DEL PEZZO SURFACES OF DEGREE ONE AND EXAMPLES OF ZARISKI MULTIPLES: COMPUTATIONAL DATA

ICHIRO SHIMADA

In the files

- `CompDataWE8.txt` (60 KB),
- `OrbitDecompbarDelta.txt` (7.4 MB),
- `OrbitDecompDelta.txt` (5.3 MB), and
- `F19model.txt` (2 KB),

we present the computational data used in the preprint

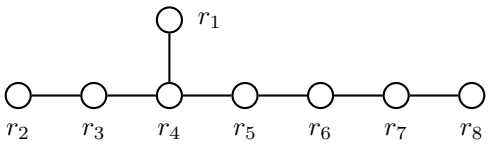
- Ichiro Shimada. Del Pezzo surfaces of degree one and examples of Zariski multiples. Preprint.

This document explains the structure and contents of the data. Some parts of the data are formatted using the `Record` format of `GAP`. In the following, we freely use the notation introduced in the preprint above.

1. Basis data in `CompDataWE8.txt`

The file "`CompDataWE8.txt`" contains the following data. Let X be a del Pezzo surface of degree 1 obtained by blowing-up $\beta\colon X \to \mathbf{P}^2$ of $\mathbf{P}^2$ at $p_1, \ldots, p_8$.

- `GramPic` is the Gram matrix of $\mathrm{Pic}(X)$ with respect to the basis $h, e_1, \ldots, e_8$, where h is the pull-back of the class of a line of $\mathbf{P}^2$ and e_i is the class of the exceptional curve over p_i.
- `GramE8` is the Gram matrix of the negative-definite root lattice $\mathbb{E}_8$ of type E_8 with respect to the basis $r_1, \ldots, r_8$ whose dual graph is the Dynkin diagram of type E_8 as below.



Supported by JSPS KAKENHI Grant Number 20H00112, 23K20209, and 23H00081.

- `EmbE8ToPic` is the 8×9 matrix such that the right multiplication of this matrix gives the embedding $\mathbb{E}_8 \hookrightarrow \mathrm{Pic}(X)$ inducing

$$\mathbb{E}_8 \cong R(X) = (\alpha_X)^\perp \subset \mathrm{Pic}(X).$$

  A vector of $R(X)$ is given by the basis of $\mathbb{E}_8$ with respect to the standard basis $r_1, \ldots, r_8$.
- `ProjPicToE8` is the 9×8 matrix such that the right multiplication of this matrix gives the orthogonal projection from $\mathrm{Pic}(X) = \mathbb{Z}\alpha_X \oplus R(X)$ to $R(X)$, where $R(X)$ is equipped with the standard basis $r_1, \ldots, r_8$ of $\mathbb{E}_8$.
- `alphaX` is the vector representation of the anti-canonical class $\alpha_X \in \mathrm{Pic}(X)$ with respect to the basis $h, e_1, \ldots, e_8$.
- `hX` is the vector representation of $h \in \mathrm{Pic}(X)$ with respect to the basis $h, e_1, \ldots, e_8$.
- `linerecs` is a list of 240 records `linerec`. Each `linerec` is a record describing a line l in X by the following items:
  - `linerec.pos` is the position of `linerec` in the list `linerecs`.
  - `linerec.vectPic` is the vector representation of the class $[l] \in \mathrm{Pic}(X)$ with respect to the basis $h, e_1, \ldots, e_8$.
  - `linerec.vectE8` is the vector representation of $[l]_R \in R(X) \cong \mathbb{E}_8$ with respect to the basis $r_1, \ldots, r_8$ of $\mathbb{E}_8$.
  - `linerec.hdeg` is the h-degree of l.
  - `linerec.mults` is the list $[\mu_1, \ldots, \mu_8]$ of multiplicities μ_i of $\beta(l)$ at p_i for $i = 1, \ldots, 8$.
- `iB` is the matrix representation of the action of the Bertini involution i_B on $\mathrm{Pic}(X)$ with respect to the basis $h, e_1, \ldots, e_8$.
- `iBpairs` is the list of i_B-pairs. It is a list of pairs $[k_1, k_2]$ of integers k_1, k_2 such that $1 \leq k_1 < k_2 \leq 240$. A pair $[k_1, k_2]$ indicates the pair of lines at the k_1th position and the k_2th position in the list `linerecs`.
- `WE8generators` is the list of 8 records `grec` corresponding to the reflections $s_i \in W(\mathbb{E}_8)$ with respect to the (-2)-vector r_i. The ith element `grec` of `WE8generators` contains the following items:
  - `grec.r` is the (-2)-vector r_i in the standard basis $r_1, \ldots, r_8$ of $\mathbb{E}_8$.
  - `grec.matE8` is the matrix representation of $s_i \in W(\mathbb{E}_8)$ with respect to the standard basis $r_1, \ldots, r_8$ of $\mathbb{E}_8$.

  - `grec.matPic` is the matrix representation with respect to the basis $h, e_1, \dots, e_8$ of $\mathrm{Pic}(X)$ of the element of $\mathrm{O}(\mathrm{Pic}(X), \alpha_X)$ corresponding to s_i by the isomorphism $\mathrm{O}(\mathrm{Pic}(X), \alpha_X) \cong W(R(X)) \cong W(\mathbb{E}_8)$.
  - `grec.permOnLines` is the permutation σ on the set of lines of X induced by s_i. A line l_k at the kth position in the list `linerecs` is mapped by s_i to the line l_{k^σ} at the k^σth position in `linerecs`.
  - `grec.permOniBpairs` is the permutation τ on the set of i_B-pairs induced by s_i. An i_B-pair at the kth position in `iBpairs` is mapped by s_i to the i_B-pair at the k^τth position in `iBpairs`.
- `WE8size` is the size 696729600 of $W(\mathbb{E}_8)$.

2. Orbits

The two files "`OrbitDecompbarDelta.txt`" and "`OrbitDecompDelta.txt`" describe the orbit decompositions of $\overline{\Delta}(\mathbb{E}_8)^{\{k\}}$ and $\Delta(\mathbb{E}_8)^{\{k\}}$ by $W(\mathbb{E}_8)$.

The file "`OrbitDecompbarDelta.txt`" contains the data `OrbitDecompbarDelta` of the orbit decomposition of the sets $\overline{\Delta}(\mathbb{E}_8)^{\{k\}}$ by the group $\overline{W}(\mathbb{E}_8)$ for $k = 1, \dots, 9$, where

$$\overline{\Delta}(\mathbb{E}_8) := \Delta(\mathbb{E}_8)/\{\pm\mathrm{id}\}, \quad \overline{W}(\mathbb{E}_8) := W(\mathbb{E}_8)/\{\pm\mathrm{id}\}.$$

This action is given by the action on the set $[120] := \{1, \dots, 120\}$ of the permutation group generated by `grec.permOniBpairs`, where `grec` runs through `WE8generators`. The kth member of `OrbitDecompbarDelta` is the list of orbits in $\overline{\Delta}(\mathbb{E}_8)^{\{k\}}$. Each orbit $o \subset [120]^{\{k\}}$ is expressed by the record `orbrec` with the following items.

- `orbrec.leng` is the length k.
- `orbrec.minrep` is a representative of o, which is the element of o that is minimum with respect to the lexicographic order on $[120]^{\{k\}}$.
- `orbrec.size` is the size $|o|$.

Analogously, the file "`OrbitDecompDelta.txt`" contains the data `OrbitDecompDelta` of the orbit decomposition of the sets $\Delta(\mathbb{E}_8)^{\{k\}}$ by the group $W(\mathbb{E}_8)$ for $k = 1, \dots, 7$. This action is given by the action on the set $[240] := \{1, \dots, 240\}$ of the permutation group generated by `grec.permOnLines`, where `grec` runs through `WE8generators`. The kth member of `OrbitDecompDelta` is the list of orbits in $\Delta(\mathbb{E}_8)^{\{k\}}$. Each orbit $o \subset [240]^{\{k\}}$ is expressed by the record `orbrec` with the following items.

- `orbrec.leng` is the length k.

- `orbrec.minrep` is a representative of o, which is the element of o that is minimum with respect to the lexicographic order on $[240]^{\{k\}}$.
- `orbrec.size` is the size $|o|$.

3. $\mathbb{F}_{19}$-MODEL

The file "`F19model.txt`" contains the data for the proof of Proposition 4.4 of the paper. An element of $\mathbb{F}_{19}$ is given by an integer a such that $0 \leq a < 19$.

- `8points` is the list $[p_1, \ldots, p_8]$ of 8 points of $\mathbf{P}^2$ written in an affine coordinate system (x, y) on $\mathbf{P}^2$.
- `9lines` is the list of 9 records `eqrec` describing the 9 lines $\ell_1, \ldots, \ell_9$. The kth member of `9lines` describes $\beta(\ell_k)$ by the following items:
  - `eqrec.pos` is the position of the line ℓ_k in the list `linerecs` of the 240 lines in $Y_{\boldsymbol{p}}$.
  - `eqrec.hdeg` is the h-degree of ℓ_k.
  - `eqrec.mults` is the list $[\mu_1, \ldots, \mu_8]$ of multiplicities μ_i of $\beta(\ell_k)$ at p_i for $i = 1, \ldots, 8$.
  - `eqrec.eq` is the equation of $\beta(\ell_k)$ with respect to the affine coordinate system (x, y) of $\mathbf{P}^2$. The variables x and y are written as "`xx`" and "`yy`", respectively. When the h-degree of ℓ_k is 0 (that is, when $k = 1$), this equation is just a string `"a point"`.

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

Email address: `ichiro-shimada@hiroshima-u.ac.jp`