ZARISKI HYPERPLANE SECTION THEOREM FOR GRASSMANNIAN VARIETIES

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ABSTRACT. Let $\phi : X \to M$ be a morphism from a smooth irreducible complex quasi-projective variety X to a Grassmannian variety M such that the image is of dimension ≥ 2. Let D be a reduced hypersurface in M, and γ a general linear automorphism of M. We show that, under a certain differential-geometric condition on $\phi(X)$ and D, the fundamental group $\pi_1((\gamma \circ \phi)^{-1}(M \setminus D))$ is isomorphic to a central extension of $\pi_1(M \setminus D) \times \pi_1(X)$ by the cokernel of $\pi_2(\phi) : \pi_2(X) \to \pi_2(M)$.

1. INTRODUCTION

Let V be a complex vector space of dimension m, and let

$$M := \mathrm{Grass}(r, V)$$

be the Grassmannian variety of all r-dimensional linear subspaces of V. Let the group $G := GL(V)$ act on M from left in the natural way. Suppose that we are given a morphism

$$\phi \,:\, X \,\to\, M$$

from a smooth irreducible quasi-projective variety X. Suppose also that a non-zero reduced effective divisor D of M is given. For $\gamma \in G$, let

$$^{\gamma}\phi \,:\, X \,\to\, M$$

denote the composite of ϕ with the action $\gamma : M \to M$ of γ on M, and let

$$^{\gamma}\Phi \,:\, {}^{\gamma}\phi^{-1}(M \setminus D) \,\to\, (M \setminus D) \times X$$

denote the morphism given by $x \mapsto ({}^{\gamma}\phi(x), x)$. We consider the homomorphism

$$^{\gamma}\Phi_* \,:\, \pi_1({}^{\gamma}\phi^{-1}(M \setminus D)) \,\to\, \pi_1(M \setminus D) \times \pi_1(X)$$

induced by ${}^{\gamma}\Phi$.

The main result of this paper states that, if $\gamma \in G$ is general, then, under a certain differential-geometric condition on $\phi(X)$ and D, the homomorphism ${}^{\gamma}\Phi_*$ gives $\pi_1({}^{\gamma}\phi^{-1}(M \setminus D))$ a structure of the central extension of $\pi_1(M \setminus D) \times \pi_1(X)$ by the cokernel of $\pi_2(\phi) : \pi_2(X) \to \pi_2(M)$. This differential-geometric condition (Condition (DG) in §2) is closely related to the problem of characterizing Chow forms among hypersurfaces in a Grassmannian variety. (See [4, Chapter 4].) In fact, if Condition (DG) is not satisfied, then $\overline{\phi(X)}$ and D or $\mathrm{Sing}\, D$ are very special subvarieties of M, and the fundamental group $\pi_1({}^{\gamma}\phi^{-1}(M \setminus D))$ is not necessarily a central extension of $\pi_1(M \setminus D) \times \pi_1(X)$ by the cokernel of $\pi_2(\phi)$. See §9 for examples.

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When M is a projective space $\mathbb{P}^{m-1}$, Condition (DG) is always satisfied. Putting ϕ to be a linear embedding of $\mathbb{P}^2$, we obtain the classical Zariski hyperplane section theorem [9], the first rigorous proof of which was given by Hamm and Lé [6]. Therefore, our result is a generalization of Zariski hyperplane section theorem to Grassmannian varieties.

This paper is organized as follows. In §2, we make some definitions, state Main Theorem, and give some remarks. In §3, we investigate the situation where Condition (DG) is not satisfied, and describe special features that $\overline{\phi(X)}$ and D possess in this situation. Sections from §4 to §8 are devoted to the proof of Main Theorem. The strategy of the proof is as follows. In §4, we extend the family of ${}^{\gamma}\phi^{-1}(M \setminus D)$ over G to a family over an affine space $\mathrm{End}(V)$, so that we can use [8, Theorem 1.3]. In §5, we prove that the fundamental group of the total space of the family over $\mathrm{End}(V)$ is a central extension of $\pi_1(M \setminus D) \times \pi_1(X)$ by the cokernel of $\pi_2(\phi)$. By [8, Theorem 1.3], it is therefore enough to show that the local monodromies on the fundamental groups of fibers of the family can be defined and are all trivial. In §6, we introduce the *transversality condition*. In §7, we prove that Condition (DG) implies the transversality condition, and in §8, we prove that the transversality condition implies the triviality of local monodromies. In §9, we present examples which show that Condition (DG) is not dispensable for the statement on the fundamental groups to hold.

Debarre [2] also found a relation between a similar differential-geometric condition on subvarieties of a Grassmannian variety and a certain connectivity theorem.

2. Statement of Main Theorem

For a point p of M, let $L(p)$ denote the linear subspace of V corresponding to p. Then we have canonical isomorphisms

$$T_pM \cong \mathrm{Hom}(L(p), V/L(p)) \quad \text{and} \quad T_p^*M \cong \mathrm{Hom}(V/L(p), L(p)), \tag{2.1}$$

where T_p^*M is the dual space of the Zariski tangent space T_pM to M at p. We define $\mathrm{rank}(\tau)$ for $\tau \in T_pM$ and $\mathrm{corank}(\omega)$ for $\omega \in T_p^*M$ to be the rank of the corresponding linear homomorphisms $L(p) \to V/L(p)$ and $V/L(p) \to L(p)$, respectively. For linear subspaces T of T_pM and N^* of T_p^*M, we put

$$\mathrm{rank}(T) := \max\{\, \mathrm{rank}(\tau) \mid \tau \in T \,\}, \quad \text{and}$$
$$\mathrm{corank}(N^*) := \max\{\, \mathrm{corank}(\omega) \mid \omega \in N^* \,\}.$$

Let Y be a reduced irreducible closed subvariety of M. We choose a general point $p \in Y$, and put

$$\mathrm{rank}\, Y := \mathrm{rank}(T_pY) \quad \text{and} \quad \mathrm{corank}\, Y := \mathrm{corank}(N_p^*Y),$$

where N_p^*Y is the co-normal space $(T_pM/T_pY)^* \subset T_p^*M$ of Y at p. Let us call them the *rank* and the *corank* of Y, respectively.

We also define a notion of *type* of a subvariety Y of M with $\mathrm{rank}\, Y = 1$ or $\mathrm{corank}\, Y = 1$ as follows.

Let A and B be finite dimensional linear spaces, and T a linear subspace of $\mathrm{Hom}(A, B)$ with $\dim T \geq 1$. Suppose that the rank of $\tau : A \to B$ is ≤ 1 for all $\tau \in T$. Then either one of the following occurs:

(I) There is a one-dimensional linear subspace B_T of B such that $\tau(A) \subseteq B_T$ for any $\tau \in T$.

(II) There is a hyperplane A_T of A such that $A_T \subseteq \operatorname{Ker} \tau$ for any $\tau \in T$.

When $\dim T = 1$, both of (I) and (II) occur, while when $\dim T \geq 2$, only one of (I) or (II) occurs.

Suppose that Y is of rank 1 (resp. of corank 1). We say that Y is of type (I) or (II) according to whether (I) or (II) holds for $T_pY \subset \operatorname{Hom}(L(p), V/L(p))$ (resp. $N_p^*Y \subset \operatorname{Hom}(V/L(p), L(p))$), where p is a general point of Y. Remark that, when Y is of corank 1 and of codimension 1 in M, then Y is both of type (I) and (II).

Let $\{D_i \mid i \in I\}$ be the set of irreducible components of the reduced hypersurface D of M, and let $\{(\operatorname{Sing} D)_j \mid j \in J^{(2)}\}$ be the set of irreducible components with codimension 2 in M of the singular locus $\operatorname{Sing} D$ of D. We consider the following conditions:

($\mathrm{a_I}$) The closure $\overline{\phi(X)}$ of $\phi(X)$ is of rank 1 with type (I).
($\mathrm{a_{II}}$) The closure $\overline{\phi(X)}$ of $\phi(X)$ is of rank 1 with type (II).
(b) For at least one $i \in I$, D_i is of corank 1.
($\mathrm{c_I}$) For at least one $j \in J^{(2)}$, $(\operatorname{Sing} D)_j$ is of corank 1 with type (I).
($\mathrm{c_{II}}$) For at least one $j \in J^{(2)}$, $(\operatorname{Sing} D)_j$ is of corank 1 with type (II).

Our differential-geometric condition (DG) is the following:

Condition (DG)**.** The Grassmannian variety M is $\mathbb{P}^{m-1}$, *or* the condition

$$(\ (\mathrm{a_I}) \text{ and } (\ (\mathrm{b}) \text{ or } (\mathrm{c_I})\ )\ ) \text{ or } (\ (\mathrm{a_{II}}) \text{ and } (\ (\mathrm{b}) \text{ or } (\mathrm{c_{II}})\ )\ )$$

is *not* satisfied.

For example, if $\overline{\phi(X)}$ is of rank > 1, *or* if all D_i $(i \in I)$ and all $(\operatorname{Sing} D)_j$ $(j \in J^{(2)})$ are of corank > 1, then Condition (DG) is satisfied. (As will be shown in §3, a subvariety of M with (co)rank 1 is of very special type.)

To describe a central extension of a fundamental group, we use the following method. Let T be an oriented connected topological manifold, and let α be an element of $H^2(T, \mathbb{Z})$. Then there exists a topological line bundle $L \to T$, unique up to isomorphisms, such that $c_1(L) = \alpha$. Let $L^\times \subset L$ be the complement to the zero section of L. We have the homotopy exact sequence

$$\longrightarrow \pi_2(T) \xrightarrow{\partial_L} \pi_1(\mathbb{C}^\times) \longrightarrow \pi_1(L^\times) \longrightarrow \pi_1(T) \longrightarrow 1$$

such that the image of $\pi_1(\mathbb{C}^\times) \to \pi_1(L^\times)$ is contained in the center. Thus we obtain a central extension of $\pi_1(T)$ by the cyclic group $\operatorname{Coker} \partial_L$, which we call *the central extension associated with* $\alpha \in H^2(T, \mathbb{Z})$.

Let $c \in H^2(M, \mathbb{Z})$ be the first Chern class of the positive generator of $\operatorname{Pic}(M)$. We define $\eta \in H^2((M \setminus D) \times X, \mathbb{Z})$ to be the cohomology class

$$-(\iota \circ \mathrm{pr}_1)^* c + (\phi \circ \mathrm{pr}_2)^* c,$$

where pr_1 and pr_2 are projections from $(M \setminus D) \times X$ to $M \setminus D$ and X, respectively, and ι is the inclusion of $M \setminus D$ into M.

Main Theorem. *Suppose that* $\dim \overline{\phi(X)} \geq 2$, *and that the condition* (DG) *is satisfied. Let* γ *be a general element of the group* G. *Then the homomorphism*

$${}^\gamma\Phi_* : \pi_1({}^\gamma\phi^{-1}(M \setminus D)) \to \pi_1(M \setminus D) \times \pi_1(X)$$

gives $\pi_1({}^\gamma\phi^{-1}(M \setminus D))$ *a structure of the central extension of* $\pi_1(M \setminus D) \times \pi_1(X)$ *by the cokernel of* $\pi_2(\phi) : \pi_2(X) \to \pi_2(M)$, *and this central extension is associated with the cohomology class* η.

Corollary 2.1. *Let $\phi : X \to \mathbb{P}^{m-1}$ be a morphism from a smooth irreducible quasi-projective variety X to $\mathbb{P}^{m-1}$, and $D \subset \mathbb{P}^{m-1}$ a reduced effective divisor. Suppose that $\dim \overline{\phi(X)} \geq 2$. If γ is a general linear automorphism of $\mathbb{P}^{m-1}$, then $\pi_1({}^\gamma\phi^{-1}(\mathbb{P}^{m-1} \setminus D))$ is isomorphic to a central extension of $\pi_1(\mathbb{P}^{m-1} \setminus D) \times \pi_1(X)$ by the cokernel of $\pi_2(\phi)$, and this central extension is associated with η.* □

Remark 2.2. We have an isomorphism $\mathrm{Grass}(r, V) \cong \mathrm{Grass}(m-r, V)$. Hence, replacing r with $m-r$ if necessary, we can assume that $r \leq m-2$. We will use this assumption in the proof of Proposition 8.4 in §8.

Remark 2.3. Since X is quasi-projective, we can embed X into a projective space $\mathbb{P}^N$. We cut X by a general linear subspace Λ of $\mathbb{P}^N$ with codimension $\dim X - 2$ to obtain a smooth surface $S := X \cap \Lambda$. Let ${}^\gamma\phi|_S : S \to M$ be the restriction of ${}^\gamma\phi$ to S. Suppose that $\gamma \in G$ is general. By Goresky and MacPherson's theorem [5, Part II, 1.1, Theorem], both of the inclusions

$$S \hookrightarrow X \quad \text{and} \quad {}^\gamma\phi|_S^{-1}(M \setminus D) \hookrightarrow {}^\gamma\phi^{-1}(M \setminus D)$$

induce isomorphisms on the fundamental groups, and the inclusion of S into X induces a surjective homomorphism $\pi_2(S) \twoheadrightarrow \pi_2(X)$. In particular, the cokernel of $\pi_2(\phi)$ is isomorphic to the cokernel of $\pi_2(\phi|_S)$. On the other hand, $\dim \overline{\phi(X)} \geq 2$ holds if and only if $\dim \overline{\phi|_S(S)} = 2$ holds. Moreover the condition $(\mathrm{a_I})$ (resp. $(\mathrm{a_{II}})$) is satisfied if and only if $(\mathrm{a_I})$ (resp. $(\mathrm{a_{II}})$) with ϕ replaced by $\phi|_S$ is satisfied. Therefore it suffices to prove Main Theorem for $\phi|_S$; that is, we can assume that $\dim X = 2$, and that $\phi : X \to M$ is a quasi-finite morphism onto its image. We will use this assumption in §8.

Remark 2.4. Let $L \to T$ and $\alpha = c_1(L) \in H^2(T, \mathbb{Z})$ be as above. We have a homomorphism between exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^2(\pi_1(T), \mathbb{Z}) \arrow[r] \arrow[d] & H^2(T, \mathbb{Z}) \arrow[r, "\pi^*"] \arrow[d] & H^2(\widetilde{T}, \mathbb{Z}) \arrow[d] \\
0 \arrow[r] & H^2(\pi_1(T), \operatorname{Coker} \partial_L) \arrow[r] & H^2(T, \operatorname{Coker} \partial_L) \arrow[r, "\pi^*"] & H^2(\widetilde{T}, \operatorname{Coker} \partial_L),
\end{tikzcd}$$

where $\pi : \widetilde{T} \to T$ is the universal covering of T (see [1]). Since

$$\pi^*(\alpha) \in H^2(\widetilde{T}, \mathbb{Z}) \cong \mathrm{Hom}(\pi_2(T), \mathbb{Z})$$

is the boundary homomorphism $\partial_L : \pi_2(T) \to \mathbb{Z}$, it becomes zero in $H^2(\widetilde{T}, \operatorname{Coker} \partial_L)$. Thus α defines an element of $H^2(\pi_1(T), \operatorname{Coker} \partial_L)$. One can easily check that this element corresponds to the central extension of $\pi_1(T)$ associated with α.

Remark 2.5. In fact, Corollary 2.1 can be easily proved directly as follows. As was remarked above, we can assume that $\dim X = 2$, and that $\phi : X \to \mathbb{P}^{m-1}$ is quasi-finite onto its image. Let γ be a general element of G. We define

$$F : X \times (\mathbb{P}^{m-1} \setminus D) \to \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$$

to be the morphism given by $F(x, y) := ({}^\gamma\phi(x), y)$. Let Δ be the diagonal of $\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}$, and let Δ_ϵ be a small tubular neighborhood of Δ. Then $F^{-1}(\Delta)$ is isomorphic to ${}^\gamma\phi^{-1}(\mathbb{P}^{m-1} \setminus D)$, and since γ is general, $F^{-1}(\Delta_\epsilon)$ is homotopic to $F^{-1}(\Delta)$. Then the result of Corollary 2.1 (except for the description of the central extension) follows from [3, Theorem 9.2 (b) with Remark 9.3].

3. Subvarieties of a Grassmannian variety with (co)rank 1

In this section, we assume that M is not a projective space $\mathbb{P}^{m-1}$.

Theorem 3.1. *Let Y be a reduced irreducible closed subvariety of M. Suppose that* $\dim Y \geq 2$.

(1) *The subvariety Y is of rank* 1 *with type* (I) *if and only if there exists a linear subspace $W \subset V$ with $\dim W = r + 1$ such that $L(p) \subset W$ for all $p \in Y$.*

(2) *The subvariety Y is of rank* 1 *with type* (II) *if and only if there exists a linear subspace $W' \subset V$ with $\dim W = r - 1$ such that $W' \subset L(p)$ for all $p \in Y$.*

Proof. The proofs of (1) and (2) are completely parallel. Therefore we will prove only (1). The 'if' part is obvious. We will prove 'only if' part.

Suppose that Y is of rank 1 with type (I). We choose a general point y_0 of Y. There exists a unique $(r+1)$-dimensional linear subspace $W(y_0)$ containing $L(y_0)$ such that $T_{y_0}Y$ is contained in the linear subspace

$$\widetilde{W}(y_0) := \{\, \tau \in \operatorname{Hom}(L(y_0), V/L(y_0)) \;\mid\; \operatorname{Im}\tau \subset W(y_0)/L(y_0) \,\}$$

of $\operatorname{Hom}(L(y_0), V/L(y_0))$ under the isomorphisms (2.1). We choose a basis

$$e_1, \dots, e_r, f_1, \dots, f_{m-r}$$

of V such that $L(y_0)$ is spanned by $e_1, \dots, e_r$, and that $W(y_0)$ is spanned by $e_1, \dots, e_r, f_1$. We define a local coordinate system $(x_{ij})_{1 \leq i \leq m-r, 1 \leq j \leq r}$ of M in such a way that the r-dimensional linear subspace $L(p)$ of V corresponding to a point $p = (x_{ij})$ is spanned by the vectors

$$e'_j(p) := e_j + \sum_{i=1}^{m-r} f_i x_{ij} \quad (j = 1, \dots, r).$$

Let d be the dimension of Y. Since Y is of type (I), we have $d \leq r$. Let $(z_1, \dots, z_d)$ be a local analytic coordinate system of Y with $y_0 = (0, \dots, 0)$ defined in a small open neighborhood U of y_0. We put

$$g_{ij}(z_1, \dots, z_d) := x_{ij}|_Y, \quad \text{and} \quad \partial_\nu g_{ij} := \frac{\partial g_{ij}}{\partial z_\nu}.$$

Then the tangent vector $(\partial/\partial z_\nu)_y \in T_y Y$ is given by an $(m-r) \times r$ matrix

$$F_\nu(y) := (\partial_\nu g_{ij}(y))$$

that expresses a linear homomorphism from $L(y)$ to $V/L(y)$ with respect to the basis $e'_1(y), \dots, e'_r(y)$ of $L(y)$ and the basis

$$f_1 \bmod L(y) \;,\; \dots \;,\; f_{m-r} \bmod L(y)$$

of $V/L(y)$. The condition that Y is of rank 1 with type (I) is equivalent to the condition that the $d \cdot r$ column vectors of the d matrices $F_1(y), \dots, F_d(y)$ are proportional to each other for any $y \in U$.

Recall that $T_{y_0}Y$ is contained in $\widetilde{W}(y_0)$. By choosing a suitable basis of V and making a linear transformation among $z_1, \dots, z_d$, we can assume that

$$\partial_\nu g_{ij}(y_0) = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = \nu, \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

holds for $\nu = 1, \dots, d$. Then, by an analytic transformation of the local coordinates $(z_1, \dots, z_d)$, we can put $g_{1\nu} \equiv z_\nu$ for $\nu = 1, \dots, d$. In particular, we have

$$\partial_\nu g_{1\nu} \equiv 1, \quad \text{and} \quad \partial_\nu g_{1j} \equiv 0 \quad (j \neq \nu). \tag{3.2}$$

Since the column vectors of the matrix $F_\nu(y)$ are proportional to each other for any $y \in U$, the equality (3.2) implies that the column vectors of $F_\nu(y)$ are zero except for the ν-th column. Hence we have $\partial_\nu g_{ij} \equiv 0$ for $j \neq \nu$; that is, g_{ij} is a function of one variable z_j. On the other hand, the μ-th column vector of $F_\mu(y)$ and the ν-th column vector of $F_\nu(y)$ are proportional to each other for any $y \in U$. Since the top entry of these column vectors is 1 by (3.2), we have $\partial_\mu g_{i\mu} \equiv \partial_\nu g_{i\nu}$ for $i = 2, \dots, m-r$. The left hand side depends only on z_μ, while the right hand side depends only on z_ν. Therefore they are constant. Since they are zero at y_0 by (3.1), we have $\partial_\nu g_{i\nu} \equiv 0$ for $i = 2, \dots, m-r$ and $\nu = 1, \dots, d$. Since $g_{i\nu}$ is zero at y_0, we have $g_{i\nu} \equiv 0$ for $i = 2, \dots, m-r$ and $\nu = 1, \dots, d$. This implies Y is contained in $\{p \in M \mid L(p) \subset W(y_0)\}$. □

Next we consider the subvariety of M with corank 1. We put

$$\mathbb{P}_*(V) := \mathrm{Grass}(1, V),$$

and consider M as the variety of all $(r-1)$-dimensional projective linear subspaces of $\mathbb{P}_*(V)$. For a point $p \in M$, let $\Pi(p) \subset \mathbb{P}_*(V)$ denote the projective linear subspace corresponding to p. Let S be a reduced irreducible closed subvariety of $\mathbb{P}_*(V)$. For a point $x \in S$, we denote by $ET_xS \subset \mathbb{P}_*(V)$ the embedded Zariski tangent space to S at x. We denote by S^{ns} the smooth locus of S, and put

$$C_k(S) := \overline{\{\, p \in M \;\mid\; \dim(\Pi(p) \cap ET_xS) = k \;\text{ for some }\; x \in \Pi(p) \cap S^{ns} \,\}},$$

where the over-line means the Zariski closure. When $k = \dim S - m + r + 1$, the subvariety $C_k(S)$ of M is the *higher associated hypersurface* defined in [4, Section 2E, Chapter 3]. Note that, if Y is a hypersurface of M, then Y is of corank 1 if and only if Y is *coisotopic* in the sense of [4, Definition 3.9, Section 3, Chapter 4]. Therefore, by Theorem 3.14 in [4, Section 3, Chapter 4], we obtain the following theorem. (See also [2, Proposition 3.3].)

Theorem 3.2. *A reduced irreducible hypersurface $Y \subset M$ is of corank 1 if and only if Y is a higher associated hypersurface $C_k(S)$ of a reduced irreducible closed subvariety $S \subset \mathbb{P}_*(V)$ with $\dim S = m - r - 1 + k$.* □

This theorem can be generalized as follows. Let M^* be the Grassmannian variety of all $(m-r)$-dimensional linear subspaces of $V^* := \mathrm{Hom}(V, \mathbb{C})$. We have a natural isomorphism

$$\delta : M^* \xrightarrow{\sim} M.$$

For a reduced irreducible closed subvariety S^* of $\mathbb{P}^*(V) := \mathbb{P}_*(V^*)$, we define the subvariety $C_k(S^*)^*$ of M^* associated to S^* in the same way.

Theorem 3.3. *Let Y be a reduced irreducible closed subvariety of M with codimension $l \geq 2$.*

(1) If Y is of corank 1 with type (I)*, then there exists a reduced irreducible closed subvariety $S \subset \mathbb{P}_*(V)$ with $\dim S = m - r - l$ such that Y coincides with $C_0(S)$.*

(2) If Y is of corank 1 with type (II)*, then there exists a reduced irreducible closed subvariety $S^* \subset \mathbb{P}^*(V)$ with $\dim S^* = r - l$ such that Y coincides with $\delta(C_0(S^*)^*)$.*

Proof. The following proof is almost same as the proof of [2, Proposition 3.3]. First note that, if $Y \subset M$ is of corank 1 with type (II), then $\delta^{-1}(Y) \subset M^*$ is of corank 1 with type (I). Therefore it is enough to prove (1).

Let Y^{ns} be a Zariski open dense subset of Y consisting of $y \in Y$ at which Y is smooth. Since $\operatorname{corank}(N_y^* Y)$ is a lower semi-continuous function of $y \in Y^{ns}$, we have $\operatorname{corank}(N_y^* Y) = 1$ for any $y \in Y^{ns}$. Let y be a point of Y^{ns}. There exists a unique one-dimensional linear subspace $B(y)$ of $L(y)$ and a linear subspace $K(y)$ of $V/L(y)$ with codimension l such that

$$T_y Y = \{ \ \tau \in \operatorname{Hom}(L(y), V/L(y)) \ \mid \ \tau(B(y)) \subset K(y) \ \}$$

under the isomorphisms (2.1). We denote by $\rho(y)$ the point of $\mathbb{P}_*(V)$ corresponding to $B(y)$. Note that $\rho(y) \in \Pi(y)$. Let Σ be the Zariski closure of $\{(y, \rho(y)) \mid y \in Y^{ns}\}$ in $Y \times \mathbb{P}_*(V)$, and let S be the image of the projection of Σ to $\mathbb{P}_*(V)$. We put

$$s := \dim S, \quad \text{and} \quad k := \dim(ET_{\rho(y_0)} S \cap \Pi(y_0)),$$

where y_0 is a general point of Y^{ns}. We then have $Y \subseteq C_k(S)$. Hence we have

$$\dim Y \ = \ (m-r)r - l \ \le \ \dim C_k(S) \ \le \ s + k(s-k) + (m-r)(r-k-1). \tag{3.3}$$

The fiber of $\Sigma \to S$ over the general point $\rho(y_0)$ of S is contained in

$$\{ \ p \in M \ \mid \ L(p) \supset B(y_0) \ \} \ \cong \ \operatorname{Grass}(r-1, m-1).$$

Hence we have

$$s \ \ge \ \dim \Sigma - (m-r)(r-1) \ = \ m - r - l. \tag{3.4}$$

Let

$$(u, v) \ \in \ \operatorname{Hom}(L(y_0), V/L(y_0)) \times \operatorname{Hom}(B(y_0), V/B(y_0))$$

be an element of

$$T_{(y_0, \rho(y_0))} \Sigma \ \subset \ T_y M \times T_{\rho(y_0)} \mathbb{P}_*(V).$$

Since $B(y) \subset L(y)$ holds for every $y \in Y^{ns}$, we have $u|_{B(y_0)} = \pi \circ v$, where π is the natural projection from $V/B(y_0)$ to $V/L(y_0)$. Since $(y_0, \rho(y_0))$ is a general point of Σ, $T_{\rho(y_0)} S$ is the image of $T_{(y_0, \rho(y_0))} \Sigma$. Therefore $T_{\rho(y_0)} S$ is contained in the linear subspace

$$\widetilde{K}(y_0) := \{ \ v \in \operatorname{Hom}(B(y_0), V/B(y_0)) \ \mid \ \operatorname{Im}(\pi \circ v) \subset K(y_0) \ \}$$

of $T_{\rho(y_0)} \mathbb{P}_*(V)$, which is of dimension $m - 1 - l$ and contains $T_{\rho(y_0)} \Pi(y_0)$. Hence we have

$$k \ \ge \ \dim T_{\rho(y_0)} S + \dim T_{\rho(y_0)} \Pi(y_0) - \dim \widetilde{K}(y_0) \ = \ s - (m - r - l). \tag{3.5}$$

Since $l \ge 2$, the pair (s, k) satisfying the inequalities (3.3), (3.4) and (3.5) is only $(m - r - l, 0)$. Therefore we have $Y = C_0(S)$ with $\dim S = m - r - l$. □

4. Construction of a family of complements over $\operatorname{End}(V)$

Hironaka's resolution of singularities gives us a smooth projective completion $\overline{X}$ of X and a morphism $\bar{\phi} : \overline{X} \to M$ such that

$$W := \overline{X} \setminus X$$

is a normal crossing divisor, and that the restriction of $\bar{\phi}$ to X coincides with ϕ. We equip W with the reduced structure so that W is a reduced divisor (possibly empty) of $\overline{X}$. For $\gamma \in G$, let ${}^{\gamma}\bar{\phi} : \overline{X} \to M$ denote the composite of $\bar{\phi}$ with the action of γ on M.

Let A denote the space $\operatorname{End}(V)$, which is an affine space of dimension m^2, and contains G as a Zariski open dense subset. We put

$$\mathcal{U} := \{\ (\gamma, p) \in A \times M \ \mid\ \dim \gamma(L(p)) = r\ \}.$$

Then the action $G \times M \to M$ of G on M extends to the morphism

$$\alpha\ :\ \mathcal{U}\ \to\ M.$$

We also put

$$\overline{\mathcal{X}} := \{\ (\gamma, x) \in A \times \overline{X}\ \mid\ (\gamma, \bar{\phi}(x)) \in \mathcal{U}\ \},$$

which is a Zariski open dense subset of $A \times \overline{X}$ containing $G \times \overline{X}$. When $(\gamma, x) \in \overline{\mathcal{X}}$, we write ${}^{\gamma}\bar{\phi}(x)$ to denote the point $\alpha(\gamma, \bar{\phi}(x))$ of M. This notation is compatible with the previous definition when $\gamma \in G$. Let

$$\psi\ :\ \overline{\mathcal{X}}\ \to\ M$$

be the morphism given by $(\gamma, x) \mapsto {}^{\gamma}\bar{\phi}(x)$, and let

$$\Psi\ :\ \overline{\mathcal{X}}\ \to\ M \times \overline{X}$$

be the morphism given by $(\gamma, x) \mapsto (\psi(x), x)$. It is easy to check that Ψ is a locally trivial fiber space in the category of complex manifolds and holomorphic maps. Every fiber of Ψ is isomorphic to

$$R := GL(r) \times \mathbb{A}^{m(m-r)}.$$

In particular, Ψ is smooth. We regard

$$(D \times \overline{X}) + (M \times W)$$

as a divisor of $M \times \overline{X}$, which is reduced because both of D and W are reduced. Since Ψ is smooth, the pull-back

$$Z' := \Psi^{-1}((D \times \overline{X}) + (M \times W)) = \psi^{-1}(D) + ((A \times W) \cap \overline{\mathcal{X}})$$

is also a reduced divisor of $\overline{\mathcal{X}}$. Let

$$\Psi'\ :\ \overline{\mathcal{X}} \setminus Z'\ \to\ (M \setminus D) \times X$$

be the restriction of Ψ. Then we have the following diagram of the fiber product

$$\begin{tikzcd} \overline{\mathcal{X}} \setminus Z' \arrow[r, hook] \arrow[d, "\Psi'"'] \arrow[dr, phantom, "\square"] & \overline{\mathcal{X}} \arrow[d, "\Psi"] \\ (M \setminus D) \times X \arrow[r, hook] & M \times \overline{X}. \end{tikzcd} \tag{4.1}$$

Let Z be the closure of Z' in $A \times \overline{X}$; that is, Z is the unique divisor of $A \times \overline{X}$ whose support is the closure of Z' and whose restriction to $\overline{\mathcal{X}}$ coincides with Z'. Then Z is again a reduced divisor. We put

$$E := (A \times \overline{X}) \setminus Z,$$

and let $f : E \to A$ be the projection.

Let $\Delta \subset A$ denote the irreducible hypersurface $A \setminus G$. For every point $p \in M$, the locus of all $\gamma \in \Delta$ such that $(\gamma, p) \notin \mathcal{U}$ is of codimension ≥ 1 in Δ. This implies that $(A \times M) \setminus \mathcal{U}$ is of codimension ≥ 2 in $A \times M$, and $(A \times \overline{X}) \setminus \overline{\mathcal{X}}$ is also of codimension ≥ 2 in $A \times \overline{X}$. Therefore the inclusion of $\overline{\mathcal{X}} \setminus Z'$ into $E = (A \times \overline{X}) \setminus Z$ induces an isomorphism

$$\pi_1(\overline{\mathcal{X}} \setminus Z')\ \xrightarrow{\sim}\ \pi_1(E).$$

For $\gamma \in A$, let F_γ denote the fiber $f^{-1}(\gamma)$, and let Z_γ be the scheme-theoretic intersection of Z with $\{\gamma\} \times \overline{X}$. We regard Z_γ as a subscheme of $\overline{X}$. If $\gamma \in G$, then we have

$$F_\gamma = \overline{X} \setminus Z_\gamma = {}^\gamma\phi^{-1}(M \setminus D),$$

and the restriction of Ψ' to $F_\gamma = {}^\gamma\phi^{-1}(M \setminus D)$ is equal to the morphism ${}^\gamma\Phi$.

Now Main Theorem follows from the following two claims.

Claim 4.1. The homomorphism $\Psi'_* : \pi_1(\overline{\mathcal{X}} \setminus Z') \to \pi_1(M \setminus D) \times \pi_1(X)$ gives $\pi_1(E) \cong \pi_1(\overline{\mathcal{X}} \setminus Z')$ a structure of the central extension of $\pi_1(M \setminus D) \times \pi_1(X)$ by the cokernel of $\pi_2(\phi)$ associated with $\eta \in H^2((M \setminus D) \times X, \mathbb{Z})$.

Claim 4.2. If the condition (DG) is satisfied, then the inclusion of $F_\gamma \hookrightarrow E$ induces an isomorphism on the fundamental groups for a general $\gamma \in G$.

5. Proof of Claim 4.1

Let $\mathcal{L} \to M$ be the universal family of r-dimensional subspaces of V. Then we have $c_1(\det \mathcal{L}) = -c$, where $c \in H^2(M, \mathbb{Z})$ is the positive generator. Let

$$\mathcal{L}_1 \to M \times \overline{X}, \quad \text{and} \quad \mathcal{L}_2 \to M \times \overline{X}$$

be the pull-backs of $\mathcal{L} \to M$ by the first projection $\mathrm{pr}_1 : M \times \overline{X} \to M$, and by the composite morphism $\bar{\phi} \circ \mathrm{pr}_2 : M \times \overline{X} \to \overline{X} \to M$, respectively. Then we have a fiber bundle

$$\mathrm{Isom}(\mathcal{L}_2, \mathcal{L}_1) \ \to \ M \times \overline{X},$$

whose fiber over (p, x) is $\mathrm{Isom}(L(\bar{\phi}(x)), L(p)) \cong GL(r)$. The $\mathbb{C}^\times$-bundle

$$\det\,(\mathrm{Isom}(\mathcal{L}_2, \mathcal{L}_1)) \ \to \ M \times \overline{X}$$

is the complement to the zero section of the line bundle $(\det \mathcal{L}_2)^{-1} \otimes \det \mathcal{L}_1$, whose first Chern class is given by

$$\bar{\eta} := -\,\mathrm{pr}_1^*\, c + (\bar{\phi} \circ \mathrm{pr}_2)^* c \ \in \ H^2(M \times \overline{X}, \mathbb{Z}).$$

If $(\gamma, x) \in \overline{\mathcal{X}}$, then $\gamma : V \to V$ induces an isomorphism from the fiber $L(\bar{\phi}(x))$ of $\mathcal{L}_2$ over $\Psi(\gamma, x) = ({}^\gamma\bar{\phi}(x), x)$ to the fiber $L({}^\gamma\bar{\phi}(x))$ of $\mathcal{L}_1$ over $\Psi(\gamma, x)$. Hence Ψ is naturally lifted to a morphism

$$\widetilde{\Psi} \,:\, \overline{\mathcal{X}} \ \to \ \det\,(\mathrm{Isom}(\mathcal{L}_2, \mathcal{L}_1)),$$

which is a fiber bundle with fibers isomorphic to $SL(r) \times \mathbb{A}^{m(m-r)}$. In particular, $\widetilde{\Psi}$ induces an isomorphism from $\pi_1(\overline{\mathcal{X}})$ to $\pi_1(\det(\mathrm{Isom}(\mathcal{L}_2, \mathcal{L}_1))$, which is a central extension of $\pi_1(M) \times \pi_1(\overline{X})$ associated with $\bar{\eta}$.

Recall that the morphism Ψ is locally trivial with fibers isomorphic to $R = GL(r) \times \mathbb{A}^{m(m-r)}$. Therefore we obtain from the diagram (4.1) a homomorphism between the homotopy exact sequences for Ψ and Ψ';

(5.1)
$$\begin{tikzcd}
\pi_2((M \setminus D) \times X) \arrow[r, "\partial"] \arrow[d] & \pi_1(R) \arrow[r] \arrow[d, equal] & \pi_1(\overline{\mathcal{X}} \setminus Z') \arrow[r, "\Psi'_*"] \arrow[d] & \pi_1((M \setminus D) \times X) \arrow[d] \\
\pi_2(M \times \overline{X}) \arrow[r] & \pi_1(R) \arrow[r] & \pi_1(\overline{\mathcal{X}}) \arrow[r] & \pi_1(M \times \overline{X}),
\end{tikzcd}$$

where vertical arrows are induced from the inclusions. Note that the morphism Ψ' factors through

$$\widetilde{\Psi}' = \widetilde{\Psi}|_{\overline{\mathcal{X}} \setminus Z'} \,:\, \overline{\mathcal{X}} \setminus Z' \ \to \ \det\,(\mathrm{Isom}(\mathcal{L}_2, \mathcal{L}_1))|_{\,(M \setminus D) \times X}.$$

Since every fiber of $\widetilde{\Psi}'$ is isomorphic to $SL(r) \times \mathbb{A}^{m(m-r)}$, this morphism $\widetilde{\Psi}'$ induces an isomorphism

$$\pi_1(\overline{\mathcal{X}} \setminus Z') \;\cong\; \pi_1(\det(\mathrm{Isom}(\mathcal{L}_2, \mathcal{L}_1))|_{(M \setminus D)\times X}),$$

so that Ψ'_* makes $\pi_1(\overline{\mathcal{X}} \setminus Z')$ the central extension of $\pi_1(M \setminus D) \times \pi_1(X)$ by the cyclic group

$$\mathrm{Coker}\ (\ \partial\ :\ \pi_2(M \setminus D) \times \pi_2(X) \to \pi_1(R)\ )$$

associated with the cohomology class $\eta = \bar{\eta}|_{(M \setminus D)\times X} \in H^2((M \setminus D) \times X, \mathbb{Z})$. Hence it is now enough to show that the cokernel of ∂ in (5.1) is isomorphic to the cokernel of $\pi_2(\phi) : \pi_2(X) \to \pi_2(M)$.

First we show that ∂ maps the first factor $\pi_2(M \setminus D)$ to zero. Because $H_2(M, \mathbb{Z})$ is an infinite cyclic group generated by the homology class of a closed algebraic curve in M, every non-zero element of $H_2(M, \mathbb{Z})$ has a non-trivial intersection number with the homology class of the ample divisor D. Hence any non-zero element of $\pi_2(M) \cong H_2(M, \mathbb{Z})$ cannot be in the image of $\pi_2(M \setminus D) \to \pi_2(M)$; that is, the homomorphism $\pi_2(M \setminus D) \to \pi_2(M)$ induced by the inclusion is a zero map. Then the commutativity of the diagram (5.1) proves the claim $\partial(\pi_2(M \setminus D) \times \{0\}) = 0$. To investigate the image of the second factor $\pi_2(X)$ by ∂, we choose a point $p_0 \in M \setminus D$ and consider the morphism

$$\Psi'_0\ :\ \{\ (\gamma, x) \in \overline{\mathcal{X}} \setminus Z'\ \mid\ {}^{\gamma}\bar{\phi}(x) = p_0\ \}\ \to\ X$$

given by $(\gamma, x) \mapsto x$, which is the pull-back of Ψ' by the inclusion

$$X \cong \{p_0\} \times X \;\hookrightarrow\; (M \setminus D) \times X.$$

Hence the boundary homomorphism $\partial_0 : \pi_2(X) \to \pi_1(R)$ associated with the locally trivial fiber space Ψ'_0 coincides with the restriction of ∂ to the second factor. On the other hand, Ψ'_0 is also obtained as the pull-back of the second projection

$$\alpha^{-1}(p_0) = \{\ (\gamma, p) \in \mathcal{U}\ \mid\ \gamma(p) = p_0\ \}\ \to\ M \tag{5.2}$$

by $\phi : X \to M$. Therefore we have a homomorphism between the homotopy exact sequences associated with Ψ'_0 and (5.2);

(5.3)
$$\begin{tikzcd}
{} \arrow[r] & \pi_2(X) \arrow[r, "\partial_0"] \arrow[d, "\phi_*"] & \pi_1(R) \arrow[r] \arrow[d, equal] & \pi_1({\Psi'}^{-1}(\{p_0\} \times X)) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi_2(M) \arrow[r, "\partial_M"] & \pi_1(R) \arrow[r] & \pi_1(\alpha^{-1}(p_0)) \arrow[r] & {} .
\end{tikzcd}$$

Thus all we have to show is that the boundary homomorphism ∂_M associated with (5.2) is an isomorphism. Since both of $\pi_2(M)$ and $\pi_1(R)$ are an infinite cyclic group, it is enough to show that $\pi_1(\alpha^{-1}(p_0))$ is trivial. Since $(A \times M) \setminus \mathcal{U}$ is of codimension ≥ 2 in $A \times M$, $\pi_1(\mathcal{U})$ is trivial. Because the morphism $\alpha : \mathcal{U} \to M$ admits a section $p \mapsto (\mathrm{id}_V, p)$, the homomorphism $\alpha_* : \pi_2(\mathcal{U}) \to \pi_2(M)$ is surjective. From the homotopy exact sequence associated with α, we see that $\pi_1(\alpha^{-1}(p_0))$ is trivial. □

6. PROOF OF CLAIM 4.2

Let $\{D_i \mid i \in I\}$ be the set of irreducible components of the reduced divisor D, and let $\{(\mathrm{Sing}\, D)_j \mid j \in J\}$ be the set of irreducible components of the singular locus $\mathrm{Sing}\, D$ of D. We regard each $(\mathrm{Sing}\, D)_j$ as a reduced subscheme of M. Let

$J^{(2)} \subset J$ be the set of all $j \in J$ such that $(\operatorname{Sing} D)_j$ is of codimension exactly 2 in M. For points $p, q \in M$ and linear subspaces $K \subset T_pM$, $L \subset T_qM$, we put

$$\begin{aligned} G(p,q) &:= \{\, \gamma \in G \mid \gamma(p) = q \,\}, \quad \text{and} \\ G(p,q;K,L) &:= \{\, \gamma \in G(p,q) \mid (d\gamma)_p(K) \subset L \,\}. \end{aligned}$$

Instead of $G(p,p)$, we write G_p. We consider the following conditions. We equip $\bar\phi(\overline{X})$ with the reduced structure.

TR1(i). Let p be a general point of $\bar\phi(\overline{X})$, and let q be a general point of D_i. Then $G(p,q;T_p\bar\phi(\overline{X}),T_qD_i)$ is of codimension ≥ 2 in $G(p,q)$.

TR2(j). Let p be a general point of $\bar\phi(\overline{X})$, and let q be a general point of $(\operatorname{Sing} D)_j$, where $j \in J^{(2)}$. Then the locus

$$\{\, \gamma \in G(p,q) \mid (d\gamma)_p(T_p\bar\phi(\overline{X})) + T_q(\operatorname{Sing} D)_j = T_qM \,\}$$

is Zariski open dense in $G(p,q)$.

We say that *the transversality condition* is satisfied if TR1(i) is satisfied for every $i \in I$ and TR2(j) is satisfied for every $j \in J^{(2)}$.

Now Claim 4.2 follows from the following two sub-claims.

Sub-claim 6.1. Suppose that the condition (DG) is satisfied. Then the transversality condition is satisfied.

Sub-claim 6.2. If the transversality condition is satisfied, then the inclusion $F_\gamma \hookrightarrow E$ induces an isomorphism $\pi_1(F_\gamma) \cong \pi_1(E)$ for a general $\gamma \in G$.

7. Proof of Sub-claim 6.1

Suppose first that $r = 1$ or $r = m-1$, i.e., that M is a projective space. For any $p \in M$, the natural representation $G_p \to GL(T_pM)$ of G_p on T_pM is surjective. Hence the assumption $\dim \bar\phi(\overline{X}) \geq 2$ implies the transversality condition.

From now on, we assume $2 \leq r \leq m-2$. Then Sub-claim 6.1 follows from the following:

(1) If TR1(i) is not satisfied, then $\bar\phi(\overline{X})$ is of rank 1 and D_i is of corank 1.

(2) If TR2(j) is not satisfied for $j \in J^{(2)}$, then $\bar\phi(\overline{X})$ is of rank 1, $(\operatorname{Sing} D)_j$ is of corank 1, and the types of $\bar\phi(\overline{X})$ and $(\operatorname{Sing} D)_j$ coincide.

Because of the definition of (co)rank, these follow immediately from Proposition 7.1 below. Let p be a point of M, and let F, H and K be linear subspaces of T_pM such that

$$\dim F \geq 2, \quad \dim H = \dim T_pM - 1, \quad \text{and} \quad \dim K = \dim T_pM - 2.$$

We denote by $G_p(F,H) \subset G_p$ the locus of all $\gamma \in G_p$ such that $(d\gamma)_p(F) \subset H$.

Proposition 7.1. (1) *Suppose that* $G_p(F,H)$ *is of codimension* ≤ 1 *in* G_p*. Then we have* $\operatorname{rank}(F) = 1$ *and* $\operatorname{corank}((T_pM/H)^*) = 1$.

(2) *Suppose that* $K + (d\gamma)_p(F)$ *fails to coincide with the total space* T_pM *for a general* $\gamma \in G_p$*. Then we have* $\operatorname{rank}(F) = 1$ *and* $\operatorname{corank}((T_pM/K)^*) = 1$*. Moreover the types of* F *and* $(T_pM/K)^*$ *coincide.*

Proof. For simplicity, we put

$$n := m - r.$$

We fix bases $\{e_1, \dots, e_r\}$ of $L(p)$ and $\{f^1, \dots, f^n\}$ of $V/L(p)$. We express, via the isomorphisms (2.1), elements $\tau \in T_pM$ (resp. $\omega \in T_p^*M$) by $r \times n$ matrices (τ_{ij}) (resp. $n \times r$ matrices (ω^{ji})), where

$$\tau(e_i) = \sum_{j=1}^{n} \tau_{ij} f^j, \quad \text{and} \quad \omega(f^j) = \sum_{i=1}^{r} \omega^{ji} e_i.$$

The canonical bilinear form $(\ ,\ ) : T_p^*M \times T_pM \to \mathbb{C}$ is then given by

$$(\omega, \tau) = \sum_{i,j} \omega^{ji} \tau_{ij}.$$

We write the natural homomorphism

$$u \, : \, G_p \, \to \, GL(L(p)) \times GL(V/L(p))$$

by $u(\gamma) = (\gamma_1^{-1}, \gamma_2)$, putting the inverse on the first factor. Let us also express elements γ_1 of $GL(L(p))$ and γ_2 of $GL(V/L(p))$ by $r \times r$ matrices $({g_i}^k)$ and $n \times n$ matrices $({h^l}_j)$, respectively;

$$\gamma_1(e_i) = \sum_{k=1}^{r} {g_i}^k e_k, \quad \gamma_2(f^l) = \sum_{j=1}^{n} {h^l}_j f^j.$$

Then the action of $\gamma \in G_p$ on T_pM is identified with the multiplication of matrices

$$(\tau_{ij}) \ \mapsto \ (\sum_{k,l} {g_i}^k \tau_{kl} {h^l}_j).$$

Now we start the proof of (1). Let α be a generator of the 1-dimensional linear subspace $(T_pM/H)^*$ of T_p^*M. We have

$$H = \{ \ \tau \in T_pM \ \mid \ (\alpha, \tau) = 0 \ \}.$$

Suppose that α is represented by an $n \times r$ matrix (α^{ji}). When $\tau \in F$ is given, the condition on $\gamma \in G_p$ for $(d\gamma)_p(\tau)$ to be contained in the hyperplane $H \subset T_pM$ is given by the quadratic equation

$$\sum_{i,j,k,l} \alpha^{ji} {g_i}^k \tau_{kl} {h^l}_j = 0,$$

where $u(\gamma) = (\gamma_1^{-1}, \gamma_2)$ and $\gamma_1 = ({g_i}^k)$, $\gamma_2 = ({h^l}_j)$. We put

$$Q(\tau) := \{ \ (({g_i}^k), ({h^l}_j)) \in \operatorname{End}(L(p)) \times \operatorname{End}(V/L(p)) \ \mid \ \sum_{i,j,k,l} \alpha^{ji} {g_i}^k \tau_{kl} {h^l}_j = 0 \ \},$$

and let $Q(\tau)^0$ be the intersection of $Q(\tau)$ with $GL(L(p)) \times GL(V/L(p))$. Then we have

$$G_p(F, H) = \bigcap_{\tau \in F} u^{-1}(Q(\tau)^0).$$

The locus $Q(\tau)$ is a quadratic hypersurface for $\tau \neq 0$. Moreover the closure of $Q(\tau)^0$ in $\operatorname{End}(L(p)) \times \operatorname{End}(V/L(p))$ is equal to $Q(\tau)$, because $Q(\tau)$ cannot possess an irreducible component in common with the complement in $\operatorname{End}(L(p)) \times \operatorname{End}(V/L(p))$ to $GL(L(p)) \times GL(V/L(p))$. It is also easy to see that, if two matrices τ_1 and τ_2 of F are linearly independent, then $Q(\tau_1)$ does not coincide with $Q(\tau_2)$. Therefore the assumption of (1) implies that, for every $\tau \in F \setminus \{0\}$, $Q(\tau)$ is a union of two hyperplanes, and all these $Q(\tau)$ contain one fixed hyperplane in common. We put

$$\rho := \operatorname{corank}((T_pM/H)^*).$$

By choosing the bases $\{e_1, \dots, e_r\}$ and $\{f^1, \dots, f^n\}$ suitably, we put the matrix (α^{ji}) into the following form:

$$\alpha^{ji} = \begin{cases} 1 & \text{if } i = j \text{ and } 1 \le i \le \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\eta = (\eta_{kl})$ be a non-zero element of F. The reducibility of $Q(\eta)$ implies that there exist $\lambda_k{}^i \in \mathbb{C}$ and $\mu^j{}_l \in \mathbb{C}$ such that

$$\sum_{i=1}^{\rho} \sum_{k,l} g_i{}^k \eta_{kl} h^l{}_i = \Big(\sum_{i,k} \lambda_k{}^i g_i{}^k \Big) \cdot \Big(\sum_{j,l} \mu^j{}_l h^l{}_j \Big);$$

that is,

$$\lambda_k{}^i \cdot \mu^j{}_l = \begin{cases} \eta_{kl} & \text{if } i = j \text{ and } 1 \le i \le \rho, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

There exists at least one (k, l) such that $\eta_{kl} \neq 0$. Hence (7.1) implies that $\rho = 1$. Moreover, we have $\lambda_k{}^i = 0$ for $i \ge 2$, $\mu^j{}_l = 0$ for $j \ge 2$, and $\eta_{kl} = \lambda_k{}^1 \cdot \mu^1{}_l$. We have $Q(\eta) = \Lambda_1 \cup \Lambda_2$, where Λ_1 and Λ_2 are hyperplanes defined by

$$\Lambda_1 = \{ \sum_k \lambda_k{}^1 g_1{}^k = 0 \}, \quad \Lambda_2 = \{ \sum_l \mu^l{}_1 h^1{}_l = 0 \}.$$

By the consideration above, either $\Lambda_1 \subset Q(\tau)$ for all $\tau \in F$ or $\Lambda_2 \subset Q(\tau)$ for all $\tau \in F$ holds. In the former case, for any $\tau \in F$, there exist scalars t_l $(l = 1, \dots, n)$ such that $\lambda_k{}^1 t_l = \tau_{kl}$. This implies that $\operatorname{Ker} \tau \subset L(p)$ contains a fixed hyperplane

$$\{ \sum_i x^i e_i \mid \sum_i x^i \lambda_i{}^1 = 0 \}$$

of $L(p)$. Thus F is of rank 1 with type (II). In the later case, for any $\tau \in F$, there exist scalars s_k $(k = 1, \dots, r)$ such that $s_k \mu^1{}_l = \tau_{kl}$. This implies that, for $k = 1, \dots, r$, the vector $\tau(e_k) \in V/L(p)$ is proportional to $\sum_{l=1}^{n} \mu^1{}_l f^l$. Thus F is of rank 1 with type (I).

Next we prove (2). We put

$$\mu := \min\{ \operatorname{corank}(\omega) \mid \omega \in (T_pM/K)^* \setminus \{0\} \}.$$

Note that μ is not the maximal rank, but the minimal one. Let $\alpha \in (T_pM/K)^*$ be an element such that $\operatorname{corank}(\alpha) = \mu$, and let $\beta \in (T_pM/K)^*$ be an element that is linearly independent with α. Then K is defined in T_pM by

$$K = \{ \tau \in T_pM \mid (\alpha, \tau) = (\beta, \tau) = 0 \}.$$

Let η and ζ be linearly independent elements of F. Then the assumption of (2) implies that

$$\det \begin{pmatrix} (\alpha, (d\gamma)_p(\eta)) & (\beta, (d\gamma)_p(\eta)) \\ (\alpha, (d\gamma)_p(\zeta)) & (\beta, (d\gamma)_p(\zeta)) \end{pmatrix} = 0 \tag{7.2}$$

holds for a general $\gamma \in G_p$, and hence for an arbitrary $\gamma \in G_p$. We write down this equation in terms of the components of the matrices $\alpha = (\alpha^{ji})$, $\beta = (\beta^{ji})$, $\eta = (\eta_{ij})$, $\zeta = (\zeta_{ij})$ and $\gamma_1 = (g_i{}^k)$, $\gamma_2 = (h^l{}_j)$, where $u(\gamma) = (\gamma_1^{-1}, \gamma_2)$. We put

$$[ji, kl : j'i', k'l'] := \alpha^{ji} \eta_{kl} \beta^{j'i'} \zeta_{k'l'}.$$

Looking at the coefficient of ${g_i}^k {h^l}_j {g_{i'}}^{k'} {h^{l'}}_{j'}$ of (7.2), we obtain the following equations:

$$\begin{aligned}
&([ji,kl:j'i',k'l'] + [j'i,kl':ji',k'l] + \\
&\qquad [ji',k'l:j'i,kl'] + [j'i',k'l':ji,kl]) - \\
&([j'i',kl:ji,k'l'] + [ji',kl':j'i,k'l] + \\
&\qquad [j'i,k'l:ji',kl'] + [ji,k'l':j'i',kl]) = 0.
\end{aligned} \tag{7.3}$$

By re-choosing the bases $\{e_1, \ldots, e_r\}$ and $\{f^1, \ldots, f^n\}$ appropriately, we get

$$\alpha^{ji} = \begin{cases} 1 & \text{if } j = i \text{ and } 1 \le i \le \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

Because $\operatorname{corank}(\alpha) = \mu$ is minimal and α and β are linearly independent, there exists (i', j') such that

$$(i' > \mu \quad \text{or} \quad j' > \mu) \quad \text{and} \quad \beta^{j'i'} \neq 0. \tag{7.5}$$

Suppose that there existed i_1 and (i_2, j_2) such that

$$\alpha^{j_2 i_1} = \alpha^{i_1 i_2} = \alpha^{j_2 i_2} = 0, \quad \alpha^{i_1 i_1} \neq 0 \quad \text{and} \quad \beta^{j_2 i_2} \neq 0.$$

Applying (7.3) to $(j, i, j', i') = (i_1, i_1, j_2, i_2)$, we would obtain $\eta_{kl}\zeta_{k'l'} - \eta_{k'l'}\zeta_{kl} = 0$ for arbitrary (k, l, k', l'). This contradicts the linear independence of η and ζ. Therefore there are no such i_1 and (i_2, j_2). This means, by (7.4) and (7.5), that

$$\mu = 1 \quad \text{and} \quad (\ \beta^{ji} \neq 0 \implies (j \le \mu \text{ or } i \le \mu)\ ).$$

Now by changing $\{e_1, \ldots, e_r\}$ and $\{f^1, \ldots, f^n\}$ again, we get

$$\alpha^{ji} = 0 \text{ unless } (j, i) = (1, 1), \quad \text{while } \alpha^{11} = 1, \quad \text{and} \tag{7.6}$$

$$\beta^{ji} = 0 \text{ unless } (i, j) = (1, 1) \text{ or } (2, 1) \text{ or } (1, 2). \tag{7.7}$$

Applying (7.3) to $(j, i, j', i') = (1, 1, 1, 2)$, we obtain

$$\beta^{12}((\eta_{kl}\zeta_{k'l'} - \eta_{k'l'}\zeta_{kl}) + (\eta_{kl'}\zeta_{k'l} - \eta_{k'l}\zeta_{kl'})) = 0$$

for arbitrary (k, l, k', l'). Putting $l = l'$, we get

$$\beta^{12}(\eta_{kl}\zeta_{k'l} - \eta_{k'l}\zeta_{kl}) = 0 \tag{7.8}$$

for arbitrary k, k' and l. Applying (7.3) to $(j, i, j', i') = (1, 1, 2, 1)$ and putting $k = k'$, we also obtain

$$\beta^{21}(\eta_{kl}\zeta_{kl'} - \eta_{kl'}\zeta_{kl}) = 0 \tag{7.9}$$

for arbitrary k and l, l'. Suppose that both of β^{12} and β^{21} were non-zero. Then (7.8) and (7.9) would imply that η and ζ should be linearly dependent, which contradicts the assumption. Hence either $(\beta^{12} \neq 0, \beta^{21} = 0)$ or $(\beta^{12} = 0, \beta^{21} \neq 0)$ holds. Combining this with (7.6), we see that $\operatorname{corank}(\omega) \le 1$ for every linear combination ω of α and β; that is, we have $\operatorname{corank}(K) = 1$.

Suppose that $\beta^{12} \neq 0$ and $\beta^{21} = 0$. Then $\operatorname{Ker}\omega \subset V/L(p)$ contains the hyperplane spanned by $f^2, \ldots, f^n$ for any $\omega \in K$. Thus K is of type (II). Moreover the fact that (7.8) holds for arbitrary elements η and ζ of F implies that there exist fixed scalars $u_1, \ldots, u_r$ such that, for any $\tau \in F$, we have $t_1, \ldots, t_n \in \mathbb{C}$ satisfying $\tau_{kl} = u_k t_l$. This implies that $\operatorname{Ker}\tau \subset L(p)$ contains a fixed hyperplane

$$\{ \sum_i x^i e_i \ \mid \ \sum_i u_i x^i = 0 \}$$

for any $\tau \in F$. Thus F is of rank 1 with type (II). Suppose that $\beta^{21} \neq 0$ and $\beta^{12} = 0$. Then $\operatorname{Im} \omega \subset L(p)$ is proportional to e_1 for any $\omega \in K$. Thus K is of type (I). Moreover, by (7.9), there exist fixed scalars $v_1, \dots, v_n$ such that, for any $\tau \in F$, we have $s_1, \dots, s_r \in \mathbb{C}$ satisfying $\tau_{kl} = s_k v_l$. This implies that $\operatorname{Im} \tau \subset V/L(p)$ is generated by a fixed vector $\sum_{j=1}^{n} v_j f^j$ for ant $\tau \in F$. Thus F is of rank 1 with type (I). □

8. Proof of Sub-claim 6.2

In order to prove Sub-claim 6.2, it is enough to show that $f : E \to A$ satisfies the conditions (T1)-(T4) in [8, Theorem 1.3].

The condition (T1) is obviously satisfied. Since f is smooth, the condition (T2) is also satisfied. For the condition (T3), it is enough to show that the locus

$$\Xi_{\emptyset} := \{ \gamma \in A \mid F_{\gamma} = \emptyset \}$$

is contained in a Zariski closed subset of codimension ≥ 2 in A. The following lemma is easy:

Lemma 8.1. *Let S be an irreducible hypersurface of M, and let p, q be two distinct points of M. Then the Zariski closed subset $\{\gamma \in G \mid \gamma(p) \in S, \gamma(q) \in S\}$ of G is of codimension ≥ 2.* □

Corollary 8.2. *If C is an irreducible Zariski closed subset of M with $\dim C \geq 1$, then the Zariski closed subset $\{\gamma \in G \mid \gamma(C) \subset D\}$ of G is of codimension ≥ 2.* □

If $\gamma \in G \cap \Xi_{\emptyset}$, then $\gamma(\bar{\phi}(\overline{X}))$ is contained in D. By Corollary 8.2, the assumption $\dim \bar{\phi}(\overline{X}) \geq 2$ implies that $G \cap \Xi_{\emptyset}$ is contained in a Zariski closed subset of codimension ≥ 2 in G.

Recall that Δ is the irreducible hypersurface $A \setminus G$ of A. Let $\Delta^{\circ} \subset \Delta$ be the Zariski open dense subset consisting of all $\gamma \in \Delta$ such that the linear homomorphism $\gamma : V \to V$ is of rank $m-1$. It is well-known that Δ° coincides with $\Delta \setminus \operatorname{Sing} \Delta$ ([7, Example 14.16]). For a point $p \in M$, we put

$$\Delta^{\circ}(p) := \{ \gamma \in \Delta^{\circ} \mid \operatorname{Ker} \gamma \not\subset L(p) \} = \{ \gamma \in \Delta^{\circ} \mid (\gamma, p) \in \mathcal{U} \},$$

which is a Zariski open dense subset of Δ°. The following lemma is obvious:

Lemma 8.3. *The morphism $\Delta^{\circ}(p) \to M$ given by $\gamma \mapsto \gamma(p)$ is surjective.* □

Let x be any point of X. By Lemma 8.3, if $\gamma \in \Delta^{\circ}(\bar{\phi}(x))$ is general, then $\gamma(\bar{\phi}(x)) \notin D$. In particular, we have $x \in \overline{X} \setminus Z_{\gamma}$. Hence $\Delta \cap \Xi_{\emptyset}$ is contained in a proper Zariski closed subset of Δ. Therefore $\Xi_{\emptyset}$ is contained in a Zariski closed subset of A with codimension ≥ 2. Thus the condition (T3) is satisfied.

Now we check the condition (T4). Let $\Sigma_f \subset A$ be the topological discriminant locus (see [8, Definition 1.2]) of $f : E \to A$, and let $\Sigma_f^{(1)}, \dots, \Sigma_f^{(k)}$ be the irreducible components of Σ_f with codimension 1 in A. If $\Delta \subset \Sigma_f$, then one of $\Sigma_f^{(i)}$ is Δ.

First let us consider the local monodromy around Δ.

Proposition 8.4. *If γ is a general point of Δ, then Z_{γ} is a reduced divisor of $\overline{X}$.*

Proof. For $\gamma \in \Delta$, we put

$$K_{\gamma} := \{ p \in M \mid \operatorname{Ker} \gamma \subset L(p) \}.$$

If $\gamma \in \Delta^\circ$, then K_γ is isomorphic to $\mathrm{Grass}(r-1, m-1)$. For $\gamma \in \Delta$, let $\overline{X}'_\gamma$ denote the fiber of the projection $\overline{\mathcal{X}} \to A$ over γ. Then we have

$$\overline{X}'_\gamma = \overline{X} \setminus \bar{\phi}^{-1}(K_\gamma).$$

First we prove that, if $\gamma \in \Delta$ is general, then $\bar{\phi}^{-1}(K_\gamma)$ is of codimension ≥ 2 in $\overline{X}$. We put

$$\mathcal{K} := \{\ (\gamma, p) \in \Delta^\circ \times M \ \mid\ p \in K_\gamma\ \}.$$

Since the projection $\mathcal{K} \to \Delta^\circ$ is smooth with fibers isomorphic to $\mathrm{Grass}(r-1, m-1)$, $\mathcal{K}$ is smooth and of dimension

$$\dim \mathcal{K} = \dim \Delta^\circ + (m-r)(r-1).$$

The group G acts of $\mathcal{K}$ from left by

$$(\gamma, p) \mapsto (\gamma \circ g^{-1},\, g(p)) \quad (g \in G).$$

The projection $\mathcal{K} \to M$ is obviously equivariant under this action. Since G acts transitively on M, the projection $\mathcal{K} \to M$ is smooth. Consider the fiber product $\mathcal{K} \times_M \overline{X}$ of the projection $\mathcal{K} \to M$ and $\bar{\phi} : \overline{X} \to M$;

$$\begin{tikzcd}
\mathcal{K} \times_M \overline{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{K} \arrow[r] \arrow[d] & \Delta^\circ \\
\overline{X} \arrow[r, "\bar{\phi}"'] & M.
\end{tikzcd}$$

The projection $\mathcal{K} \times_M \overline{X} \to \overline{X}$ is smooth and of relative dimension equal to $\dim \mathcal{K} - \dim M$. Hence we have

$$\dim(\mathcal{K} \times_M \overline{X}) = \dim \overline{X} + \dim \mathcal{K} - \dim M = \dim \overline{X} + \dim \Delta^\circ - (m-r).$$

Let $q : \mathcal{K} \times_M \overline{X} \to \Delta^\circ$ be the composite of the projections $\mathcal{K} \times_M \overline{X} \to \mathcal{K}$ and $\mathcal{K} \to \Delta^\circ$. By construction, $\bar{\phi}^{-1}(K_\gamma)$ is isomorphic to $q^{-1}(\gamma)$. Therefore, if $\gamma \in \Delta^\circ$ is general, we have

$$\dim \bar{\phi}^{-1}(K_\gamma) \ \le\ \dim(\mathcal{K} \times_M \overline{X}) - \dim \Delta^\circ \ =\ \dim \overline{X} - (m-r).$$

Since we have assumed $r \le m-2$ (see Remark 2.2), the codimension of $\bar{\phi}^{-1}(K_\gamma)$ in $\overline{X}$ is at least 2 for a general $\gamma \in \Delta$.

Let Z'_γ denote the scheme-theoretic intersection of $\overline{X}'_\gamma$ and the divisor Z' of $\overline{\mathcal{X}}$. If $\gamma \in \Delta$ is general, then $\overline{X} \setminus \overline{X}'_\gamma$ is of codimension ≥ 2 in $\overline{X}$, and hence Z_γ coincides with the closure of Z'_γ in $\overline{X}$. Therefore it is enough to show that Z'_γ is a reduced divisor of $\overline{X}'_\gamma$ for a general $\gamma \in \Delta$. We put

$$\overline{\mathcal{X}}_{\Delta^\circ} := (\Delta^\circ \times \overline{X}) \cap \overline{\mathcal{X}},$$

and let Z'_{Δ° be the scheme-theoretic intersection of Z' and $\overline{\mathcal{X}}_{\Delta^\circ}$. For $\gamma \in \Delta^\circ$, we denote by

$$\psi'_{\Delta^\circ} \ :\ \overline{\mathcal{X}}_{\Delta^\circ} \ \to\ M \quad \text{and} \quad \psi'_\gamma \ :\ \overline{X}'_\gamma \ \to\ M$$

the restrictions of $\psi : \overline{\mathcal{X}} \to M$ to $\overline{\mathcal{X}}_{\Delta^\circ}$ and to $\overline{X}'_\gamma$, respectively. Then we have

$$Z'_{\Delta^\circ} = {\psi'}^{-1}_{\Delta^\circ}(D) \ +\ (\Delta^\circ \times W) \cap \overline{\mathcal{X}}_{\Delta^\circ},$$

and, for $\gamma \in \Delta^\circ$, the divisor

$$Z'_\gamma = {\psi'}^{-1}_\gamma(D) \ +\ W \cap \overline{X}'_\gamma$$

of $\overline{X}'_\gamma$ is the scheme-theoretic intersection of Z'_{Δ° and $\overline{X}'_\gamma$ in $\overline{\mathcal{X}}_{\Delta^\circ}$. Note that G acts on $\overline{\mathcal{X}}_{\Delta^\circ}$ by

$$(\gamma, x) \mapsto (g \circ \gamma, x) \quad (g \in G),$$

and that ψ'_{Δ° is equivariant under the action of G. Since G acts on M transitively, ψ'_{Δ° is smooth. Therefore $\psi'^{-1}_{\Delta^\circ}(D)$ is a reduced divisor of $\overline{\mathcal{X}}_{\Delta^\circ}$. Hence, if $\gamma \in \Delta^\circ$ is general, then $\psi'^{-1}_\gamma(D)$ is a reduced divisor of $\overline{X}'_\gamma$.

Let $W_1, \dots, W_m$ be the irreducible components of W. We choose a general point w_i of W_i for each i. If $\gamma \in \Delta^\circ$ is general, then $w_i \notin \bar{\phi}^{-1}(K_\gamma)$ and $\gamma(\bar{\phi}(w_i)) \notin D$ by Lemma 8.3. Hence $W \cap \overline{X}'_\gamma$ and $\psi'^{-1}_\gamma(D)$ have no common irreducible components. Thus Z'_γ is a reduced divisor of $\overline{X}'_\gamma$ for a general $\gamma \in \Delta^\circ$. □

Let B_Δ be a Zariski open dense subset of A containing the generic point of Δ such that $B_\Delta \cap \Sigma_f \subset \Delta$. Let $f_\Delta : E_\Delta \to B_\Delta$ be the restriction of f to $E_\Delta := f^{-1}(B_\Delta)$. By Proposition 8.4, we see that the conditions (B1) and (B2) of [8, Proposition 4.3] are satisfied by f_Δ. Hence the local monodromy around Δ is trivial.

Next we consider the local monodromy μ_i around $\Sigma_f^{(i)}$ that is not Δ. From now on, we will assume that $\dim X = 2$, and that $\bar{\phi}$ is quasi-finite onto its image (see Remark 2.3). We put

$$E_G := f^{-1}(G),$$

and let $f_G : E_G \to G$ be the restriction of f to E_G. Then we are exactly in the situation of [8, §5]. Indeed, the restriction of the morphism $\psi : \overline{\mathcal{X}} \to M$ to E_G coincides with

$$\bar{g} : G \times \overline{X} \to M$$

in [8, §5]. (Note that we put $B := G$ in [8, §5].) Recalling the definition of the divisor Z of $A \times \overline{X}$, we see that E_G is the complement in $G \times \overline{X}$ to

$$Z_G := (G \times W) + \bar{g}^{-1}(D).$$

Therefore we can prove the triviality of the local monodromy μ_i around $\Sigma_f^{(i)}$ by showing that the conditions (G1)-(G3) of [8, Proposition 5.1] are satisfied.

Recall, from [8, §5], that $\overline{Y} = \bar{\phi}(\overline{X})$. The condition (G1) is satisfied because of our assumption. The condition (G2) follows from Corollary 8.2. The condition (G3) follows from the following proposition, in which we use the assumption in Sub-claim 6.2 that the transversality condition is satisfied. Recall, from [8, §5], that $\mathrm{Sing}(\gamma(\overline{Y}) \cap D)$ is the locus consisting of all points $y \in \gamma(\overline{Y}) \cap D$ such that either $\gamma(\overline{Y})$ is singular at y, or D is singular at y, or $T_y\gamma(\overline{Y}) + T_yD \neq T_yM$.

Proposition 8.5. *Suppose that the transversality condition is satisfied. Then the locus* $\{\gamma \in G \mid \dim \mathrm{Sing}(\gamma(\overline{Y}) \cap D) > 0\}$ *is contained in a Zariski closed subset of codimension* ≥ 2 *in* G.

Proof. We assume that there exists an irreducible hypersurface Ξ of G such that $\dim \mathrm{Sing}(\xi(\overline{Y}) \cap D) > 0$ for a general point $\xi \in \Xi$, and derive a contradiction.

Let $Q \subset \overline{Y}$ be the minimal Zariski closed subset such that the quasi-finite morphism $\bar{\phi} : \overline{X} \to \overline{Y}$ is étale over $\overline{Y} \setminus Q$. We put

$$\overline{X}_0 := \overline{X} \setminus \bar{\phi}^{-1}(Q).$$

By Corollary 8.2, the locus $\{\gamma \in G \mid \dim(\gamma(Q) \cap D) = 1\}$ is of codimension ≥ 2 in G. Because ξ is a general point of the hypersurface Ξ, we have $\dim(\xi(Q) \cap D) = 0$

or $\xi(Q) \cap D = \emptyset$. Therefore the assumption $\dim \operatorname{Sing}(\xi(\overline{Y}) \cap D) \geq 1$ would imply that the locus

$$\{ \; x \in \overline{X}_0 \;\mid\; p := {}^{\xi}\bar{\phi}(x) \in D \quad \text{and} \quad (d\,{}^{\xi}\bar{\phi})_x(T_x\overline{X}) \subset T_pD \; \}$$

should contain a curve. We consider the incident variety

$$\Omega := \{ \; (\gamma, x, p) \in G \times \overline{X}_0 \times D \;\mid\; {}^{\gamma}\bar{\phi}(x) = p \;\text{ and }\; (d\,{}^{\gamma}\bar{\phi})_x(T_x\overline{X}) \subset T_pD \; \}.$$

Then the dimension of the fiber of the projection $\mathrm{pr}_G : \Omega \to G$ over the general point ξ of Ξ should be ≥ 1. Thus $\mathrm{pr}_G^{-1}(\Xi)$ would contain an irreducible component with dimension $\geq \dim G$.

For $i \in I$, we put

$$D_i^{ns} := D_i \setminus (D_i \cap \operatorname{Sing} D).$$

Let $\mathrm{pr}_D : \Omega \to D$ be the projection. Then we have

$$\Omega = \coprod_{i \in I} \Omega_i^{ns} \amalg \bigcup_{j \in J} \Omega_j^{sing},$$

where

$$\Omega_i^{ns} := \mathrm{pr}_D^{-1}(D_i^{ns}) \quad \text{and} \quad \Omega_j^{sing} := \mathrm{pr}_D^{-1}((\operatorname{Sing} D)_j).$$

First we show that

$$\dim \Omega_i^{ns} \leq \dim G - 1 \quad \text{for all } i \in I. \tag{8.1}$$

The fiber of the projection

$$\Omega_i^{ns} \;\to\; \overline{X}_0 \times D_i^{ns}$$

over $(x, p) \in \overline{X}_0 \times D_i^{ns}$ is the subvariety

$$G(\; \bar{\phi}(x), \; p \; ; \; (d\,\bar{\phi})_x(T_x\overline{X}), \; T_pD_i^{ns} \;) \tag{8.2}$$

of G. This fiber is of codimension ≥ 1 in $G(\bar{\phi}(x), p)$ for every $(x, p) \in \overline{X}_0 \times D_i^{ns}$, because the action of the stabilizer subgroup G_p on T_pM is an irreducible representation. On the other hand, the condition TR1(i) implies that the fiber (8.2) is of codimension ≥ 2 in $G(\bar{\phi}(x), p)$ for a general point (x, p) of $\overline{X}_0 \times D_i^{ns}$. Thus we obtain (8.1) by easy dimension counts. Next we show that

$$\dim(\mathrm{pr}_G^{-1}(\Xi) \cap \Omega_j^{sing}) \leq \dim G - 1 \tag{8.3}$$

for all $j \in J$. If $p \in \operatorname{Sing} D$, then $T_pD = T_pM$. Therefore the fiber of the projection

$$\Omega_j^{sing} \;\to\; \overline{X}_0 \times (\operatorname{Sing} D)_j$$

over $(x, p) \in \overline{X}_0 \times (\operatorname{Sing} D)_j$ is $G(\bar{\phi}(x), p)$. Since $G(\bar{\phi}(x), p) \cong G_p$ is irreducible, Ω_j^{sing} is also irreducible, and

$$\dim \Omega_j^{sing} = \dim(\operatorname{Sing} D)_j + \dim G_p + \dim X \leq \dim G,$$

where the equality holds if and only if $\dim(\operatorname{Sing} D)_j = \dim M - 2$; that is, $j \in J^{(2)}$. Therefore (8.3) holds for any $j \in J \setminus J^{(2)}$. Suppose that $j \in J^{(2)}$. The condition TR2(j) implies that there exist an element $\gamma_0 \in G$ and a point $p \in {}^{\gamma_0}\bar{\phi}(\overline{X}_0) \cap (\operatorname{Sing} D)_j$ such that ${}^{\gamma_0}\bar{\phi}(\overline{X}_0)$ and $(\operatorname{Sing} D)_j$ are smooth at p and intersect transversely at p. Then the locus of all $\gamma \in G$ such that ${}^{\gamma}\bar{\phi}(\overline{X}_0) \cap (\operatorname{Sing} D)_j \neq \emptyset$ is a Zariski open subset of G containing γ_0. This implies that the projection $\Omega_j^{sing} \to G$ is dominant. Hence $\mathrm{pr}_G^{-1}(\Xi) \cap \Omega_j^{sing}$ must be of codimension ≥ 1 in the irreducible variety Ω_j^{sing}. Thus (8.3) is proved for all $j \in J$. Combining (8.1) and (8.3), we see that $\dim \mathrm{pr}_G^{-1}(\Xi) \le \dim G - 1$, which yields a contradiction. $\square$

9. Examples

We consider the case when $m = 4$ and $r = 2$; that is, $M = \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3)$. For a point $Q \in \mathbb{P}^3$ and a plane $H \subset \mathbb{P}^3$, we put

$$X_Q := \{\, p \in M \mid Q \in \Pi(p) \,\} \quad \text{and} \quad Y_H := \{\, p \in M \mid \Pi(p) \subset H \,\}.$$

Let $f_Q : X_Q \hookrightarrow M$ and $g_H : Y_H \hookrightarrow M$ be the inclusions, both of which induce isomorphisms on the second homotopy groups. Let $C \subset \mathbb{P}^3$ be a closed curve. We put

$$D_C := \{\, p \in M \mid C \cap \Pi(p) \neq \emptyset \,\},$$

which is a hypersurface of M. We choose $Q \in \mathbb{P}^3$ and $H \subset \mathbb{P}^3$ in general positions with respect to C, and consider the three fundamental groups

$$\pi_1(M \setminus D_C), \quad \pi_1(f_Q^{-1}(M \setminus D_C)), \quad \text{and} \quad \pi_1(g_H^{-1}(M \setminus D_C)).$$

Note that $f_Q^{-1}(M \setminus D_C)$ is isomorphic to $\mathbb{P}^2 \setminus p_Q(C)$, where $p_Q : C \to \mathbb{P}^2$ is the projection with the center Q. Note also that $g_H^{-1}(M \setminus D_C)$ is isomorphic to

$$H^\vee \setminus \bigcup_{x \in H \cap C} l_x,$$

where $H^\vee$ is the dual projective plane of H and $l_x \subset H^\vee$ is the line corresponding to a point $x \in H$.

- Suppose that C consists of d lines passing through a point of $\mathbb{P}^3$ such that no three of them are on a plane. Then we have

  $$\pi_1(f_Q^{-1}(M \setminus D_C)) \cong F_{d-1} \quad \text{and} \quad \pi_1(g_H^{-1}(M \setminus D_C)) \cong \mathbb{Z}^{\oplus(d-1)},$$

  where F_{d-1} is the free group of rank $d-1$. In this case, we can easily prove that $\pi_1(M \setminus D_C)$ is isomorphic to $\mathbb{Z}^{\oplus(d-1)}$.
- Suppose that C is a smooth curve of degree d on a plane in $\mathbb{P}^3$. Then we have

  $$\pi_1(f_Q^{-1}(M \setminus D_C)) \cong \mathbb{Z}/d\mathbb{Z} \quad \text{and} \quad \pi_1(g_H^{-1}(M \setminus D_C)) \cong F_{d-1}.$$

  In this case, we can show that $\pi_1(M \setminus D_C) \cong \mathbb{Z}/d\mathbb{Z}$.

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