DEL PEZZO SURFACES OF DEGREE ONE AND EXAMPLES OF ZARISKI MULTIPLES

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ABSTRACT. We construct examples of Zariski N-tuples with large N using the monodromy action of the Weyl group of type E_8 on the set of 240 lines in a del Pezzo surface of degree one.

1. INTRODUCTION

We work over the field of complex numbers. A *plane curve* means a reduced, possibly reducible, projective plane curve.

Zariski [20, 21] demonstrated that an equisingular family of plane curves may fail to be connected by providing an example of a pair of 6-cuspidal plane sextics such that their complements have non-isomorphic fundamental groups. Artal Bartolo [2] revisited Zariski's work, and defined a *Zariski pair* as a pair of plane curves that have the same combinatorial type of singularities but differ in their embedding topologies. More generally, a collection of N plane curves is called a *Zariski N-tuple* if any two in the collection form a Zariski pair.

Since [2], many Zariski multiples have been constructed using a wide variety of methods. The construction of Zariski multiples has served as a good testing ground for various techniques in the study of embedding topology of plane curves. See, for example, the survey paper [3].

In this paper, we employ the lattice structure of the middle homology group of the double plane branching along the plane curve as an invariant to distinguish topological types [16]. We enumerate the topological types in our Zariski multiples by a monodromy argument initiated in [15]. Del Pezzo surfaces of degree 1 and the Weyl group of type E_8 play an important role in our investigation.

As the main result, we obtain the following:

Theorem 1.1. *For each integer k satisfying $1 < k < 119$, there exists a Zariski $N(k)$-tuple $\mathcal{Z}_k$ consisting of plane curves of degree $7 + 2k$, where*

$$N(k) \geq \frac{1}{348364800}\binom{120}{k}. \tag{1.1}$$

See Section 2 for the precise description of the combinatorial type of the plane curves in $\mathcal{Z}_k$, and the exact values of $N(k)$. We have $N(120 - k) = N(k)$, and the values of $N(k)$ for small k are as follows:

(1.2)

k	1	2	3	4	5	6	7	8	9
$N(k)$	1	2	5	15	48	212	1116	7388	56946 .

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Putting $k = 60$, we obtain an example of a Zariski $N(60)$-tuple with

$$N(60) > 2.77 \times 10^{26}.$$

In our previous paper [17], we constructed Zariski multiples by using the monodromy of the Weyl group of type E_7 for a family of del Pezzo surfaces of degree 2. For example, we obtained a Zariski 105-tuple in which each member is a union of a smooth quartic curve and 14 lines chosen from its 28 bitangents. These examples extended the works [4, 6, 5] by Bannai et al. We also constructed a series of Zariski N-tuple with $N \to \infty$ by means of 4-tangent conics of a smooth quartic curve.

It is a natural step to extend this argument to the Weyl group of type E_8 and del Pezzo surfaces of degree 1. A smooth quartic curve in [17] is now replaced by a t_3-sextic (Definition 2.4), and its 28 bitangents are replaced by 120 special tangent conics (Definition 2.6). The choices of k conics from the 120 special tangent conics yield the Zariski $N(k)$-tuple $\mathcal{Z}_k$. It seems to be an interesting problem to ask what becomes of the 4-tangent conics of a smooth quartic curve in the context of a t_3-sextic. See Remark 6.2.

This paper contains two improvements compared with [17]. One is that we present a unified method to compute the monodromy group for families of del Pezzo surfaces. In fact, the proof of [17, Theorem 3.1] on the monodromy of a family of del Pezzo surfaces of degree 2 was incomplete, and we give a rectified argument in the present paper. Another improvement is that we use a simpler reasoning to distinguish topological types in our Zariski multiples. This simplification is based on an observation (Lemma 6.1) by Artal Bartolo.

In several parts of the proofs, we rely on brute-force computations carried out by a computer. For this purpose, we employ `GAP` [9]. A detailed computational data is available from [18].

The plan of this paper is as follows. In Section 2, we precisely state our main result in Theorem 2.12. We define the combinatorial type of our Zariski $N(k)$-tuples, and give the exact value of $N(k)$. In Section 3, we develop a general theory to compute the monodromy group for families of del Pezzo surfaces. In Section 4, we study the lines in a del Pezzo surface of degree 1 more closely. In Section 5, we relate the theory of del Pezzo surfaces to that of singular plane curves, and deduce some properties of t_3-sextics from our discussion about del Pezzo surfaces of degree 1. In Section 6, we investigate the embedding topology of t_3-sextics and prove Theorem 2.12. We conclude this paper by a remark on the work [14] about $K3$ surfaces obtained as double covers of del Pezzo surfaces of degree 1.

Notation.

1. For a set M and a non-negative integer k, we denote by $M^{\{k\}} := \binom{M}{k}$ the set of subsets $S \subset M$ of size $|S| = k$.
2. The orthogonal group $\mathrm{O}(L)$ of a lattice L acts on L from the right.
3. For a topological space T, we write $H_i(T)$ and $H^i(T)$ for $H_i(T; \mathbb{Z})$ and $H^i(T; \mathbb{Z})$, respectively.

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2. Main result

In Section 2.1, we define the combinatorial type σ_k of plane curves in our Zariski $N(k)$-tuple $\mathcal{Z}_k$. In Section 2.2, we define the number $N(k)$, and state our main result Theorem 2.12.

2.1. Combinatorial type σ_k.

Definition 2.1. Two plane curves D and D' are said to have the *same combinatorial type* if there exist tubular neighborhoods $T \subset \mathbb{P}^2$ of D and $T' \subset \mathbb{P}^2$ of D' such that $(\mathbb{P}^2, T)$ and $(\mathbb{P}^2, T')$ are homeomorphic.

See [3, Remark 3] for another formulation of the notion of combinatorial type.

Definition 2.2. A germ $(C, \mathbf{0})$ of isolated plane curve singularity is said to be a *t_m-singularity* if C consists of m smooth local branches and each pair of the local branches has intersection number 2.

Note that a t_2-singularity is an ordinary tacnode (an a_3-singularity).

Definition 2.3. Let $C \subset \mathbb{P}^2$ be a plane curve with a t_m-singularity at $A \in C$. The common tangent line $\Lambda \subset \mathbb{P}^2$ to the local branches of C at A is called the *tangent line to C at A*.

Definition 2.4. A plane curve C of degree 6 is called a *t_3-sextic* if the singular locus of C consists of a single point A, and $A \in C$ is a t_3-singular point.

Note that a t_3-sextic is irreducible. We fix a point $A \in \mathbb{P}^2$ and a line $\Lambda \subset \mathbb{P}^2$ passing through A.

Definition 2.5. A t_3-sextic with the singular point A and the tangent line Λ at A is called a t_3-sextic *in the frame* (A, Λ).

Let C be a t_3-sextic in the frame (A, Λ).

Definition 2.6. A smooth conic Γ is said to be a *special tangent conic* of C if the following hold;

- the conic Γ passes through A, and $C + \Gamma$ has t_4-singularity at A, and
- at every intersection point of C and Γ other than A, the intersection multiplicity is even.

For a special tangent conic Γ, we put

$$\mathrm{Tac}\,(\Gamma) := \mathrm{Sing}(C + \Gamma) \setminus \{A\}.$$

Definition 2.7. We say that a set $\{\Gamma_1, \dots, \Gamma_k\}$ consisting of k special tangent conics of C is said to be *"in a general position"* if the following conditions hold.

- Any two of $\Gamma_1, \dots, \Gamma_k$ have local intersection number 2 at A.
- Each $\mathrm{Tac}\,(\Gamma_i)$ consists of 3 tacnodes of $C + \Gamma_i$.
- The sets $\mathrm{Tac}\,(\Gamma_1), \dots, \mathrm{Tac}\,(\Gamma_k)$ are disjoint to each other.
- The singular points of the union $C + \Gamma_1 + \cdots + \Gamma_k$ other than A and the tacnodes in $\mathrm{Tac}\,(\Gamma_1), \dots, \mathrm{Tac}\,(\Gamma_k)$ are ordinary nodes.

In Section 5.1, we prove the following:

Proposition 2.8. *All t_3-sextics in the frame (A, Λ) are parameterized by a Zariski open subset $\mathcal{T}$ of a 15-dimensional linear subspace of $|\mathcal{O}_{\mathbb{P}^2}(6)|$.*

For a point $t \in \mathcal{T}$, we denote by C_t the corresponding t_3-sextic. In Section 5.2, we prove the following:

Proposition 2.9. *Let t be a general point of $\mathcal{T}$. Then C_t has exactly 120 special tangent conics, and they are in a general position.*

For $t \in \mathcal{T}$, we denote by $G(C_t)$ the set of special tangent conics of C_t. For $s = \{\Gamma_1, \dots, \Gamma_k\} \in G(C_t)^{\{k\}}$, we put

$$D_{t,s} := C_t + \Lambda + \Gamma_1 + \cdots + \Gamma_k.$$

Definition 2.10. By Proposition 2.9, the combinatorial type of $D_{t,s}$ does not depend on the choice of $t \in \mathcal{T}$ and $s \in G(C_t)^{\{k\}}$, provided that t is general in $\mathcal{T}$. We denote this combinatorial type by σ_k.

The curve of combinatorial type σ_k is of degree $7+2k$, and its singularities consist of one t_{4+k}-singular point, $3k$ tacnodes, and $k(k-1)$ ordinary nodes. It should be noted that the information given by a combinatorial type includes, not only the types of singular points, but also more detailed data such as which irreducible components correspond to which local branch of each singular point. See [3, Remark 3].

2.2. **Main Theorem.** Let $\mathbb{E}_8$ be the root lattice of type E_8, that is, $\mathbb{E}_8$ is an even unimodular *negative-definite* lattice of rank 8 generated by vectors of square norm -2. (Note that we adopt the sign convention opposite to the standard one.) We denote by $\Delta(\mathbb{E}_8)$ be the set of vectors of square norm -2 in $\mathbb{E}_8$, which is of size 240. Let $W(\mathbb{E}_8)$ be the Weyl group of $\mathbb{E}_8$, that is, the subgroup of $\mathrm{O}(\mathbb{E}_8)$ generated by reflections with respect to vectors of square norm -2. In fact, we have an equality $W(\mathbb{E}_8) = \mathrm{O}(\mathbb{E}_8)$. See (3.1). We then put

$$\overline{W} := W(\mathbb{E}_8)/\{\pm\mathrm{id}\}, \qquad \overline{\Delta} := \Delta(\mathbb{E}_8)/\{\pm\mathrm{id}\}.$$

Then $\overline{W}$ acts on $\overline{\Delta}$ and hence on the set $\overline{\Delta}^{\{k\}}$ of k-element subsets of $\overline{\Delta}$. We define

$$N(k) := \text{the number of } \overline{W}\text{-orbits in } \overline{\Delta}^{\{k\}}.$$

Finally, we define the topological types of plane curves as follows.

Definition 2.11. Two plane curves D and D' are said to have *the same embedding topology* if there exists a homeomorphism between $(\mathbb{P}^2, D)$ and $(\mathbb{P}^2, D')$.

Then our main result is stated as follows:

Theorem 2.12. *Let o be a general point of $\mathcal{T}$. We consider the plane curves $D_{o,s}$ of combinatorial type σ_k, where s runs through $G(C_o)^{\{k\}}$. Then, classifying these curves by the embedding topology yields exactly $N(k)$ classes.*

Since $|\overline{\Delta}| = 120$ and $|\overline{W}| = 348364800$, we have the inequality (1.1). Therefore Theorem 2.12 implies Theorem 1.1.

3. Del Pezzo surfaces

In this section, we investigate del Pezzo surfaces. For the classical results about del Pezzo surfaces, we refer the reader to [8, 13]. In Section 3.1, we study the Picard lattice of a del Pezzo surface. In Section 3.2, we present a simple method (Corollary 3.3) for studying the monodromy of a family of del Pezzo surfaces. In Sections 3.3–3.5, we apply this method to natural families of del Pezzo surfaces of degree 3, 2, and 1, respectively.

d	n	τ_n	$\lvert W(R(X))\rvert$	$\lvert \mathrm{Aut}(\tau_n)\rvert$	$\lvert L(X)\rvert$	$\lvert L^{[n]}(X)'\rvert$
6	3	A_1+A_2	$2^2\cdot 3$	2	6	2
5	4	A_4	$2^3\cdot 3\cdot 5$	2	10	5
4	5	D_5	$2^7\cdot 3\cdot 5$	2	16	16
3	6	E_6	$2^7\cdot 3^4\cdot 5$	2	27	72
2	7	E_7	$2^{10}\cdot 3^4\cdot 5\cdot 7$	1	56	576
1	8	E_8	$2^{14}\cdot 3^5\cdot 5^2\cdot 7$	1	240	17280

Table 3.1. Lattice theoretic data

3.1. **Picard lattice.** Let d be a positive integer ≤ 6. We put

$$n := 9-d.$$

Let X be a del Pezzo surface of degree d, that is, a smooth surface whose anti-canonical class $\alpha_X := [-K_X]$ is ample of self-intersection number d. The Picard lattice $\mathrm{Pic}(X)$ of X is of rank $n+1$, and is canonically isomorphic to $H^2(X)$. There exists a birational morphism

$$\beta\colon X \to \mathbf{P}^2$$

that is a blowing-up at distinct n points on $\mathbf{P}^2$, and $\mathrm{Pic}(X)$ has a basis $h, e_1, \dots, e_n$, where h is the class of the pullback of a line on $\mathbf{P}^2$, and $e_1, \dots, e_n$ are the classes of exceptional curves. With respect to this basis, the Gram matrix of $\mathrm{Pic}(X)$ is the diagonal matrix with diagonal entries $1, -1, \dots, -1$, and the anti-canonical class $\alpha_X \in \mathrm{Pic}(X)$ is written as

$$\alpha_X = (3, -1, \dots, -1).$$

The orthogonal complement

$$R(X) := (\alpha_X)^\perp$$

of α_X in $\mathrm{Pic}(X)$ is a negative-definite root lattice of type τ_n, where τ_n is given in Table 3.1. Let $W(R(X))$ denote the Weyl group of the lattice $R(X)$. The order of $W(R(X))$ is obtained from the ADE-type τ_n. The root lattice $R(X)$ has a basis $r_1, \dots, r_n$ consisting of (-2)-vectors whose dual graph is the ordinary Dynkin diagram of type τ_n. We have

$$\mathrm{O}(R(X)) = W(R(X)) \rtimes \mathrm{Aut}(\tau_n), \tag{3.1}$$

where $\mathrm{Aut}(\tau_n)$ is the group of symmetries of the root system $\{r_1, \dots, r_n\}$. We put

$$\mathrm{O}(\mathrm{Pic}(X), \alpha_X) := \{\, g \in \mathrm{O}(\mathrm{Pic}(X)) \mid \alpha_X^g = \alpha_X \,\}.$$

Then we have a natural homomorphism

$$\mathrm{O}(\mathrm{Pic}(X), \alpha_X) \to \mathrm{O}(R(X)) \tag{3.2}$$

given by the restriction $g \mapsto g|R(X)$. It is obvious that (3.2) is injective.

Proposition 3.1. *The image of the homomorphism (3.2) is equal to $W(R(X))$.*

Proof. We put $A := \mathbb{Z}\alpha_X$, $R := R(X)$, and consider their discriminant groups

$$d_A := A^\vee/A, \quad d_R := R^\vee/R,$$

where $A^\vee$ and $R^\vee$ are the dual lattices of A and R, respectively. The unimodular lattice $\mathrm{Pic}(X)$, which is a submodule of $A^\vee \oplus R^\vee$ containing $A \oplus R$, gives rise to the graph

$$\mathrm{Pic}(X)/(A \oplus R) \;\subset\; d_A \times d_R$$

of an isomorphism $d_A \cong d_R$. Hence an isometry $g \in \mathrm{O}(R)$ extends to an isometry of $\mathrm{Pic}(X)$ that acts on A trivially if and only if g acts on d_R trivially. It is easy to see that a reflection with respect to a (-2)-vector acts trivially on the discriminant group. On the other hand, a direct calculation shows that a non-trivial element of $\mathrm{Aut}(\tau_n)$ (if there exists any) acts non-trivially on d_R. Hence Proposition 3.1 follows from (3.1). □

A smooth rational curve l on X is called a *line* if $\langle l, \alpha_X\rangle = 1$. Every line has a self-intersection number -1, and the set of lines in X is identified with

$$L(X) := \{\, \lambda \in \mathrm{Pic}(X) \mid \langle \lambda, \lambda\rangle = -1,\ \langle \lambda, \alpha_X\rangle = 1 \,\}.$$

By abuse of language, we sometimes call an element of $L(X)$ a line. We can enumerate all the elements of $L(X)$ explicitly. Let $L^{[n]}(X)$ denote the set of all ordered n-tuples

$$\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_n]$$

of lines such that $\langle \lambda_i, \lambda_j\rangle = 0$ for any i, j with $i \neq j$. By Proposition 3.1 and [8, Proposition II-4], we have the following:

Corollary 3.2. *The natural action of* $\mathrm{O}(\mathrm{Pic}(X), \alpha_X) \cong W(R(X))$ *on* $L^{[n]}(X)$ *is free and transitive.* □

For $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_n] \in L^{[n]}(X)$, we have a birational morphism to a projective plane that is the contraction of the lines $l_1, \dots, l_n$ whose classes are $\lambda_1, \dots, \lambda_n$. We denote this blowing-down morphism by

$$\beta_{\boldsymbol{\lambda}} \,:\, X \;\to\; \boldsymbol{P}(X/\boldsymbol{\lambda}). \tag{3.3}$$

3.2. **Monodromy.** Let $f\colon \mathcal{X} \to \mathcal{U}$ be a family of del Pezzo surfaces of degree d. We assume that the parameter space $\mathcal{U}$ is smooth and irreducible. For a point u of $\mathcal{U}$, we denote by X_u the fiber $f^{-1}(u)$. Instead of α_{X_u}, we denote by $\alpha_u \in \mathrm{Pic}(X_u)$ the anti-canonical class of X_u. We choose a base point $b \in \mathcal{U}$. The local system

$$R^2 f_* \mathbb{Z} \;\to\; \mathcal{U}$$

is a family of the lattices $\mathrm{Pic}(X_u) \cong H^2(X_u)$, and it has a section $u \mapsto \alpha_u$. Hence we obtain a monodromy homomorphism

$$\Phi \,:\, \pi_1(\mathcal{U}, b) \;\to\; \mathrm{O}(\mathrm{Pic}(X_b), \alpha_b) \cong W(R(X_b)). \tag{3.4}$$

We investigate the surjectivity of the monodromy homomorphism Φ.

We consider the étale covering

$$\mathcal{L}^{[n]} \to \mathcal{U}$$

whose fiber over $u \in \mathcal{U}$ is the set $L^{[n]}(X_u)$ of ordered sets of disjoint n lines in X_u. The associated monodromy action of $\pi_1(\mathcal{U}, b)$ on $L^{[n]}(X_b)$ is given by

$$[\gamma] \;\mapsto\; (\boldsymbol{\lambda} \;\mapsto\; \boldsymbol{\lambda}^{\Phi([\gamma])})$$

for $[\gamma] \in \pi_1(\mathcal{U}, b)$, where $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}^{\Phi([\gamma])}$ denotes the action of the element $\Phi([\gamma])$ of $\mathrm{O}(\mathrm{Pic}(X_b), \alpha_b)$ on $L^{[n]}(X_b)$. The orbits of the monodromy action on $L^{[n]}(X_b)$ correspond bijectively to the connected components of $\mathcal{L}^{[n]}$. Hence, by Corollary 3.2, we obtain the following:

Corollary 3.3. *The index of the image of the monodromy homomorphism Φ in $W(R(X_b))$ is equal to the number of the connected components of $\mathcal{L}^{[n]}$.* $\square$

We fix a projective plane $\mathbf{P}^2$. For an ordered set $\boldsymbol{p} = [p_1, \dots, p_n] \in (\mathbf{P}^2)^n$ of distinct n points of $\mathbf{P}^2$, we denote by

$$\beta_{\boldsymbol{p}} \ : \ Y_{\boldsymbol{p}} \ \to \ \mathbf{P}^2$$

the blowing-up at the points $p_1, \dots, p_n$.

Definition 3.4. Let $\mathcal{P}_n$ be the Zariski open subset of $(\mathbf{P}^2)^n$ consisting of all ordered sets $\boldsymbol{p} = [p_1, \dots, p_n]$ of distinct n points of $\mathbf{P}^2$ such that

(i) no three points in $\boldsymbol{p}$ are on a line,
(ii) no six points in $\boldsymbol{p}$ are on a conic, and
(iii) there exists no cubic curve passing through 7 points in $\boldsymbol{p}$ and having a double point at the 8th point.

Theorem 3.5 (Théorème II-1 in [8])**.** *The surface $Y_{\boldsymbol{p}}$ is a del Pezzo surface of degree $d = 9 - n$ if and only if $\boldsymbol{p} \in \mathcal{P}_n$.* $\square$

For $\boldsymbol{p} = [p_1, \dots, p_n] \in \mathcal{P}_n$, we have a distinguished element

$$\boldsymbol{\rho}_{\boldsymbol{p}} = [\rho_1, \dots, \rho_n]$$

of $L^{[n]}(Y_{\boldsymbol{p}})$, where ρ_i is the class of the exceptional curve over p_i. We consider the incidence variety

$$\mathcal{I} := \left\{ \ (u, \boldsymbol{p}, g) \ \middle| \ u \in \mathcal{U},\ \boldsymbol{p} \in \mathcal{P}_n, \text{ and } g \text{ is an isomorphism } X_u \xrightarrow{\sim} Y_{\boldsymbol{p}} \ \right\}$$

with the projections

$$\mathcal{U} \ \xleftarrow{\varpi_1} \ \mathcal{I} \ \xrightarrow{\varpi_2} \ \mathcal{P}_n.$$

A point of $\mathcal{L}^{[n]}$ is written as $(u, \boldsymbol{\lambda})$, where $u \in \mathcal{U}$ and $\boldsymbol{\lambda} \in L^{[n]}(X_u)$. Then we can lift $\varpi_1 \colon \mathcal{I} \to \mathcal{U}$ to $\varpi_1^{\mathcal{L}} \colon \mathcal{I} \to \mathcal{L}^{[n]}$ by setting

$$\varpi_1^{\mathcal{L}}(u, \boldsymbol{p}, g) := (u, \boldsymbol{\lambda}),$$

where $\boldsymbol{\lambda}$ is the element of $L^{[n]}(X_u)$ that is mapped to $\boldsymbol{\rho}_{\boldsymbol{p}} \in L^{[n]}(Y_{\boldsymbol{p}})$ by the bijection $L^{[n]}(X_u) \xrightarrow{\sim} L^{[n]}(Y_{\boldsymbol{p}})$ induced by $g \colon X_u \xrightarrow{\sim} Y_{\boldsymbol{p}}$:

$$\begin{tikzcd}
\mathcal{L}^{[n]} \arrow[d] & \mathcal{I} \arrow[l, "\varpi_1^{\mathcal{L}}"'] \arrow[r, "\varpi_2"] \arrow[dl, "\varpi_1"] & \mathcal{P}_n \\
\mathcal{U} \, . & &
\end{tikzcd}$$

The fiber of $\varpi_1^{\mathcal{L}}$ over $(u, \boldsymbol{\lambda}) \in \mathcal{L}^{[n]}$ is the variety of all isomorphisms

$$\boldsymbol{P}(X_u/\boldsymbol{\lambda}) \ \xrightarrow{\sim} \ \mathbf{P}^2,$$

where $\boldsymbol{P}(X_u/\boldsymbol{\lambda})$ is defined in (3.3). Indeed, if we have $\varpi_1^{\mathcal{L}}(u, \boldsymbol{p}, g) = (u, \boldsymbol{\lambda})$, then the isomorphism $g \colon X_u \xrightarrow{\sim} Y_{\boldsymbol{p}}$ maps the exceptional curves of the blowing-down $\beta_{\boldsymbol{\lambda}} \colon X_u \to \boldsymbol{P}(X_u/\boldsymbol{\lambda})$ to the exceptional curves of the blowing-down $\beta_{\boldsymbol{p}} \colon Y_{\boldsymbol{p}} \to \mathbf{P}^2$, and hence g induces an isomorphism $\boldsymbol{P}(X_u/\boldsymbol{\lambda}) \xrightarrow{\sim} \mathbf{P}^2$. Conversely, if we are given a point $(u, \boldsymbol{\lambda})$ of $\mathcal{L}^{[n]}$ and an isomorphism $\bar{g} \colon \boldsymbol{P}(X_u/\boldsymbol{\lambda}) \xrightarrow{\sim} \mathbf{P}^2$, then, setting $\boldsymbol{p}$ to be the image of the centers of $\beta_{\boldsymbol{\lambda}}$ by $\bar{g}$ and lifting $\bar{g}$ to the isomorphism $g\colon X_u \xrightarrow{\sim} Y_{\boldsymbol{p}}$ between their blow-ups, we obtain a point $(u, \boldsymbol{p}, g)$ in the fiber of $\varpi_1^{\mathcal{L}}$ over $(u, \boldsymbol{\lambda})$.

The variety of isomorphisms $\boldsymbol{P}(X_u/\boldsymbol{\lambda}) \cong \mathbf{P}^2$ is isomorphic to $\mathrm{PGL}(3, \mathbb{C})$. Hence $\varpi_1^{\mathcal{L}}$ gives a bijection from the set of connected components of $\mathcal{I}$ to that of $\mathcal{L}^{[n]}$. We investigate the connected components of $\mathcal{I}$ by the second projection $\varpi_2\colon \mathcal{I} \to \mathcal{P}_n$.

3.3. **Family of cubic surfaces.** The anti-canonical model of a del Pezzo surface of degree $d = 3$ is a smooth cubic surface. We fix a projective space $\mathbb{P}^3$, and consider the family $\mathcal{X} \to \mathcal{U}$ of smooth cubic surfaces, where $\mathcal{U}$ is the Zariski open subset of $|\mathcal{O}_{\mathbb{P}^3}(3)| \cong \mathbb{P}^{19}$ parameterizing all smooth cubic surfaces.

The following reproduces the result of Harris [10] on the Galois group of 27 lines in a smooth cubic surface.

Proposition 3.6. *For the family $\mathcal{X} \to \mathcal{U}$ of smooth cubic surfaces, the monodromy homomorphism Φ is surjective onto the Weyl group $W(R(X_b))$ of type E_6.*

Proof. For any point $\boldsymbol{p} \in \mathcal{P}_6$, the fiber of ϖ_2 over $\boldsymbol{p}$ is the variety of all isomorphisms between $\mathbb{P}^3$ and the projective space

$$|\alpha_{\boldsymbol{p}}|^{\vee} = \mathbb{P}^*(H^0(Y_{\boldsymbol{p}}, \mathcal{O}(\alpha_{\boldsymbol{p}}))),$$

where $\alpha_{\boldsymbol{p}}$ is the anti-canonical class of $Y_{\boldsymbol{p}}$. This variety is isomorphic to $\mathrm{PGL}(4, \mathbb{C})$. Hence $\mathcal{L}^{[6]}$ is connected. Therefore Φ is surjective by Corollary 3.3. □

See [19] for an application of Proposition 3.6.

3.4. **Family of quartic double planes.** The anti-canonical model of a del Pezzo surface X of degree $d = 2$ is a double plane $X \to \mathbb{P}^2$ branching along a smooth quartic curve. We fix a projective plane $\mathbb{P}^2$. Let $\mathcal{U}$ be the Zariski open subset of $|\mathcal{O}_{\mathbb{P}^2}(4)| \cong \mathbb{P}^{14}$ that parameterizes all smooth quartic curves in $\mathbb{P}^2$. For each $u \in \mathcal{U}$, we denote by $B_u \subset \mathbb{P}^2$ the corresponding quartic curve. We consider the family $\mathcal{X} \to \mathcal{U}$ of smooth quartic double planes, that is, $\mathcal{X}$ is the double cover of $\mathbb{P}^2 \times \mathcal{U}$ with the projection $\mathcal{X} \to \mathcal{U}$ whose fiber X_u over $u \in \mathcal{U}$ is the double plane $X_u \to \mathbb{P}^2$ branching along B_u.

The following proposition was stated in [17], but the proof was incomplete. A weaker result concerning the Galois group of the 28 bitangents of a smooth quartic curve had been proved in Harris [10].

Proposition 3.7. *For the family $\mathcal{X} \to \mathcal{U}$ of smooth quartic double planes, the monodromy homomorphism Φ is surjective onto the Weyl group $W(R(X_b))$ of type E_7.*

Proof. Let $\boldsymbol{p}$ be a point of $\mathcal{P}_7$, and let $Y_{\boldsymbol{p}} \to \mathsf{P}^2_{\boldsymbol{p}}$ be the anti-canonical model of $Y_{\boldsymbol{p}}$. Let $\mathsf{B}_{\boldsymbol{p}} \subset \mathsf{P}^2_{\boldsymbol{p}}$ be the branch curve of $Y_{\boldsymbol{p}} \to \mathsf{P}^2_{\boldsymbol{p}}$. The fiber of $\varpi_2\colon \mathcal{I} \to \mathcal{P}_7$ over $\boldsymbol{p}$ consists of pairs (u, g), where u is a point of $\mathcal{U}$ and g is an isomorphism from X_u to $Y_{\boldsymbol{p}}$. An isomorphism g from X_u to $Y_{\boldsymbol{p}}$ induces an isomorphism

$$\bar{g} \;:\; \mathbb{P}^2 \xrightarrow{\sim} \mathsf{P}^2_{\boldsymbol{p}}.$$

Conversely, suppose that an isomorphism $\gamma\colon \mathbb{P}^2 \xrightarrow{\sim} \mathsf{P}^2_{\boldsymbol{p}}$ is given. Let $u \in \mathcal{U}$ be the point such that $B_u = \gamma^{-1}(\mathsf{B}_{\boldsymbol{p}})$. Then γ admits exactly *two* liftings

$$g_1 \;:\; X_u \xrightarrow{\sim} Y_{\boldsymbol{p}}, \quad g_2 \;:\; X_u \xrightarrow{\sim} Y_{\boldsymbol{p}},$$

which differ by the deck-transformation of X_u over $\mathbb{P}^2$. Since $\mathrm{PGL}(3, \mathbb{C})$ is smooth and irreducible, we see that the fiber of $\varpi_2\colon \mathcal{I} \to \mathcal{P}_7$ over $\boldsymbol{p}$ has at most two connected components. Therefore $\mathcal{I}$ has at most two connected components, and so does $\mathcal{L}^{[7]}$. Consequently, the index $[W : \Gamma]$ of the image

$$\Gamma := \mathrm{Image}(\Phi)$$

of Φ in $W := W(R(X_b))$ is at most 2.

We assume

$$[W : \Gamma] = 2, \tag{3.5}$$

and derive a contradiction. Note that the Weyl group W of type E_7 contains a simple group G as the kernel of $\det\colon W \to \{\pm 1\}$. See [7, page 46]. We show in the next paragraph that Γ contains an element of determinant -1. By assumption (3.5), we see that $G \cap \Gamma$ is a normal subgroup of G with index 2, which contradicts the simplicity of G.

Let $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \cong \mathbb{P}^{14}$ be the hypersurface that parameterizes singular quartic curves. Then $\mathcal{H}$ is irreducible. Let q be a general point of $\mathcal{H}$. Then the corresponding quartic curve $B_q \subset \mathbb{P}^2$ has an ordinary node as its only singularity, and hence the double plane $X_q \to \mathbb{P}^2$ branching along B_q has an ordinary double point as its only singularity. We choose a sufficiently small closed disc $D \subset |\mathcal{O}_{\mathbb{P}^2}(4)|$ intersecting $\mathcal{H}$ at q transversely, and let $\gamma\colon [0,1] \to \mathcal{U}$ be a loop that goes from the base point b to a point $q' \in \partial D$ along a path τ, makes a round trip along ∂D in positive-direction, and retraces the path τ backwards to b. The monodromy action on $H^2(X_b)$ by $[\gamma] \in \pi_1(\mathcal{U}, b)$ is calculated by the *local Picard–Lefschetz formula*. (See, for example, [12].) We have a *vanishing cycle* $v \in H^2(X_b)$ corresponding to the ordinary double point of X_q, which satisfies $\langle \alpha_b, v\rangle = 0$ and $\langle v, v\rangle = -2$, and the monodromy on $H^2(X_b) \cong \mathrm{Pic}(X_b)$ by $[\gamma]$ is the reflection $x \mapsto x + \langle v, x\rangle v$ with respect to v. Hence we have $\det(\Phi([\gamma])) = -1$. Note that the identification $\mathrm{O}(\mathrm{Pic}(X_b), \alpha_b) \cong W(R(X_b))$ preserves the determinant. Therefore Γ contains an element of determinant -1. □

3.5. **Family of bi-anti-canonical models of del Pezzo surfaces of degree 1.** Let X be a del Pezzo surface of degree $d = 1$. Then the complete linear system $|2\alpha_X|$ of bi-anti-canonical divisors of X gives rise to a double covering

$$X \to Q \subset \mathbb{P}^3$$

of a singular quadric surface Q of rank 3 (a quadric cone) that branches along $B \cup \{V\}$, where B is a smooth member of $|\mathcal{O}_Q(3)|$ and $V \in Q$ is the vertex. Conversely, for a quadric cone Q with the vertex $V \in Q$ and a smooth member B of $|\mathcal{O}_Q(3)|$, the double cover $X \to Q$ branching along $B \cup \{V\}$ is the bi-anti-canonical model of a del Pezzo surface X of degree 1.

We fix a quadric cone Q with the vertex $V \in Q$. Let $\mathcal{U}$ denote the Zariski open subset of $|\mathcal{O}_Q(3)| \cong \mathbb{P}^{15}$ that parameterizes all smooth members. For $u \in \mathcal{U}$, we denote by $B_u \subset Q$ the corresponding curve. We consider the family $\mathcal{X} \to \mathcal{U}$ of del Pezzo surfaces of degree 1 such that $\mathcal{X}$ is the double cover of $Q \times \mathcal{U}$ with the projection $\mathcal{X} \to \mathcal{U}$ whose fiber X_u over $u \in \mathcal{U}$ is the double cover $X_u \to Q$ branching along $B_u \cup \{V\}$.

Proposition 3.8. *For the family $\mathcal{X} \to \mathcal{U}$ of bi-anti-canonical models of del Pezzo surfaces of degree 1, the monodromy homomorphism Φ is surjective onto the Weyl group $W(R(X_b))$ of type E_8.*

Proof. The first half of the proof is almost the same as that of Proposition 3.7. Let $\boldsymbol{p}$ be a point of $\mathcal{P}_8$, and let $Y_{\boldsymbol{p}} \to \mathsf{Q}_{\boldsymbol{p}}$ be the bi-anti-canonical model of $Y_{\boldsymbol{p}}$. Let $\mathsf{B}_{\boldsymbol{p}} \subset \mathsf{Q}_{\boldsymbol{p}}$ be the curve component of the branch locus of $Y_{\boldsymbol{p}} \to \mathsf{Q}_{\boldsymbol{p}}$. The fiber of $\varpi_2 \colon \mathcal{I} \to \mathcal{P}_8$ over $\boldsymbol{p}$ consists of pairs (u, g), where u is a point of $\mathcal{U}$ and g is an isomorphism from X_u to $Y_{\boldsymbol{p}}$. An isomorphism g from X_u to $Y_{\boldsymbol{p}}$ induces an isomorphism $\bar{g} \colon Q \xrightarrow{\sim} \mathsf{Q}_{\boldsymbol{p}}$. Conversely, suppose that an isomorphism $\gamma \colon Q \xrightarrow{\sim} \mathsf{Q}_{\boldsymbol{p}}$ is given. Let $u \in \mathcal{U}$ be the point such that $B_u = \gamma^{-1}(\mathsf{B}_{\boldsymbol{p}})$. Then γ lifts to exactly *two* isomorphisms from X_u to $Y_{\boldsymbol{p}}$. Since the variety of isomorphisms from Q to $\mathsf{Q}_{\boldsymbol{p}}$ is smooth and irreducible, the fiber of ϖ_2 over $\boldsymbol{p}$ has at most two connected components. Therefore $\mathcal{I}$ has at most two connected components, and hence the index $[W : \Gamma]$ of the image $\Gamma := \mathrm{Image}(\Phi)$ of the monodromy Φ in $W := W(R(X_b))$ is at most 2. We assume $[W : \Gamma] = 2$, and derive a contradiction.

Note that the Weyl group W of type E_8 has the structure $2.G.2$, where

$$2.G = \mathrm{Ker}\,(\det \colon W \to \{\pm 1\}), \quad G.2 = W/\{\pm \mathrm{id}\},$$

and G is a simple group. See [7, page 85]. By the assumption $[W : \Gamma] = 2$, we have $|\Gamma| = |2.G| = |G.2|$. Let Δ be the set of (-2)-vectors in the root lattice $R(X_b)$. For $r \in \Delta$, let $s_r \in W$ denote the reflection with respect to r.

There exists a member B_q of $|\mathcal{O}_Q(3)|$ that does not pass through V and has an ordinary node as its only singularity. By the argument using the local Picard–Lefschetz formula as in the proof of Proposition 3.7, we see that there exists a vanishing cycle $v \in \Delta$ corresponding to the ordinary double point of X_q such that Γ contains the reflection s_v. Since $\det s_v = -1$, we see that $\Gamma \cap (2.G)$ is of index 2 in $2.G$. If $-\mathrm{id} \in \Gamma$, then $-\mathrm{id} \in \Gamma \cap (2.G)$ and hence $(\Gamma \cap (2.G))/\{\pm \mathrm{id}\}$ would be a subgroup of $(2.G)/\{\pm \mathrm{id}\} = G$ with index 2. Since G is simple, we have $-\mathrm{id} \notin \Gamma$. Hence Γ is mapped isomorphically onto $G.2 = W/\{\pm \mathrm{id}\}$, which acts on $\Delta/\{\pm \mathrm{id}\}$ transitively. Since $s_v \in \Gamma$ and $g^{-1} \cdot s_v \cdot g = s_{v^g} = s_{-v^g}$, we see that $s_r \in \Gamma$ for any $r \in \Delta$. Thus we obtain $\Gamma = W$, which is a contradiction. □

4. LINES IN A DEL PEZZO SURFACE OF DEGREE ONE

We choose a *general* point b of the parameter space $\mathcal{U}$ of the family $\mathcal{X} \to \mathcal{U}$ of bi-anti-canonical models of del Pezzo surfaces of degree 1 treated in Section 3.5. The purpose of this section is to investigate the configuration of lines in the del Pezzo surface X_b.

In Section 4.1, we introduce the notions of *Bertini involution* and *tangent plane sections*. In Section 4.2, using Proposition 3.8, we describe the orbit decompositions of the set of lines in X_b by the monodromy action. In Section 4.3, we describe the lines in X_b as plane curves on the projective plane $\mathbf{P}^2$ obtained by contracting 8 disjoint lines in X_b. In Section 4.4, we confirm that the union of lines in X_b has only ordinary double points as its singularities.

4.1. Bertini involution.

We have an orthogonal direct-sum decomposition

$$\mathrm{Pic}(X_b) = \mathbb{Z}\alpha_b \oplus R(X_b),$$

where α_b is the anti-canonical class of X_b. The orthogonal projection from $\mathrm{Pic}(X_b)$ to $R(X_b)$ induces a bijection

$$L(X_b) \cong \Delta(R(X_b)) \tag{4.1}$$

between the set $L(X_b)$ of lines in X_b and the set $\Delta(R(X_b))$ of (-2)-vectors of the root lattice $R(X_b)$ of type E_8. For a line $l \in L(X_b)$, we denote by

$$[l]_R := [l] - \alpha_b \ \in \ \Delta(R(X_b))$$

the corresponding (-2)-vector. Let

$$\varphi \colon X_b \to Q$$

be the bi-anti-canonical model of X_b, where $Q \subset \mathbb{P}^3$ is a quadric cone with the vertex $V \in Q$, and let $B_b \subset Q$ be the curve component of the branch locus of φ. Then B_b is a smooth member of $|\mathcal{O}_Q(3)|$. The deck-transformation of φ is called the *Bertini involution*, and is denoted by

$$i_B \colon X_b \xrightarrow{\sim} X_b.$$

We call a pair $\{l, l'\}$ of lines in X_b an *i_B-pair* if $l' = i_B(l)$, and say that $l' = i_B(l)$ is the *i_B-partner of l*. It is easy to see that, for lines l, l' in X_b, the following are equivalent: (i) the pair $\{l, l'\}$ is an i_B-pair, (ii) $\langle l, l' \rangle = 3$, (iii) $[l] + [l'] = 2\alpha_b$, and (iv) $[l]_R + [l']_R = 0$.

Definition 4.1. A plane section $H \cap Q$ of Q, where H is a linear plane in $\mathbb{P}^3$, is called a *tangent plane section for B_b* if H does not pass through the vertex V and the local intersection number at each intersection point of H and B_b is even. We put

$$S(B_b) := \text{the set of tangent plane sections for } B_b.$$

The image $\varphi(l)$ of a line $l \subset X_b$ by φ is a tangent plane section for B_b. Conversely, the pullback by φ of a tangent plane section is the union of a line and its i_B-partner. Hence we have natural identifications

$$\overline{L}(X_b) := L(X_b)/\langle i_B \rangle \ \cong \ \overline{\Delta}(R(X_b)) := \Delta(R(X_b))/\{\pm \mathrm{id}\} \ \cong \ S(B_b). \tag{4.2}$$

In particular, there exist exactly 120 tangent plane sections for B_b.

Remark 4.2. The smooth $(2,3)$-complete intersection $B_b \subset \mathbb{P}^3$ is the canonical model of a genus 4 curve with a vanishing theta constant, and the tangent plane sections for B_b are in one-to-one correspondence with the odd theta-characteristics of B_b. See Chapter IV and Appendix B of [1].

4.2. **Orbit decomposition by the monodromy.** By Proposition 3.8, we can compute the monodromy actions of $\pi_1(\mathcal{U}, b)$ on the sets in (4.1) and (4.2) explicitly. We describe the orbit decompositions of $L(X_b)^{\{k\}}$ and $\overline{L}(X_b)^{\{k\}}$ under these monodromy actions.

4.2.1. *The action on $L(X_b)^{\{k\}}$.* The monodromy action on the set $L(X_b)^{\{k\}}$ of k-element subsets of $L(X_b)$ for small k is as follows. The numbers of orbits are given as follows:

k	1	2	3	4	5	6	7
	1	4	12	62	378	3557	45282 .

- The action on $L(X_b)^{\{1\}} = L(X_b)$ is transitive.
- The action on $L(X_b)^{\{2\}}$ has four orbits. For an orbit $o \subset L(X_b)^{\{2\}}$, let $m(o)$ denote the intersection number $\langle l_1, l_2 \rangle$ of lines, where $\{l_1, l_2\} \in o$. Then the four orbits are distinguished by $m(o)$ as follows:

$m(o)$	0	1	2	3
$\lvert o \rvert$	6720	15120	6720	120 .

- The action on $L(X_b)^{\{3\}}$ has 12 orbits. For an orbit $o \subset L(X_b)^{\{3\}}$, let $t(o)$ denote the non-decreasing sequence of the intersection numbers $\langle l_i, l_j\rangle$ for $1 \le i < j \le 3$, where $\{l_1, l_2, l_3\} \in o$. Then the 12 orbits are described as follows:

$t(o)$	$\lvert o\rvert$	$t(o)$	$\lvert o\rvert$
$[0,0,0]$	60480	$[0,2,3]$	13440
$[0,0,1]$	181440	$[1,1,1]$	302400
$[0,0,2]$	6720	$[1,1,2]$	483840
$[0,1,1]$	483840	$[1,1,3]$	15120
$[0,1,2]$	362880	$[1,2,2]$	181440
$[0,2,2]$,	181440	$[2,2,2]$	2240 .

Let $\mathcal{L}^{\{k\}} \to \mathcal{U}$ be the étale covering whose fiber over $u \in \mathcal{U}$ is $L(X_u)^{\{k\}}$.

Corollary 4.3. (1) *The space* $\mathcal{L}^{\{2\}}$ *consists of exactly* 4 *irreducible components* $\mathcal{L}_0^{\{2\}}, \dots, \mathcal{L}_3^{\{2\}}$, *where* $\mathcal{L}_m^{\{2\}} \to \mathcal{U}$ *be the family of pairs* $\{l_1, l_2\}$ *of lines in* X_u *such that* $\langle l_1, l_2\rangle = m$. (2) *The space* $\mathcal{L}^{\{3\}}$ *consists of exactly* 12 *irreducible components* $\mathcal{L}_t^{\{3\}}$, *where* $t = [t_1, t_2, t_3]$ *runs through the list*

$$\begin{array}{l} [0,0,0],\ [0,0,1],\ [0,0,2],\ [0,1,1],\ [0,1,2],\ [0,2,2],\ [0,2,3],\\ [1,1,1],\ [1,1,2],\ [1,1,3],\ [1,2,2],\ [2,2,2]. \end{array} \tag{4.3}$$

The étale covering $\mathcal{L}_t^{\{3\}} \to \mathcal{U}$ *is the family of triples* $\{l_1, l_2, l_3\}$ *of lines in* X_u *such that* $[\langle l_1, l_2\rangle, \langle l_2, l_3\rangle, \langle l_1, l_3\rangle]$ *is equal to* t *up to order.* □

4.2.2. *The action on* $\overline{L}(X_b)^{\{k\}}$. The action on the set $\overline{L}(X_b)^{\{k\}}$ is identified with the action of $\overline{W}$ on $\overline{\Delta}^{\{k\}}$ defined in Section 2.2. Therefore the numbers of orbits are $N(k)$ and, for small k, they are given in Table 1.2.

- The action on the set $\overline{L}(X_b)^{\{1\}} = \overline{L}(X_b)$ of i_B-pairs is transitive.
- The action decomposes $\overline{L}(X_b)^{\{2\}}$ into two orbits of size 3360 and 3780. These two orbits are distinguished by Figure 4.1, where a line is denoted by a circle ○, an i_B-pair is denoted by ○━○, and the intersection number of distinct two lines is given by the number of line-segments connecting the corresponding circles.
- The action decomposes $\overline{L}(X_b)^{\{3\}}$ into five orbits. These orbits are depicted in Figure 4.2.

4.3. **Lines in the birational model $\mathbf{P}^2$.** We choose disjoint 8 lines $l_1, \dots, l_8$ in X_b and consider the contraction $\beta\colon X_b \to \mathbf{P}^2$ of these lines. Let $p_i \in \mathbf{P}^2$ be the point $\beta(l_i)$ for $i = 1, \dots, 8$, and let $h \in \operatorname{Pic}(X_b)$ be the class of the pullback of a line in $\mathbf{P}^2$. Since

$$3h = (\alpha_b + l_1 + \cdots + l_8),$$

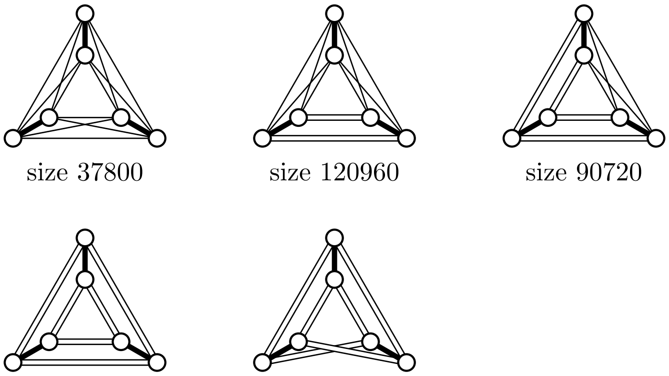
and we have $\ell + i_B(\ell) = 2\alpha_b$, it follows that

$$\langle h, \ell\rangle + \langle h, i_B(\ell)\rangle = 6$$

holds for any line ℓ. We investigate the images of the lines ℓ by β. Calculating the intersection numbers with the exceptional lines $l_1, \dots, l_8$ of β, we obtain the following. (See also [13, Theorem 26.2].) In the following, the phrase "C passes through $p \in \mathbf{P}^2$ *once*" means that p is a smooth point of C.

size 3360 size 3780

Figure 4.1. Orbits in $\overline{L}(X_b)^{\{2\}}$



size 37800 size 120960 size 90720

size 1120 size 30240

Figure 4.2. Orbits in $\overline{L}(X_b)^{\{3\}}$

- There exist exactly 8 lines l_i with h-degree 0. Their i_B-partners are of h-degree 6: the sextic curve $\beta(i_B(l_i)) \subset \mathbf{P}^2$ has a triple point at p_i, and double points at the 7 points in $\{p_1, \dots, p_8\} \setminus \{p_i\}$.
- There exist exactly 28 lines l_{ij} with h-degree 1, where $1 \le i < j \le 8$. The line l_{ij} is mapped by β to the line in $\mathbf{P}^2$ passing through p_i and p_j. Their i_B-partners are of h-degree 5: the quintic curve $\beta(i_B(l_{ij})) \subset \mathbf{P}^2$ passes through p_i and p_j once, and has double points at the 6 points in $\{p_1, \dots, p_8\} \setminus \{p_i, p_j\}$.
- There exist exactly 56 lines $l_{\overline{ijk}}$ with h-degree 2, where $1 \le i < j < k \le 8$. The line $l_{\overline{ijk}}$ is mapped by β to the conic in $\mathbf{P}^2$ passing through the five points in $\{p_1, \dots, p_8\} \setminus \{p_i, p_j, p_k\}$. Their i_B-partners are of h-degree 4: the quartic curve $\beta(i_B(l_{\overline{ijk}}))$ has double points at p_i, p_j, p_k and passes through the 5 points in $\{p_1, \dots, p_8\} \setminus \{p_i, p_j, p_k\}$ once.
- There exist exactly 56 lines $l_{i,j}$ with h-degree 3, where $1 \le i, j \le 8$ and $i \ne j$. We have $i_B(l_{i,j}) = l_{j,i}$. The cubic curve $\beta(i_B(l_{i,j}))$ passes through the 6 points in $\{p_1, \dots, p_8\} \setminus \{p_i, p_j\}$ once, has a double point at p_i, and does not pass through p_j.

4.4. **Union of lines.** In this section, we confirm the following result, which must be well known, but of which we could not find a reference.

Proposition 4.4. *The union of the* 240 *lines in* X_b *has only ordinary points as its singularities.*

h-degree $\langle h, \ell\rangle$	multiplicities $\langle l_i, \ell\rangle$	number
0	$(-1)^1 0^7$	8
1	$0^6 1^2$	28
2	$0^3 1^5$	56
3	$0^1 1^6 2^1$	56
4	$1^5 2^3$	56
5	$1^2 2^6$	28
6	$2^7 3^1$	8

Table 4.1. 240 curves in $\mathbf{P}^2$

Proof. Recall that X_b is a *general* member of the family $\mathcal{X} \to \mathcal{U}$. By Corollary 4.3, it is enough to prove the following.

(m) For $m = 2$ and $m = 3$, there exist a point $u \in \mathcal{U}$ and lines ℓ_1, ℓ_2 in X_u such that $\langle \ell_1, \ell_2\rangle = m$, and that ℓ_1 and ℓ_2 intersect at distinct m points.

(t) For each $t = [t_1, t_2, t_3]$ in the second line of (4.3), there exist a point $u \in \mathcal{U}$ and lines ℓ_1, ℓ_2, ℓ_3 in X_u such that $[\langle \ell_1, \ell_2\rangle, \langle \ell_2, \ell_3\rangle, \langle \ell_1, \ell_3\rangle]$ is equal to t up to order, and that $\ell_1 \cap \ell_2 \cap \ell_3$ is empty.

We find such a del Pezzo surface X_u by choosing 8 points $p_1, \dots, p_8$ on $\mathbf{P}^2$ satisfying the conditions in Definition 3.4. Let $Y_{\boldsymbol{p}} \to \mathbf{P}^2$ be the blowing-up at these points. The lines in $Y_{\boldsymbol{p}}$ can be calculated by the description given in Section 4.3, and we search for lines satisfying the conditions in (m) and (t). It is enough to find an example over a finite field.

We give an example over $\mathbb{F}_{19}$. We choose the following 8 points:

$$\begin{aligned} &p_1 = (0, 0), \quad p_2 = (1, 0), \quad p_3 = (0, 1), \quad p_4 = (1, 1), \\ &p_5 = (2, 15), \quad p_6 = (15, 4), \quad p_7 = (11, 15), \quad p_8 = (12, 16), \end{aligned}$$

where we use affine coordinates of $\mathbf{P}^2$. It is easy to confirm that these points satisfy the conditions in Definition 3.4. A line ℓ in $Y_{\boldsymbol{p}}$ is denoted as $[d; \mu_1, \dots, \mu_8]$, where d is the h-degree and μ_i is the multiplicity $\langle l_i, \ell\rangle$ of $\beta(\ell)$ at p_i. We consider the following lines ℓ_i, and calculate the defining equations of $\beta(\ell_i)$ in $\mathbf{P}^2$:

$$\begin{array}{lll@{\qquad}lll} \ell_1 & := & [0; -1, 0, 0, 0, 0, 0, 0, 0], & \ell_6 & := & [3; 2, 0, 1, 1, 1, 1, 1, 1], \\ \ell_2 & := & [3; 2, 1, 1, 1, 1, 1, 1, 0], & \ell_7 & := & [2; 1, 1, 1, 1, 1, 0, 0, 0], \\ \ell_3 & := & [6; 3, 2, 2, 2, 2, 2, 2, 2], & \ell_8 & := & [6; 2, 3, 2, 2, 2, 2, 2, 2], \\ \ell_4 & := & [1; 1, 1, 0, 0, 0, 0, 0, 0], & \ell_9 & := & [6; 2, 2, 3, 2, 2, 2, 2, 2]. \\ \ell_5 & := & [2; 1, 0, 1, 1, 1, 1, 0, 0], & & & \end{array}$$

Then we confirm that the pair ℓ_1, ℓ_2 (resp. the pair ℓ_1, ℓ_3) satisfies condition (m) for $m = 2$ (resp. $m = 3$). Moreover, the triple $\tau = \{\ell_i, \ell_j, \ell_k\}$ satisfies condition (t) for $t = [t_1, t_2, t_3]$, where

τ	t	τ	t
ℓ_1, ℓ_4, ℓ_5	$[1, 1, 1]$	ℓ_1, ℓ_6, ℓ_9	$[1, 2, 2]$
ℓ_1, ℓ_4, ℓ_6	$[1, 1, 2]$	ℓ_1, ℓ_6, ℓ_8	$[2, 2, 2]$.
ℓ_1, ℓ_3, ℓ_7	$[1, 1, 3]$		

The defining equations of $\beta(\ell_i)$ can be found in [18]. □

Corollary 4.5. (1) *Every tangent plane section* $H \cap Q$ *for* B_b *intersects* B_b *at distinct three points.* (2) *Suppose that* $H_1 \cap Q$ *and* $H_2 \cap Q$ *are distinct tangent plane sections for* B_b*. Then* $H_1 \cap B_b$ *and* $H_2 \cap B_b$ *are disjoint, and* $H_1 \cap H_2 \cap Q$ *consists of distinct two points.* (3) *Suppose that* $H_1 \cap Q, H_2 \cap Q,$ *and* $H_3 \cap Q$ *are distinct tangent plane sections for* B_b*. Then* $H_1 \cap H_2 \cap H_3 \cap Q$ *is empty.* □

5. Del Pezzo surfaces of degree one and t_3-sextics

In this section, we relate t_3-sextics with del Pezzo surfaces of degree 1. In Section 5.1, we prove Propositions 2.8 and exhibit the parameter space $\mathcal{T}$ of t_3-sextics in the frame (A, Λ). In Section 5.2, we describe a birational map π_p from Q to $\mathbb{P}^2$, which gives a proof of Propositions 2.9, and induces a birational map between $\mathcal{U}$ and $\mathcal{T}$.

5.1. **Plane curves with t_m-singularity.** We fix a point $A \in \mathbb{P}^2$ and a line $\Lambda \subset \mathbb{P}^2$ passing through A. Let (x, y) be affine coordinates of $\mathbb{P}^2$ such that $A = (0,0)$ and $\Lambda = \{y = 0\}$. We consider a plane curve $C \subset \mathbb{P}^2$ of degree d defined by

$$f(x,y) = \sum_{\mu+\nu\le d} a_{\mu\nu} x^{\mu} y^{\nu} = 0.$$

Proposition 5.1. *Suppose that* $d \ge 2m$*. The plane curve* $C = \{f = 0\}$ *has a* t_m*-singularity at* A *with the tangent line* Λ *if and only if the following holds:*

(i) $a_{\mu\nu} = 0$ *if* $\mu + 2\nu < 2m$*, and*
(ii) *the following equation has distinct* m *roots:*

$$a_{0,m}\, z^m + a_{2,m-1}\, z^{m-1} + \cdots + a_{2m-2,2}\, z + a_{2m,0} = 0.$$

Proof. We consider the blowing-up

$$(u, v) \quad \mapsto \quad (x, y) = (u, uv)$$

of $\mathbb{P}^2$ at A. The strict transform of Λ is given by $v = 0$, and it intersects with the exceptional divisor $E = \{u = 0\}$ at the point $(u, v) = (0, 0)$. Then C has a t_m-singularity at A with the tangent line Λ if and only if the total transform

$$f(u, uv) = 0$$

of C contains E with multiplicity m and the strict transform

$$\sum_{\mu+\nu\le d} a_{\mu\nu} u^{\mu+\nu-m} v^{\nu} = 0$$

of C has an ordinary m-fold point at $(u, v) = (0, 0)$ with each local branch intersecting E transversely. This condition is equivalent to conditions (i) and (ii) in the statement. □

Proposition 2.8 follows from Proposition 5.1 immediately. In the following, for a point $t \in \mathcal{T}$, we denote by $C_t \subset \mathbb{P}^2$ the corresponding t_3-sextic.

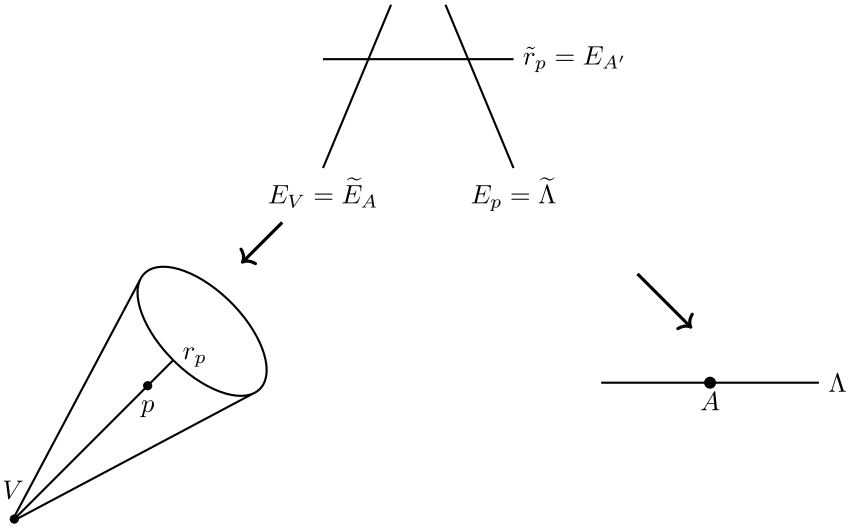


FIGURE 5.1. Birational map π_p

5.2. **Birational map π_p.** Recall that $Q \subset \mathbb{P}^3$ is a quadric cone with the vertex $V \in Q$. Then Q is ruled by lines passing through V. We choose a smooth point $p \in Q \setminus \{V\}$. Then the projection from p induces a birational map

$$\pi_p \colon Q \dashrightarrow \mathbb{P}^2.$$

This birational map π_p defines a frame (A, Λ) in $\mathbb{P}^2$ as follows.

We describe π_p in detail. Let $r_p \subset Q$ be the line in the ruling passing through p, and let $\widetilde{Q} \to Q$ be the composite of the minimal desingularization of Q and the blowing-up at p. Let E_V, E_p, and $\tilde{r}_p$ be the exceptional curve over V, the exceptional curve over p, and the strict transform of r_p in $\widetilde{Q}$. Then E_V, E_p, $\tilde{r}_p$ are smooth rational curves on $\widetilde{Q}$ with self-intersection number -2, -1, -1, respectively. We blow down $\tilde{r}_p$ to a point, and then we blow down the image E'_V of E_V to a point. The resulting surface is the target plane $\mathbb{P}^2$ of π_p. The curves E_V and $\tilde{r}_p$ are mapped to a point $A \in \mathbb{P}^2$, and the curve E_p is mapped to a line Λ passing through A. Every member of the ruling of Q other than r_p is mapped to a line passing through A. A plane section $H \cap Q$ not containing V and p is mapped to a smooth conic that is passing through A and is tangent to Λ at A.

The inverse of π_p is given as follows. Let $(\mathbb{P}^2)^{\sim} \to \mathbb{P}^2$ be the blowing up at A, and let $\widetilde{Q} \to (\mathbb{P}^2)^{\sim}$ be the blowing up at the intersection point A' of the exceptional curve E_A over A and the strict transform of Λ. Then the strict transform $\widetilde{E}_A$ of E_A (resp. $\widetilde{\Lambda}$ of Λ) in $\widetilde{Q}$ is a smooth rational curve of self-intersection number -2 (resp. -1). We contract these two curves, and let $\widetilde{Q} \to Q$ denote the contraction map. Then $\widetilde{E}_A$ is contracted to the singular point V of Q, and $\widetilde{\Lambda}$ is contracted to the center p of the projection. The exceptional curve $E_{A'} \subset \widetilde{Q}$ over A' is mapped to the line r_p in the ruling.

5.3. **Birational map Π_p.** Recall that $\mathcal{U} \subset |\mathcal{O}_Q(3)|$ is the parameter space of the family $\mathcal{X} \to \mathcal{U}$ of bi-anti-canonical models of del Pezzo surfaces of degree 1, and that $\mathcal{T}$ is the parameter space of the family of t_3-sextics $C_t \subset \mathbb{P}^2$ in the frame (A, Λ). Note that both of $\mathcal{T}$ and $\mathcal{U}$ are of dimension 15.

We have chosen a point $p \in Q \setminus \{V\}$. Suppose that u is a general point of $\mathcal{U}$. In particular, the curve B_u does not contain p and the ruling line r_p intersects B_u at three distinct points. Then the image of B_u by π_p is a t_3-sextic $C_{t(u)}$ in the frame (A, Λ), where $t(u) \in \mathcal{T}$. Conversely, if t is a general point of $\mathcal{T}$, then the image of the t_3-sextic C_t by the inverse of π_p is a member $B_{u(t)}$ of $\mathcal{U}$. Therefore the birational map π_p between Q and $\mathbb{P}^2$ induces a birational map

$$\Pi_p \colon \mathcal{U} \dashrightarrow \mathcal{T}. \tag{5.1}$$

Proof of Proposition 2.9. Recall that $G(C_t)$ denotes the set of special tangent conics of the t_3-sextic C_t for $t \in \mathcal{T}$, and that $S(B_u)$ denotes the set of tangent plane sections for B_u for $u \in \mathcal{U}$. Suppose that $u \in \mathcal{U}$ is general. Then the birational map π_p induces a bijection

$$S(B_u) \;\cong\; G(C_{t(u)}).$$

Moreover, the properties of the tangent plane sections for B_u given in Corollary 4.5 carry over to properties of the special tangent conics of C_t, ensuring that they are in a general position. □

6. EMBEDDING TOPOLOGY

In this section, we prove Theorem 2.12. The following observation is due to Artal Bartolo.

Lemma 6.1. *Any self-homeomorphism of $\mathbb{P}^2$ preserves the orientation.*

Proof. Note that the intersection form on $H_2(\mathbb{P}^2)$ depends on the choice of an orientation. For an orientation ξ of $\mathbb{P}^2$, let $\langle \quad \rangle_\xi$ denote the corresponding intersection form on $H_2(\mathbb{P}^2)$. Then we have $\langle \quad \rangle_{-\xi} = -\langle \quad \rangle_\xi$. Let η be the orientation coming from the complex structure of $\mathbb{P}^2$, which satisfies $\langle \gamma, \gamma \rangle_\eta \geq 0$ for any $\gamma \in H_2(\mathbb{P}^2)$. Suppose that a self-homeomorphism f of $\mathbb{P}^2$ satisfies $f_*\eta = -\eta$. Then we have

$$\langle \gamma, \gamma \rangle_\eta = \langle f_*\gamma, f_*\gamma \rangle_{f_*\eta} = -\langle f_*\gamma, f_*\gamma \rangle_\eta,$$

which is a contradiction. □

Now we prove our main result.

Proof of Theorem 2.12. Let $\mathcal{U}$ and $\mathcal{T}$ be as in Section 5.3. We choose Zariski open subsets $\mathcal{T}^0 \subset \mathcal{T}$ and $\mathcal{U}^0 \subset \mathcal{U}$ such that $\mathcal{T}^0$ and $\mathcal{U}^0$ are isomorphic via the birational map $\Pi_p \colon \mathcal{U} \dashrightarrow \mathcal{T}$, and such that the special tangent conics of C_t are in general position for any $t \in \mathcal{T}_0$. We fix a point $o \in \mathcal{T}^0$, and let $b \in \mathcal{U}^0$ be the point corresponding to o via Π_p. Consider the family $\mathcal{G}^0 \to \mathcal{T}^0$ whose fiber over $t \in \mathcal{T}^0$ is the set $G(C_t)$ of special tangent conics. Then $\mathcal{G}^0 \to \mathcal{T}^0$ is isomorphic to the pullback of the family $\mathcal{S} \to \mathcal{U}$ of the sets $S(B_u)$ of tangent plane sections via the morphism

$$\mathcal{T}^0 \;\cong\; \mathcal{U}^0 \;\hookrightarrow\; \mathcal{U}. \tag{6.1}$$

In particular, the monodromy action of $\pi_1(\mathcal{T}^0, o)$ on $G(C_o)$ is induced by the monodromy action of $\pi_1(\mathcal{U}, b)$ on $S(B_b)$ via the bijection $S(B_b) \cong G(C_o)$ given by π_p and the surjective homomorphism

$$\pi_1(\mathcal{T}^0, o) \twoheadrightarrow \pi_1(\mathcal{U}, b)$$

induced by (6.1). Recall from Proposition 3.8 that, under the identification of $S(B_b)$ with $\overline{\Delta}(R(X_b))$ in (4.2), the monodromy action of $\pi_1(\mathcal{U}, b)$ on $S(B_b)$ factors through the surjective homomorphism (3.4) to the Weyl group $W(R(X_b))$.

We consider the family $\mathcal{G}^{0\{k\}} \to \mathcal{T}^0$ whose fiber over t is the set $G(C_t)^{\{k\}}$. By the above discussion on the monodromy and the definition of $N(k)$, the space $\mathcal{G}^{0\{k\}}$ has exactly $N(k)$ connected components. If two points s, s' of $G(C_o)^{\{k\}}$ are in the same connected component of $\mathcal{G}^{0\{k\}}$, then we can deform $D_{o,s}$ to $D_{o,s'}$ in $\mathbb{P}^2$ without changing the embedding topology along a path in $\mathcal{G}^{0\{k\}}$ connecting s and s'.

To complete the proof of Theorem 2.12, it is enough to show that, if $D_{o,s}$ and $D_{o,s'}$ have the same embedding topology, then s and s' belong to the same connected component of $\mathcal{G}^{0\{k\}}$. For a special tangent conic $\Gamma \in G(C_o)$ of C_o, let $\delta_\Gamma \in \overline{\Delta}(R(X_b))$ denote the pair $\{[l_\Gamma]_R, -[l_\Gamma]_R\} \subset R(X_b)$, where $\{l_\Gamma, i_B(l_\Gamma)\}$ is the i_B-pair obtained from the tangent plane section for B_b corresponding to Γ via π_p. We then put

$$\delta(s) := \{\, \delta_\Gamma \mid \Gamma \in s \,\} \quad \in \quad \overline{\Delta}(R(X_b))^{\{k\}}.$$

To show that s and s' are in the same connected component of $\mathcal{G}^{0\{k\}}$, it is enough to find an isometry $g \in W(R(X_b))$ of the lattice $R(X_b)$ such that the self-bijection of $\overline{\Delta}(R(X_b))^{\{k\}}$ induced by g maps $\delta(s)$ to $\delta(s')$.

Suppose that $D_{o,s}$ and $D_{o,s'}$ have the same embedding topology. We have a homeomorphism

$$\Psi \;:\; (\mathbb{P}^2, D_{o,s}) \xrightarrow{\sim} (\mathbb{P}^2, D_{o,s'}).$$

Then Ψ induces a self-homeomorphism Ψ_M of the complement

$$M_o := \mathbb{P}^2 - (C_o + \Lambda).$$

Since $H_1(M_o) \cong \mathbb{Z}$, there exists a unique double covering

$$\varphi_M \colon Z_o \to M_o$$

of M_o by a connected surface Z_o, and Ψ_M lifts to a self-homeomorphism Ψ_Z of Z_o. Note that the lift Ψ_Z is unique up to the deck-transformation of Z_o over M_o. Since Ψ_M is the restriction of a self-homeomorphism of $\mathbb{P}^2$, Lemma 6.1 implies that Ψ_M preserves the orientation of M_o, and hence Ψ_Z is an orientation-preserving homeomorphism of Z_o. Consequently, the automorphism

$$g_Z(\Psi) \colon H_2(Z_o) \xrightarrow{\sim} H_2(Z_o)$$

of the $\mathbb{Z}$-module $H_2(Z_o)$ induced by Ψ_Z preserves the intersection form $\langle \quad \rangle_Z$ given by the complex structure of Z_o. We put

$$\mathrm{Ker} \langle \quad \rangle_Z := \{\, x \in H_2(Z_o) \mid \langle x, y \rangle_Z = 0 \text{ for any } y \in H_2(Z_o) \,\}.$$

Then $g_Z(\Psi)$ gives rise to an isometry of the lattice

$$\overline{H}_2(Z_o) := H_2(Z_o) / \mathrm{Ker} \langle \quad \rangle_Z.$$

Note that π_p induces an isomorphism from M_o to

$$Q_b^0 := Q - (B_b + r_p).$$

Let $2\widetilde{B}_b$ and $(r_p)^{\sim}$ be the pullback of B_b and r_p by the double covering $X_b \to Q$. Then the double covering Z_o of M_o can be identified with

$$X_b^0 := X_b - (\widetilde{B}_b + (r_p)^{\sim}).$$

Note that this identification is unique up to the deck-transformation of X_b^0 over Q_b^0. In the following, we regard Z_o as a Zariski open subset of X_b. We also identify $H_2(X_b)$ with $H^2(X_b) = \operatorname{Pic}(X_b)$ by the Poincaré duality. Since the homology classes of $\widetilde{B}_b$ and $(r_p)^{\sim}$ in $H_2(X_b)$ are $3\alpha_b$ and α_b, respectively, and both of $\widetilde{B}_b$ and $(r_p)^{\sim}$ are irreducible, the Poincaré-Lefschetz duality implies that the inclusion $Z_o \hookrightarrow X_b$ yields a surjective homomorphism

$$H_2(Z_o) \twoheadrightarrow (\alpha_b)^{\perp} \subset H_2(X_b) \tag{6.2}$$

that preserves the intersection form, where $(\alpha_b)^{\perp}$ is the orthogonal complement of α_b. By the identification $H_2(X_b) = \operatorname{Pic}(X_b)$, we have $(\alpha_b)^{\perp} = R(X_b)$, and hence the intersection form on $(\alpha_b)^{\perp}$ is non-degenerate. Therefore the kernel of (6.2) is equal to $\operatorname{Ker}\langle\ \ \rangle_Z$. In particular, the lattice $\overline{H}_2(Z_o)$ is isomorphic to the lattice $R(X_b)$. Consequently, the homeomorphism Ψ defines an isometry

$$g_X(\Psi) \in W(R(X_b)).$$

Note that $g_X(\Psi)$ is uniquely determined by Ψ up to ±id.

We will show that the self-bijection of $\overline{\Delta}(R(X_b))^{\{k\}}$ induced by $g_X(\Psi)$ sends $\delta(s) = \{\, \delta_\Gamma \mid \Gamma \in s \,\}$ to $\delta(s')$. Since Ψ maps the elements of s to the elements of s' bijectively, it suffices to show that, if Ψ maps a special tangent conic Γ to a special tangent conic Γ', then $g_X(\Psi)$ maps δ_Γ to $\delta_{\Gamma'}$.

Suppose that $\Psi(\Gamma) = \Gamma'$. The curve $\varphi_M^{-1}(\Gamma \cap M_o)$ in Z_o has two connected components Γ^+ and Γ^-. The closure of Γ^+ in X_b is a line l_Γ, and the closure of Γ^- is its i_B-partner. (Recall that we have $Z_o \subset X_b$.) These two components, viewed as locally finite topological cycles, give rise to linear forms

$$\gamma^+ \colon H_2(Z_o) \to \mathbb{Z}, \quad \gamma^- \colon H_2(Z_o) \to \mathbb{Z},$$

by the intersection paring. By definition, each of these linear forms factors through the quotient homomorphism

$$H_2(Z_o) \twoheadrightarrow (\alpha_b)^{\perp} = R(X_b)$$

given by $Z_o \hookrightarrow X_b$, and the induced linear form $(\alpha_b)^{\perp} \to \mathbb{Z}$ coincides with the intersection pairing with $[l_\Gamma]_R \in R(X_b)$ and with $[i_B(l_\Gamma)]_R = -[l_\Gamma]_R$, respectively. The connected components of the curve $\varphi_M^{-1}(\Gamma' \cap M_o)$ are $\Psi_Z(\Gamma^+)$ and $\Psi_Z(\Gamma^-)$. The linear forms on $H_2(Z_o)$ given by these locally finite topological cycles are equal to $\gamma^+ \circ g_Z(\Psi)^{-1}$ and $\gamma^- \circ g_Z(\Psi)^{-1}$, respectively. Each of these linear forms yields a linear form on $(\alpha_b)^{\perp} = R(X_b)$, which is the intersection pairing with $[l_{\Gamma'}]_R$ and with $-[l_{\Gamma'}]_R$, respectively, where $l_{\Gamma'}$ is the closure of $\Psi_Z(\Gamma^+)$ in X_b. Hence we obtain $g_X(\Psi)([l_\Gamma]_R) = [l_{\Gamma'}]_R$. □

Remark 6.2. The double covering W of X_b branching along a general member of $|2\alpha_b|$ is a $K3$ surface. In [14], it is proved that the automorphism group of W is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, that W contains exactly 240 smooth rational curves, and that W has a structure of the double plane $W \to \mathbb{P}^2$ whose branch curve is a smooth sextic possessing 120 conics that are 6-tangent. This $K3$ surface had been discovered in [11].

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