$ . By further abuse of notation, we often say that  $S_\varepsilon$  is a displacement of  $S$  in  $T$ .

If  $S = S_0$  is equipped with an orientation  $\sigma_0$ , then a displacement  $S_\varepsilon$  of  $S$  is also oriented by  $\sigma_0$  via the homeomorphism  $d_\varepsilon: S \rightarrow S_\varepsilon$ . If  $S$  is a topological cycle, then  $S_\varepsilon$  is also a topological cycle, and their classes in the homology group of  $T$  are the same.

We say that a displacement  $d_t: S \rightarrow T$  *preserves a subspace  $R$  of  $T$*  if  $d_t(Q) \in R$  holds for any  $Q \in S \cap R$  and any  $t \in I_\varepsilon$ . We say that  $d$  is *stationary* on a subspace  $S' \subset S$  if  $d(Q, t) = Q$  holds for any  $(Q, t) \in S' \times I_\varepsilon$ . A *trivial displacement* is a displacement that is stationary on the whole space  $S$ .

**5.2. C-displacement.** Recall from Section 2.2 that  $T(\mathbb{A}^2(\mathbb{R}))$  is the  $\mathbb{R}$ -vector space of translations of  $\mathbb{A}^2(\mathbb{R})$ , and that  $T[\lambda(\mathbb{R})]$  is the subspace of translations preserving a real affine line  $\lambda(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R})$ . Let  $C$  be a bounded chamber, and let

$$\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$$

be a continuous function.

**Definition 5.2.** Suppose that  $\ell_i(\mathbb{R}) \in \mathcal{A}$  defines an edge  $C \cap \ell_i(\mathbb{R})$  of  $C$ . We say that  $\delta$  satisfies the *e-condition* for the edge  $C \cap \ell_i(\mathbb{R})$  if  $\delta(Q) \in T[\ell_i(\mathbb{R})]$  holds for any  $Q \in C \cap \ell_i(\mathbb{R})$ .

*Remark 5.3.* If  $\delta$  satisfies the *e-condition* for every edge of  $C$ , then we have  $\delta(P) = 0$  for every  $P \in \text{Vert}(C)$ , because  $T[\ell_i(\mathbb{R})] \cap T[\ell_j(\mathbb{R})] = \{0\}$  holds if  $\ell_i(\mathbb{R}) \cap \ell_j(\mathbb{R}) = \{P\}$ .

Suppose that  $\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$  satisfies the *e-condition* for every edge of  $C$ . We define  $d: C \times I_\varepsilon \rightarrow \mathbb{A}^2(\mathbb{C})$  by

$$(5.1) \quad d(Q, t) := Q + \sqrt{-1} \delta(Q)t.$$

Note that we have

$$(5.2) \quad \text{pr}_{\mathbb{R}}(d(Q, t)) = Q \quad \text{for any } (Q, t) \in C \times I_\varepsilon,$$

where  $\text{pr}_{\mathbb{R}}: \mathbb{A}^2(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{R})$  is the projection that takes the real part of points of  $\mathbb{A}^2(\mathbb{C})$  (see Section 2.2). This implies that  $d_t := d|(C \times \{t\})$  is a homeomorphism from  $C$  to

$$C_t := d_t(C)$$

for any  $t \in I_\varepsilon$ , and hence  $d$  is a displacement of  $C$  in  $\mathbb{A}^2(\mathbb{C})$ . By the  $e$ -condition, the displacement  $d$  preserves  $\ell_i(\mathbb{C})$  for each  $\ell_i(\mathbb{R}) \in \mathcal{A}$  defining edges of  $C$ . Since  $\mathcal{P} \subset \mathbb{A}^2(\mathbb{R})$  and  $\text{pr}_{\mathbb{R}}(B(\mathbb{C})) = B(\mathbb{R})$ , we see from (5.2) that

$$(5.3) \quad Q \in C^\bullet = C \setminus \text{Vert}(C) \implies d(Q, t) \notin \mathcal{P},$$

$$(5.4) \quad Q \in C^\circ = C \setminus (C \cap B(\mathbb{R})) \implies d(Q, t) \notin B(\mathbb{C}).$$

Note that  $\beta: Y(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{C})$  induces an isomorphism from  $\beta^{-1}(C^\bullet)$  to  $C^\bullet$ . By (5.3), there exists a unique continuous map

$$d^\bullet: \beta^{-1}(C^\bullet) \times I_\varepsilon \rightarrow Y(\mathbb{C})$$

that fits in the commutative diagram

$$(5.5) \quad \begin{array}{ccc} \beta^{-1}(C^\bullet) \times I_\varepsilon & \xrightarrow{d^\bullet} & Y(\mathbb{C}) \\ \beta \times \text{id} \downarrow & & \downarrow \beta \\ C \times I_\varepsilon & \xrightarrow{d} & \mathbb{A}^2(\mathbb{C}). \end{array}$$

It is obvious that  $d^\bullet$  is a displacement of  $\beta^{-1}(C^\bullet)$  in  $Y(\mathbb{C})$ . We define a map

$$d^\sharp: \beta^\sharp C \times I_\varepsilon \rightarrow Y(\mathbb{C})$$

by the following:

$$d^\sharp(Q, t) := \begin{cases} d^\bullet(Q, t) & \text{if } Q \in \beta^{-1}(C^\bullet), \\ Q & \text{if } Q \in \beta^\sharp C \setminus \beta^{-1}(C^\bullet) = \bigcup_{P \in \text{Vert}(C)} J_{C,P}. \end{cases}$$

**Definition 5.4.** We say that  $\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$  is *regular at*  $P \in \text{Vert}(C)$  if  $d^\sharp$  is continuous at  $(Q, t)$  for any  $Q \in J_{C,P}$  and any  $t \in I_\varepsilon$ . We simply say that  $\delta$  is *regular* if  $\delta$  is regular at every vertex  $P$  of  $C$ , that is, if  $d^\sharp$  is continuous.

If  $\delta$  is regular, then  $d^\sharp$  is a displacement of  $\beta^\sharp C$  in  $Y(\mathbb{C})$  that is stationary on each  $J_{C,P}$ , and that is preserving all  $\beta^\sharp \ell_i(\mathbb{C}) \subset \beta^\sharp Y(\mathbb{C})$ , where  $\ell_i(\mathbb{R}) \in \mathcal{A}$  defines an edge of  $C$ . In particular, the displacement  $d^\sharp$  preserves the intersection  $\beta^\sharp C \cap \beta^\sharp B(\mathbb{C})$ . Since the branch locus of  $\phi: X \rightarrow Y(\mathbb{C})$  is  $\beta^\sharp B(\mathbb{C})$ , it follows that  $d^\sharp$  lifts to a displacement

$$d^\Delta: \Delta(C) \times I_\varepsilon \rightarrow X$$

of  $\Delta(C)$  in  $X$  that is stationary on each  $S_{C,P} \subset \partial \Delta(C)$ . We have

$$\Delta(C)_t = \phi^{-1}((\beta^\sharp C)_t)$$

with the orientation given by the standard orientation  $\sigma_C$  of  $\phi^{-1}(\beta^\sharp C)$ . Gluing  $d^\Delta$  with the trivial displacements of the capping hemispheres  $H_{C,P}$ , we obtain a displacement

$$d^\Sigma: \Sigma(C) \times I_\varepsilon \rightarrow X$$

of  $\Sigma(C)$  in  $X$  that is stationary on each  $H_{C,P}$ .

**Definition 5.5.** These displacements  $d$ ,  $d^\sharp$ ,  $d^\Delta$ , and  $d^\Sigma$  are called the  $C$ -displacements associated with a regular continuous map  $\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$  satisfying the  $e$ -condition for every edge of  $C$ .



We introduce an operation of  $q_\rho$ -modification, which is useful in obtaining a regular continuous map  $\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$ . We use the good local coordinate systems  $(\xi, \eta)$  on  $\mathbb{A}^2(\mathbb{C})$ ,  $(\tilde{\xi}, \mu)$  on  $Y(\mathbb{C})$ ,  $(x, y)$  on  $\mathbb{A}^2(\mathbb{R})$ , and  $(\tilde{x}, m)$  on  $Y(\mathbb{R})$ , that are defined in Section 4.1. We can assume, without loss of generality, that we are in Case (1) in (4.3), so that there exist real numbers  $a, b$  with  $a < b$  such that  $C$  is given by  $ax \leq y \leq bx$  locally around  $P = (0, 0)$  in  $\mathbb{A}^2(\mathbb{R})$ . We define real-valued functions  $\delta_x(x, y)$  and  $\delta_y(x, y)$  defined in a small neighborhood of the origin  $P$  in  $\{(x, y) \mid ax \leq y \leq bx\}$  by

$$(5.6) \quad \delta(Q) = \delta_x(x(Q), y(Q)) \cdot e_x + \delta_y(x(Q), y(Q)) \cdot e_y,$$

where  $(x(Q), y(Q))$  are the coordinates of  $Q \in C$ , and  $e_x, e_y$  are the basis of  $T(\mathbb{A}^2(\mathbb{R}))$  given by (2.4). Because  $\beta$  restricted to  $Y(\mathbb{R})$  is written as

$$(\tilde{x}, m) \mapsto (x, y) = (\tilde{x}, m\tilde{x}),$$

the displacement  $d^\bullet: \beta^{-1}(C^\bullet) \times I_\varepsilon \rightarrow Y(\mathbb{C})$  defined by the diagram (5.5) is written in terms of the coordinate systems  $(\tilde{x}, m)$  and  $(\tilde{\xi}, \mu)$  as

$$(5.7) \quad ((\tilde{x}, m), t) \mapsto (\tilde{\xi}, \mu) = \left( \tilde{x} + \sqrt{-1} \delta_x(\tilde{x}, m\tilde{x}) \cdot t, \frac{m\tilde{x} + \sqrt{-1} \delta_y(\tilde{x}, m\tilde{x}) \cdot t}{\tilde{x} + \sqrt{-1} \delta_x(\tilde{x}, m\tilde{x}) \cdot t} \right).$$

We introduce an inner-product on the  $\mathbb{R}$ -vector space  $T(\mathbb{A}^2(\mathbb{R}))$ . Then we have a distance on  $\mathbb{A}^2(\mathbb{R})$ . For points  $Q, Q' \in \mathbb{A}^2(\mathbb{R})$ , we denote by  $|QQ'|$  the distance between  $Q$  and  $Q'$ . Let  $\rho$  be a small positive real number such that  $|PP'| > 2\rho$  holds for any pair of distinct vertexes  $P, P'$  of  $C$ . For each vertex  $P \in \text{Vert}(C)$ , we put

$$(5.8) \quad U_{P,\rho} := \{Q \in C \mid |PQ| \leq \rho\}.$$

Then each point of  $C$  belongs to at most one of  $U_{P,\rho}$ . For  $Q \in U_{P,\rho}$ , let  $q_{P,\rho}(Q)$  denote the unique point on the line segment  $PQ$  such that the distance  $|Pq_{P,\rho}(Q)|$  between  $P$  and  $q_{P,\rho}(Q)$  satisfies

$$|Pq_{P,\rho}(Q)| = |PQ|^2/\rho.$$

Since  $u \mapsto u^2/\rho$  is a bijection from the closed interval  $[0, \rho] \subset \mathbb{R}$  to itself, we see that  $q_{P,\rho}$  is a self-homeomorphism of  $U_{P,\rho} \subset C$ . We construct  $q_\rho: C \rightarrow C$  by

$$q_\rho(Q) := \begin{cases} q_{P,\rho}(Q) & \text{if } Q \in U_{P,\rho} \text{ for some } P \in \text{Vert}(C), \\ Q & \text{otherwise.} \end{cases}$$

Since  $q_{P,\rho}(Q) = Q$  for  $Q \in U_{P,\rho}$  with  $|PQ| = \rho$ , we see that  $q_\rho$  is continuous on the whole  $C$ , and hence is a self-homeomorphism of  $C$ .

**Definition 5.6.** Let  $\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$  be a continuous map. We call the composite  $\delta \circ q_\rho: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$  of  $q_\rho$  and  $\delta$  the  $q_\rho$ -modification of  $\delta$ .

Note that  $q_\rho: C \rightarrow C$  preserves each edge of  $C$ . Hence, if  $\delta$  satisfies the  $e$ -condition for every edge of  $C$ , then so does its  $q_\rho$ -modification  $\delta \circ q_\rho$ .

**Lemma 5.7.** Let  $\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$  be a continuous map that satisfies the  $e$ -condition for every edge of  $C$ . Suppose that  $\delta$  is affine-linear in a small neighborhood of each vertex  $P$  of  $C$ . Then the  $q_\rho$ -modification  $\delta \circ q_\rho$  is regular.

*Proof.* We define continuous functions  $\delta_x \circ q_\rho(x, y)$  and  $\delta_y \circ q_\rho(x, y)$  on  $U_{P, \rho}$  by

$$\delta \circ q_\rho(Q) = (\delta_x \circ q_\rho)(x(Q), y(Q)) \cdot e_x + (\delta_y \circ q_\rho)(x(Q), y(Q)) \cdot e_y.$$

Note that the function  $m$  in the coordinates  $(\tilde{x}, m)$  of  $Y(\mathbb{R})$  is bounded in a neighborhood of  $J_{C, P}$  in  $\beta^\sharp C$ . By the assumption that  $\delta$  is affine-linear locally around  $P$ , it follows that

$$|\delta_x(\tilde{x}, m\tilde{x})|/|\tilde{x}| \quad \text{and} \quad |\delta_y(\tilde{x}, m\tilde{x})|/|\tilde{x}|$$

are bounded in a neighborhood of  $J_{C, P}$  in  $\beta^\sharp C$ . On the other hand, since  $|u^2/\rho|$  tends to 0 faster than  $|u|$  does as  $u \rightarrow 0$ , we have

$$|(\delta_x \circ q_\rho)(\tilde{x}, m\tilde{x})|/|\tilde{x}| \rightarrow 0, \quad |(\delta_y \circ q_\rho)(\tilde{x}, m\tilde{x})|/|\tilde{x}| \rightarrow 0,$$

as  $\tilde{x} \rightarrow 0$ . Hence we have

$$\frac{m\tilde{x} + \sqrt{-1}(\delta_y \circ q_\rho)(\tilde{x}, m\tilde{x}) \cdot t}{\tilde{x} + \sqrt{-1}(\delta_x \circ q_\rho)(\tilde{x}, m\tilde{x}) \cdot t} \rightarrow m$$

as  $\tilde{x} \rightarrow 0$ . Therefore by (5.7) with  $\delta_x$  and  $\delta_y$  replaced by  $\delta_x \circ q_\rho$  and  $\delta_y \circ q_\rho$  respectively, we see that, if  $\delta$  is  $q_\rho$ -modified, then  $d^\bullet(Q, t)$  tends to  $Q$  as  $Q \in \beta^{-1}(C^\bullet)$  approaches a point of  $J_{C, P} \subset \beta^\sharp C$ , and hence  $\delta \circ q_\rho$  is regular at  $P$ .  $\square$

**5.3.  $E$ -displacement.** Next we define  $E$ -displacements of various subspaces in  $Y(\mathbb{C})$  and  $X$ . Let  $C$  be a bounded chamber, and  $P$  a vertex of  $C$ .

**Definition 5.8.** For a point  $Q \in E_P$ , we denote by  $\lambda_Q(\mathbb{C})$  the complex affine line in  $\mathbb{A}^2(\mathbb{C})$  passing through  $P$  such that its strict transform  $\beta^\sharp \lambda_Q(\mathbb{C})$  in  $Y(\mathbb{C})$  intersects  $E_P$  at  $Q$ . A *tubular neighborhood* of  $E_P$  is an open neighborhood  $\mathcal{N}_P \subset Y(\mathbb{C})$  of  $E_P$  in  $Y(\mathbb{C})$  equipped with a continuous map

$$p_{\mathcal{N}}: \mathcal{N}_P \rightarrow E_P$$

such that, for any  $Q \in E_P$ , the fiber  $p_{\mathcal{N}}^{-1}(Q)$  is an open disk of  $\beta^\sharp \lambda_Q(\mathbb{C})$  with center  $Q$ . We call  $p_{\mathcal{N}}$  the *projection* of the tubular neighborhood  $\mathcal{N}_P$ .

Let  $p_{\mathcal{N}}: \mathcal{N}_P \rightarrow E_P$  be a tubular neighborhood of  $E_P$ . We consider the inclusion  $E_P \hookrightarrow \mathcal{N}_P$  as a section of  $p_{\mathcal{N}}$ , which we call the *zero section*. For  $Q' \in p_{\mathcal{N}}^{-1}(Q)$  and  $t \in I := [0, 1]$ , let  $tQ'$  denote the unique point on  $p_{\mathcal{N}}^{-1}(Q) \subset \beta^\sharp \lambda_Q(\mathbb{C})$  such that

$$\beta(tQ') = P + t \cdot \tau_{P, \beta(Q')},$$

where  $\tau_{P, \beta(Q')} \in T(\mathbb{A}^2(\mathbb{C}))$  is the translation that maps  $P$  to  $\beta(Q')$ .

We describe a tubular neighborhood in terms of local coordinates. Let  $(\xi, \eta)$ ,  $(\tilde{\xi}, \mu)$ ,  $(x, y)$ ,  $(\tilde{x}, m)$  be good local coordinate systems of  $\mathbb{A}^2(\mathbb{C})$ ,  $Y(\mathbb{C})$ ,  $\mathbb{A}^2(\mathbb{R})$ ,  $Y(\mathbb{R})$ , respectively, given in Section 4.1. We can assume, without loss of generality, that the location of  $C$  is Case (1) in (4.3), so that there exist real numbers  $a, b$  with  $a < b$  such that  $\beta^\sharp C$  is defined by  $\tilde{x} \geq 0$  and  $a \leq m \leq b$  in the chart of  $(\tilde{x}, m)$  on  $Y(\mathbb{R})$ . By definition, in terms of the coordinates  $(\tilde{\xi}, \mu)$  of  $Y(\mathbb{C})$ , the projection  $p_{\mathcal{N}}$  is given by

$$\mathcal{N}_P \ni (\tilde{\xi}, \mu) \mapsto (0, \mu) \in E_P,$$

and the map  $Q' \mapsto tQ'$  is given by

$$(\tilde{\xi}, \mu) \mapsto (t\tilde{\xi}, \mu).$$

**Definition 5.9.** A continuous section

$$s: E_P \rightarrow \mathcal{N}_P$$

of  $p_{\mathcal{N}}$  is said to be *admissible with respect to  $C$*  if it satisfies the following.

- (s1) There exists one and only one point  $Q_0 \in E_P$  such that  $s(Q_0) \in E_P$ .
- (s2) The point  $Q_0 \in E_P$  is not on the arc  $J_{C,P} = E_P \cap \beta^{\sharp}C$ .
- (s3) If  $Q \in J_{C,P}$ , then  $s(Q) \in \beta^{\sharp}C$ .

Suppose that  $s: E_P \rightarrow \mathcal{N}_P$  is a section admissible with respect to  $C$ . We can write  $s$  as

$$\mu \mapsto (\tilde{\xi}, \mu) = (\sigma(\mu), \mu)$$

in a neighborhood of  $J_{C,P}$  in  $E_P$ , where  $\sigma$  is a continuous function such that, if  $m$  is a real number in the closed interval  $[a, b]$ , then  $\sigma(m) \in \mathbb{R}_{>0}$ .

**Example 5.10.** Let  $c$  be a sufficiently small positive real number. Then we have a section  $s: E_P \rightarrow \mathcal{N}_P$  admissible with respect to  $C$  such that the function  $\sigma(\mu)$  is constantly equal to  $c$  in a small neighborhood of  $J_{C,P}$  in  $E_P$ .

Note that the image  $s(E_P)$  of  $s$  and  $E_P$  intersect only at  $Q_0$ , and, since  $s(E_P)$  and  $E_P$  are homologous in  $Y(\mathbb{C})$ , the local intersection number at  $Q_0$  is equal to the self-intersection number  $-1$  of  $E_P$ .

For  $t \in I$ , we denote by

$$ts: E_P \rightarrow \mathcal{N}_P \subset Y(\mathbb{C})$$

the continuous map  $Q \mapsto t \cdot s(Q)$ . Let  $\varepsilon'$  be a sufficiently small positive real number. For  $t \in I_{\varepsilon'}$ , let  $(E_P)_t \subset \mathcal{N}_P$  be the image of  $E_P$  by  $ts$ , and let  $(J_{C,P})_t$  be the image of  $J_{C,P}$  by  $ts$ . Note that we have

$$(J_{C,P})_t = (E_P)_t \cap \beta^{\sharp}C.$$

Let  $Q_a$  (resp.  $Q_b$ ) be the intersection point of  $E_P$  and  $\beta^{\sharp}l_a(\mathbb{C})$  (resp.  $\beta^{\sharp}l_b(\mathbb{C})$ ). Then  $(J_{C,P})_t$  is an arc on  $(E_P)_t$  connecting  $ts(Q_a)$  and  $ts(Q_b)$ . We put

$$(D_P)_t := \phi^{-1}((E_P)_t).$$

Then  $\phi|(D_P)_t: (D_P)_t \rightarrow (E_P)_t$  is a double covering whose branch points are the end points  $ts(Q_a)$  and  $ts(Q_b)$  of the arc  $(J_{C,P})_t$ . Therefore

$$(S_{C,P})_t := \phi^{-1}((J_{C,P})_t)$$

is a circle on  $(D_P)_t$ . Note that  $(S_{C,P})_t$  decomposes  $(D_P)_t$  into the union of two closed hemispheres. Let  $(H_{C,P})_t$  denote the hemisphere obtained from  $(H_{C,P})_0 = H_{C,P}$  by continuity. Thus, from the section  $s: E_P \rightarrow \mathcal{N}_P$  admissible with respect to  $C$ , we obtain displacements

$$\begin{aligned} & \{ (J_{C,P})_t \mid t \in I_{\varepsilon'} \} \text{ of } J_{C,P} \text{ in } \beta^{\sharp}C, \\ & \{ (E_P)_t \mid t \in I_{\varepsilon'} \} \text{ of } E_P \text{ in } Y(\mathbb{C}), \\ & \{ (D_P)_t \mid t \in I_{\varepsilon'} \} \text{ of } D_P \text{ in } X, \\ & \{ (S_{C,P})_t \mid t \in I_{\varepsilon'} \} \text{ of } S_{C,P} \text{ in } \phi^{-1}(\beta^{\sharp}C), \text{ and} \\ & \{ (H_{C,P})_t \mid t \in I_{\varepsilon'} \} \text{ of } H_{C,P} \text{ in } X, \end{aligned}$$

which we call *E-displacements* associated with  $s: E_P \rightarrow \mathcal{N}_P$ .

For positive real numbers  $\alpha, \beta$  with  $\alpha < \beta$ , let  $g_{\alpha, \beta}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  denote the continuous function defined by

$$g_{\alpha, \beta}(u) := \begin{cases} \alpha + (\beta - \alpha)u/\beta & \text{if } 0 \leq u \leq \beta, \\ u & \text{if } u \geq \beta. \end{cases}$$

Then  $g_{\alpha, \beta}$  is a homeomorphism from  $\mathbb{R}_{\geq 0}$  to  $\mathbb{R}_{\geq \alpha}$  that is the identity on  $\mathbb{R}_{\geq \beta}$ . We choose  $\varepsilon'$  so small that the point  $(\tilde{x}, m) = (2\varepsilon'\sigma(m), m)$  is in the chart of the coordinate system  $(\tilde{\xi}, \mu)$  for all  $m \in [a, b]$ . Let  $U$  denote the chart. We then define a displacement

$$(5.9) \quad \beta^\sharp C \times I_{\varepsilon'} \rightarrow \beta^\sharp C$$

of  $\beta^\sharp C$  in  $\beta^\sharp C$  by

$$(Q', t) \mapsto \begin{cases} (g_{t\sigma(m), 2t\sigma(m)}(\tilde{x}), m) & \text{if } Q' = (\tilde{x}, m) \in U \cap \beta^\sharp C, \\ Q' & \text{otherwise.} \end{cases}$$

Note that this map is continuous because we have  $g_{t\sigma(m), 2t\sigma(m)}(\tilde{x}) = \tilde{x}$  for  $\tilde{x} \geq 2t\sigma(m)$ . Let  $(\beta^\sharp C)_t$  be the closed subset of  $\beta^\sharp C$  obtained by removing

$$\{(\tilde{x}, m) \mid a \leq m \leq b, \ 0 \leq \tilde{x} < t\sigma(m)\}$$

from  $\beta^\sharp C$ . Then the displacement (5.9) is a family of maps that shrink  $\beta^\sharp C$  to  $(\beta^\sharp C)_t \subset \beta^\sharp C$  homeomorphically. Note that the displacement (5.9) preserves the subspaces  $\beta^\sharp C \cap \beta^\sharp \ell_a(\mathbb{C})$  and  $\beta^\sharp C \cap \beta^\sharp \ell_b(\mathbb{C})$ . Putting

$$\Delta(C)_t := \phi^{-1}((\beta^\sharp C)_t)$$

with the orientation  $\sigma_C$ , we obtain a displacement  $\{\Delta(C)_t \mid t \in I_{\varepsilon'}\}$  of  $\Delta(C)$  in  $X$ . We have

$$\partial \Delta(C)_t = (S_{C, P})_t \sqcup \bigsqcup_{P' \in \text{Vert}(C) \setminus \{P\}} S_{C, P'}.$$

Therefore we have a displacement

$$\Sigma(C)_t := \Delta(C)_t \cup \left( (H_{C, P})_t \sqcup \bigsqcup_{P' \in \text{Vert}(C) \setminus \{P\}} H_{C, P'} \right)$$

of the topological 2-cycle  $\Sigma(C)_0 = \Sigma(C)$  in  $X$ . These displacements

$$(5.10) \quad \begin{aligned} & \{(\beta^\sharp C)_t \mid t \in I_{\varepsilon'}\} && \text{of } \beta^\sharp C \text{ in } \beta^\sharp C, \\ & \{\Delta(C)_t \mid t \in I_{\varepsilon'}\} && \text{of } \Delta(C) \text{ in } \Delta(C), \text{ and} \\ & \{\Sigma(C)_t \mid t \in I_{\varepsilon'}\} && \text{of } \Sigma(C) \text{ in } X \end{aligned}$$

are also called the *E-displacements* associated with the section  $s: E_P \rightarrow \mathcal{N}_P$ .

We can easily extend the definition of *E-displacements* to the case where we are given sections at several vertexes. Note that the displacements in (5.10) are stationary outside small neighborhoods of  $J_{C, P}$  in  $\beta^\sharp C$ , of  $S_{C, P}$  in  $\Delta(C)$ , and of  $H_{C, P}$  in  $\Sigma(C)$ , respectively. By choosing sufficiently small  $\varepsilon'$ , we can make these neighborhoods arbitrarily small. Therefore, if we are given sections

$$s_i: E_{P_i} \rightarrow \mathcal{N}_{P_i} \quad (i = 1, \dots, k)$$

admissible with respect to  $C$  for distinct vertexes  $P_1, \dots, P_k$  of  $C$ , we can glue the displacements associated with these  $s_i$  together. For example, let  $\mathcal{U}(P_i)$  be a

neighborhood of  $H_{C,P}$  in  $\Sigma(C)$  such that the  $E$ -displacement  $d_{s_i}: \Sigma(C) \times I_{\varepsilon'} \rightarrow X$  associated with  $s_i$  is stationary outside of  $\mathcal{U}(P_i)$ . Then we can define a displacement

$$d := d_{s_1, \dots, s_k}: \Sigma(C) \times I_{\varepsilon'} \rightarrow X$$

by the following:

$$d(Q, t) := \begin{cases} d_{s_i}(Q, t) & \text{if } Q \in \mathcal{U}(P_i) \text{ for some } i, \\ Q & \text{if } Q \notin \mathcal{U}(P_i) \text{ for any } i. \end{cases}$$

**Definition 5.11.** These displacements of  $\beta^\sharp C$  in  $\beta^\sharp C$ , of  $\Delta(C)$  in  $\Delta(C)$ , and of  $\Sigma(C)$  in  $X$ , are called the  $E$ -displacements associated with the continuous sections  $s_i: E_{P_i} \rightarrow \mathcal{N}_{P_i}$ .

By construction, the  $E$ -displacements associated with these sections  $s_i: E_{P_i} \rightarrow \mathcal{N}_{P_i}$  ( $i = 1, \dots, k$ ) satisfy the following:

$$(5.11) \quad (\beta^\sharp C)_{\varepsilon'} \cap E_{P_i} = \emptyset \text{ and hence } \Delta(C)_{\varepsilon'} \cap D_{P_i} = \emptyset \text{ for } i = 1, \dots, k.$$

## 6. PROOF OF THEOREM 1.7 (1), (2), (3)

We prove assertions (1), (2), (3) of Theorem 1.7 for the standard orientations  $\sigma_C$  fixed in Section 2.1.

**6.1. Proof of assertion (1).** The case where  $P \notin \text{Vert}(C)$  is obvious. Suppose that  $P \in \text{Vert}(C)$ . As in Section 5.3, we choose a section  $s: E_P \rightarrow \mathcal{N}_P$  of a tubular neighborhood  $\mathcal{N}_P \rightarrow E_P$  admissible with respect to  $C$ , and construct the  $E$ -displacement  $\{\Sigma(C)_t \mid t \in I_{\varepsilon'}\}$  associated with  $s$ . Note that  $\Delta(C)_{\varepsilon'}$  is disjoint from  $D_P$ . (See (5.11).) Recall that  $(E_P)_{\varepsilon'}$  and  $E_P$  intersect only at one point  $Q_0$ , and the local intersection number at  $Q_0$  is  $-1$ . Since  $Q_0 \notin J_{C,P}$ , we see that  $(D_P)_{\varepsilon'}$  and  $D_P$  intersect only at the two points in  $\phi^{-1}(Q_0)$  and that the local intersection number at each point is  $-1$ . Assuming  $\varepsilon'$  to be small enough, we have that one of the two points in  $\phi^{-1}(Q_0)$  is on  $(H_{C,P})_{\varepsilon'}$ , whereas the other does not belong to  $(H_{C,P})_{\varepsilon'}$ . Hence  $\Sigma(C)_{\varepsilon'}$  and  $D_P$  intersect only at one point, at which the local intersection number is  $-1$ . Thus assertion (1) is proved.  $\square$

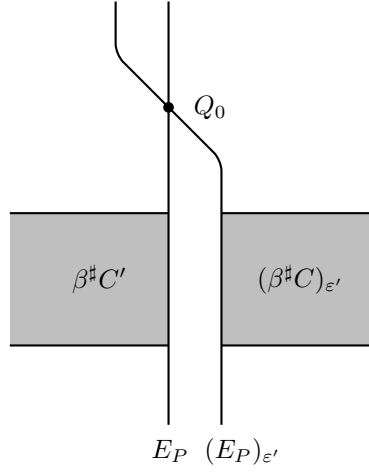
**6.2. Proof of assertion (2).** Obvious.  $\square$

**6.3. Proof of assertion (3).** Suppose that  $C \cap C'$  consists of a single point  $P$ . Let  $(\xi, \eta)$ ,  $(\tilde{\xi}, \mu)$ ,  $(x, y)$ ,  $(\tilde{x}, m)$  be good local coordinates on  $\mathbb{A}^2(\mathbb{C})$ ,  $Y(\mathbb{C})$ ,  $\mathbb{A}^2(\mathbb{R})$ ,  $Y(\mathbb{R})$ , respectively, given in Section 4.1. Without loss of generality, we can assume that the location of  $C$  is Case (1) and the location of  $C'$  is Case (3) in (4.3). Note that we have

$$J_{C,P} = J_{C',P} \text{ on } E_P, \quad S_{C,P} = S_{C',P} \text{ on } D_P.$$

By Corollary 4.7, we see that  $H_{C,P}$  and  $H_{C',P}$  are the closures of *distinct* connected components of  $D_P \setminus S_{C,P} = D_P \setminus S_{C',P}$ .

As in Section 5.3, we choose a section  $s: E_P \rightarrow \mathcal{N}_P$  of a tubular neighborhood  $\mathcal{N}_P \rightarrow E_P$  admissible with respect to  $C$ , and construct the  $E$ -displacement  $\Sigma(C)_{\varepsilon'}$  associated with  $s$ . We illustrate the  $E$ -displacements of  $\beta^\sharp C$  and  $E_P$  in  $Y(\mathbb{C})$  in Figure 6.1, where  $Q_0$  is the zero of the section  $s$ . Since  $\beta^\sharp C'$  and  $(\beta^\sharp C)_{\varepsilon'}$  are disjoint, it follows that  $\Delta(C')$  and  $\Delta(C)_{\varepsilon'}$  are disjoint. Since  $E_P$  and  $(\beta^\sharp C)_{\varepsilon'}$  are disjoint, we see that  $H_{C',P}$  and  $\Delta(C)_{\varepsilon'}$  are disjoint. Since  $\beta^\sharp C'$  and  $(E_P)_{\varepsilon'}$  are disjoint, it follows that  $\Delta(C')$  and  $(H_{C,P})_{\varepsilon'}$  are disjoint. Note that  $D_P$  and  $(D_P)_{\varepsilon'}$

FIGURE 6.1.  $E$ -displacement for the proof of assertion (3)

intersect only at the two points in  $\phi^{-1}(Q_0)$ . Since  $H_{C,P}$  and  $H_{C',P}$  are distinct hemispheres of  $D_P$  given by  $S_{C,P} = S_{C',P}$ , and  $\epsilon'$  is sufficiently small, neither of the two points of  $\phi^{-1}(Q_0)$  is on  $H_{C',P} \cap (H_{C,P})_{\epsilon'}$ . Therefore  $\Sigma(C')$  and  $(\Sigma(C))_{\epsilon'}$  are disjoint.  $\square$

## 7. PROOF OF THEOREM 1.7 (4)

We prove assertion (4) of Theorem 1.7 for the standard orientations  $\sigma_C$  fixed in Section 2.1. We first prove that the intersection number  $\langle [\Sigma(C)], [\Sigma(C')] \rangle$  for distinct bounded chambers  $C$  and  $C'$  sharing a common edge does not depend on  $\mathcal{A}$ . See Lemma 7.1. We then compare the intersection number  $\langle [\Sigma(C)], [\Sigma(C')] \rangle$  with the intersection number  $\langle [\Sigma^-(C)], [\Sigma^-(C')] \rangle^-$  that is calculated by replacing the defining polynomial  $f$  of  $\mathcal{A}$  with  $-f$ . Using (1.3), we obtain a proof of assertion (4) of Theorem 1.7.

**7.1. A preliminary lemma.** We consider two nodal real line arrangements  $\mathcal{A}_1$  and  $\mathcal{A}_2$ . Let  $j$  be in  $\{1, 2\}$ . Let  $\mathcal{A}_j$  be a nodal real line arrangement with a defining polynomial  $f_j$ . We consider the morphisms

$$X_j \xrightarrow{\rho_j} W_j \xrightarrow{\pi_j} \mathbb{A}^2(\mathbb{C}).$$

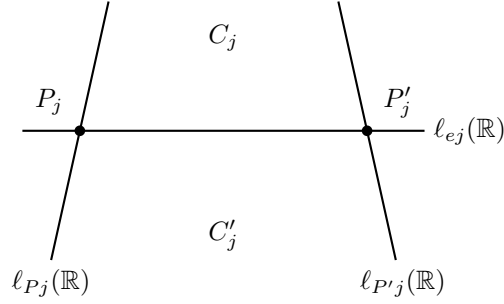
Here  $W_j$  is defined in  $\mathbb{C} \times \mathbb{A}^2(\mathbb{C})$  by

$$\omega_j^2 = f_j,$$

where  $\omega_j$  is an affine coordinate of  $\mathbb{C}$ , and  $\pi_j$  denotes the double covering  $(\omega_j, Q) \mapsto Q$ . The morphism  $\rho_j: X_j \rightarrow W_j$  is the minimal desingularization. Then, for a bounded chamber  $C$  of  $\mathcal{A}_j$ , we have a vanishing cycle  $\Sigma(C)$  in  $X_j$  with the standard orientation determined by  $\sigma_{\mathbb{A}}$  and  $f_j$ . We denote by

$$\langle \rangle_j: H_2(X_j; \mathbb{Z}) \times H_2(X_j; \mathbb{Z}) \rightarrow \mathbb{Z}$$

the intersection form on  $X_j$ .


 FIGURE 7.1.  $C_j$  and  $C'_j$ 

**Lemma 7.1.** *Let  $C_j$  and  $C'_j$  be distinct bounded chambers of  $\mathcal{A}_j$  that share a common edge. Then we have*

$$(7.1) \quad \langle [\Sigma(C_1)], [\Sigma(C'_1)] \rangle_1 = \langle [\Sigma(C_2)], [\Sigma(C'_2)] \rangle_2.$$

*Proof.* The idea is to construct a diffeomorphism from an open neighborhood  $U_{X,1}$  of  $\Sigma(C_1) \cap \Sigma(C'_1)$  in  $X_1$  to an open neighborhood  $U_{X,2}$  of  $\Sigma(C_2) \cap \Sigma(C'_2)$  in  $X_2$  that induces orientation-preserving isomorphisms

$$(7.2) \quad U_{X,1} \cap \Sigma(C_1) \cong U_{X,2} \cap \Sigma(C_2) \quad \text{and} \quad U_{X,1} \cap \Sigma(C'_1) \cong U_{X,2} \cap \Sigma(C'_2).$$

Since the intersection number of topological cycles is determined by the geometric situation in a neighborhood of the set-theoretic intersection of the topological cycles, this diffeomorphism proves (7.1). In fact, we will construct a biholomorphic isomorphism  $\psi_X: U_{X,1} \xrightarrow{\sim} U_{X,2}$  explicitly.

Since  $C_j$  and  $C'_j$  are adjacent, the signs of  $f_j$  on  $C_j$  and on  $C'_j$  are opposite. Interchanging  $C_j$  and  $C'_j$  if necessary, we can assume without loss of generality that

$$f_j(C_j) \subset \mathbb{R}_{\geq 0}, \quad f_j(C'_j) \subset \mathbb{R}_{\leq 0}.$$

Let  $e_j := P_j P'_j$  be the common edge of  $C_j$  and  $C'_j$ , where we have

$$\text{Vert}(C_j) \cap \text{Vert}(C'_j) = \{P_j, P'_j\}.$$

Let  $\ell_{e_j}(\mathbb{R}) \in \mathcal{A}_j$  be the line defining the edge  $P_j P'_j$ . Let  $(\xi_j, \eta_j)$  be good affine coordinates of  $\mathbb{A}^2(\mathbb{C})$  with  $(x_j, y_j) = (\text{Re } \xi_j, \text{Re } \eta_j)$  regarded as affine coordinates of  $\mathbb{A}^2(\mathbb{R})$  such that  $\ell_{e_j}(\mathbb{R})$  is defined by  $y_j = 0$ , and such that  $\{P_j, P'_j\}$  is  $\{(\pm 1, 0)\}$ . See Section 2.3. Note that, if  $(\xi_j, \eta_j)$  satisfies these properties, then so does  $(-\xi_j, -\eta_j)$ . Replacing  $(\xi_j, \eta_j)$  with  $(-\xi_j, -\eta_j)$  if necessary, we can assume that

$$C_j \subset \{y_j \geq 0\}, \quad C'_j \subset \{y_j \leq 0\}.$$

Interchanging  $P_j$  and  $P'_j$  if necessary, we can further assume that

$$P_j = (-1, 0), \quad P'_j = (1, 0)$$

in terms of  $(x_j, y_j)$ . Let  $\ell_{P_j}(\mathbb{R})$  (resp.  $\ell_{P'_j}(\mathbb{R})$ ) be the line in  $\mathcal{A}_j$  other than  $\ell_{e_j}(\mathbb{R})$  that passes through  $P_j$  (resp.  $P'_j$ ). Then there exist real numbers  $g_j, g'_j$  such that

$$\ell_{P_j}(\mathbb{R}) : x_j = -1 + g_j y_j, \quad \ell_{P'_j}(\mathbb{R}) : x_j = 1 + g'_j y_j.$$

See Figure 7.1.

We prepare another copy of  $\mathbb{A}^2(\mathbb{C})$  with good affine coordinates  $(\xi_0, \eta_0)$ , and we regard  $(x_0, y_0) = (\operatorname{Re} \xi_0, \operatorname{Re} \eta_0)$  as affine coordinates of  $\mathbb{A}^2(\mathbb{R})$ . We put

$$P_0 := (-1, 0), \quad P'_0 := (1, 0), \quad e_0 := P_0 P'_0,$$

and define three lines  $\ell_{e_0}(\mathbb{R}), \ell_{P_0}(\mathbb{R}), \ell_{P'_0}(\mathbb{R})$  by

$$\ell_{e_0}(\mathbb{R}) : y_0 = 0, \quad \ell_{P_0}(\mathbb{R}) : x_0 = -1, \quad \ell_{P'_0}(\mathbb{R}) : x_0 = 1.$$

We then put

$$C_0 := \{ (x_0, y_0) \mid -1 \leq x_0 \leq 1, 0 \leq y_0 \},$$

$$C'_0 := \{ (x_0, y_0) \mid -1 \leq x_0 \leq 1, 0 \geq y_0 \}.$$

Finally, we put

$$f_0 := y_0(1 - x_0)(1 + x_0),$$

which is a defining polynomial of the nodal real line arrangement  $\mathcal{A}_0$  consisting of three lines  $\ell_{e_0}(\mathbb{R}), \ell_{P_0}(\mathbb{R}), \ell_{P'_0}(\mathbb{R})$ . Note that we have

$$f_0(C_0) \subset \mathbb{R}_{\geq 0}, \quad f_0(C'_0) \subset \mathbb{R}_{\leq 0}.$$

We define a quadratic map  $\psi_j : \mathbb{A}^2(\mathbb{C}) \rightarrow \mathbb{A}^2(\mathbb{C})$  by

$$(\xi_0, \eta_0) \mapsto (\xi_j, \eta_j) = \left( \xi_0 + \left( \frac{1 - \xi_0}{2} g_j + \frac{1 + \xi_0}{2} g'_j \right) \eta_0, \eta_0 \right),$$

which maps  $P_0$  to  $P_j$  and  $P'_0$  to  $P'_j$ . Let  $U_0$  be a sufficiently small open neighborhood of  $e_0$  in  $\mathbb{A}^2(\mathbb{C})$ . Then there exists an open neighborhood  $U_j$  of  $e_j$  in  $\mathbb{A}^2(\mathbb{C})$  such that  $\psi_j$  induces a biholomorphic isomorphism

$$\psi_{jU} : U_0 \xrightarrow{\sim} U_j.$$

The inverse of  $\psi_{jU}$  is given by

$$(\xi_j, \eta_j) \mapsto (\xi_0, \eta_0) = \left( \frac{2\xi_j - (g_j + g'_j)\eta_j}{2 - (g_j - g'_j)\eta_j}, \eta_j \right).$$

We can easily check that  $\psi_{jU}$  maps  $U_0 \cap \mathbb{A}^2(\mathbb{R})$  to  $U_j \cap \mathbb{A}^2(\mathbb{R})$  isomorphically and orientation-preservingly, and that it also maps isomorphically

$$\begin{aligned} U_0 \cap C_0 & \quad \text{to} \quad U_j \cap C_j, \\ U_0 \cap C'_0 & \quad \text{to} \quad U_j \cap C'_j, \\ U_0 \cap \ell_{e_0}(\mathbb{C}) & \quad \text{to} \quad U_j \cap \ell_{e_j}(\mathbb{C}), \\ U_0 \cap \ell_{P_0}(\mathbb{C}) & \quad \text{to} \quad U_j \cap \ell_{P_j}(\mathbb{C}), \\ U_0 \cap \ell_{P'_0}(\mathbb{C}) & \quad \text{to} \quad U_j \cap \ell_{P'_j}(\mathbb{C}). \end{aligned}$$

We make  $U_0$  so small that  $U_j$  does not intersect any element of the complexified line arrangement  $\mathcal{A}_{j\mathbb{C}}$  other than  $\ell_{e_j}(\mathbb{C}), \ell_{P_j}(\mathbb{C}), \ell_{P'_j}(\mathbb{C})$ . Then

$$u_j := f_0 / \psi_{jU}^* f_j$$

is a holomorphic function on  $U_0$  that does *not* have any zero. Moreover we have  $u_j(Q) \in \mathbb{R}_{>0}$  for any  $Q \in C_0 \cup C'_0$ . We can assume that  $U_0$  is simply-connected. Then we can define a holomorphic function  $v_j$  on  $U_0$  such that  $v_j^2 = u_j$  (in particular  $v_j$  does not have any zero) and that

$$(7.3) \quad v_j(Q) \in \mathbb{R}_{>0} \quad \text{for any } Q \in C_0 \cup C'_0.$$



Let  $W_0$  be defined in  $\mathbb{C} \times \mathbb{A}^2(\mathbb{C})$  by  $\omega_0^2 = f_0$ , where  $\omega_0$  is an affine coordinate of  $\mathbb{C}$ , and let  $\pi_0: W_0 \rightarrow \mathbb{A}^2(\mathbb{C})$  be the double covering  $(\omega_0, Q_0) \mapsto Q_0$ . We put

$$U_{W,0} := \pi_0^{-1}(U_0), \quad U_{W,j} := \pi_j^{-1}(U_j).$$

We define an isomorphism

$$\psi_{jW}: U_{W,0} \xrightarrow{\sim} U_{W,j}$$

that covers the isomorphism  $\psi_{jU}: U_0 \xrightarrow{\sim} U_j$  by

$$(7.4) \quad (\omega_0, Q_0) \mapsto (\omega_j, Q_j) = (\omega_0/v_j(Q_0), \psi_{jU}(Q_0)).$$

We put

$$\psi_W := \psi_{2W} \circ \psi_{1W}^{-1}: U_{W,1} \xrightarrow{\sim} U_{W,2},$$

which induces isomorphisms from  $U_{W,1} \cap \pi_1^{-1}(C_1)$  to  $U_{W,2} \cap \pi_2^{-1}(C_2)$  and from  $U_{W,1} \cap \pi_1^{-1}(C'_1)$  to  $U_{W,2} \cap \pi_2^{-1}(C'_2)$ . Recall from Definition 2.3 that the positive-sheet and the negative-sheet are defined by the sign of  $\omega \in \mathbb{R}$  or  $\omega/\sqrt{-1} \in \mathbb{R}$ . Therefore it follows from (7.3) and (7.4) that  $\psi_W$  maps the positive-sheet of  $U_{W,1} \cap \pi_1^{-1}(C_1^\circ)$  to the positive-sheet of  $U_{W,2} \cap \pi_2^{-1}(C_2^\circ)$ , and the positive-sheet of  $U_{W,1} \cap \pi_1^{-1}(C_1^{\circ\circ})$  to the positive-sheet of  $U_{W,2} \cap \pi_2^{-1}(C_2^{\circ\circ})$ . Hence the isomorphism by  $\psi_W$  from  $U_{W,1} \cap \pi_1^{-1}(C_1)$  to  $U_{W,2} \cap \pi_2^{-1}(C_2)$  is orientation-preserving, where orientations are the standard orientations defined by  $\sigma_{\mathbb{A}}$  and  $f_j$ . We put

$$U_{X,j} := \rho_j^{-1}(U_{W,j}).$$

Since the minimal resolution of an ordinary node of an algebraic surface is unique, the isomorphism  $\psi_W$  induces an isomorphism

$$\psi_X: U_{X,1} \xrightarrow{\sim} U_{X,2}$$

that maps  $D_{P_1}$  to  $D_{P_2}$  and  $D_{P'_1}$  to  $D_{P'_2}$ . Since  $\psi_X$  induces orientation-preserving isomorphisms

$$U_{X,1} \cap \Delta(C_1) \cong U_{X,2} \cap \Delta(C_2), \quad U_{X,1} \cap \Delta(C'_1) \cong U_{X,2} \cap \Delta(C'_2),$$

it follows that  $\psi_X$  induces isomorphisms of the capping hemispheres

$$\begin{aligned} H_{C_1, P_1} &\xrightarrow{\sim} H_{C_2, P_2}, & H_{C_1, P'_1} &\xrightarrow{\sim} H_{C_2, P'_2}, \\ H_{C'_1, P_1} &\xrightarrow{\sim} H_{C'_2, P_2}, & H_{C'_1, P'_1} &\xrightarrow{\sim} H_{C'_2, P'_2}. \end{aligned}$$

Therefore  $\psi_X$  induces orientation-preserving isomorphisms (7.2).

Note that  $\Sigma(C_j) \cap \Sigma(C'_j)$  is contained in  $U_{X,j}$ . The intersection number of  $\Sigma(C_1)$  and  $\Sigma(C'_1)$  is calculated by making a small displacement  $\Sigma(C'_1)_\varepsilon$  of  $\Sigma(C'_1)$  that is stationary outside  $U_{X,1}$  so that  $\Sigma(C_1)$  and  $\Sigma(C'_1)_\varepsilon$  intersect transversely. Transplanting this procedure into  $U_{X,2}$  by  $\psi_X$ , we see that  $\Sigma(C_2)$  and a displacement  $\Sigma(C'_2)_\varepsilon$  of  $\Sigma(C'_2)$  intersect in the same way as  $\Sigma(C_1)$  and  $\Sigma(C'_1)_\varepsilon$ . Therefore we obtain (7.1).  $\square$

**7.2. Proof of assertion (4) of Theorem 1.7.** Interchanging  $C$  and  $C'$  if necessary, we can assume that

$$(7.5) \quad f(C) \subset \mathbb{R}_{\geq 0}, \quad f(C') \subset \mathbb{R}_{\leq 0}.$$

We construct morphisms (1.1) with  $f$  replaced by  $-f$ , and denote them by

$$(7.6) \quad X^- \xrightarrow{\rho^-} W^- \xrightarrow{\pi^-} \mathbb{A}^2(\mathbb{C}),$$

where  $W^-$  is defined in  $\mathbb{C} \times \mathbb{A}^2(\mathbb{C})$  by

$$\omega^2 = -f$$

with  $\pi^-(\omega, Q) = Q$ , and  $\rho^-: X^- \rightarrow W^-$  is the minimal desingularization. For a bounded chamber  $C''$ , let  $\Sigma^-(C'')$  denote the vanishing cycle over  $C''$  in  $X^-$  equipped with the standard orientation  $\sigma_{C''}^-$  defined by  $\sigma_{\mathbb{A}}$  and  $-f$ . Let

$$\langle \quad \rangle^-: H_2(X^-; \mathbb{Z}) \times H_2(X^-; \mathbb{Z}) \rightarrow \mathbb{Z}$$

denote the intersection form on  $X^-$ . By Lemma 7.1, we have

$$(7.7) \quad \langle [\Sigma(C)], [\Sigma(C')] \rangle = \langle [\Sigma^-(C)], [\Sigma^-(C')] \rangle^-.$$

Let  $\Phi_W: W \xrightarrow{\sim} W^-$  be the isomorphism defined by

$$(\omega, Q) \mapsto (\sqrt{-1}\omega, Q).$$

Note that we have

$$\omega \in \mathbb{R}_{\geq 0} \implies \sqrt{-1}\omega \in \sqrt{-1}\mathbb{R}_{\geq 0}, \quad \omega \in \sqrt{-1}\mathbb{R}_{\geq 0} \implies \sqrt{-1}\omega \in -\mathbb{R}_{\geq 0}.$$

Therefore, by (7.5), we see that  $\Phi_W$  maps the positive-sheet of  $\pi^{-1}(C^\circ)$  to the positive-sheet of  $(\pi^-)^{-1}(C^\circ)$ , whereas it maps the positive-sheet of  $\pi^{-1}(C'^\circ)$  to the *negative*-sheet of  $(\pi^-)^{-1}(C'^\circ)$ . Therefore the isomorphism  $\Phi_X: X \xrightarrow{\sim} X^-$  induced by  $\Phi_W$  induces an isomorphism  $\Sigma(C, \sigma_C) \cong \Sigma^-(C, \sigma_C^-)$ , whereas it induces an isomorphism  $\Sigma(C', \sigma_{C'}) \cong \Sigma^-(C', -\sigma_{C'}^-)$ . Hence the homomorphism  $\Phi_{X*}: H_2(X; \mathbb{Z}) \xrightarrow{\sim} H_2(X^-; \mathbb{Z})$  satisfies

$$(7.8) \quad \Phi_{X*}([\Sigma(C)]) = [\Sigma^-(C)],$$

$$(7.9) \quad \Phi_{X*}([\Sigma(C')]) = \sum_{P'' \in \text{Vert}(C')} [D_{P''}] - [\Sigma^-(C')],$$

where (7.9) is derived from (1.3). Since  $\Phi_{X*}$  preserves the intersection form, we have

$$(7.10) \quad \langle [\Sigma(C)], [\Sigma(C')] \rangle = \langle \Phi_{X*}([\Sigma(C)]), \Phi_{X*}([\Sigma(C')]) \rangle^-.$$

Comparing (7.7), (7.8), (7.9), (7.10), we obtain

$$2 \langle [\Sigma(C)], [\Sigma(C')] \rangle = \sum_{P'' \in \text{Vert}(C')} \langle [\Sigma^-(C)], [D_{P''}] \rangle^- = -2,$$

where we use assertion (1) of Theorem 1.7 and  $|\text{Vert}(C) \cap \text{Vert}(C')| = 2$  in the second equality. Thus Theorem 1.7 (4) is proved.  $\square$

## 8. PROOF OF THEOREM 1.7 (5)

We prove assertion (5) of Theorem 1.7 for the standard orientations  $\sigma_C$  fixed in Section 2.1.

**8.1. Target displacements.** We construct an  $E$ -displacement

$$(8.1) \quad \Sigma(C)_{\varepsilon'} = \Delta(C)_{\varepsilon'} \cup \bigsqcup_{P \in \text{Vert}(C)} (H_{C,P})_{\varepsilon'}$$

of  $\Sigma(C)$  in  $X$  associated with sections  $s_P: E_P \rightarrow \mathcal{N}_P$  for all  $P \in \text{Vert}(C)$ , and a  $C$ -displacement

$$(8.2) \quad \Sigma(C)_\varepsilon = \Delta(C)_\varepsilon \cup \bigsqcup_{P \in \text{Vert}(C)} H_{C,P}$$

of  $\Sigma(C)$  in  $X$  associated with a continuous function  $\delta: C \rightarrow T(\mathbb{A}^2(\mathbb{R}))$ . By the definition of  $E$ -displacements (see (5.11)), for any  $P \in \text{Vert}(C)$ , we have that

$\Delta(C)_{\varepsilon'}$  and  $H_{C,P}$  are disjoint, and that  $(H_{C,P})_{\varepsilon'}$  and  $H_{C,P}$  intersect only at one point, at which the local intersection number is  $-1$ . By choosing a sufficiently small  $\varepsilon'$ , we have that  $(H_{C,P})_{\varepsilon'}$  and  $H_{C,P'}$  are disjoint for  $P \neq P'$ . We construct these displacements (8.1) and (8.2) in such a way that they have the following additional properties.

- (EC1) The two spaces  $\Delta(C)_{\varepsilon'}$  and  $\Delta(C)_{\varepsilon}$  intersect only at two points, each intersection point is in the interior of  $\Delta(C)_{\varepsilon'}$  and of  $\Delta(C)_{\varepsilon}$ , and the local intersection number is  $-1$  at each intersection point.
- (EC2) For any  $P \in \text{Vert}(C)$ , we have that  $(H_{C,P})_{\varepsilon'}$  and  $\Delta(C)_{\varepsilon}$  intersect only at one point, that the intersection point is in the interior of  $(H_{C,P})_{\varepsilon'}$  and of  $\Delta(C)_{\varepsilon}$ , and that the local intersection number is  $1$ .

Then we have

$$\begin{aligned} \langle [\Sigma(C)], [\Sigma(C)] \rangle &= \langle [\Sigma(C)_{\varepsilon'}], [\Sigma(C)_{\varepsilon}] \rangle = \\ &= \sum_{P \in \text{Vert}(C)} (-1) + \sum_{P \in \text{Vert}(C)} 1 + (-2) = -2, \end{aligned}$$

where the first sum comes from the intersections of  $(H_{C,P})_{\varepsilon'}$  and  $H_{C,P}$ , the second sum comes from the intersections of  $(H_{C,P})_{\varepsilon'}$  and  $\Delta(C)_{\varepsilon}$ , and the third term  $(-2)$  comes from the intersection of  $\Delta(C)_{\varepsilon'}$  and  $\Delta(C)_{\varepsilon}$ . Thus assertion (5) of Theorem 1.7 will be proved.

**8.2. Construction of the  $E$ -displacement.** For each  $P \in \text{Vert}(C)$ , we fix good local coordinates  $(\xi_P, \eta_P)$ ,  $(x_P, y_P)$ ,  $(\tilde{\xi}_P, \mu_P)$ , and  $(\tilde{x}_P, m_P)$  on  $\mathbb{A}^2(\mathbb{C})$ ,  $\mathbb{A}^2(\mathbb{R})$ ,  $Y(\mathbb{C})$ , and  $Y(\mathbb{R})$ , given in Section 4.1. We can assume, without loss of generality, that  $C$  is located as Case (1) in (4.3) so that there exist real numbers  $a_P, b_P$  with  $a_P < b_P$  such that  $\beta^\sharp C$  is given by  $\tilde{x}_P \geq 0$  and  $a_P \leq m_P \leq b_P$  in the chart of  $(\tilde{x}_P, m_P)$  on  $Y(\mathbb{R})$ . To make the computation easy, we choose  $(\xi_P, \eta_P)$  in such a way that we have

$$(8.3) \quad a_P = -1, \quad b_P = 1.$$

See Remark 4.1. We choose a sufficiently small real number  $c$ , and choose a section  $s_P: E_P \rightarrow \mathcal{N}_P$  of a tubular neighborhood  $\mathcal{N}_P \rightarrow E_P$  such that, in a neighborhood of  $J_{C,P}$  in  $E_P$ , the section  $s_P$  is written as

$$(0, \mu) \mapsto (c, \mu)$$

in terms of  $(\tilde{\xi}_P, \mu_P)$ . Then  $s_P$  is admissible with respect to  $C$ . See Example 5.10. Now we construct the  $E$ -displacement (8.1) of  $\Sigma(C)$  in  $X$  associated with all these sections  $s_P$ .

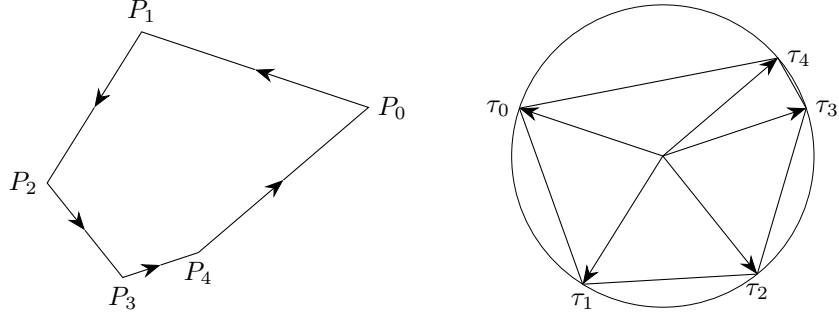
We denote by  $\Gamma_P$  the image by  $\beta$  of the region

$$\{ (\tilde{x}_P, m_P) \mid 0 \leq \tilde{x}_P \leq c\varepsilon', -1 \leq m_P \leq 1 \}$$

in  $\beta^\sharp C$ . In terms of the coordinates  $(x_P, y_P)$  of  $\mathbb{A}^2(\mathbb{R})$ , we have

$$(8.4) \quad \Gamma_P = \{ (x_P, y_P) \mid 0 \leq x_P \leq c\varepsilon', -x_P \leq y_P \leq x_P \} \subset C.$$

**8.3. Construction of the  $C$ -displacement.** As in Section 5.2, we introduce an inner-product on the  $\mathbb{R}$ -vector space  $T(\mathbb{A}^2(\mathbb{R}))$ . In particular, we have a distance and angles on  $\mathbb{A}^2(\mathbb{R})$ .

FIGURE 8.1.  $C$  and  $\Theta$ 

Let  $n$  be the size of  $\text{Vert}(C)$ . We index the vertexes of  $C$  by elements  $\nu$  of the cyclic group  $\mathbb{Z}/n\mathbb{Z}$  as

$$\text{Vert}(C) = \{ P_\nu \mid \nu \in \mathbb{Z}/n\mathbb{Z} \}$$

in such a way that, for each  $\nu \in \mathbb{Z}/n\mathbb{Z}$ , the line segment  $P_\nu P_{\nu+1}$  is an edge of  $C$ , and that the cyclic sequence  $P_0, P_1, \dots, P_{n-1}, P_n = P_0$  of vertexes goes around  $C$  in the positive-direction with respect to the fixed orientation  $\sigma_{\mathbb{A}}$  of  $\mathbb{A}^2(\mathbb{R})$ . Let  $\ell_\nu(\mathbb{R}) \in \mathcal{A}$  be the line defining the edge  $P_\nu P_{\nu+1}$ , and let  $\overrightarrow{P_\nu P_{\nu+1}} \in T(\mathbb{A}^2(\mathbb{R}))$  be the translation such that

$$P_\nu + \overrightarrow{P_\nu P_{\nu+1}} = P_{\nu+1}.$$

Obviously we have  $\overrightarrow{P_\nu P_{\nu+1}} \in T[\ell_\nu(\mathbb{R})]$ . We define a unit vector  $\tau_\nu \in T[\ell_\nu(\mathbb{R})]$  by

$$\tau_\nu := \overrightarrow{P_\nu P_{\nu+1}} / |P_\nu P_{\nu+1}|.$$

Let  $o \in T(\mathbb{A}^2(\mathbb{R}))$  be the zero vector. Then  $\tau_{\nu+1}$  is obtained by rotating  $\tau_\nu$  in a positive-direction by the angle

$$\theta_\nu := \angle \tau_\nu o \tau_{\nu+1} = \pi - \angle P_\nu P_{\nu+1} P_{\nu+2},$$

where the positive-direction is with respect to the orientation of  $T(\mathbb{A}^2(\mathbb{R}))$  induced by  $\sigma_{\mathbb{A}}$ . We denote by  $\Theta_\nu$  the triangle  $\tau_\nu o \tau_{\nu+1}$  in  $T(\mathbb{A}^2(\mathbb{R}))$ . Then the interiors of these  $n$  triangles are disjoint, and, since the sum of  $\theta_0, \dots, \theta_{n-1}$  is equal to  $2\pi$ , their union

$$\Theta := \bigcup_{\nu \in \mathbb{Z}/n\mathbb{Z}} \Theta_\nu$$

contains  $o$  in its interior. The cyclic sequence  $\tau_0, \dots, \tau_{n-1}, \tau_n = \tau_0$  of the vertexes of the  $n$ -gon  $\Theta$  goes in a positive-direction around  $\Theta$ . See Figure 8.1.

We define a continuous function

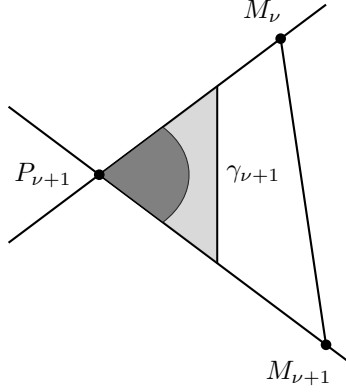
$$\delta: C \rightarrow \Theta \subset T(\mathbb{A}^2(\mathbb{R}))$$

as follows. Let  $M_\nu$  be the mid-point of the edge  $P_\nu P_{\nu+1}$ , and let  $M$  be the  $n$ -gon  $M_0 M_1 \dots M_{n-1} M_0$ . Then we have a homeomorphism

$$\delta_M: M \xrightarrow{\sim} \Theta$$

with the following properties.

- For each  $\nu \in \mathbb{Z}/n\mathbb{Z}$ , we have  $\delta_M(M_\nu) = \tau_\nu$ . Moreover, the restriction of  $\delta_M$  to  $M_\nu M_{\nu+1}$  is an affine-linear isomorphism of line segments from  $M_\nu M_{\nu+1}$  to  $\tau_\nu \tau_{\nu+1}$ . In particular,  $\delta_M$  is orientation-preserving.


 FIGURE 8.2.  $\triangle M_{\nu}P_{\nu+1}M_{\nu+1}$ ,  $\Gamma_{P_{\nu+1}}$ , and  $U_{P_{\nu+1},\rho}$ 

- The inverse-image  $\delta_M^{-1}(o)$  consists of a single point  $Q_0$  in the interior of  $M$ ,  $\delta_M$  is differentiable at the point  $Q_0$ , and  $Q_0$  is not a critical point of  $\delta_M$ .

For each  $\nu \in \mathbb{Z}/n\mathbb{Z}$ , there exists a unique affine-linear isomorphism

$$\delta_{\nu}: \triangle M_{\nu}P_{\nu+1}M_{\nu+1} \xrightarrow{\sim} \Theta_{\nu} = \triangle \tau_{\nu}o\tau_{\nu+1}$$

that maps  $M_{\nu}$  to  $\tau_{\nu}$ ,  $P_{\nu+1}$  to  $o$ , and  $M_{\nu+1}$  to  $\tau_{\nu+1}$ , respectively. In particular, the map  $\delta_{\nu}$  is orientation-reversing. Then the restrictions to the line segment  $M_{\nu}M_{\nu+1}$  of  $\delta_M$  and of  $\delta_{\nu}$  coincide. Thus, gluing  $\delta_M$  and  $\delta_{\nu}$  along  $M_{\nu}M_{\nu+1}$  for  $\nu \in \mathbb{Z}/n\mathbb{Z}$ , we obtain a continuous map

$$\delta': C \rightarrow \Theta \subset T(\mathbb{A}^2(\mathbb{R})).$$

Note that  $\delta'$  maps the edge  $P_{\nu}P_{\nu+1}$  to the line-segment  $o\tau_{\nu} \subset T[\ell_{\nu}(\mathbb{R})]$ . Hence  $\delta'$  satisfies the  $e$ -condition for every edge of  $C$ .

We assume that we have chosen  $c \in \mathbb{R}_{>0}$  and  $\varepsilon'$  in the construction of the  $E$ -displacement so small that, for each  $P_{\nu+1} \in \text{Vert}(C)$ , the triangle  $\Gamma_{P_{\nu+1}}$  defined by (8.4) is contained in  $\triangle M_{\nu}P_{\nu+1}M_{\nu+1}$  and that the edge

$$\gamma_{\nu+1} := C \cap \{x_{P_{\nu+1}} = c\varepsilon'\}$$

of  $\Gamma_{P_{\nu+1}}$  is disjoint from the edge  $M_{\nu}M_{\nu+1}$  of  $\triangle M_{\nu}P_{\nu+1}M_{\nu+1}$ . We choose a sufficiently small positive real number  $\rho$  so that, for each  $P_{\nu+1} \in \text{Vert}(C)$ , the closed region  $U_{P_{\nu+1},\rho}$  defined by (5.8) is contained in  $\Gamma_{P_{\nu+1}}$  and disjoint from the edge  $\gamma_{\nu+1} \subset \Gamma_{P_{\nu+1}}$ . (See Figure 8.2.) We execute the  $q_{\rho}$ -modification to  $\delta'$  at each  $P_{\nu+1} \in \text{Vert}(C)$ , and obtain a continuous function

$$\delta: C \rightarrow \Theta \subset T(\mathbb{A}^2(\mathbb{R}))$$

that satisfies the  $e$ -condition for every edge of  $C$  and regular at each vertex of  $C$ . See Lemma 5.7. Note that, since this  $q_{\rho}$ -modification does not affect the restriction of  $\delta'$  to a small neighborhood of the line segment  $\gamma_{\nu+1}$  in  $C$ , the function  $\delta$  restricted to a small neighborhood of  $\gamma_{\nu+1}$  in  $C$  is equal to the unique affine-linear map that maps  $M_{\nu}$ ,  $P_{\nu+1}$ ,  $M_{\nu+1}$  to  $\tau_{\nu}$ ,  $o$ ,  $\tau_{\nu+1}$ , respectively.

8.4. **Intersection of  $\Delta(C)_{\varepsilon'}$  and  $\Delta(C)_{\varepsilon}$ .** We prove that the  $E$ -displacement and the  $C$ -displacement thus constructed satisfy condition (EC1). Recall that

$$C_{\varepsilon} = \{ Q + \sqrt{-1} \delta(Q) \varepsilon \mid Q \in C \}.$$

Hence we have

$$C \cap C_{\varepsilon} := \{ Q \in C \mid \delta(Q) = 0 \} = \{ Q_0 \} \sqcup \text{Vert}(C),$$

where  $\{Q_0\} = \delta_M^{-1}(o)$ . Let  $Q'_0 \in Y(\mathbb{R})$  be the point such that  $\beta(Q'_0) = Q_0$ . We have

$$\beta^{\sharp} C \cap (\beta^{\sharp} C)_{\varepsilon} = \{Q'_0\} \sqcup \bigsqcup_{P \in \text{Vert}(C)} J_{C,P}.$$

Since  $(\beta^{\sharp} C)_{\varepsilon'} \subset \beta^{\sharp} C$  and  $(\beta^{\sharp} C)_{\varepsilon'} \cap J_{C,P} = \emptyset$ , we obtain

$$(\beta^{\sharp} C)_{\varepsilon'} \cap (\beta^{\sharp} C)_{\varepsilon} = \{Q'_0\}.$$

Recall that  $\Delta(C)_{\varepsilon'} = \phi^{-1}((\beta^{\sharp} C)_{\varepsilon'})$  and  $\Delta(C)_{\varepsilon} = \phi^{-1}((\beta^{\sharp} C)_{\varepsilon})$ , each equipped with the standard orientation  $\sigma_C$ . We put

$$\phi_{\Delta, \varepsilon'} := \phi|_{\Delta(C)_{\varepsilon'}} : \Delta(C)_{\varepsilon'} \rightarrow (\beta^{\sharp} C)_{\varepsilon'}, \quad \phi_{\Delta, \varepsilon} := \phi|_{\Delta(C)_{\varepsilon}} : \Delta(C)_{\varepsilon} \rightarrow (\beta^{\sharp} C)_{\varepsilon}.$$

Let  $Q'_{0,+}$  and  $Q'_{0,-}$  be the points of  $X$  such that  $\phi^{-1}(Q'_0) = \{Q'_{0,+}, Q'_{0,-}\}$ . We then have

$$\Delta(C)_{\varepsilon'} \cap \Delta(C)_{\varepsilon} = \{Q'_{0,+}, Q'_{0,-}\}.$$

Interchanging  $Q'_{0,+}$  and  $Q'_{0,-}$  if necessary, we can assume that the restrictions  $\phi_{\Delta, \varepsilon'}$  and  $\phi_{\Delta, \varepsilon}$  of  $\phi$  to  $\Delta(C)_{\varepsilon'}$  and to  $\Delta(C)_{\varepsilon}$  are both orientation-preserving at  $Q'_{0,+}$ , and that they are both orientation-reversing at  $Q'_{0,-}$ . In particular, the local intersection number  $(\Delta(C)_{\varepsilon'}, \Delta(C)_{\varepsilon})_{Q'_{0,+}}$  of  $\Delta(C)_{\varepsilon'}$  and  $\Delta(C)_{\varepsilon}$  at  $Q'_{0,+}$  is equal to their local intersection number  $(\Delta(C)_{\varepsilon'}, \Delta(C)_{\varepsilon})_{Q'_{0,-}}$  at  $Q'_{0,-}$ . Since  $\beta \circ \phi$  is a local isomorphism locally around each of the points  $Q'_{0,\pm}$  of  $X$ , we see that

$$(\Delta(C)_{\varepsilon'}, \Delta(C)_{\varepsilon})_{Q'_{0,+}} = (\Delta(C)_{\varepsilon'}, \Delta(C)_{\varepsilon})_{Q'_{0,-}} = (C, C_{\varepsilon})_{Q_0},$$

where  $(C, C_{\varepsilon})_{Q_0}$  is the local intersection number of  $C$  and  $C_{\varepsilon}$  at  $Q_0$ . We show that  $(C, C_{\varepsilon})_{Q_0} = -1$ .

We choose a coordinate system  $(\xi, \eta)$  of  $\mathbb{A}^2(\mathbb{C})$  that is good at  $Q_0$ . (See Section 4.1). We then put

$$(x, y) = (\text{Re } \xi, \text{Re } \eta), \quad (u, v) = (\text{Im } \xi, \text{Im } \eta).$$

Then an ordered basis of the real tangent space at  $Q_0$  of  $\mathbb{A}^2(\mathbb{C})$  oriented by the complex structure is

$$(8.5) \quad \partial/\partial x, \quad \partial/\partial u, \quad \partial/\partial y, \quad \partial/\partial v.$$

We write tangent vectors at  $Q_0$  in terms of this basis. The tangent space at  $Q_0$  of  $C$  oriented by  $\sigma_{\mathbb{A}}$  has the ordered basis

$$\mathbf{a}_1 := (1, 0, 0, 0), \quad \mathbf{a}_2 := (0, 0, 1, 0).$$

Recall that  $C_{\varepsilon}$  is given by the equations

$$u = \varepsilon \delta_x(x, y), \quad v = \varepsilon \delta_y(x, y),$$

where  $\delta_x$  and  $\delta_y$  are functions defined by (5.6), and that  $C_\varepsilon$  is oriented by  $\sigma_{\mathbb{A}}$  via the homeomorphism  $\text{pr}_{\mathbb{R}}|_{C_\varepsilon}: C_\varepsilon \cong C$ . Therefore the tangent space at  $Q_0$  of  $C_\varepsilon$  with this orientation has the ordered basis

$$\mathbf{a}_3 := \left( 1, \varepsilon \frac{\partial \delta_x}{\partial x}(Q_0), 0, \varepsilon \frac{\partial \delta_y}{\partial x}(Q_0) \right), \quad \mathbf{a}_4 := \left( 0, \varepsilon \frac{\partial \delta_x}{\partial y}(Q_0), 1, \varepsilon \frac{\partial \delta_y}{\partial y}(Q_0) \right).$$

To prove  $(C, C_\varepsilon)_{Q_0} = -1$ , it is enough to show  $\det A < 0$ , where  $A$  is the  $4 \times 4$  matrix whose  $i$ th row vector is  $\mathbf{a}_i$  defined above for  $i = 1, \dots, 4$ . (See [5, Chapter 3].) Since  $\delta$  is orientation-preserving at  $Q_0$ , its Jacobian at  $Q_0$  is  $> 0$ :

$$\frac{\partial(\delta_x, \delta_y)}{\partial(x, y)}(Q_0) > 0.$$

Therefore we have

$$\det A = -\varepsilon^2 \frac{\partial(\delta_x, \delta_y)}{\partial(x, y)}(Q_0) < 0.$$

**8.5. Intersection of  $(H_{C,P})_{\varepsilon'}$  and  $\Delta(C)_\varepsilon$ .** Let  $P$  be a vertex of  $C$ . We investigate the intersection of  $(H_{C,P})_{\varepsilon'}$  and  $\Delta(C)_\varepsilon$ , and show that condition (EC2) is satisfied. Recall that we have fixed good local coordinate systems  $(\xi_P, \eta_P)$ ,  $(x_P, y_P)$ ,  $(\tilde{\xi}_P, \mu_P)$ ,  $(\tilde{x}_P, m_P)$  at  $P$  (see Definition 4.2) such that  $a_P = -1$  and  $b_P = 1$  hold (see (8.3)), and that the location of  $C$  is Case (1) of (4.3). For simplicity, we omit the subscript  $P$  from these coordinates. Let  $\nu + 1 \in \mathbb{Z}/n\mathbb{Z}$  be the index of  $P$ . By (8.3), there exist positive real numbers  $r, r', r'', r'''$  such that

$$M_\nu = r(1, 1), \quad M_{\nu+1} = r'(1, -1), \quad \tau_\nu = -r''(e_x + e_y), \quad \tau_{\nu+1} = r'''(e_x - e_y).$$

in terms of  $(x, y)$ , where  $e_x, e_y$  is the basis of  $T(\mathbb{A}^2(\mathbb{R}))$  given by (2.4). Recall that, in a small neighborhood  $U_\gamma$  of the line-segment  $\gamma_{\nu+1} = \{x = c\varepsilon'\} \cap C$  in  $C$ , the function  $\delta$  is equal to the affine-linear map such that  $\delta(P) = o$ ,  $\delta(M_\nu) = \tau_\nu$ ,  $\delta(M_{\nu+1}) = \tau_{\nu+1}$ . Hence there exist real numbers  $g, h$  satisfying

$$(8.6) \quad -h < g < h$$

such that, on  $U_\gamma$ , we have

$$(8.7) \quad \delta_x(x, y) = gx - hy, \quad \delta_y(x, y) = -hx + gy,$$

where  $\delta_x$  and  $\delta_y$  are defined by (5.6). Then, by the formula (5.7), the  $C$ -displacement  $d_\varepsilon^\sharp: \beta^\sharp C \rightarrow (\beta^\sharp C)_\varepsilon$  associated with  $\delta$  is given in a small neighborhood  $\beta^{-1}(U_\gamma)$  of  $\beta^{-1}(\gamma_{\nu+1})$  in  $\beta^\sharp C$  by

$$(\tilde{x}, m) \mapsto \left( \tilde{x}(1 + \sqrt{-1}\varepsilon(g - hm)), \frac{m + \sqrt{-1}\varepsilon(-h + gm)}{1 + \sqrt{-1}\varepsilon(g - hm)} \right) \in (\beta^\sharp C)_\varepsilon.$$

Recall that the  $E$ -displacement  $(E_P)_{\varepsilon'}$  is given by  $\tilde{\xi} = c\varepsilon'$  in a small neighborhood of  $\beta^{-1}(\gamma_{\nu+1})$  in  $Y(\mathbb{C})$ . Since  $c\varepsilon' \in \mathbb{R}_{>0}$ , it follows that  $(E_P)_{\varepsilon'}$  and  $(\beta^\sharp C)_\varepsilon$  intersect only at one point, which we write as

$$R = d_\varepsilon^\sharp(R_0),$$

where  $R_0$  is the point

$$(\tilde{x}, m) = (c\varepsilon', g/h).$$

The coordinates of the point  $R$  in terms of  $(\tilde{\xi}, \mu)$  is

$$(\tilde{\xi}_R, \mu_R) = \left( c\varepsilon', \frac{g}{h} + \sqrt{-1}\varepsilon \left( -h + \frac{g^2}{h} \right) \right).$$

By (8.6) and  $\varepsilon$  being very small, the value  $\mu_R$  of  $\mu$  at  $R$  satisfies

$$(8.8) \quad -1 < \operatorname{Re} \mu_R < 1, \quad \operatorname{Im} \mu_R < 0, \quad |\operatorname{Im} \mu_R| \ll 1.$$

More precisely, we can make  $|\operatorname{Im} \mu_R|$  arbitrarily small by choosing a sufficiently small  $\varepsilon$ . Then  $(D_P)_{\varepsilon'} = \phi^{-1}((E_P)_{\varepsilon'})$  and  $\Delta(C)_\varepsilon$  intersect at two points in  $\phi^{-1}(R)$ . Let  $\zeta$  be the function on  $X$  defined by (4.6). In the current situation, we have

$$\zeta^2 = \frac{\mu + 1}{\mu - 1}.$$

We calculate the value of  $\zeta$  at the points of  $\phi^{-1}(R)$ . Note that, by (8.8), making  $\varepsilon$  smaller if necessary, we can assume that

$$\operatorname{Re} \frac{\mu_R + 1}{\mu_R - 1} < 0, \quad 0 < \operatorname{Im} \frac{\mu_R + 1}{\mu_R - 1} \ll 1.$$

Hence one solution  $\zeta_R$  of the equation  $\zeta^2 = (\mu_R + 1)/(\mu_R - 1)$  satisfies

$$(8.9) \quad 0 < \operatorname{Re} \zeta_R \ll 1, \quad 0 < \operatorname{Im} \zeta_R,$$

and the other solution is  $-\zeta_R$ . Let  $\tilde{R}_+$  and  $\tilde{R}_-$  be the points of  $\phi^{-1}(R)$  such that  $\zeta(\tilde{R}_+) = \zeta_R$  and  $\zeta(\tilde{R}_-) = -\zeta_R$ .

Recall from Table 4.1 that, because we are in Case (1) of (4.3), the circle  $S_{C,P}$  is given by  $\operatorname{Re} \zeta = 0$  on  $D_P = \{\phi^* \tilde{\xi} = 0\}$ . Hence one and only one of  $\tilde{R}_+$  or  $\tilde{R}_-$  belongs to  $(H_{C,P})_{\varepsilon'}$ . In particular, the set-theoretic intersection of  $(H_{C,P})_{\varepsilon'}$  and  $(\Delta(C))_\varepsilon$  consists of a single point. We will show that the local intersection number at this point is 1.

Suppose that  $\tilde{R}_+ \in (H_{C,P})_{\varepsilon'}$ . By (8.9), the hemisphere  $H_{C,P}$  is given by  $\operatorname{Re} \zeta|_{D_P} \geq 0$ , that is, we are in the second row of Table 4.2. The orientation  $\operatorname{ori}_H$  on  $S_{C,P} = \partial H_{C,P}$  given by the complex structure of  $H_{C,P}$  is  $\downarrow$ . Hence the orientation  $\operatorname{ori}_C$  on  $S_{C,P} = \partial \Delta(C)$  given by the standard orientation on  $\Delta(C)$  is  $\uparrow$ . Since  $\phi$  is given by

$$\mu = \frac{\zeta^2 + 1}{\zeta^2 - 1},$$

and since  $\operatorname{Im} \zeta_R > 0$  by (8.9), the image by  $\phi$  of the orientation  $\operatorname{ori}_C$  of  $S_{C,P} = \partial \Delta(C)$  near  $\tilde{R}_+$  is the  $m$ -increasing direction on  $J_{C,P}$ . Indeed, when  $\zeta$  moves from 0 to  $\infty\sqrt{-1}$  along  $\sqrt{-1}\mathbb{R}_{>0}$ , then  $\mu$  moves from  $-1$  to 1 along  $\mathbb{R}$ . Recall that the standard orientation of  $\beta^{\sharp}C$  induces the  $m$ -decreasing orientation on  $J_{C,P}$ . (See Corollary 4.4 and Figure 4.3.) Therefore  $\tilde{R}_+$  is on the *negative*-sheet of  $(\Delta(C))_\varepsilon$ .

Suppose that  $\tilde{R}_- \in (H_{C,P})_{\varepsilon'}$ . By (8.9), we see that  $H_{C,P}$  is given by  $\operatorname{Re} \zeta \leq 0$ , and hence we are in the first row of Table 4.2. The orientation  $\operatorname{ori}_H$  on  $S_{C,P} = \partial H_{C,P}$  is  $\uparrow$ . Hence the orientation  $\operatorname{ori}_C$  on  $S_{C,P} = \partial \Delta(C)$  is  $\downarrow$ . Since  $-\operatorname{Im} \zeta_R < 0$  by (8.9), the image by  $\phi$  of the orientation  $\operatorname{ori}_C$  of  $S_{C,P} = \partial \Delta(C)$  near  $\tilde{R}_-$  is the  $m$ -increasing orientation on  $J_{C,P}$ . Therefore  $\tilde{R}_-$  is also on the *negative*-sheet of  $(\Delta(C))_\varepsilon$ .

In any case, the restriction of  $\beta \circ \phi$  to a neighborhood in  $\Delta(C)_\varepsilon$  of the intersection point  $\tilde{R}_+$  or  $\tilde{R}_-$  of  $(H_{C,P})_{\varepsilon'}$  and  $\Delta(C)_\varepsilon$  is an orientation-reversing local isomorphism to an open neighborhood of  $\beta(R)$  in  $C_\varepsilon$ . Hence, to show that the local



intersection number of  $(H_{C,P})_{\varepsilon'}$  and  $(\Delta(C))_{\varepsilon}$  is 1, it is enough to prove that, in  $\mathbb{A}^2(\mathbb{C})$ , the local intersection number at  $\beta(R)$  of

$$\beta((E_P)_{\varepsilon'}) = \{\xi = c\varepsilon'\}$$

with the orientation coming from the complex structure and the space  $C_{\varepsilon}$  with the orientation induced by  $\sigma_{\mathbb{A}} = -1$ . Note that  $C_{\varepsilon}$  is given locally around  $\beta(R)$  by

$$\begin{aligned}\xi &= x + \sqrt{-1}\varepsilon\delta_x(x, y) = x + \sqrt{-1}\varepsilon(gx - hy), \\ \eta &= y + \sqrt{-1}\varepsilon\delta_y(x, y) = y + \sqrt{-1}\varepsilon(-hx + gy).\end{aligned}$$

We use the ordered basis (8.5) of the real tangent space of  $\mathbb{A}^2(\mathbb{C})$ . Then the tangent space of  $\beta((E_P)_{\varepsilon'})$  oriented by the complex structure has an ordered basis

$$\mathbf{b}_1 := (0, 0, 1, 0), \quad \mathbf{b}_2 := (0, 0, 0, 1).$$

On the other hand, the oriented tangent space of  $C_{\varepsilon}$  has an ordered basis

$$\begin{aligned}\mathbf{b}_3 &:= \left(1, \varepsilon \frac{\partial \delta_x}{\partial x}, 0, \varepsilon \frac{\partial \delta_y}{\partial x}\right) = (1, \varepsilon g, 0, -\varepsilon h), \\ \mathbf{b}_4 &:= \left(0, \varepsilon \frac{\partial \delta_x}{\partial y}, 1, \varepsilon \frac{\partial \delta_y}{\partial y}\right) = (0, -\varepsilon h, 1, \varepsilon g).\end{aligned}$$

Let  $B$  be the  $4 \times 4$  matrix whose  $i$ th row vector is  $\mathbf{b}_i$  defined above for  $i = 1, \dots, 4$ . Since  $h > 0$  by (8.6), we have  $\det B = -\varepsilon h < 0$ . Therefore the local intersection number at  $\beta(R)$  of  $\beta((E_P)_{\varepsilon'})$  and  $C_{\varepsilon}$  is  $-1$ . (See [5, Chapter 3].) Thus (EC2) holds at  $P$ .  $\square$

## 9. EXAMPLES

Using a smooth projective completion  $\tilde{X}$  of  $X$ , we check certain consequences of Theorem 1.7 in several examples.

**9.1. Lattice.** A *lattice* is a free  $\mathbb{Z}$ -module  $L$  of finite rank with a *non-degenerate* symmetric bilinear form  $\langle \cdot \cdot \rangle : L \times L \rightarrow \mathbb{Z}$ . Let  $L$  be a lattice. The *signature* of  $L$  is the pair  $(s_+, s_-)$  of numbers of positive and negative eigenvalues of the Gram matrix of  $L$ . A lattice  $L$  is embedded naturally into its dual  $L^\vee := \text{Hom}(L, \mathbb{Z})$ . The *discriminant group* of  $L$  is the finite abelian group

$$\text{disc}(L) := L^\vee / L.$$

We say that  $L$  is *unimodular* if its discriminant group is trivial. By definition, we have the following:

**Proposition 9.1.** *Let  $L'$  be a sublattice of a lattice  $L$  with a finite index  $m$ . Then  $\text{disc}(L)$  is a sub-quotient of  $\text{disc}(L')$ , and  $|\text{disc}(L)|$  is equal to  $|\text{disc}(L')|/m^2$ .  $\square$*

The following is well known. See, for example, [9].

**Proposition 9.2.** *Let  $L$  be a sublattice of a unimodular lattice  $M$ . Suppose that  $M/L$  is torsion-free. We put*

$$(9.1) \quad L^\perp := \{x \in M \mid \langle x, y \rangle = 0 \text{ for all } y \in L\}.$$

*Then we have  $\text{disc}(L) \cong \text{disc}(L^\perp)$ .  $\square$*

Combining Propositions 9.1 and 9.2, we obtain the following:

**Corollary 9.3.** *Let  $L$  be a sublattice of a unimodular lattice  $M$ . Then  $\text{disc}(L^\perp)$  is a sub-quotient of  $\text{disc}(L)$ .  $\square$*

Let  $\mathcal{A}$  be a nodal real line arrangement, and let  $\langle \cdot \rangle$  be the intersection form on  $H_2(X; \mathbb{Z})$ . We put

$$(9.2) \quad \text{Ker} \langle \cdot \rangle := \{x \in H_2(X; \mathbb{Z}) \mid \langle x, y \rangle = 0 \text{ for any } y \in H_2(X; \mathbb{Z})\}.$$

Then  $\langle \cdot \rangle$  yields a natural structure of the lattice on

$$\overline{H}(X) := H_2(X; \mathbb{Z}) / \text{Ker} \langle \cdot \rangle,$$

and we can calculate the Gram matrix of  $\overline{H}(X)$  by Theorem 1.7. In particular, we can calculate the signature and the discriminant group of  $\overline{H}(X)$  by Theorem 1.7.

**9.2. A projective completion of  $X$ .** Let  $\mathbb{P}^2(\mathbb{C})$  be the complex projective plane containing  $\mathbb{A}^2(\mathbb{C})$  as an affine part. We put

$$\tilde{\ell}_\infty(\mathbb{C}) := \mathbb{P}^2(\mathbb{C}) \setminus \mathbb{A}^2(\mathbb{C}).$$

For  $\ell_i(\mathbb{C}) \in \mathcal{A}_\mathbb{C}$ , we denote by  $\tilde{\ell}_i(\mathbb{C})$  the projective completion of  $\ell_i(\mathbb{C})$  in  $\mathbb{P}^2(\mathbb{C})$ , and put

$$\tilde{\mathcal{A}}_\mathbb{C} := \begin{cases} \{\tilde{\ell}_i(\mathbb{C}) \mid \ell_i(\mathbb{C}) \in \mathcal{A}_\mathbb{C}\} \cup \{\tilde{\ell}_\infty(\mathbb{C})\} & \text{if } |\mathcal{A}| \text{ is odd,} \\ \{\tilde{\ell}_i(\mathbb{C}) \mid \ell_i(\mathbb{C}) \in \mathcal{A}_\mathbb{C}\} & \text{if } |\mathcal{A}| \text{ is even.} \end{cases}$$

Let  $\tilde{B}(\mathbb{C})$  be the union of the complex projective lines in  $\tilde{\mathcal{A}}_\mathbb{C}$ . Then  $\deg \tilde{B}(\mathbb{C})$  is even. We consider the morphisms

$$(9.3) \quad \tilde{X} \xrightarrow{\tilde{\rho}} \tilde{W} \xrightarrow{\tilde{\pi}} \mathbb{P}^2(\mathbb{C}),$$

where  $\tilde{\pi}: \tilde{W} \rightarrow \mathbb{P}^2(\mathbb{C})$  is the double covering whose branch locus is equal to  $\tilde{B}(\mathbb{C})$ , and  $\tilde{\rho}: \tilde{X} \rightarrow \tilde{W}$  is the minimal desingularization. We put

$$\Lambda_\infty := \tilde{\rho}^{-1}(\tilde{\pi}^{-1}(\tilde{\ell}_\infty(\mathbb{C}))) \subset \tilde{X}.$$

Then we have  $X = \tilde{X} \setminus \Lambda_\infty$ , and the inclusion  $\iota: X \hookrightarrow \tilde{X}$  induces natural homomorphisms

$$\iota_* : H_2(X; \mathbb{Z}) \rightarrow H_2(\tilde{X}; \mathbb{Z}) \rightarrow H_2(\tilde{X}; \mathbb{Z})_{\text{fr}} := H_2(\tilde{X}; \mathbb{Z}) / \text{torsion},$$

which preserves the intersection form. Since  $\tilde{X}$  is smooth and compact, the intersection form makes the free  $\mathbb{Z}$ -module  $H_2(\tilde{X}; \mathbb{Z})_{\text{fr}}$  a unimodular lattice. Let

$$H_\infty \subset H_2(\tilde{X}; \mathbb{Z})_{\text{fr}}$$

be the submodule generated by the classes of irreducible components of  $\Lambda_\infty$ . Then the image of  $\iota_*$  is equal to the orthogonal complement  $H_\infty^\perp$  of  $H_\infty$  in  $H_2(\tilde{X}; \mathbb{Z})_{\text{fr}}$  defined by (9.1). Suppose that  $H_\infty$  is a sublattice of  $H_2(\tilde{X}; \mathbb{Z})$ , that is, the intersection form restricted to  $H_\infty$  is non-degenerate. Then the intersection form restricted to the image  $H_\infty$  of  $\iota_*$  is also non-degenerate, and hence the kernel of  $\iota_*$  is equal to  $\text{Ker} \langle \cdot \rangle$  defined by (9.2). Therefore the two lattices  $\overline{H}(X)$  and  $H_\infty^\perp$  are isomorphic:

$$(9.4) \quad \overline{H}(X) \cong H_\infty^\perp.$$

**9.3. The lattices  $H_2(\tilde{X}; \mathbb{Z})$  and  $H_\infty^\perp$ .** In this section, we consider arrangements  $\mathcal{A}$  with the following property:

(9.5) For any  $\lambda_i(\mathbb{R}) \in \mathcal{A}$ , at most one line in  $\mathcal{A} \setminus \{\lambda_i(\mathbb{R})\}$  is parallel to  $\lambda_i(\mathbb{R})$ .

We put  $N := |\mathcal{A}|$ , and assume  $N \geq 3$ . We then put

$$\tilde{N} := |\tilde{\mathcal{A}}_{\mathbb{C}}| = \begin{cases} N + 1 & \text{if } N \text{ is odd,} \\ N & \text{if } N \text{ is even.} \end{cases}$$

Suppose that  $\mathcal{A}$  has exactly  $p$  parallel pairs, where  $2p \leq N$ . Then we have

$$(9.6) \quad |\mathbf{Ch}_b| = \frac{(N-1)(N-2)}{2} - p, \quad |\mathcal{P}| = \frac{N(N-1)}{2} - p.$$

**Proposition 9.4.** *Suppose that (9.5) holds and that  $N \geq 3$ . Let  $p$  be the number of parallel pairs in  $\mathcal{A}$ .*

(1) *The homology group  $H_2(\tilde{X}; \mathbb{Z})$  is torsion-free, and the unimodular lattice  $H_2(\tilde{X}; \mathbb{Z})$  is of rank  $\tilde{N}^2 - 3\tilde{N} + 4$  with signature*

$$\left( \frac{1}{4}\tilde{N}^2 - \frac{3}{2}\tilde{N} + 3, \frac{3}{4}\tilde{N}^2 - \frac{3}{2}\tilde{N} + 1 \right).$$

(2) *The submodule  $H_\infty$  of  $H_2(\tilde{X}; \mathbb{Z})$  is a sublattice of rank*

$$r_\infty := \text{rank } H_\infty = \begin{cases} 1 + N + 2p & \text{if } N \text{ is odd,} \\ 1 + p & \text{if } N \text{ is even and } N \neq 2p, \\ 2 + p & \text{if } N = 2p, \end{cases}$$

*with the signature  $(1, r_\infty - 1)$ . The discriminant group is given by*

$$\text{disc}(H_\infty) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{N-1} & \text{if } N \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^{p+1} & \text{if } N \text{ is even and } N \neq 2p, \\ (\mathbb{Z}/2\mathbb{Z})^{p-1} \times (\mathbb{Z}/2(p-1)\mathbb{Z}) & \text{if } N = 2p. \end{cases}$$

By Theorem 1.3, we have

$$\text{rank } H_2(X; \mathbb{Z}) = |\mathbf{Ch}_b| + |\mathcal{P}| = (N-1)^2 - 2p.$$

Suppose that  $N$  is odd  $\geq 3$ . By Proposition 9.4, we have

$$\text{rank } H_2(\tilde{X}; \mathbb{Z}) - \text{rank } H_\infty = \tilde{N}^2 - 3\tilde{N} + 4 - (1 + N + 2p).$$

Since these two are equal, the equality (9.4) implies  $\text{Ker} \langle \quad \rangle = 0$ . Therefore we obtain the following:

**Corollary 9.5.** *Suppose that (9.5) holds and that  $N$  is odd  $\geq 3$ . Then the intersection form  $\langle \quad \rangle$  on  $H_2(X; \mathbb{Z})$  is non-degenerate.  $\square$*

*Proof of Proposition 9.4.* Let  $\mathcal{P}_\infty$  denote the set of singular points of  $\tilde{B}(\mathbb{C})$  located on  $\tilde{\ell}_\infty(\mathbb{C})$ . By assumption (9.5), it follows that  $\mathcal{P}_\infty$  consists of simple singular points of type  $a_1$  (ordinary double points) and/or of type  $d_4$  (ordinary triple points). Let

$$\mathcal{P}_\infty = \mathcal{P}_{\infty, a1} \sqcup \mathcal{P}_{\infty, d4}$$

be the decomposition of  $\mathcal{P}_\infty$  according to the types. We have the following.

- If  $N$  is odd, then  $|\mathcal{P}_{\infty, a1}| = N - 2p$  and  $|\mathcal{P}_{\infty, d4}| = p$ .
- If  $N$  is even, then  $|\mathcal{P}_{\infty, a1}| = p$  and  $|\mathcal{P}_{\infty, d4}| = 0$ .

In particular, the branch curve  $\widetilde{B}(\mathbb{C})$  of the double covering  $\tilde{\pi}: \widetilde{W} \rightarrow \mathbb{P}^2(\mathbb{C})$  has only simple singularities, and hence  $\widetilde{W}$  has only rational double points as its singularities. By the theory of simultaneous resolution of rational double points of algebraic surfaces, we see that  $\widetilde{X}$  is diffeomorphic to the double cover  $\widetilde{X}'$  of  $\mathbb{P}^2(\mathbb{C})$  branching along a *smooth* projective plane curve of degree  $\widetilde{N}$ . Since  $H_2(\widetilde{X}'; \mathbb{Z})$  is torsion-free, so is  $H_2(\widetilde{X}; \mathbb{Z})$ . We can calculate the second Betti number  $b_2$  and the geometric genus  $p_g$  of  $\widetilde{X}'$  easily. Note that the signature of  $H_2(\widetilde{X}'; \mathbb{Z})$  is

$$(1 + 2p_g, b_2 - 1 - 2p_g).$$

Therefore we obtain a proof of assertion (1).

For  $P \in \mathcal{P}_{\infty, a1}$ , let  $a(P)$  denote the smooth rational curve on  $\widetilde{X}$  that is contracted to  $P$ . We have  $\langle a(P), a(P) \rangle = -2$ . For  $Q \in \mathcal{P}_{\infty, d4}$ , there exist four smooth rational curves

$$d_1(Q), d_2(Q), d_3(Q), d_4(Q)$$

on  $\widetilde{X}$  that are contracted to  $Q$ . These curves have the self-intersection  $(-2)$ , and form the Coxeter-Dynkin diagram of type  $D_4$  as the dual graph. Let  $R_\infty$  denote the reduced part of the strict transform of  $\tilde{\ell}_\infty(\mathbb{C})$  in  $\widetilde{X}$ . Note that  $R_\infty$  is irreducible if and only if  $N \neq 2p$ . Then  $H_\infty$  is generated by

- (a) the classes of  $a(P)$ , where  $P$  runs through  $\mathcal{P}_{\infty, a1}$ ,
- (d) the classes of  $d_1(Q), \dots, d_4(Q)$ , where  $Q$  runs through  $\mathcal{P}_{\infty, d4}$ , and
- (R) the classes of irreducible components of  $R_\infty$ .

Let  $H'_\infty$  be the submodule of  $H_\infty$  generated by the classes in (a) and (d) above, and

- (L) the class of the total transform  $\mathcal{L}_\infty$  of  $\tilde{\ell}_\infty(\mathbb{C})$  in  $\widetilde{X}$ .

Since we have  $\langle \mathcal{L}_\infty, \mathcal{L}_\infty \rangle = 2$  and  $\mathcal{L}_\infty$  is orthogonal to the classes of exceptional curves in (a) and (d) above, it follows that  $H'_\infty$  is a lattice with

$$\text{rank } H'_\infty = 1 + |\mathcal{P}_{\infty, a1}| + 4|\mathcal{P}_{\infty, d4}|, \quad \text{disc } H'_\infty \cong (\mathbb{Z}/2\mathbb{Z})^{1+|\mathcal{P}_{\infty, a1}|+2|\mathcal{P}_{\infty, d4}|}.$$

Suppose that  $N$  is odd. Then  $R_\infty$  is irreducible and the multiplicity of  $R_\infty$  in  $\mathcal{L}_\infty$  is 2. Hence  $H'_\infty$  is a submodule of index 2 in  $H_\infty$ . Therefore  $H_\infty$  is a lattice, and the rank and the discriminant group of  $H_\infty$  can be derived from those of  $H'_\infty$ .

Suppose that  $N$  is even and  $2p < N$ . Then  $R_\infty$  is irreducible and the multiplicity of  $R_\infty$  in  $\mathcal{L}_\infty$  is 1. Hence we have  $H'_\infty = H_\infty$ .

Suppose that  $N = 2p$ . Then  $R_\infty$  is a disjoint union of two smooth rational curves  $R_{\infty,+}$  and  $R_{\infty,-}$ . For  $P \in \mathcal{P}_{\infty, a1}$ , we have  $\langle a(P), R_{\infty,+} \rangle = \langle a(P), R_{\infty,-} \rangle = 1$ . Using

$$\mathcal{L}_\infty = R_{\infty,+} + R_{\infty,-} + \sum_{P \in \mathcal{P}_{\infty, a1}} a(P)$$

and  $\langle \mathcal{L}_\infty, \mathcal{L}_\infty \rangle = 2$ , we obtain

$$\langle R_{\infty,+}, R_{\infty,+} \rangle = \langle R_{\infty,-}, R_{\infty,-} \rangle = 1 - p.$$

By these formulas, we see that  $H_\infty$  is a lattice of rank  $2 + |\mathcal{P}_{\infty, a1}|$ . From the Gram matrix of  $H_\infty$ , we can compute the discriminant group. The computation is left to the reader.

Since  $H_\infty$  is contained in the Hodge part  $H^{1,1}(\widetilde{X})$  of  $H^2(\widetilde{X}; \mathbb{C})$  and  $\mathcal{L}_\infty \in H_\infty$  is an ample class, we see that the signature of  $H_\infty$  is  $(1, r_\infty - 1)$ .  $\square$

**9.4. Experiments.** For fixed  $N \geq 3$  and  $p \leq N/2$ , we randomly generate a nodal real line arrangement  $\mathcal{A}$  satisfying (9.5), and calculate the signature and the discriminant group of  $\overline{H}(X)$  topologically by Theorem 1.7. We compare the result with the signature and the discriminant group of  $H_\infty$  computed by Proposition 9.4. We see that the equality of signatures holds for  $\overline{H}(X)$  and  $H_\infty^\perp$ , as is expected from (9.4), and that  $\text{disc}(\overline{H}(X))$  is a sub-quotient of  $\text{disc}(H_\infty)$ , as is expected by (9.4) and Corollary 9.3.

*Remark 9.6.* In fact, for all examples we computed, we obtain an isomorphism  $\text{disc}(\overline{H}(X)) \cong \text{disc}(H_\infty)$ , and hence, in these cases, the sublattice  $H_\infty$  is primitive in  $H_2(\tilde{X}; \mathbb{Z})$ .

**Example 9.7.** Suppose that  $N = 6$  and  $p = 0$ . We have

$$|\mathbf{Ch}_b| = 10, \quad |\mathcal{P}| = 15,$$

and hence  $H_2(X; \mathbb{Z})$  is of rank 25. On the other hand, the unimodular lattice  $H_2(\tilde{X}; \mathbb{Z})$  is of rank 22 with signature (3, 19), and the sublattice  $H_\infty$  is of rank 1 with signature (1, 0) and  $\text{disc}(H_\infty) \cong \mathbb{Z}/2\mathbb{Z}$ . Hence  $H_\infty^\perp$  is of rank 21 with signature (2, 19).

For randomly generated arrangements satisfying (9.5) with  $N = 6$  and  $p = 0$ , we checked that  $\overline{H}(X)$  is of rank 21 with signature (2, 19) and  $\text{disc}(\overline{H}(X)) \cong \mathbb{Z}/2\mathbb{Z}$ . Remark that there exist several combinatorial structures of nodal arrangements of six real lines with no parallel pairs. For example, the numbers of  $n$ -gons in  $\mathbf{Ch}_b$  can vary as follows:

$n$	3	4	5	6	$n$	3	4	5	6
	4	4	2	0		5	4	0	1
	4	5	1	0		6	2	2	0
	4	5	0	1		6	3	1	0
	4	6	0	0		6	3	0	1
	5	3	2	0		7	0	3	0
	5	4	1	0					

**Example 9.8.** Suppose that  $N = 6$  and  $p = 3$ . We have

$$|\mathbf{Ch}_b| = 7, \quad |\mathcal{P}| = 12,$$

and hence  $H_2(X; \mathbb{Z})$  is of rank 19. On the other hand, the unimodular lattice  $H_2(\tilde{X}; \mathbb{Z})$  is of rank 22 with signature (3, 19), and the lattice  $H_\infty$  is of rank 5 with signature (1, 4) and  $\text{disc}(H_\infty) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$ . Hence  $H_\infty^\perp$  is of rank 17 with signature (2, 15).

For randomly generated such arrangements, we checked that  $\overline{H}(X)$  is of rank 17 with signature (2, 15) and  $\text{disc}(\overline{H}(X)) \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$ , regardless of combinatorial structures of  $\mathbf{Ch}_b$ .

**Example 9.9.** Suppose that  $N = 24$  and  $p = 10$ . We have

$$|\mathbf{Ch}_b| = 243, \quad |\mathcal{P}| = 266,$$

and hence  $H_2(X; \mathbb{Z})$  is of rank 509. On the other hand, the unimodular lattice  $H_2(\tilde{X}; \mathbb{Z})$  is of rank 508 with signature (111, 397), and the sublattice  $H_\infty$  is of rank 11 with signature (1, 10) and  $\text{disc}(H_\infty) \cong (\mathbb{Z}/2\mathbb{Z})^{11}$ . Hence  $H_\infty^\perp$  is of rank 497 with signature (110, 387). For randomly generated such arrangements, we obtained expected signature (110, 387), and  $\text{disc}(\overline{H}(X)) \cong (\mathbb{Z}/2\mathbb{Z})^{11}$ .

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