ZARISKI PAIRS ON CUBIC SURFACES

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Abstract. A line arrangement of a smooth cubic surface is a subset of the set of lines on the cubic surface. We define a notion of Zariski pairs of line arrangements on general cubic surfaces, and make the complete list of these Zariski pairs.

1. Introduction

We work over the complex number field $\mathbb{C}$. Cayley and Salmon showed in 1849 that every smooth cubic surface contains exactly 27 lines. The configuration of these 27 lines is a beautiful historical topic of algebraic geometry. In this paper, we investigate this configuration from the viewpoint of *Zariski pairs*.

By a *plane curve*, we mean a reduced, possibly reducible, projective plane curve. We say that a pair (C_1, C_2) of plane curves is a *Zariski pair* if C_1 and C_2 have the same combinatorial type of singularities, but have different embedded topologies in the projective plane. This notion of Zariski pairs was formulated in Artal-Bartolo's seminal paper [1], in which he investigated a pair of 6-cuspidal sextics discovered by Zariski in 1929, and presented some new examples. Since then, many authors have studied Zariski pairs of plane curves from various points of view. See, for example, the survey [2].

We introduce a notion of *Zariski pairs of line arrangements on general cubic surfaces*.

Definition 1.1. A point Q of a smooth cubic surface X is said to be an *Eckardt point* if three lines on X pass through Q.

A general cubic surface has no Eckardt points. Let $X \subset \mathbb{P}^3$ be a smooth cubic surface with no Eckardt points, and let $L(X)$ denote the set of lines on X. We express the configuration of lines on X by the intersection form

$$\langle \ell, \ell' \rangle := \begin{cases} -1 & \text{if } \ell = \ell', \\ 0 & \text{if } \ell \neq \ell', \text{ and } \ell \text{ and } \ell' \text{ are disjoint}, \\ 1 & \text{if } \ell \neq \ell', \text{ and } \ell \text{ and } \ell' \text{ intersect} \end{cases}$$

for $\ell, \ell' \in L(X)$.

Definition 1.2. A *line arrangement on a general cubic surface* is a pair $[S, X]$ of a smooth cubic surface X with no Eckardt points and a subset S of $L(X)$. In this situation, we say that S is a *line arrangement on* X. We denote by $\mathcal{A}$ the set of line arrangements on general cubic surfaces.

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We introduce three equivalence relations $\sim_d$, $\sim_c$, and $\sim_t$ on $\mathcal{A}$.

Definition 1.3. Let $[S, X]$ and $[S', X']$ be elements of $\mathcal{A}$.

- We say that $[S, X]$ and $[S', X']$ are *deformation equivalent* and write
$$[S, X] \sim_d [S', X']$$
if there exists a continuous family $\mathcal{X} := \{ X_t \mid t \in [0,1] \}$ of smooth cubic surfaces with no Eckardt points connecting $X = X_0$ and $X' = X_1$ such that S is deformed continuously to S' along $\mathcal{X}$. We denote by $[S, X]_d$ the equivalence class containing $[S, X]$ under the equivalence relation $\sim_d$.
- We say that $[S, X]$ and $[S', X']$ have the *same embedded topology* and write
$$[S, X] \sim_t [S', X']$$
if there exists a homeomorphism $X \xrightarrow{\sim} X'$ that maps the union $\Lambda(S) \subset X$ of lines in S to the union $\Lambda(S') \subset X'$ of lines in S'. We denote by $[S, X]_t$ the equivalence class containing $[S, X]$ under the equivalence relation $\sim_t$.
- We say that $[S, X]$ and $[S', X']$ have the *same combinatorial type* and write
$$[S, X] \sim_c [S', X']$$
if there exists a bijection between S and S' that preserves the intersection form $\langle\ ,\ \rangle$. We denote by $[S, X]_c$ the equivalence class containing $[S, X]$ under the equivalence relation $\sim_c$.

It is obvious that we have the following implications:
$$[S, X] \sim_d [S', X'] \implies [S, X] \sim_t [S', X'] \implies [S, X] \sim_c [S', X'].$$
Therefore we have natural surjections
$$\mathcal{A}/\!\sim_d \;\twoheadrightarrow\; \mathcal{A}/\!\sim_t \;\twoheadrightarrow\; \mathcal{A}/\!\sim_c.$$
Following the definition of Zariski pairs of plane curves, we make the following:

Definition 1.4. We say that two equivalence classes $[S, X]_d$ and $[S', X']_d$ form a *Zariski pair of line arrangements on general cubic surfaces* (a Zariski pair in $\mathcal{A}$, for short) if $[S, X]$ and $[S', X']$ have the same combinatorial type, but have different embedded topologies.

We choose and fix a smooth cubic surface X with no Eckardt points, and denote by $\mathcal{A}_X := 2^{L(X)}$ the set of line arrangements on X. Since smooth cubic surfaces with no Eckardt points are parameterized by a smooth connected variety, the inclusion $\mathcal{A}_X \hookrightarrow \mathcal{A}$ induces a bijection
$$\mathcal{A}_X/\!\sim_d \;\cong\; \mathcal{A}/\!\sim_d.$$
Since $\mathcal{A}_X$ is finite, we can regard Zariski pairs in $\mathcal{A}$ as a *toy model* of classical Zariski pairs of plane curves. In fact, we can enumerate all Zariski pairs in $\mathcal{A}$ by a brute force method. This complete list is the main result of this note.

To distinguish embedded topologies, we use the lattice structure on the middle cohomology group $H^2(X, \mathbb{Z})$ of a smooth cubic surface X. The cup-product $\langle\ ,\ \rangle$ makes $H^2(X, \mathbb{Z})$ a unimodular lattice of rank 7. For a line arrangement S on X, let $H(S) \subset H^2(X, \mathbb{Z})$ denote the submodule generated by the classes of lines in S, and we put
$$H(S)^\perp := \{ x \in H^2(X, \mathbb{Z}) \mid \langle x, y \rangle = 0 \text{ for all } y \in H(S) \}.$$

Recall that a lattice M is said to be *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ holds for all $x \in M$, and to be *odd* otherwise.

Our result is as follows:

Theorem 1.5. *There exist exactly two Zariski pairs*

$$([S_1, X]_d,\ \ [S_2, X]_d) \qquad \textit{and} \qquad ([T_1, X]_d,\ \ [T_2, X]_d)$$

of line arrangements on general cubic surfaces.

(1) *The combinatorial type of S_i is as follows. We have $|S_i| = 5$, and any distinct lines $\ell, \ell' \in S_i$ are disjoint. The embedded topologies of S_1 and S_2 are distinguished by the fact that $H(S_1)^\perp$ is odd, whereas $H(S_2)^\perp$ is even.*

(2) *The combinatorial type of T_i is as follows. We have $|T_i| = 6$, and, for $\mu \neq \nu$,*

$$\langle \ell_\mu, \ell_\nu\rangle = \begin{cases} 1 & \text{if } \mu = 0 \text{ or } \nu = 0, \\ 0 & \text{if } \mu \neq 0 \text{ and } \nu \neq 0 \end{cases}$$

holds under a suitable numbering $\ell_0, \dots, \ell_5$ of elements of T_i. The embedded topologies of T_1 and T_2 are distinguished by the fact that $H_1(X \setminus \Lambda(T_1), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_1(X \setminus \Lambda(T_2), \mathbb{Z}) = 0$.

The main ingredient of the proof is the result of Harris [5] on the Galois group of the 27 lines on a cubic surface X. We write the Galois action on $L(X)$ explicitly, and calculate the orbit-decomposition of $\mathcal{A}_X = 2^{L(X)}$. Comparing the combinatorial types and the embedded topologies of these orbits, we obtain Theorem 1.5.

In [8], by the result of Harris [5] on the Galois group of the 28 bitangents of a smooth quartic plane curve, we obtained many Zariski multiples of plane curves. For general methods to distinguish embedded topologies by lattices, see [7].

For the actual computation, we used `GAP` [4]. In [9], we present a detailed computation data.

Convention. The orthogonal group $\mathrm{O}(M)$ of a lattice M acts on M from the right. The symmetric group $\mathfrak{S}(T)$ of a finite set T also acts on T from the right.

2. The 27 lines on a cubic surface

In this section, we recall some basic facts about cubic surfaces and review the result of Harris [5]. For a general theory of cubic surfaces, we refer the reader to Chapter 9 of Dolgachev [3]. Harris [5] showed that the Galois group of the 27 lines on a cubic surface is isomorphic to the odd orthogonal group $\mathrm{O}^-(6, \mathbb{F}_2)$. We rewrite the statement in terms of the Weyl group $W(E_6)$ of type E_6, which is isomorphic to $\mathrm{O}^-(6, \mathbb{F}_2)$, and describe the Galois action on the 27 lines *explicitly*.

2.1. **Action of $W(E_6)$ on the 27 lines.** Let $P_1, \dots, P_6$ be general six points of $\mathbb{P}^2$, and let $X \to \mathbb{P}^2$ be the blowing-up at $P_1, \dots, P_6$. For a divisor D on X, let $[D] \in H^2(X, \mathbb{Z})$ denote its class. Then $D \mapsto [D]$ induces an isomorphism from the Picard group $\operatorname{Pic} X$ with the intersection form to $H^2(X, \mathbb{Z})$ with the cup-product $\langle\ ,\ \rangle$. From now on, we identify $\operatorname{Pic}(X)$ with $H^2(X, \mathbb{Z})$. Let $h \in H^2(X, \mathbb{Z})$ be the class of the pull-back of a line on $\mathbb{P}^2$, and let $e_i := [E_i]$ be the class of the exceptional curve E_i over P_i for $i = 1, \dots, 6$. The lattice $H^2(X, \mathbb{Z})$ is of rank 7 with a basis $h, e_1, \dots, e_6$, under which the Gram matrix is given by the diagonal matrix

$$\operatorname{diag}(1, -1, -1, -1, -1, -1, -1).$$

We express elements of $H^2(X, \mathbb{Z})$ as vectors with respect to this basis. The class of the anti-canonical line bundle $-K_X$ is

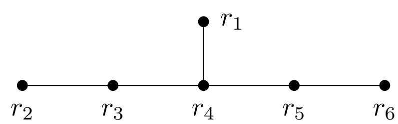
$$[-K_X] = (3, -1, -1, -1, -1, -1, -1).$$

We have $\langle -K_X, -K_X \rangle = 3$, and the complete linear system $|-K_X|$ embeds X into $\mathbb{P}^3$ as a smooth cubic surface. We denote by K the sublattice of $H^2(X, \mathbb{Z})$ generated by $[-K_X]$, and by V the orthogonal complement of K in $H^2(X, \mathbb{Z})$. Then V is a negative-definite root lattice of type E_6. Indeed, the (-2)-vectors

$$\begin{aligned} r_1 &:= (-1, 0, 0, 0, 1, 1, 1), \\ r_2 &:= (0, 1, -1, 0, 0, 0, 0), \\ r_3 &:= (0, 0, 1, -1, 0, 0, 0), \\ r_4 &:= (0, 0, 0, 1, -1, 0, 0), \\ r_5 &:= (0, 0, 0, 0, 1, -1, 0), \\ r_6 &:= (0, 0, 0, 0, 0, 1, -1) \end{aligned}$$

constitute a basis of the lattice V, and form the dual graph

(2.1)



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which is the Dynkin diagram of type E_6. Hence we have

$$\mathrm{O}(V) = W(E_6) \rtimes \langle g_0 \rangle,$$

where $W(E_6) \subset \mathrm{O}(V)$ is the Weyl group of type E_6 generated by the reflections

$$\sigma_\nu \colon x \mapsto x + \langle x, r_\nu \rangle r_\nu \qquad (\nu = 1, \dots, 6)$$

with respect to the roots $r_1, \dots, r_6 \in V$, and g_0 is the involution of V given by

$$r_1 \leftrightarrow r_1, \quad r_2 \leftrightarrow r_6, \quad r_3 \leftrightarrow r_5, \quad r_4 \leftrightarrow r_4,$$

which corresponds to the automorphism of the graph (2.1). Note that the order of $W(E_6)$ is 51840.

Our goal is to construct an action

$$W(E_6) \to \mathfrak{S}(L(X)) \tag{2.2}$$

of the subgroup $W(E_6) \subset \mathrm{O}(V)$ on the set $L(X)$ of lines on X.

Definition 2.1. For a lattice M, we denote by $M^\vee$ the *dual lattice*

$$\{\, x \in M \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z} \ \text{ for all } \ y \in M \,\},$$

and by $\mathrm{disc}(M)$ the cokernel $M^\vee / M$ of the natural embedding $M \hookrightarrow M^\vee$. We call $\mathrm{disc}(M)$ the *discriminant group* of M.

We refer the reader to [6] for a general theory of discriminant groups. The discriminant group $\mathrm{disc}(K)$ of the lattice $K = \mathbb{Z}[-K_X]$ of rank 1 is a cyclic group of order 3 generated by $[-K_X]/3 \pmod{K}$. We have inclusions

$$K \oplus V \ \subset \ H^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})^\vee \ \subset \ K^\vee \oplus V^\vee.$$

Since K and V are primitive in $H^2(X, \mathbb{Z})$, the submodule

$$H^2(X, \mathbb{Z}) / (K \oplus V) \ \subset \ \mathrm{disc}(K) \times \mathrm{disc}(V)$$

is the graph of an isomorphism

$$\gamma_H \colon \operatorname{disc}(V) \xrightarrow{\sim} \operatorname{disc}(K).$$

The normal subgroup $W(E_6)$ of $\mathrm{O}(V)$ is the kernel of the natural homomorphism

$$\mathrm{O}(V) \to \operatorname{Aut}(\operatorname{disc}(V)) = \{\pm 1\}.$$

Indeed, the reflections $\sigma_1, \dots, \sigma_6 \in \mathrm{O}(V)$ act on $\operatorname{disc}(V)$ trivially, whereas the involution $g_0 \in \mathrm{O}(V)$ acts on $\operatorname{disc}(V)$ by -1. Therefore we have

$$W(E_6) = \left\{ g \in \mathrm{O}(V) \;\middle|\; \begin{array}{l} \text{the isometry } g \text{ extends to an isometry } \tilde{g} \text{ of} \\ H^2(X,\mathbb{Z}) \text{ that acts on } K \text{ trivially} \end{array} \right\}.$$

By $g \mapsto \tilde{g}$, we obtain an isomorphism

$$W(E_6) \xrightarrow{\sim} \{\, \tilde{g} \in \mathrm{O}(H^2(X,\mathbb{Z})) \mid [-K_X]^{\tilde{g}} = [-K_X] \,\}. \tag{2.3}$$

By $\ell \mapsto [\ell]$, we embed $L(X)$ into $H^2(X,\mathbb{Z})$. Its image is

$$L' := \{\, x \in H^2(X,\mathbb{Z}) \mid \ \langle x, x\rangle = -1, \ \ \langle x, [-K_X]\rangle = 1 \ \},$$

on which $W(E_6)$ acts by (2.3). Thus we obtain a homomorphism (2.2). By the projection $H^2(X,\mathbb{Z}) \to V^\vee$, the set L' is mapped bijectively to the set

$$L'' := \{\, v \in V^\vee \mid \ \langle v, v\rangle = -4/3, \quad \gamma_H(v \bmod V) = [-K_X]/3 \pmod{K} \ \},$$

where $\gamma_H \colon \operatorname{disc}(V) \xrightarrow{\sim} \operatorname{disc}(K)$ is the isomorphism defined above. Calculating the action on L'' of the reflections $\sigma_\nu \in W(E_6)$, we compute (2.2) explicitly as follows.

For $i = 1, \dots, 6$, let $\ell[i]$ denote the exceptional curve E_i over P_i. For i, j with $1 \le i < j \le 6$, let $\ell[ij]$ denote the strict transform of the line on $\mathbb{P}^2$ passing through P_i and P_j. For $k = 1, \dots, 6$, let $\ell[\bar{k}]$ denote the strict transform of the conic on $\mathbb{P}^2$ passing through the 5 points $\{P_1, \dots, P_6\} \setminus \{P_k\}$. The set $L(X)$ consists of these smooth rational curves. Their classes are

$$[\ell[i]] = e_i, \quad [\ell[ij]] = h - e_i - e_j, \quad [\ell[\bar{k}]] = 2h - (e_1 + \dots + e_6) + e_k.$$

We number elements of $L(X) = \{\ell_1, \dots, \ell_{27}\}$ as follows:

$$\begin{aligned}
&\ell_1 := \ell[1], \ \dots, \ \ell_6 := \ell[6], \\
&\ell_7 := \ell[12], \ \ell_8 := \ell[13], \ \ell_9 := \ell[14], \ \ell_{10} := \ell[15], \ \ell_{11} := \ell[16], \\
&\ell_{12} := \ell[23], \ \ell_{13} := \ell[24], \ \ell_{14} := \ell[25], \ \ell_{15} := \ell[26], \\
&\ell_{16} := \ell[34], \ \ell_{17} := \ell[35], \ \ell_{18} := \ell[36], \\
&\ell_{19} := \ell[45], \ \ell_{20} := \ell[46], \ \ell_{21} := \ell[56], \\
&\ell_{22} := \ell[\bar{1}], \ \dots, \ \ell_{27} := \ell[\bar{6}].
\end{aligned} \tag{2.4}$$

We let $\tau \in \mathfrak{S}_{27}$ act on $L(X)$ as $(\ell_i)^\tau := \ell_{(i^\tau)}$. Then the reflections σ_ν act on $L(X)$ by the following permutations.

$$\begin{array}{lll}
\sigma_1 & \mapsto & (4,21)(5,20)(6,19)(7,24)(8,23)(12,22), \\
\sigma_2 & \mapsto & (1,2)(8,12)(9,13)(10,14)(11,15)(22,23), \\
\sigma_3 & \mapsto & (2,3)(7,8)(13,16)(14,17)(15,18)(23,24), \\
\sigma_4 & \mapsto & (3,4)(8,9)(12,13)(17,19)(18,20)(24,25), \\
\sigma_5 & \mapsto & (4,5)(9,10)(13,14)(16,17)(20,21)(25,26), \\
\sigma_6 & \mapsto & (5,6)(10,11)(14,15)(17,18)(19,20)(26,27).
\end{array} \tag{2.5}$$

2.2. **Monodromy action on the 27 lines.** All cubic surfaces are parameterized by the projective space $\mathbb{P}^{19} = \mathbb{P}_*(H^0(\mathbb{P}^3, \mathcal{O}(3)))$. For $t \in \mathbb{P}^{19}$, let $X_t \subset \mathbb{P}^3$ denote the corresponding cubic surface. We put

$$\mathcal{U} := \{\, t \in \mathbb{P}^{19} \mid X_t \text{ is smooth}\,\}, \quad \mathcal{U}^0 := \{\, t \in \mathcal{U} \mid X_t \text{ has no Eckardt points}\,\},$$

which are Zariski open subsets of $\mathbb{P}^{19}$. We then put

$$\mathcal{L} := \{\, (t, \ell) \mid \ell \subset X_t \,\} \quad \subset \quad \mathcal{U} \times \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3),$$

where $\mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3)$ is the Grassmannian variety of lines in $\mathbb{P}^3$. The first projection $\pi_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{U}$ is an étale covering of degree 27, and the fiber L_t of $\pi_{\mathcal{L}}$ over $t \in \mathcal{U}$ is the set $L(X_t)$ of lines on the cubic surface X_t. Let $b \in \mathcal{U}^0$ be the point such that X_b is the cubic surface X fixed in the previous subsection. We have $X_b = X$ and $L_b = L(X)$.

Theorem 2.2 (Harris [5])**.** *The image of the monodromy action*

$$\mu_L \colon \pi_1(\mathcal{U}, b) \longrightarrow \mathfrak{S}(L_b)$$

associated with $\pi_{\mathcal{L}}$ is equal to the image of the homomorphism (2.2).

Proof. We consider the universal family

$$\pi \,:\, \mathcal{X} := \{\, (t, x) \in \mathcal{U} \times \mathbb{P}^3 \mid x \in X_t \,\} \;\to\; \mathcal{U}$$

of smooth cubic surfaces. We have a local constant system $R^2\pi_*\mathbb{Z}$ on $\mathcal{U}$, and $\ell \mapsto [\ell]$ gives an embedding $\mathcal{L} \hookrightarrow R^2\pi_*\mathbb{Z}$ over $\mathcal{U}$. The monodromy action

$$\mu_H \colon \pi_1(\mathcal{U}, b) \longrightarrow \mathrm{GL}(H^2(X_b, \mathbb{Z}))$$

associated with $R^2\pi_*\mathbb{Z}$ is compatible with μ_L via $L_b \hookrightarrow H^2(X_b, \mathbb{Z})$, and the image of μ_H is contained in the right-hand side of (2.3), because $[-K_{X_t}]$ is the class of a hyperplane section of $X_t \subset \mathbb{P}^3$ and hence is invariant under the monodromy. Therefore the image of μ_L is contained in the image of the embedding (2.2). Harris [5] proved that the size of the image of μ_L is $|\mathrm{O}^-(6, \mathbb{F}_2)| = 51840 = |W(E_6)|$. Hence the image of μ_L coincides with the image of (2.2). □

The inclusion $\mathcal{U}^0 \hookrightarrow \mathcal{U}$ induces a surjective homomorphism $\pi_1(\mathcal{U}^0, b) \twoheadrightarrow \pi_1(\mathcal{U}, b)$. Therefore, for $S_1, S_2 \in \mathcal{A}_{X_b} = 2^{L(X_b)}$, we see that $[S_1, X_b] \sim_d [S_2, X_b]$ holds if and only if S_1 and S_2 belong to the same $W(E_6)$-orbit under the action of $W(E_6)$ on $2^{L(X_b)}$ induced by (2.2).

Remark 2.3. The image of μ_L is maximal in the sense that

$$\mathrm{Im}\,\mu_L = \{\, \tau \in \mathfrak{S}(L_b) \mid \langle \ell, \ell' \rangle = \langle \ell^\tau, \ell'^\tau \rangle \;\text{ for all }\; \ell, \ell' \in L_b \,\}.$$

3. Orbit decomposition and Zariski pairs

3.1. $W(E_6)$**-orbits.** Recall that $X = X_b$. We calculate the orbit decomposition of $\mathcal{A}_X = 2^{L(X)}$ under the action of $W(E_6)$. By the numbering (2.4), a line arrangement on X is expressed as a subset of $\{1, \dots, 27\}$. We write a line arrangement $S \subset L(X)$ as an *increasing* sequence $[s_1, \dots, s_n]$ of integers in $\{1, \dots, 27\}$. In particular, for $S = [s_1, \dots, s_n]$ and $\gamma \in \mathfrak{S}_{27}$, we denote by S^γ the increasing sequence of integers obtained by *sorting* the set $\{s_1^\gamma, \dots, s_n^\gamma\}$. Let $\mathcal{C}_n \subset \mathcal{A}_X = 2^{L(X)}$ be the set of line arrangements consisting of n lines. We introduce the lexicographic order $\prec$ on each $\mathcal{C}_n$, that is, if

$$S^{(0)} := [s_1^{(0)}, \dots, s_n^{(0)}] \quad \text{and} \quad S^{(1)} := [s_1^{(1)}, \dots, s_n^{(1)}]$$

```
procedure IsMinimal (a line arrangement S = [s_1, ..., s_n])
    flag := true
    for all γ ∈ W(E_6) do
        if S^γ ≺ S then
            flag := false, and break from the for-loop
        end if
    end for
    if flag then
        Append S to minreps
        if n = 0 then m := 0
        else  m := s_n
        end if
        for all t from m + 1 to 27 do
            S' := [s_1, ..., s_n, t]
            IsMinimal(S')
        end for
    end if
end procedure
```

PROCEDURE 3.1. `IsMinimal`

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
|orbits|	1	1	2	4	8	18	39	73	135	234	363	509	641	715

n	14	15	16	17	18	19	20	21	22	23	24	25	26	27
|orbits|	715	641	509	363	234	135	73	39	18	8	4	2	1	1

TABLE 3.1. Number of orbits $|\mathcal{C}_n/{\sim_d}|$

are distinct elements of $\mathcal{C}_n$, then we say $S^{(0)} \prec S^{(1)}$ if and only if, for the smallest index i such that $s_i^{(0)} \neq s_i^{(1)}$, we have $s_i^{(0)} < s_i^{(1)}$. A line arrangement $S \subset L(X)$ is said to be *minimal* if S is minimal with respect to $\prec$ in the orbit

$$o(S) := \{\, S^\gamma \mid \gamma \in W(E_6)\,\}.$$

Every $W(E_6)$-orbit in $\mathcal{A}_X$ contains a unique minimal element. Note that, if $S = [s_1, \dots, s_n]$ is minimal, then so is $[s_1, \dots, s_m]$ for any $m < n$. Therefore, by setting `minreps` := [ ] and putting the empty line arrangement [ ] to the recursive procedure `IsMinimal` given in Procedure 3.1, we obtain the complete list `minreps` of minimal representatives of $W(E_6)$-orbits. Thus we obtain 5486 orbits as is given in Table 3.1.

3.2. **Zariski pairs.** We compare the combinatorial types of all $W(E_6)$-orbits in $\mathcal{A}_X$. It turns out that the natural surjection $\mathcal{A}_X/{\sim_d} \twoheadrightarrow \mathcal{A}_X/{\sim_c}$ has exactly two fibers of size > 1. Each of them is of size 2, and the two elements in the fiber have different embedded topologies, as is shown in Sections 3.2.1 and 3.2.2 below. As a corollary, we obtain the following:

Corollary 3.1. *The equivalence relations $\sim_d$ and $\sim_t$ on $\mathcal{A}$ are the same.* □

3.2.1. *A pair of arrangements of* 5 *lines.* Let $o(S_1)$ and $o(S_2)$ be the $W(E_6)$-orbits whose minimal representatives are

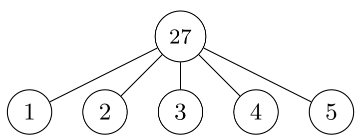
$$S_1 = [1, 2, 3, 4, 5] \quad \text{and} \quad S_2 = [1, 2, 3, 4, 21],$$

respectively. We have $|o(S_1)| = 432$ and $|o(S_2)| = 216$. Each of these line arrangements consists of disjoint 5 lines, and hence they have the same combinatorial type. (Recall that $\ell_{21} = \ell[56]$.) On the other hand, we see that $H(S_1)^\perp$ is an odd lattice and $H(S_2)^\perp$ is an even lattice. Therefore they have different embedded topologies.

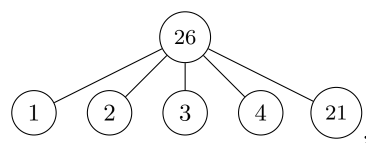
3.2.2. *A pair of arrangements of* 6 *lines.* Let $o(T_1)$ and $o(T_2)$ be the $W(E_6)$-orbits whose minimal representatives are

$$T_1 = [1, 2, 3, 4, 5, 27] \quad \text{and} \quad T_2 = [1, 2, 3, 4, 21, 26],$$

respectively. We have $|o(T_1)| = |o(T_2)| = 432$. Their combinatorial types are given by the dual graphs



and



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respectively, and hence they have the same combinatorial type. (Recall that $\ell_{21} = \ell[56]$, $\ell_{26} = \ell[\bar{5}]$ and $\ell_{27} = \ell[\bar{6}]$.) We have

$$H_1(X \backslash \Lambda(T_i)) \cong H^3(X, \Lambda(T_i)) \cong \operatorname{Coker}(H^2(X) \to \bigoplus_{\ell \in T_i} H^2(\ell)) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } i = 1, \\ 0 & \text{for } i = 2. \end{cases}$$

Here we omit $\mathbb{Z}$ in the (co)homology groups. Therefore they have different embedded topologies.

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