ON ZARISKI-VAN KAMPEN THEOREM

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Abstract. Let $f : E \to B$ be a dominant morphism, where E and B are smooth irreducible complex quasi-projective varieties, and let F_b be a general fiber of f. We present conditions under which the homomorphism $\pi_1(F_b) \to \pi_1(E)$ induced by the inclusion is injective.

1. Introduction

We work over the complex number field $\mathbb{C}$.

Let E and B be smooth irreducible quasi-projective varieties, and let

$$f : E \to B$$

be a dominant morphism. For a point $a \in B$, we denote by F_a the fiber $f^{-1}(a)$. We choose a general point b of B, and a point $\tilde{b}$ of F_b. Let

$$i : F_b \hookrightarrow E$$

denote the inclusion morphism.

In [5], Nori proved the following:

Proposition 1.1 ([5], Lemma 1.5 (C)). *Suppose that there exists a Zariski closed subset Ξ of B with codimension ≥ 2 such that, if $a \in B \setminus \Xi$, then F_a is irreducible and possesses at least one point at which f is smooth. Then the sequence*

$$\pi_1(F_b, \tilde{b}) \xrightarrow{i_*} \pi_1(E, \tilde{b}) \xrightarrow{f_*} \pi_1(B, b) \longrightarrow 1$$

is exact. □

We will study the kernel of i_*. When f has a global section, the classical Zariski-van Kampen theorem describes $\operatorname{Ker} i_*$ in terms of the monodromy relations in $\pi_1(F_b)$. The purpose of this paper is to investigate $\operatorname{Ker} i_*$ in a situation where only local monodromies are available. More precisely, we will show that, in some cases, the triviality of the local monodromies on the fundamental groups of fibers implies the injectivity of i_*.

In order to define the local monodromy on the fundamental group of a fiber, let us assume that the the condition in Proposition 1.1 is satisfied.

Definition 1.2 ([5], Lemma 1.5 (A)). The *topological discriminant locus* Σ_f of f is the minimal Zariski closed subset of B among the Zariski closed subsets Σ of B with the following properties;

- Σ contains the locus $f(\operatorname{Sing} f)$ of critical values of f, and
- f is locally trivial over $B \setminus \Sigma$ as a continuous map in the complex topology.

1991 *Mathematics Subject Classification.* 14F35.

Let $\Sigma_f^{(1)}, \ldots, \Sigma_f^{(k)}$ be the irreducible components of Σ_f with codimension 1 in B. Let x_i be a general point of $\Sigma_f^{(i)}$, and U_i a sufficiently small open ball in B with the center x_i. Since Ξ is of codimension ≥ 2 in B, we have $\Sigma_f^{(i)} \not\subset \Xi$, and hence $x_i \notin \Xi$. Therefore we have a holomorphic local section

$$s_i \,:\, U_i \,\to\, f^{-1}(U_i)$$

of f defined on U_i. We fix local coordinates $(z_1, \ldots, z_m)$ on U_i with the origin x_i such that Σ_f is defined by $z_1 = 0$. We put

$$a_i := (\varepsilon, 0, \ldots, 0) \in U_i \setminus (U_i \cap \Sigma_f),$$

where ε is a sufficiently small positive real number, and consider the loop

$$\lambda_i \,:\, (I, \partial I) \,\to\, (U_i \setminus (U_i \cap \Sigma_f), a_i)$$

defined by

$$\lambda_i(t) := (\varepsilon \exp(2\pi\sqrt{-1}t), 0, \ldots, 0),$$

which we will call a simple loop around the hypersurface $\Sigma_f^{(i)}$. Using the local section s_i, we can define the monodromy

$$\mu_i \,:\, \pi_1(F_{a_i}, s_i(a_i)) \,\xrightarrow{\sim}\, \pi_1(F_{a_i}, s_i(a_i))$$

along the loop λ_i. We call μ_i a *local monodromy* around $\Sigma_f^{(i)}$. In §2, we will show that the condition for μ_i to be trivial does not depend on the choice of the local section s_i (Corollary 2.5).

Theorem 1.3. *Suppose that the following conditions are satisfied:*

- (T1) *The quasi-projective variety B is either a non-compact Riemann surface or an affine space $\mathbb{A}^N$.*
- (T2) *The morphism f is flat.*
- (T3) *There exists a Zariski closed subset Ξ of B with codimension ≥ 2 such that, if $a \in B \setminus \Xi$, then F_a is irreducible and possesses at least one point at which f is smooth.*
- (T4) *The local monodromy μ_i around $\Sigma_f^{(i)}$ is trivial for $i = 1, \ldots, k$.*

Then, for a general point b of B, the sequence

$$1 \;\longrightarrow\; \pi_1(F_b, \tilde{b}) \;\xrightarrow{i_*}\; \pi_1(E, \tilde{b}) \;\xrightarrow{f_*}\; \pi_1(B, b) \;\longrightarrow\; 1 \tag{1.1}$$

is exact.

By Proposition 1.1, the condition (T3) implies that the sequence (1.1) is exact except for the injectivity of i_*. Hence all we have to show is that i_* is injective.

The condition (T1) of Theorem 1.3 suggests the following:

Problem 1.4. Suppose that the conditions (T2)-(T4) of Theorem 1.3 are satisfied. Can one define a homomorphism $\partial : \pi_2(B) \to \pi_1(F_b)$ such that $\operatorname{Ker} i_* = \operatorname{Im} \partial$ holds?

In [11], we studied the homotopy lifting property of f, and gave a partial answer to Problem 1.4.

In view of Theorem 1.3, it is important to know whether a given local monodromy is trivial or not. In the second half of this paper, we present some algebro-geometric conditions under which a given local monodromy is trivial. As a corollary, we obtain a simple proof of [6, Theorem 1], many applications of which have been given ([7], [8], [9], [10]). As another application of the results in this paper, we will prove in

[12] a hyperplane section theorem of Zariski type for the fundamental groups of Zariski open subsets of Grassmannian varieties.

This paper is organized as follows. In §2, we review the classical Zariski-van Kampen theorem; that is, we study $\operatorname{Ker} i_*$ in a situation where a global section exists ([13], [14], see also [2] and [4]). In §3, we prove Theorem 1.3. In §4, we study, in various settings, the problem when a local monodromy is trivial. In §5, we apply Situation (C) in §4 to a morphism from a smooth irreducible quasi-projective surface to a variety on which an algebraic group acts.

Notation and terminologies.

(1) Let $\alpha : I \to X$ and $\beta : I \to X$ be paths on a topological space X. We define the order of the conjunction of paths in such a way that the path $\alpha\beta$ is defined only when $\alpha(1) = \beta(0)$. By this notation, the monodromy action of the fundamental group of the base space on the fundamental group of a fiber is from *right*.

(2) For a morphism $\phi : X \to Y$ with X and Y smooth, we denote by $\operatorname{Sing}\phi \subset X$ the Zariski closed subset of critical points of ϕ.

(3) We say that a morphism $\phi : X \to Y$ is locally trivial if it is locally trivial as a continuous map in the complex topology.

2. The classical Zariski-van Kampen theorem

For a subset S of a group G, we denote by $\mathrm{NC}_G(S)$ the normal closure of S in G; that is, $\mathrm{NC}_G(S)$ is the smallest normal subgroup of G containing S.

Suppose that a group H acts on a group N from *right*. We write this action by

$$n \mapsto n^h \qquad (n \in N, h \in H).$$

The semi-direct product $N \rtimes H$ is the set $N \times H$ equipped with a structure of the group by

$$(n_1, h_1)(n_2, h_2) := (n_1 n_2^{(h_1^{-1})}, h_1 h_2).$$

We have a natural exact sequence

$$1 \longrightarrow N \xrightarrow{\iota} N \rtimes H \xrightarrow{\rho} H \longrightarrow 1$$

with a natural section $\sigma : H \to N \rtimes H$ of ρ. Conversely, suppose that an exact sequence

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\rho} H \longrightarrow 1$$

and a section $\sigma : H \to G$ of ρ are given. Then an action of H on N from right is defined by

$$\iota(n^h) := \sigma(h)^{-1}\iota(n)\sigma(h) \qquad (n \in N, h \in H),$$

and G is isomorphic to the semi-direct product $N \rtimes H$ constructed by this action.

For a subset T of H, we put

$$\mathrm{Rel}(T) := \{\, n^{-1}n^h \;\mid\; n \in N,\, h \in T \,\} \;\subset\; N.$$

The following is easy to prove:

Lemma 2.1. *Let T be a subset of H. Then $\iota^{-1}(\mathrm{NC}_{N\rtimes H}(\sigma(T)))$ coincides with $\mathrm{NC}_N(\mathrm{Rel}(\mathrm{NC}_H(T)))$.* □

Let X be a path-connected topological space, and b a point of X. We denote by $[S^1, X]$ the set of homotopy classes of continuous maps from the circle S^1 to X. Then there exists a natural bijection between $[S^1, X]$ and the set $\mathrm{Conj}(\pi_1(X, b))$ of conjugate classes of $\pi_1(X, b)$.

Let M be a connected complex manifold, and D a reduced hypersurface of M. Let D_i be an irreducible component of D, and let p be a point of D_i not contained in $\operatorname{Sing} D$. There exist local coordinates $(z_1, \dots, z_m)$ of M with the origin p such that D is defined by $z_1 = 0$ locally around p. Let

$$u : S^1 \to M \setminus D$$

be a continuous map given by $(\cos t, \sin t) \mapsto (\varepsilon \exp(2\pi\sqrt{-1}t), 0, \dots, 0)$ in terms of the local coordinates $(z_1, \dots, z_m)$, where ε is a sufficiently small positive real number. The homotopy class $[u] \in [S^1, M \setminus D]$ of this continuous map does not depend on the choice of p, the local coordinates, and ε. We call $[u] \in [S^1, M \setminus D]$ the homotopy class of simple free loops around D_i.

Definition 2.2. Let $\tilde{b}$ be a point of $M \setminus D$. A loop

$$v : (I, \partial I) \to (M \setminus D, \tilde{b})$$

with the base point $\tilde{b}$ is called *a simple loop around* D_i if its homotopy class $[v] \in \pi_1(M \setminus D, \tilde{b})$ is contained in the conjugate class that corresponds to the homotopy class of simple free loops around D_i via the natural bijection between $[S^1, M \setminus D]$ and $\operatorname{Conj}(\pi_1(M \setminus D, \tilde{b}))$.

We will consider the homomorphism $j_* : \pi_1(M \setminus D, \tilde{b}) \to \pi_1(M, \tilde{b})$ induced by the inclusion $j : M \setminus D \hookrightarrow M$. The following lemma is well-known:

Lemma 2.3. *Suppose that D consists of a finite number of irreducible components $D_1, \dots, D_k$. Let v_i be a simple loop around D_i with the base point $\tilde{b}$, and let V be the subset $\{[v_1], \dots, [v_k]\}$ of $\pi_1(M \setminus D, \tilde{b})$. Then $\operatorname{Ker} j_*$ coincides with the normal closure $\mathrm{NC}_{\pi_1(M\setminus D, \tilde{b})}(V)$ of V.* □

Let U be a complex manifold, and let

$$g : M \to U$$

be a surjective holomorphic map. For a point a of U, we denote by G_a the fiber $g^{-1}(a)$. Suppose that there exists a hypersurface Γ of U such that g is locally trivial over $U \setminus \Gamma$ as a continuous map. Suppose also that Γ consists of a finite number of irreducible components $\Gamma_1, \dots, \Gamma_k$. We assume that there exists a continuous global section

$$s : U \to M$$

of g. We choose a point $b \in U \setminus \Gamma$, and put

$$\tilde{b} := s(b) \in G_b.$$

Using the section s, we can define the monodromy action of $\pi_1(U \setminus \Gamma, b)$ on $\pi_1(G_b, \tilde{b})$ from right. For each irreducible component Γ_i of Γ, we choose a simple loop

$$w_i : (I, \partial I) \to (U \setminus \Gamma, b)$$

around Γ_i, and put

$$W := \{[w_1], \dots, [w_k]\} \subset \pi_1(U \setminus \Gamma, b), \quad \text{and} \quad \widetilde{W} := \mathrm{NC}_{\pi_1(U\setminus\Gamma, b)}(W).$$

Recall that $\operatorname{Rel}(\widetilde{W})$ is the subset $\{n^{-1}n^h \mid n \in \pi_1(G_b, \tilde{b}),\ h \in \widetilde{W}\}$ of $\pi_1(G_b, \tilde{b})$, which is called the set of *monodromy relations*.

Proposition 2.4. *Suppose that s is holomorphic at each point of Γ. Suppose also that $g^{-1}(\Gamma_i)$ is an irreducible hypersurface of M for $i = 1, \dots, k$. Then the kernel of the homomorphism $i_* : \pi_1(G_b, \tilde{b}) \to \pi_1(M, \tilde{b})$ induced by the inclusion $i : G_b \hookrightarrow M$ coincides with the normal closure*

$$\mathrm{NC}_{\pi_1(G_b,\tilde{b})}(\mathrm{Rel}(\widetilde{W}))$$

of the set of monodromy relations in $\pi_1(G_b, \tilde{b})$.

Proof. We put $U^\circ := U \setminus \Gamma$ and $M^\circ := M \setminus g^{-1}(\Gamma)$. Let $g^\circ : M^\circ \to U^\circ$ and $s^\circ : U^\circ \to M^\circ$ denote the restrictions of g and s, respectively. We also denote by $i^\circ : G_b \hookrightarrow M^\circ$ the inclusion. Because of the section s° of g°, we get a short exact sequence

$$1 \longrightarrow \pi_1(G_b, \tilde{b}) \xrightarrow{i^\circ_*} \pi_1(M^\circ, \tilde{b}) \xrightarrow{g^\circ_*} \pi_1(U^\circ, b) \longrightarrow 1$$

with the section s°_* of g°_* from the homotopy exact sequence of the locally trivial fiber space g°. Note that the monodromy action of $\pi_1(U^\circ, b)$ on $\pi_1(G_b, \tilde{b})$ coincides with the composite of s°_* and the inner-automorphism of $\pi_1(M^\circ, \tilde{b})$; that is, we have

$$i^\circ_*([u]^{[v]}) = [s^\circ \circ v]^{-1} \cdot [u] \cdot [s^\circ \circ v] \quad \text{in } \pi_1(M^\circ, \tilde{b}),$$

where u is a loop in G_b with the base point $\tilde{b}$, and v is a loop in U° with the base point b. Hence $\pi_1(M^\circ, \tilde{b})$ is canonically isomorphic to the semi-direct product $\pi_1(G_b, \tilde{b}) \rtimes \pi_1(U^\circ, b)$ constructed from the monodromy action of $\pi_1(U^\circ, b)$ on $\pi_1(G_b, \tilde{b})$. Since s is holomorphic at each point of Γ, the loop $s^\circ \circ w_i$ in M° is a simple loop around the irreducible hypersurface $g^{-1}(\Gamma_i)$ of M. Therefore the kernel of the homomorphism

$$j_* : \pi_1(M^\circ, \tilde{b}) \to \pi_1(M, \tilde{b})$$

induced by the inclusion $j : M^\circ \hookrightarrow M$ coincides with $\mathrm{NC}_{\pi_1(M^\circ,\tilde{b})}(s^\circ_*(W))$ by Lemma 2.3. Since $i = j \circ i^\circ$, and i°_* is injective, we have

$$\mathrm{Ker}\, i_* = (i^\circ_*)^{-1}(\mathrm{Ker}\, j_*) = \mathrm{NC}_{\pi_1(G_b,\tilde{b})}(\mathrm{Rel}(\widetilde{W}))$$

by Lemma 2.1. □

When U is simply connected, the normal closure $\widetilde{W}$ of W in $\pi_1(U \setminus \Gamma, b)$ coincides with $\pi_1(U \setminus \Gamma, b)$. Hence we obtain the following:

Corollary 2.5. *Suppose that $g^{-1}(\Gamma_i)$ is irreducible for $i = 1, \dots, k$, that s is holomorphic at each point of Γ, and that U is simply connected. Then the following two conditions are equivalent:*

(i) *The monodromy action of $\pi_1(U \setminus \Gamma, b)$ on $\pi_1(G_b, \tilde{b})$ associated to the section s is trivial.*
(ii) *The inclusion $G_b \hookrightarrow M$ induces an injective homomorphism from $\pi_1(G_b, \tilde{b})$ to $\pi_1(M, \tilde{b})$.*

In particular, if the monodromy action of $\pi_1(U \setminus \Gamma, b)$ on $\pi_1(G_b, \tilde{b})$ is trivial for one section, then it is trivial for any section. □

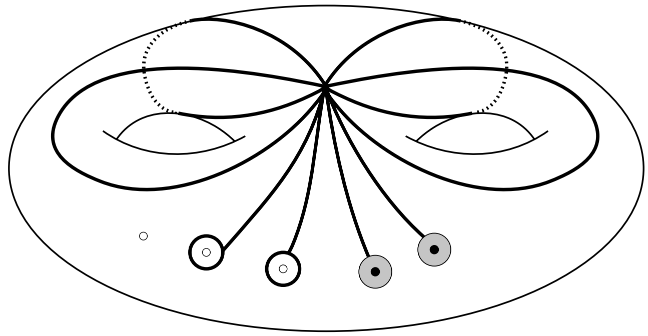

Figure 3.1. The one-dimensional CW-complex K

3. Proof of Theorem 1.3

3.1. The case of a non-compact Riemann surface. Suppose that B is a non-compact Riemann surface. Let $\overline{B}$ be the smooth compactification of B. We put

$$P := \overline{B} \setminus B.$$

In this case, the topological discriminant locus Σ_f consists of a finite number of points of B. We put

$$\Sigma_f = \{q_1, \dots, q_k\}.$$

For each $q_i \in \Sigma_f$, we put a sufficiently small closed disc Δ_i on B with the center q_i. We have a finite one-dimensional CW-complex K on $\overline{B} \setminus (\Sigma_f \cup P)$ containing b such that $K \cap \Delta_i$ consists of a single point $r_i \in \partial\Delta_i$ for $i = 1, \dots, k$, and that $K \cup \partial\Delta_1 \cup \dots \cup \partial\Delta_k$ is a strong deformation retract of $\overline{B} \setminus (\Sigma_f \cup P)$. Figure 3.1 illustrates K by thick lines and Δ_i by shaded discs, in a situation where $\overline{B}$ is of genus 2, P consists of three points indicated by $\circ$, and Σ_f consists of two points indicated by $\bullet$. Let L be the union of K and Δ_i $(i = 1, \dots, k)$. Then L is a strong deformation retract of B containing Σ_f in its interior. By the condition (T3) of Theorem 1.3, we have a local section

$$s_i : \Delta_i \to f^{-1}(\Delta_i)$$

of f defined on Δ_i that is holomorphic in the interior of Δ_i. Since the restriction

$$f \mid f^{-1}(K) : f^{-1}(K) \to K$$

of f to $f^{-1}(K)$ is a locally trivial fiber space with a connected fiber, and K is of real dimension 1, there exists a continuous section

$$s_K : K \to f^{-1}(K)$$

of f defined on K such that $s_K(r_i) = s_i(r_i)$ holds for each i. Gluing s_K and s_i $(i = 1, \dots k)$ together, we obtain a section

$$s_L : L \to f^{-1}(L)$$

of f defined over L. There is an open subset U of B that is containing L as a strong deformation retract, and is a strong deformation retract of B. Then we can extend s_L to a continuous section

$$s_U : U \to f^{-1}(U)$$

of f defined over U. Note that s_U is holomorphic at each point q_i of Σ_f. By the condition (T3), $F_{q_i} = f^{-1}(q_i)$ is irreducible for each $q_i \in \Sigma_f$. Hence we can apply Proposition 2.4 to the restriction $f_U : E_U \to U$ of f to

$$E_U := f^{-1}(U).$$

Using the condition (T4), we conclude that the inclusion $F_b \hookrightarrow E_U$ induces an injective homomorphism

$$\pi_1(F_b, \tilde{b}) \;\hookrightarrow\; \pi_1(E_U, \tilde{b}),$$

where $\tilde{b} := s_U(b)$. On the other hand, since f is locally trivial over $B \setminus U$ and U is a strong deformation retract of B, the inclusion $E_U \hookrightarrow E$ induces an isomorphism

$$\pi_1(E_U, \tilde{b}) \;\cong\; \pi_1(E, \tilde{b}).$$

Hence i_* is injective.

3.2. **The case of an affine space.** Next we treat the case where B is an affine space $\mathbb{A}^N$ by induction on N. The case where $B = \mathbb{A}^1$ is proved above. Suppose that $N > 1$. Let

$$\rho \,:\, B \,\to\, A$$

be a general affine projection, where A is a one-dimensional affine line $\mathbb{A}^1$, and let

$$g \,:\, E \,\to\, A$$

be the composite of f and ρ. For a point $t \in A$, we put

$$B_t := \rho^{-1}(t) \cong \mathbb{A}^{N-1}, \quad E_t := g^{-1}(t) = f^{-1}(B_t),$$

and denote by

$$f_t \,:\, E_t \,\to\, B_t$$

the restriction of f to E_t.

The strategy of the proof is as follows:

Step 1. We show that, when $t \in A$ is general, f_t satisfies the four conditions in Theorem 1.3. Combining this with the induction hypothesis, we see that, if $b \in B$ is general, then the inclusion $F_b \hookrightarrow E_{\rho(b)}$ induces an isomorphism on the fundamental groups.

Step 2. We show that g satisfies the four conditions in Theorem 1.3, and hence, if $b \in B$ is general, the inclusion $E_{\rho(b)} \hookrightarrow E$ induces an isomorphism on the fundamental groups. Combining this with Step 1 above, we complete the proof of Theorem 1.3.

Step 1. First note that E_t is irreducible for every $t \in A$. Indeed, since B_t is of codimension 1 in B, the condition (T3) implies that a general fiber of f_t is irreducible. Hence, if E_t were reducible, there should exist an irreducible component of E_t whose image by f_t is contained in a proper Zariski closed subset of B_t. Since every irreducible component of E_t is of dimension equal to $\dim E - 1$, we get a contradiction with the condition (T2).

Let Ξ_0 denote the singular locus $\operatorname{Sing} \Sigma_f$ of Σ_f, where Σ_f is regarded with the reduced structure, and let Ξ_1 denote the union of all irreducible components of Σ_f with codimension ≥ 2 in B. Recall that $\Sigma_f^{(i)}$ $(i = 1, \ldots, k)$ are the irreducible components of Σ_f with codimension 1 in B. Since ρ is general, there exists a proper

Zariski closed subset $\Xi^{(i)}$ of $\Sigma_f^{(i)}$ containing $\operatorname{Sing}\Sigma_f^{(i)}$ such that the restriction $\rho\,|\,\Sigma_f^{(i)}$ of ρ to $\Sigma_f^{(i)}$ is smooth at every point of $\Sigma_f^{(i)}\setminus\Xi^{(i)}$. We put

$$\Xi' := \Xi\cup\Xi_0\cup\Xi_1\cup\Xi^{(1)}\cup\cdots\cup\Xi^{(k)},$$

where the first Ξ is the Zariski closed subset that appears in the condition (T3). Then Ξ' is a Zariski closed subset of B with codimension ≥ 2. There exists a finite set Γ of points of A such that, if $t\in A\setminus\Gamma$, then E_t is smooth, and $B_t\cap\Xi'$ is of codimension ≥ 2 in B_t.

Let t be a point of $A\setminus\Gamma$. We show that $f_t : E_t\to B_t$ satisfies the four conditions in Theorem 1.3. The condition (T1) is obvious. The condition (T2) for f_t follows from the condition (T2) for f. Since $B_t\cap\Xi$ is of codimension ≥ 2 in B_t, it follows that f_t satisfies the condition (T3).

Since ρ is a *general* affine projection, the intersection

$$\Sigma_f(t) := B_t\cap\Sigma_f$$

of B_t and Σ_f is a proper Zariski closed subset of B_t. Note that, if f is smooth at a point $z\in E_t$, then so is f_t. Therefore $\Sigma_f(t)$ contains the set $f_t(\operatorname{Sing} f_t)$ of critical values of f_t, and hence $\Sigma_f(t)$ contains the topological discriminant locus $\Sigma_{f_t}\subset B_t$ of f_t. Let $\Sigma_f(t)^{(j)}$ be an irreducible component of $\Sigma_f(t)$ with codimension 1 in B_t, and let y be a general point of $\Sigma_f(t)^{(j)}$. Since $B_t\cap\Xi'$ is of codimension ≥ 2 in B_t, there exists a unique $\Sigma_f^{(i)}$ among $\Sigma_f^{(1)}, \dots, \Sigma_f^{(k)}$ such that

- $\Sigma_f(t)^{(j)}$ is an irreducible component of the intersection of B_t with $\Sigma_f^{(i)}$,
- $\Sigma_f^{(i)}$ is smooth at y, and intersects B_t transversely at y.

Let $U_{t,y}$ be a small open neighborhood of y in B_t, and let a be a point of $U_{t,y}\setminus(U_{t,y}\cap\Sigma_f(t))$. Then a simple loop

$$\lambda : (I,\partial I)\to(U_{t,y}\setminus(U_{t,y}\cap\Sigma_f(t)),a)$$

in $B_t\setminus\Sigma_f(t)$ around $\Sigma_f(t)^{(j)}$ can be regarded as a simple loop in $B\setminus\Sigma_f$ around $\Sigma_f^{(i)}$. A holomorphic local section $s_{t,y}$ of f_t can be defined on $U_{t,y}$ by restricting a holomorphic local section of f around y. Hence the local monodromy on $\pi_1(F_a, s_{t,y}(a))$ along the loop λ associated to the holomorphic local section $s_{t,y}$ is trivial by the condition (T4) for f. Thus the condition (T4) for f_t is satisfied.

Step 2. The conditions (T1) and (T2) are obvious. Since E_t is irreducible, as was shown in Step 1, and $B_t\setminus(B_t\cap\Xi)$ is non-empty for any $t\in A$, the condition (T3) is satisfied by g. Let q be a point of the topological discriminant locus $\Sigma_g\subset A$ of g. We choose a sufficiently small open disc $D\subset A$ with the center q, and a point $t_0\in D\setminus\{q\}$. Let r_0 be a general point of B_{t_0}. Since ρ is a general affine projection, we have a holomorphic local section

$$s_1 : D\to\rho^{-1}(D)$$

of ρ such that $s_1(D)\cap\Sigma_f=\emptyset$, and that $s_1(t_0)=r_0$. Then there exists a holomorphic local section

$$s_2 : s_1(D)\to f^{-1}(s_1(D))$$

of $f\,|\,f^{-1}(s_1(D))$. We can define a holomorphic local section s of g by

$$s := s_2\circ s_1 : D\to g^{-1}(D).$$

We will show that the local monodromy on $\pi_1(E_{t_0}, s(t_0))$ along the simple loop

$$\lambda' : (I, \partial I) \to (D \setminus \{q\}, t_0)$$

around q associated with the section s is trivial. The intersection $D \cap \Gamma$ is either empty or consisting of the single point q. In particular, we have $t_0 \notin \Gamma$. Since r_0 is general in B_{t_0}, we obtain from Step 1 an isomorphism

$$\pi_1(F_{r_0}, s(t_0)) \cong \pi_1(E_{t_0}, s(t_0)) \tag{3.1}$$

induced by the inclusion $F_{r_0} \hookrightarrow E_{t_0}$. Since $s_1(D) \cap \Sigma_f = \emptyset$, f is locally trivial over $s_1(D)$, and hence the local monodromy on $\pi_1(F_{r_0}, s(t_0))$ along a simple loop

$$s_1 \circ \lambda' : (I, \partial I) \to (s_1(D) \setminus \{s_1(q)\}, r_0)$$

around $s_1(q)$ associated with the section s_2 is trivial. From the isomorphism (3.1) induced by the inclusion, we see that the local monodromy on $\pi_1(E_{t_0}, s(t_0))$ along the loop λ' is also trivial. Thus the condition (T4) for g is also satisfied. □

4. Local monodromies

In this section, we present, in the following three situations, sufficient conditions for the local monodromy μ_i around $\Sigma_f^{(i)}$ to be trivial.

Recall that x_i is a general point of the irreducible hypersurface $\Sigma_f^{(i)}$ in B, and λ_i is a simple loop around $\Sigma_f^{(i)}$ in a sufficiently small open ball U_i in B with the center x_i. If a holomorphic local section s_i of f is defined on U_i, then the local monodromy on $\pi_1(F_{a_i}, s_i(a_i))$ along λ_i is defined, where a_i is the base point of the loop λ_i.

Remark 4.1. Since the local monodromy μ_i to be trivial is a local property on B, we can replace B by a small Zariski open neighborhood of x_i when we use the following propositions. For example, removing all irreducible components of Σ_f except for $\Sigma_f^{(i)}$, we can assume that Σ_f is an irreducible hypersurface in B.

Situation (A). Let Y be a smooth irreducible quasi-projective variety. Suppose that we are given a morphism

$$\phi : E \to Y.$$

We denote by

$$\Phi : E \to B \times Y$$

the morphism defined by $\Phi(x) := (f(x), \phi(x))$.

Proposition 4.2. *Suppose that the following conditions hold;*

- (A1) *Y is simply connected,*
- (A2) *Φ is dominant, and its general fiber is connected,*
- (A3) *there exists an open neighborhood W of x_i in B such that, for any $(a, y) \in W \times Y$, the fiber $\Phi^{-1}(a, y)$ has at least one point at which Φ is smooth, and*
- (A4) *the topological discriminant locus $\Sigma_\Phi \subset B \times Y$ of Φ does not contain $\{x_i\} \times Y$.*

Then the local monodromy μ_i is defined and trivial.

Proof. Note that $\Phi^{-1}(\{a\} \times Y) = F_a$ for every $a \in B$. We denote by

$$\phi_a : F_a \to Y$$

the restriction of ϕ to F_a. We have a diagram of the fiber product

$$\begin{array}{ccc} F_a & \hookrightarrow & E \\ {\scriptstyle \phi_a} \downarrow & \square & \downarrow {\scriptstyle \Phi} \\ Y & \hookrightarrow & B \times Y, \end{array}$$

where the lower inclusion is given by $y \mapsto (a, y)$. Let η be a general point of Y. Then $(x_i, \eta) \in B \times Y$ is not contained in Σ_Φ by the condition (A4). Hence there exist sufficiently small open balls U in B with the center x_i and V in Y with the center η such that the following hold:

- The open subset W in the condition (A3) contains U. Hence, for any $a \in U$, every fiber of ϕ_a possesses at least one point at which ϕ_a is smooth.
- The product $U \times V$ is disjoint from Σ_Φ. In particular, if $(a, y) \in U \times V$, then $\Phi^{-1}(a, y) = \phi_a^{-1}(y)$ is smooth and, by the condition (A2), irreducible.

It follows that ϕ_a satisfies the condition in Proposition 1.1 for any $a \in U$, and that V is disjoint from the topological discriminant locus $\Sigma_{\phi_a} \subset Y$ of ϕ_a for any $a \in U$. There exists a holomorphic local section

$$\tilde{s} \,:\, U \times V \;\to\; \Phi^{-1}(U \times V)$$

of Φ defined on $U \times V$. Putting

$$s(a) := \tilde{s}(a, \eta),$$

we obtain a holomorphic local section

$$s \,:\, U \;\to\; f^{-1}(U)$$

of f defined on U such that $s(a) \in \phi_a^{-1}(\eta)$ for any $a \in U$. Hence the local monodromy μ_i is defined.

Applying Proposition 1.1 to ϕ_a $(a \in U)$ and using the condition (A1), we see that the inclusion $\phi_a^{-1}(\eta) \hookrightarrow F_a$ induces a surjective homomorphism

$$\pi_1(\phi_a^{-1}(\eta), s(a)) \;\twoheadrightarrow\; \pi_1(F_a, s(a))$$

for any $a \in U$. We draw the simple loop λ_i around $\Sigma_f^{(i)}$ in $U \setminus (U \cap \Sigma_f)$. Let

$$\tilde{\lambda}_i \,:\, (I, \partial I) \;\to\; (U \times V, (a_i, \eta))$$

be the loop defined by

$$\tilde{\lambda}_i(t) := (\lambda_i(t), \eta).$$

Since Φ is locally trivial over $U \times V$, which is simply connected, the monodromy action

$$\tilde{\mu}_i \,:\, \pi_1(\Phi^{-1}(a_i, \eta), \tilde{s}(a_i, \eta)) \;\xrightarrow{\sim}\; \pi_1(\Phi^{-1}(a_i, \eta), \tilde{s}(a_i, \eta))$$

along the loop $\tilde{\lambda}_i$ associated with the section $\tilde{s}$ is trivial. The diagram

$$\begin{array}{ccc} \pi_1(\phi_{a_i}^{-1}(\eta), s(a_i)) & \xrightarrow{\tilde{\mu}_i} & \pi_1(\phi_{a_i}^{-1}(\eta), s(a_i)) \\ \downarrow & & \downarrow \\ \pi_1(F_{a_i}, s(a_i)) & \xrightarrow{\mu_i} & \pi_1(F_{a_i}, s(a_i)) \end{array}$$

is commutative, where the vertical arrows are induced by the inclusion of $\phi_{a_i}^{-1}(\eta) = \Phi^{-1}(a_i, \eta)$ into F_{a_i}. Since $\tilde{\mu}_i$ is trivial and the vertical homomorphisms are surjective, we see that μ_i is also trivial. $\square$

Situation (B). Suppose that there exists a smooth projective morphism

$$\bar{f} : \overline{E} \to B$$

from a quasi-projective variety $\overline{E}$ such that E is the complement $\overline{E} \setminus Z$ to a reduced divisor (possibly empty) Z of $\overline{E}$, and that f is the restriction of $\bar{f}$ to E. For a point $a \in B$, let us denote by $\overline{F}_a$ the fiber $\bar{f}^{-1}(a)$, and by Z_a the *scheme-theoretic* intersection of Z and $\overline{F}_a$. We have $F_a = \overline{F}_a \setminus Z_a$. Note that $\bar{f}$ is locally trivial over B, because it is smooth and projective.

Proposition 4.3. *Suppose that the following conditions hold:*

- (B1) *The fiber $\overline{F}_a$ of $\bar{f}$ is connected.*
- (B2) *There exists a Zariski closed subset Ξ of B with codimension ≥ 2 such that Z_a is a reduced divisor of $\overline{F}_a$ for any $a \in B \setminus \Xi$.*

Then the local monodromy μ_i is defined and trivial.

Proof. The morphism f is smooth and dominant, and its general fiber is connected by the conditions (B1). By the condition (B2), the locus $\{a \in B \mid F_a = \emptyset\}$ is contained in a Zariski closed subset of codimension ≥ 2 in B. Since x_i is a general point of the hypersurface $\Sigma_f^{(i)}$ of B, a holomorphic local section of f is defined in a small open neighborhood of x_i. Therefore the local monodromy μ_i is defined.

We embed $\overline{E}$ into a projective space $\mathbb{P}^M$. Let L be a general linear subspace of $\mathbb{P}^M$ with

$$\dim L = M - (\dim \overline{E} - \dim B) + 1.$$

We put

$$\overline{E}_L := \overline{E} \cap L, \quad E_L := E \cap L$$

and denote by

$$\bar{f}_L : \overline{E}_L \to B \quad \text{and} \quad f_L : E_L \to B$$

the restrictions of $\bar{f}$ and f, respectively. Let U be a sufficiently small open ball in B with the center x_i. Since L is general, the scheme-theoretic intersection $\overline{F}_{x_i} \cap L$ is a connected smooth curve, and hence $\bar{f}_L$ is smooth and locally trivial over U with fibers being compact Riemann surfaces. Moreover, by the condition (B2), the scheme-theoretic intersection $Z_{x_i} \cap L$ is a reduced divisor of the compact Riemann surface $\bar{f}_L^{-1}(x_i) = \overline{F}_{x_i} \cap L$. Then f_L is locally trivial over U with fibers being punctured Riemann surfaces, because the number of the punctured points $Z_a \cap L$ does not vary when a moves on U.

There exists a Zariski closed subset $\Sigma_{(f,f_L)}$ of B with codimension ≥ 1 such that the pair

$$(f, f_L) : (E, E_L) \to B$$

is locally trivial over $B \setminus \Sigma_{(f,f_L)}$ as a pair of continuous maps in the complex topology. Let $\Sigma'_{(f,f_L)}$ be the union of all irreducible components of $\Sigma_{(f,f_L)}$ that are not contained in Σ_f. Then $\Sigma'_{(f,f_L)} \cap \Sigma_f$ is of codimension ≥ 2 in B. Since x_i is a general point of the hypersurface $\Sigma_f^{(i)}$, and U is sufficiently small, we have

$$U \setminus (U \cap \Sigma_f) \subset B \setminus \Sigma_{(f,f_L)}.$$

Since f_L is smooth over U, we have a holomorphic local section

$$s : U \to f_L^{-1}(U)$$

of f_L. We draw the simple loop λ_i around $\Sigma_f^{(i)}$ in $U\setminus(U\cap\Sigma_f)$. The local monodromy

$$\mu_i' \,:\, \pi_1(f_L^{-1}(a_i), s(a_i)) \xrightarrow{\sim} \pi_1(f_L^{-1}(a_i), s(a_i))$$

along λ_i associated with the section s is trivial, because f_L is locally trivial over U and U is simply connected. Since (f, f_L) is locally trivial over $U \setminus (U \cap \Sigma_f)$, Deligne's theorem [1] [3, Theorem 1.1 (B)] implies that the inclusion of $f_L^{-1}(a_i)$ into F_{a_i} induces a surjective homomorphism

$$\pi_1(f_L^{-1}(a_i), s(a_i)) \twoheadrightarrow \pi_1(F_{a_i}, s(a_i)).$$

Therefore the triviality of the local monodromy μ_i on $\pi_1(F_{a_i}, s(a_i))$ follows from the triviality of μ_i'. □

Combining Theorem 1.3 and Proposition 4.3, we obtain the following. Let $\overline{F}$ be a smooth irreducible projective variety, and Z a reduced hypersurface of $\mathbb{A}^N \times \overline{F}$. For a point $a \in \mathbb{A}^N$, we denote by Z_a the scheme-theoretic intersection of Z and $\{a\} \times \overline{F}$, and regard it as a Zariski closed subset of $\overline{F}$.

Corollary 4.4 ([6], Theorem 1)**.** *Suppose that there exists a Zariski closed subset Ξ of $\mathbb{A}^N$ with codimension ≥ 2 such that Z_a is a reduced divisor of $\overline{F}$ for any $a \in \mathbb{A}^N \setminus \Xi$. Then the inclusion of $\overline{F} \setminus Z_a$ into $(\mathbb{A}^N \times \overline{F}) \setminus Z$ induces an isomorphism of the fundamental groups for a general $a \in \mathbb{A}^N$.* □

Situation (C). Let $\overline{X}$ be a smooth irreducible projective variety, and W a reduced divisor (possibly empty) of $\overline{X}$. We put

$$X := \overline{X} \setminus W.$$

Let M be a smooth irreducible projective variety, and D a very ample divisor of M. Suppose that we are given a morphism

$$\bar{g} \,:\, B \times \overline{X} \to M$$

such that $\bar{g}(B \times \overline{X}) \not\subset D$. We put

$$Z := (B \times W) + \bar{g}^{-1}(D),$$

which is a divisor of $B \times \overline{X}$. We consider the situation where

$$E = (B \times \overline{X}) \setminus Z,$$

and $f : E \to B$ is the projection.

We denote by

$$g \,:\, B \times X \to M \quad \text{and} \quad g_E \,:\, E \to M \setminus D$$

the restrictions of $\bar{g}$ to the Zariski open subsets $B \times X$ and E of $B \times \overline{X}$, respectively. For $a \in B$, we denote by

$$\bar{g}_a \,:\, \overline{X} \to M \quad \text{and} \quad g_a \,:\, X \to M$$

the restrictions of $\bar{g}$ and g to $\{a\} \times \overline{X}$ and $\{a\} \times X$, respectively. Then we have

$$F_a \;:=\; f^{-1}(a) \;=\; X \setminus g_a^{-1}(D) \;=\; \overline{X} \setminus (W \cup \bar{g}_a^{-1}(D)).$$

Let P denote the projective space $\mathbb{P}_* H^0(M, \mathcal{O}_M(D))$, which parameterizes all effective divisors in the complete linear system $|D|$. For a point $p \in P$, let D_p denote the corresponding divisor of M. We put

$$\mathcal{H} := \{ \, (a, x, p) \in B \times X \times P \;\mid\; (a, x) \in E, \; g_a(x) \in D_p \, \},$$

and let

$$\rho : \mathcal{H} \to B \times P$$

be the natural projection. We have a natural identification

$$\rho^{-1}(a,p) \;=\; g_a^{-1}(D_p) \cap F_a \;=\; \bar{g}_a^{-1}(D_p) \setminus (\bar{g}_a^{-1}(D_p) \cap (W \cup \bar{g}_a^{-1}(D)))$$

for any $(a,p) \in B \times P$.

Proposition 4.5. *Suppose that the following conditions hold:*

(C1) *The Zariski closed subset $\{a \in B \mid \bar{g}_a(\overline{X}) \subset D\}$ of B is of codimension ≥ 2.*
(C2) *For any $a \in B$, the dimension of $\bar{g}_a(\overline{X})$ is ≥ 2.*
(C3) *The topological discriminant locus $\Sigma_\rho \subset B \times P$ of ρ does not contain $\{x_i\} \times P$.*

Then the local monodromy μ_i is defined and trivial.

Proof. By the condition (C1), the locus $\{a \in B \mid F_a = \emptyset\}$ is contained in a Zariski closed subset of codimension ≥ 2 in B. Since f is smooth, we have a local holomorphic section of f defined in a small open neighborhood of x_i. Therefore the local monodromy μ_i is defined.

We denote by ∞ the point of P corresponding to the divisor $D \in |D|$ given at the outset, and write D_∞ instead of D. We put

$$P^\times := P \setminus \{\infty\}.$$

Let Q be the projective space that parameterizes the projective lines of P passing through ∞, and let

$$\alpha : P^\times \to Q$$

be the natural projection, which is locally trivial in the Zariski topology with fibers isomorphic to the affine line $\mathbb{A}^1$. For a point $q \in Q$, let $A_q \subset P^\times$ denote the fiber $\alpha^{-1}(q)$. If $y \in M \setminus D_\infty$ and $q \in Q$, then there exists a unique point $\gamma_q(y)$ of A_q such that $y \in D_{\gamma_q(y)}$. Hence, for each $q \in Q$, we have a natural morphism

$$\gamma_q : M \setminus D_\infty \to A_q,$$

whose fibers over $p \in A_q$ is $D_p \setminus (D_p \cap D_\infty)$. Let

$$\phi_q : E \to A_q$$

be the composite of $g_E : E \to M \setminus D_\infty$ and $\gamma_q : M \setminus D_\infty \to A_q$. Let

$$\Phi_q : E \to B \times A_q$$

be the morphism defined by $\Phi_q(a,x) := (a, \phi_q(a,x))$. We put

$$\mathcal{H}^\times := \rho^{-1}(B \times P^\times).$$

Note that the restriction

$$\rho^\times : \mathcal{H}^\times \to B \times P^\times$$

of ρ to $\mathcal{H}^\times$ is the universal family of Φ_q ($q \in Q$); that is, we have a diagram of the fiber product

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{H}^\times \\ {\scriptstyle \Phi_q}\downarrow & \square & \downarrow{\scriptstyle \rho^\times} \\ B \times A_q & \hookrightarrow & B \times P^\times \end{array}$$

for any $q \in Q$, where the upper horizontal arrow is the inclusion given by

$$(a,x) \mapsto (a, x, \phi_q(a,x)).$$

We will prove the triviality of μ_i by showing that, when $q \in Q$ is chosen generally, the morphisms ϕ_q and Φ_q satisfy the four conditions in Proposition 4.2 with $Y = A_q$.

The condition (A1) is obvious. Since ρ is dominant by the condition (C1), Φ_q is dominant for a general $q \in Q$. For any $a \in B$ and a general $p \in P$, the condition (C2) implies that $\bar{g}_a^{-1}(D_p)$ is smooth and connected by Bertini's theorem ([3, Theorem 1.1]). Hence $\rho^{-1}(a,p)$ is connected for a general $(a,p) \in B \times P$, because $\rho^{-1}(a,p)$ is a Zariski open dense subset of $\bar{g}_a^{-1}(D_p)$. Therefore, if $q \in Q$ is general, Φ_q satisfies the condition (A2). Since the topological discriminant locus $\Sigma_{\Phi_q} \subset B \times A_q$ of Φ_q is contained in the intersection of $B \times A_q \subset B \times P$ and $\Sigma_\rho \subset B \times P$, the condition (C3) implies that Φ_q satisfies the condition (A4), when $q \in Q$ is general.

In order to check the condition (A3), we put

$$\Gamma := \{ \, (a,p) \in B \times P \;\mid\; \rho^{-1}(a,p) \setminus (\rho^{-1}(a,p) \cap \operatorname{Sing} \rho) = \emptyset \, \}.$$

Since Γ is the the complement in $B \times P$ to $\rho(\mathcal{H} \setminus \operatorname{Sing} \rho)$, it is a constructible set. Hence it is a finite disjoint union of locally Zariski closed subsets. Let

$$\mathrm{pr}_B \,:\, B \times P \,\to\, B$$

be the natural projection. We define $\overline{\Gamma}$ to be the Zariski closure in $B \times P$ of Γ, and Ξ_B to be the Zariski closed subset

$$\{ \, a \in B \;\mid\; \dim (\mathrm{pr}_B^{-1}(a) \cap \overline{\Gamma}) \geq \dim P - 1 \, \}$$

of B. In order to show that Φ_q satisfies the condition (A3) for a general $q \in Q$, it is enough to prove that Ξ_B is of codimension ≥ 2 in B. Indeed, let

$$\beta \,:\, \widetilde{P} \,\to\, P$$

be the blowing up of P at $\infty \in P$, and let

$$\tilde{\alpha} \,:\, \widetilde{P} \,\to\, Q$$

be the natural projection, which coincides with α on $P^\times$. We denote by

$$\overline{\Gamma}^\sim \;\subset\; B \times \widetilde{P}$$

the strict transform of $\overline{\Gamma}$ by $\mathrm{id}_B \times \beta$. We put

$$\overline{\Gamma}_Q := (\mathrm{id}_B \times \tilde{\alpha})(\overline{\Gamma}^\sim) \;\subset\; B \times Q.$$

Since $\mathrm{id}_B \times \tilde{\alpha}$ is a smooth projective morphism of relative dimension 1, $\overline{\Gamma}_Q$ is a Zariski closed subset of $B \times Q$, and if $a \in B \setminus \Xi_B$, then $(\{a\} \times Q) \cap \overline{\Gamma}_Q$ is of codimension≥ 1 in $\{a\} \times Q$. Because x_i is a general point of the hypersurface $\Sigma_f^{(i)}$ of B, this point x_i is not contained in Ξ_B. Therefore $\{x_i\} \times Q$ is not contained in $\overline{\Gamma}_Q$. Let q be a general point of Q. Then we have $(x_i, q) \notin \overline{\Gamma}_Q$. Hence there exist open neighborhoods W of x_i in B and W' of q in Q such that

$$(W \times W') \;\cap\; \overline{\Gamma}_Q = \emptyset.$$

This implies

$$(W \times \tilde{\alpha}^{-1}(W')) \;\cap\; \overline{\Gamma}^\sim = \emptyset.$$

In particular, we have

$$(W \times A_q) \;\cap\; \Gamma = \emptyset.$$

Since Φ_q is the pull-back of ρ by the inclusion $B \times A_q \hookrightarrow B \times P$, the fiber of Φ_q over any point of $W \times A_q$ possesses at least one smooth point. Hence Φ_q satisfies the condition (A3).

Now we assume that Ξ_B is of codimension ≤ 1 in B, and derive a contradiction. By the assumption, there exists an irreducible locally Zariski closed subset Γ' of $B \times P$ contained in Γ such that its Zariski closure $\overline{\Gamma}'$ has the following property; $\mathrm{pr}_B(\overline{\Gamma}')$ is of codimension ≤ 1 in B, and a general fiber of

$$\mathrm{pr}_B \mid \overline{\Gamma}' \,:\, \overline{\Gamma}' \,\to\, \mathrm{pr}_B(\overline{\Gamma}'),$$

which is regarded as a Zariski closed subset of P, is of codimension ≤ 1.

We fix a general point a_0 of $\mathrm{pr}_B(\overline{\Gamma}')$, and let $\overline{\Gamma}'_0$ be an irreducible component of $(\mathrm{pr}_B \mid \overline{\Gamma}')^{-1}(a_0)$. Then $\overline{\Gamma}'_0$ is of codimension ≤ 1 in P. We write

$$g_0 \,:\, X \,\to\, M \quad \text{and} \quad \bar{g}_0 \,:\, \overline{X} \,\to\, M$$

instead of g_{a_0} and $\bar{g}_{a_0}$ for simplicity. Since Γ' is locally Zariski closed, it is Zariski open dense in $\overline{\Gamma}'$. Let p_0 be a general point of $\overline{\Gamma}'_0$. Then (a_0, p_0) be a point of $\Gamma' \subset \Gamma$, and hence $\rho^{-1}(a_0, p_0)$ is either empty or contained in $\mathrm{Sing}\,\rho$. Suppose that $\rho^{-1}(a_0, p_0) = \emptyset$. We put

$$\overline{Y}_0 := \bar{g}_0(\overline{X}).$$

We denote by C the Zariski closure in $\overline{Y}_0$ of

$$\overline{Y}_0 \setminus g_E(F_{a_0}) = (\overline{Y}_0 \cap D_\infty) \cup (\overline{Y}_0 \setminus g_0(X)).$$

Since $\mathrm{pr}_B(\overline{\Gamma}')$ is of codimension ≤ 1 in B, the condition (C1) implies $\overline{Y}_0 \not\subset D_\infty$. Hence C is a proper Zariski closed subset of $\overline{Y}_0$. Let $C_1, \dots, C_k$ be the irreducible components of C with codimension 1 in $\overline{Y}_0$. We put

$$\Delta_j := \{\, p \in P \ \mid\ C_j \subset D_p \,\}.$$

By the condition (C2), we have $\dim C_j \ge 1$. Hence Δ_j is a linear subspace of codimension ≥ 2 in P. For $p \in P$, $\rho^{-1}(a_0, p)$ is empty only if $\overline{Y}_0 \cap D_p \subseteq C$, which is equivalent to

$$\overline{Y}_0 \cap D_p \ \subseteq\ \cup_{j=1}^{k} C_j. \tag{4.1}$$

Note that, if (4.1) holds, then there exists at least one C_j among $C_1, \dots, C_k$ such that $C_j \subset \overline{Y}_0 \cap D_p$. Therefore we have $\overline{\Gamma}'_0 \subset \cup_j \Delta_j$, which contradicts to the fact that $\overline{\Gamma}'_0$ is of codimension ≤ 1 in P. Therefore $\rho^{-1}(a_0, p_0)$ is non-empty and hence is contained in $\mathrm{Sing}\,\rho$ for a general $p_0 \in \overline{\Gamma}'_0$.

We put

$$\overline{\mathcal{H}}_0 := \{\, (x, p) \in \overline{X} \times P \ \mid\ \bar{g}_0(x) \in D_p \,\},$$

and let

$$\bar{\rho}_0 \,:\, \overline{\mathcal{H}}_0 \,\to\, P \quad \text{and} \quad \bar{\sigma}_0 \,:\, \overline{\mathcal{H}}_0 \,\to\, \overline{X}$$

be the projections. Note that $\bar{\sigma}_0$ is a smooth projective morphism with fibers being hyperplanes of P. We put

$$\overline{\mathcal{S}}_0 := \mathrm{Sing}\,\bar{\rho}_0.$$

By the consideration above, $\bar{\rho}_0^{-1}(p_0) \cong \bar{\rho}^{-1}(a_0, p_0)$ has at least one irreducible component that is contained in $\overline{\mathcal{S}}_0$ for a general $p_0 \in \overline{\Gamma}'_0$. Since $\overline{\Gamma}'_0$ is of codimension ≤ 1 in P, there exists an irreducible component $\overline{\mathcal{S}}'_0$ of $\overline{\mathcal{S}}_0$ with codimension 1 in $\overline{\mathcal{H}}_0$ such that $\overline{\mathcal{S}}'_0$ is contained in $\overline{\mathcal{S}}_0 \cap \bar{\rho}_0^{-1}(\overline{\Gamma}'_0)$, and that $\bar{\rho}_0(\overline{\mathcal{S}}'_0)$ coincides with $\overline{\Gamma}'_0$.

Let (x, p) be a point of $\overline{\mathcal{H}}_0$. Then the Zariski tangent space to $\bar{\rho}_0^{-1}(p) \cong \bar{\rho}^{-1}(a_0, p)$ at (x, p) is canonically identified with the subspace $(d\bar{g}_0)_x^{-1}(T_{\bar{g}_0(x)} D_p)$ of $T_x \overline{X}$ with codimension ≤ 1. Hence (x, p) is contained in $\overline{\mathcal{S}}_0$ if and only if

$$\operatorname{Im}(d\bar{g}_0)_x \ \subseteq\ T_{\bar{g}_0(x)} D_p.$$

For non-negative integers ν, we put

$$(\overline{X})_\nu := \{\ x \in \overline{X} \ \mid\ \operatorname{rank}(d\bar{g}_0)_x \le \nu\ \}.$$

If $x \in (\overline{X})_0$, then $\bar{\sigma}_0^{-1}(x) \cap \overline{\mathcal{S}}_0$ coincides with $\bar{\sigma}_0^{-1}(x)$, which is isomorphic to the hyperplane

$$H(x) := \{\ p \in P \ \mid\ \bar{g}_0(x) \in D_p\ \}$$

of P. We also have

$$\begin{aligned} x \in \overline{X} \setminus (\overline{X})_0 \quad &\Longrightarrow \quad \bar{\sigma}_0^{-1}(x) \cap \overline{\mathcal{S}}_0 \text{ is of codimension} \ge 1 \text{ in } \bar{\sigma}_0^{-1}(x), \\ x \in \overline{X} \setminus (\overline{X})_1 \quad &\Longrightarrow \quad \bar{\sigma}_0^{-1}(x) \cap \overline{\mathcal{S}}_0 \text{ is of codimension} \ge 2 \text{ in } \bar{\sigma}_0^{-1}(x). \end{aligned}$$

By the condition (C2), $(\overline{X})_1$ is a proper Zariski closed subset of $\overline{X}$. Since $\overline{\mathcal{S}}'_0$ is of codimension 1 in $\overline{\mathcal{H}}_0$, $\bar{\sigma}_0(\overline{\mathcal{S}}'_0)$ must be contained in $(\overline{X})_0$, and the fiber of

$$\bar{\sigma}_0 \,|\, \overline{\mathcal{S}}'_0 \,:\, \overline{\mathcal{S}}'_0 \,\to\, \bar{\sigma}(\overline{\mathcal{S}}'_0)$$

over an arbitrary point $x \in \bar{\sigma}_0(\overline{\mathcal{S}}'_0)$ coincides with $\bar{\sigma}_0^{-1}(x) \cong H(x)$.

Let (x_1, p_1) be a general point of $\overline{\mathcal{S}}'_0$. Then p_1 is a general point of the linear system $H(x_1)$. Since $\dim H(x_1) > 0$, Bertini's theorem implies that the divisor

$$\rho^{-1}(a_0, p_1) = g_0^{-1}(D_{p_1})$$

of X has at least one smooth point. On the other hand, because p_1 is a general point of $\bar{\rho}_0(\overline{\mathcal{S}}'_0) = \overline{\Gamma}'_0$, the divisor $\rho^{-1}(a_0, p_1)$ must be contained in $\operatorname{Sing}\rho$. Thus we get a contradiction. $\square$

5. Action of an algebraic group

Let $\overline{X}$ be a smooth irreducible projective *surface*, and W a reduced divisor of $\overline{X}$ (possibly empty). We put

$$X := \overline{X} \setminus W.$$

Let M be a smooth irreducible projective variety on which a connected algebraic group G acts from left, and let D be a very ample divisor of M. Suppose that a morphism

$$\bar{\phi} \,:\, \overline{X} \,\to\, M$$

is given. We denote by $\phi : X \to M$ the restriction of $\bar{\phi}$ to X. For $\gamma \in G$, let

$${}^{\gamma}\bar{\phi} \,:\, \overline{X} \,\to\, M \quad \text{and} \quad {}^{\gamma}\phi \,:\, X \,\to\, M$$

be the composites of $\bar{\phi}$ and ϕ with the action $\gamma : M \to M$ of γ, respectively. We assume that there exists at least one $\gamma \in G$ such that ${}^{\gamma}\bar{\phi}(\overline{X}) \not\subset D$. Putting

$$B := G,$$

we obtain a morphism

$$\bar{g} \,:\, B \times \overline{X} \,\to\, M$$

defined by $\bar{g}(\gamma, x) := {}^{\gamma}\bar{\phi}(x)$. We put

$$Z := (B \times W) + \bar{g}^{-1}(D), \quad E := (B \times \overline{X}) \setminus Z,$$

and let $f : E \to B$ be the projection. By the assumption above, $\bar{g}^{-1}(D)$ is a divisor of $B \times \overline{X}$, and hence f is a dominant morphism. By definition, we have a natural identification

$$F_\gamma := f^{-1}(\gamma) = {}^\gamma\phi^{-1}(M \setminus D)$$

for any $\gamma \in G = B$. We put

$$\overline{Y} := \bar{\phi}(\overline{X}),$$

and equip $\overline{Y}$ with the reduced structure. For $\gamma \in G$, let $(\gamma(\overline{Y}) \cap D)^\circ$ denote the Zariski open subset of $\gamma(\overline{Y}) \cap D$ consisting of all points $y \in \gamma(\overline{Y}) \cap D$ at which $\gamma(\overline{Y})$ and D are smooth and intersecting transversely. We then put

$$\operatorname{Sing}(\gamma(\overline{Y}) \cap D) := (\gamma(\overline{Y}) \cap D) \setminus (\gamma(\overline{Y}) \cap D)^\circ.$$

As before, let $\Sigma_f \subset B = G$ be the topological discriminant locus of f, and $\Sigma_f^{(i)}$ an irreducible component of Σ_f with codimension 1 in B. Let γ_i be a general point of $\Sigma_f^{(i)}$, and λ_i a simple loop around in $\Sigma_f^{(i)}$ in a sufficiently small open ball U_i around γ_i with the base point $a_i \in U_i \setminus (U_i \cap \Sigma_f)$.

Proposition 5.1. *Suppose that the following conditions hold;*

- (G1) $\dim \overline{Y} = 2$, *so that $\bar{\phi}$ is quasi-finite onto its image,*
- (G2) *for any irreducible Zariski closed subset C of $\overline{Y}$ with $\dim C > 0$, the Zariski closed subset $\{\gamma \in G \mid \gamma(C) \subset D\}$ of G is of codimension ≥ 2, and*
- (G3) *the locus $\{\gamma \in G \mid \dim \operatorname{Sing}(\gamma(\overline{Y}) \cap D) > 0\}$ is contained in a Zariski closed subset of codimension ≥ 2 in G.*

Then the local monodromy μ_i on $\pi_1({}^{a_i}\phi^{-1}(M \setminus D))$ along the loop λ_i is defined and trivial.

Remark 5.2. Suppose that D has a non-reduced irreducible component D'. By the definition of $\operatorname{Sing}(\gamma(\overline{Y}) \cap D)$, if the conditions (G1) and (G3) are satisfied, then we have $\gamma(\overline{Y}) \cap D' = \emptyset$ for a general $\gamma \in G$. In particular, D' is not ample.

Proof. As in Situation (C) in the previous section, we denote by P the the projective space $\mathbb{P}_* H^0(M, \mathcal{O}_M(D))$, and by D_p the divisor corresponding to a point $p \in P$. We put

$$\overline{\mathcal{H}} := \{\ (\gamma, x, p) \in B \times \overline{X} \times P \ \mid\ {}^\gamma\bar{\phi}(x) \in D_p\ \}, \quad \mathcal{H} := \overline{\mathcal{H}} \cap (E \times P),$$

and let

$$\bar{\rho} : \overline{\mathcal{H}} \to B \times P, \quad \rho : \mathcal{H} \to B \times P$$

be the natural projections. We will check that, in this situation, the three conditions in Proposition 4.5 are satisfied. The condition (C1) follows from the conditions (G1) and (G2). The condition (C2) follows from the condition (G1). Therefore all we have to show is that the topological discriminant locus $\Sigma_\rho \subset B \times P$ of ρ does not contain $\{\gamma_i\} \times P$.

Let $\Sigma_{\bar{\rho}} \subset B \times P$ be the topological discriminant locus of $\bar{\rho}$. By Bertini's theorem and the condition (G1), the general fiber of $\bar{\rho}$ is a connected compact Riemann surface, and, for any $\gamma \in B$, there exists a point $p \in P$ such that ${}^\gamma\bar{\phi}^{-1}(D_p)$ is a smooth irreducible curve on $\overline{X}$. Hence the intersection of $\Sigma_{\bar{\rho}}$ with $\{\gamma\} \times P$ is of codimension ≥ 1 in $\{\gamma\} \times P$ for any $\gamma \in B$. On the other hand, the general fiber of ρ is a punctured Riemann surface. Hence, if $(\gamma_0, p_0) \in (B \times P) \setminus \Sigma_{\bar{\rho}}$, then $(\gamma_0, p_0)$is not contained in Σ_ρ if and only if the number of the punctured points $\bar{\rho}^{-1}(\gamma, p) \setminus \rho^{-1}(\gamma, p)$ on the compact Riemann surface $\bar{\rho}^{-1}(\gamma, p)$ does not vary locally around (γ_0, p_0).

As before, we write D_∞ instead of D. We choose a general point $p_0 \in P$, and write D_0 instead of D_{p_0}. We have $(\gamma_i, p_0) \notin \Sigma_{\bar\rho}$. Let $U(\gamma_i)$ and $U(p_0)$ be sufficiently small open neighborhoods of γ_i in B and of p_0 in P, respectively. We put

$$U := U(\gamma_i) \times U(p_0).$$

For $(\gamma, p) \in U$, we put

$$T_W(\gamma, p) := W \cap {}^\gamma\bar\phi^{-1}(D_p) \quad \text{and} \quad T_\infty(\gamma, p) := {}^\gamma\bar\phi^{-1}(D_\infty \cap D_p).$$

Then we have

$$\bar\rho^{-1}(\gamma, p) \setminus \rho^{-1}(\gamma, p) = T_W(\gamma, p) \ \cup\ T_\infty(\gamma, p).$$

Therefore, in order to show that the condition (C3) is satisfied, it suffices to prove that the cardinality $|T_W(\gamma, p) \cup T_\infty(\gamma, p)|$ is constant when (γ, p) moves on U.

First remark that the condition (G2) implies the following. If R is a Zariski closed subset of $\overline{X}$ with $\dim R \le 1$, then we have

$$\gamma_i(\bar\phi(R)) \cap D_\infty = \emptyset \quad \text{or} \quad \dim(\gamma_i(\bar\phi(R)) \cap D_\infty) = 0.$$

Indeed, if $\bar\phi(R)$ has an irreducible component $\bar\phi(R)'$ with $\dim \bar\phi(R)' = 1$, then $\gamma_i(\bar\phi(R)') \not\subset D_\infty$ by the condition (G2), because γ_i is a general point of the irreducible hypersurface $\Sigma_f^{(i)}$ of G.

By definition, $T_W(\gamma, p)$ and $T_\infty(\gamma, p)$ are disjoint if and only if

$$\gamma(\bar\phi(W)) \cap D_\infty \cap D_p \ = \ \emptyset. \tag{5.1}$$

By the remark above, we have

$$\gamma_i(\bar\phi(W)) \cap D_\infty = \emptyset \quad \text{or} \quad \dim(\gamma_i(\bar\phi(W)) \cap D_\infty) = 0.$$

Since p_0 is general in P, we see that (5.1) holds for any $(\gamma, p) \in U$.

Let $W_1, \dots, W_k$ be the irreducible components of W such that

$$\dim \bar\phi(W_\mu) = 1 \qquad (\mu = 1, \dots, k),$$

and let $W_{k+1}, \dots, W_m$ be the other irreducible components. Obviously we have

$$\dim \bar\phi(W_\nu) = 0 \qquad (\nu = k+1, \dots, m).$$

For $\mu = 1, \dots, k$, let e_μ be the mapping degree of

$$\bar\phi \,|\, W_\mu \,:\, W_\mu \,\to\, \bar\phi(W_\mu),$$

and let $\bar\phi(W_\mu)^\circ$ be the Zariski open dense subset of $\bar\phi(W_\mu)$ such that $\bar\phi(W_\mu)$ is smooth at every point of $\bar\phi(W_\mu)^\circ$, and that $\bar\phi \,|\, W_\mu$ is étale over $\bar\phi(W_\mu)^\circ$. We denote by d_μ the degree of the curve $\bar\phi(W_\mu)$ (with the reduced structure) with respect to the very ample line bundle $\mathcal{O}_M(D_\infty)$. Let $\gamma \in G$ be an arbitrary element. If $p \in P$ is general, D_p intersects $\gamma(\bar\phi(W_\mu))$ transversely at distinct d_μ points in $\gamma(\bar\phi(W_\mu)^\circ)$. Moreover we have

$$D_p \,\cap\, \gamma(\bar\phi(W_\nu)) \ = \ \emptyset \quad (\nu = k+1, \dots, m).$$

Since p_0 is general, $|T_W(\gamma, p)|$ is constantly equal to $\sum_{\mu=1}^k d_\mu e_\mu$ for any $(\gamma, p) \in U$.

There exists a Zariski open dense subset $\overline{Y}^\circ$ of $\overline{Y}$ such that

$$\bar\phi \,|\, \bar\phi^{-1}(\overline{Y}^\circ) \,:\, \bar\phi^{-1}(\overline{Y}^\circ) \,\to\, \overline{Y}^\circ$$

is étale. Let ϵ be the mapping degree of this étale morphism. We put

$$Q := \overline{Y} \setminus \overline{Y}^\circ,$$

which is a Zariski closed subset of $\overline{Y}$ with $\dim Q \le 1$. Then, by the remark above, we have

$$\gamma_i(\bar{\phi}(Q)) \cap D_\infty = \emptyset \quad \text{or} \quad \dim(\gamma_i(\bar{\phi}(Q)) \cap D_\infty) = 0.$$

Since $p_0 \in P$ is general, we have

$$\gamma(Q) \cap D_\infty \cap D_p = \emptyset \quad \text{for any } (\gamma, p) \in U. \tag{5.2}$$

By the condition (G3), we have

$$\mathrm{Sing}(\gamma_i(\overline{Y}) \cap D_\infty) = \emptyset \quad \text{or} \quad \dim(\mathrm{Sing}(\gamma_i(\overline{Y}) \cap D_\infty)) = 0.$$

Since $p_0 \in P$ is general, we have

$$\mathrm{Sing}(\gamma_i(\overline{Y}) \cap D_\infty) \;\cap\; D_0 = \emptyset,$$

and the intersection $\gamma_i(\overline{Y}) \cap D_\infty \cap D_0$ is transverse; that is, at every point $y \in \gamma_i(\overline{Y}) \cap D_\infty \cap D_0$, all of $\gamma_i(\overline{Y})$, D_∞ and D_0 are smooth, and the intersection of their Zariski tangent spaces in $T_y M$ is of dimension 0. Hence, for any $(\gamma, p) \in U$,

$$\gamma(\overline{Y}) \cap D_\infty \cap D_p$$

consists of distinct δ points, where δ is the degree of $\overline{Y}$ with respect to the line bundle $\mathcal{O}_M(D_\infty)$, and over each point of this intersection, ${}^\gamma\bar{\phi} : \overline{X} \to \gamma(\overline{Y})$ is étale by (5.2). Hence $|T_\infty(\gamma, p)|$ is constantly equal to $\delta\epsilon$ when (γ, p) moves on U.

Combining the previous three paragraphs, we conclude that the number of the punctured points $T_W(\gamma, p) \cup T_\infty(\gamma, p)$ is constant locally around (γ_i, p_0), and hence (γ_i, p_0) is not contained in Σ_ρ. □

Remark 5.3. Proposition 5.1 plays a crucial role in the proof of Zariski hyperplane section theorem for Grassmannian varieties in [12].

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