INVOLUTIONS IN THE AFFINE CONWAY GROUP

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Abstract. The affine Conway group is the group of affine isometries of the Leech lattice. This group is isomorphic to the automorphism group of a standard fundamental domain for the action of the Weyl group on the even hyperbolic lattice $\mathrm{II}_{1,25}$ of rank 26.

In this paper, we show that the affine Conway group has exactly nine conjugacy classes of involutions. We investigate their properties and show that, among these nine classes, one class can be regarded as an analogue of the class of Enriques involutions of K3 surfaces.

Motivated by possible applications to K3 and Enriques surfaces, we investigate in detail the orthogonal groups of the hyperbolic lattices arising as the invariant sublattices of involutions in $\mathrm{II}_{1,25}$. This computation is carried out using the Borcherds method. Unlike the examples considered previously, the induced chambers possess infinitely many walls.

1. Introduction

1.1. **Main results.** The *affine Conway group* $\cdot\infty$ is the group of all affine isometries of the Leech lattice Λ, and is equal to the semidirect product $\cdot 0 \ltimes \Lambda$ of $\cdot 0$ and Λ, where $\cdot 0 := \mathrm{O}(\Lambda)$ is the orthogonal group of Λ acting on Λ from the right, and $(g, \tau) \in \cdot 0 \ltimes \Lambda$ acts on Λ as

$$\lambda \;\mapsto\; \lambda^g + \tau.$$

In this paper, we classify the involutions in $\cdot\infty$ and investigate their properties.

An interesting outcome of our study is that several distinguished lattices appear naturally. In particular, the unimodular hyperbolic lattices $\mathrm{I}_{1,9}$ and $\mathrm{II}_{1,9}$ of rank 10 arise as the invariant sublattices of involutions of the natural action of $\cdot\infty$ on the even hyperbolic lattice $L_{26} := \mathrm{II}_{1,25}$ of rank 26.

Another interesting aspect of this work lies in its connection with algebraic geometry. Via the Borcherds method [1, 2], the action of $\cdot\infty$ on L_{26} has found applications in the study of automorphism groups of K3 surfaces, Enriques surfaces, and, more recently, irreducible holomorphic symplectic (IHS) manifolds (see, for example, [12]). Our results can be interpreted as geometric results concerning some *virtual* K3 surface $\mathbb{X}$ and *virtual* Enriques surface $\mathbb{Y}$. See Section 1.2.

Our first main result is as follows.

Theorem 1.1. *There exist exactly* $9 = 1+1+3+4$ *conjugacy classes of involutions in the affine Conway group* $\cdot\infty$.

Key words and phrases. Conway group, Leech lattice, hyperbolic lattice, involution.

The natural homomorphism $\cdot\infty \to \cdot 0$ maps the conjugacy class $[\gamma]$ of an involution $\gamma = (g, \tau)$ in $\cdot\infty$ to the conjugacy class $[g]$ of the involution g in $\cdot 0$. Conway [8, Chapter 10, Section 3.7] showed that there exist exactly four conjugacy classes $[\varepsilon_8], [\varepsilon_{12}], [\varepsilon_{16}], [\varepsilon_{24}]$ of involutions in $\cdot 0$, where ε_n is an involution such that the rank of the invariant sublattice $\mathrm{Ker}(\mathrm{id} - \varepsilon_n)$ of the Leech lattice Λ is $24 - n$. The decomposition $9 = 1 + 1 + 3 + 4$ in Theorem 1.1 is intended to show the number of conjugacy classes of involutions in $\cdot\infty$ over each of the conjugacy classes of involutions in $\cdot 0$.

A lattice of rank $n > 1$ is said to be *hyperbolic* if its associated real quadratic space has signature $(1, n-1)$. (Since our applications are to algebraic surfaces, we use the convention $(1, n-1)$ instead of $(n-1, 1)$.) A *positive cone* $\mathcal{P}$ of a hyperbolic lattice L is one of the two connected components of $\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}$.

The group $\cdot\infty$ was considered by Conway [6] in his study of the orthogonal group of an even unimodular hyperbolic lattice $\mathrm{II}_{1,25}$. Note that $\mathrm{II}_{1,25}$ is isomorphic to

$$L_{26} := U \oplus \Lambda(-1),$$

where U is the hyperbolic plane with the basis $\boldsymbol{w}_0, \boldsymbol{w}_1$ satisfying

$$\langle \boldsymbol{w}_0, \boldsymbol{w}_0 \rangle = \langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle = 0, \quad \langle \boldsymbol{w}_0, \boldsymbol{w}_1 \rangle = 1, \tag{1.1}$$

and $\Lambda(-1)$ is the *negative-definite* Leech lattice. Let $\mathcal{P}_{26}$ be the positive cone of L_{26} containing $\boldsymbol{w}_0$ in its closure, let $\mathrm{O}(L_{26}, \mathcal{P}_{26})$ be the stabilizer subgroup of $\mathcal{P}_{26}$ in $\mathrm{O}(L_{26})$, and let $W(L_{26})$ denote the Weyl group of L_{26} (see Section 2.1 for the terminology). Then $\cdot\infty$ is embedded into $\mathrm{O}(L_{26}, \mathcal{P}_{26})$ as the stabilizer of $\boldsymbol{w}_0$. Conway [6] showed that this subgroup is the automorphism group of a standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}_{26}$. In particular, we have

$$\mathrm{O}(L_{26}, \mathcal{P}_{26}) = W(L_{26}) \rtimes \cdot\infty.$$

To state our next main result, we use the following notation. Let M be a lattice. For an involution $g \in \mathrm{O}(M)$, we put

$$M(g, +) := \{ x \in M \mid x^g = x \}, \quad M(g, -) := \{ x \in M \mid x^g = -x \}. \tag{1.2}$$

For a non-zero integer m, let $M(m)$ denote the lattice obtained from M by multiplying its bilinear form by m. For a group G and an element $g \in G$, we denote by $\mathrm{Cen}(G, g)$ the centralizer of g in G. Then Theorem 1.1 is refined as follows:

Theorem 1.2. *Suppose that $\gamma = (g, \tau)$ is an involution in $\cdot\infty$, and let n be the rank of the anti-invariant sublattice $\Lambda(g, -)$ of the involution $g \in \cdot 0$ on Λ. Then, regarding γ as an involution in $\mathrm{O}(L_{26}, \mathcal{P}_{26})$, we see that the sublattice $L_{26}(\gamma, -)$ of L_{26} is of rank n and is isomorphic to a sublattice of $\Lambda(-1)$, whereas $L_{26}(\gamma, +)$ is a hyperbolic lattice of rank $26 - n$.*

The nine conjugacy classes in Theorem 1.1 are distinguished by the isomorphism classes of $L_{26}(\gamma, +)$ and $L_{26}(\gamma, -)$ as in Table 1.1.

In Table 1.1, we use the following notation.

$\gamma = (g, \tau)$	$L_{26}(\gamma, +)$	$\lvert L_{26}(\gamma, -)_4\rvert$
$\gamma_8 = (\varepsilon_8, 0)$	$U \oplus \Lambda_{16}(-1)$	240
$\gamma_{12} = (\varepsilon_{12}, 0)$	$U \oplus \Sigma_{12}(-2)$	264
$\gamma_{16,1} = (\varepsilon_{16}, 0)$	$U \oplus E_8(-2)$	4320
$\gamma_{16,120} = (\varepsilon_{16}, \tau_{120})$	$\mathrm{I}_{1,9}(2)$	2016
$\gamma_{16,135} = (\varepsilon_{16}, \tau_{135})$	$\mathrm{II}_{1,9}(2)$	2272
$\gamma_{24,0} = (\varepsilon_{24}, 0)$	U	196560
$\gamma_{24,4} = (\varepsilon_{24}, t_4)$	V_4	102352
$\gamma_{24,6} = (\varepsilon_{24}, t_6)$	V_6	98256
$\gamma_{24,8} = (\varepsilon_{24}, t_8)$	V_8	98256

TABLE 1.1. $L_{26}(\gamma, +)$ and $L_{26}(\gamma, -)$

- We denote by $|L_{26}(\gamma, -)_4|$ the size of the set of vectors of norm -4 in the negative-definite lattice $L_{26}(\gamma, -)$. Note that, as a sublattice of $\Lambda(-1)$, the even lattice $L_{26}(\gamma, -)$ has no vectors of norm -2.
- Λ_n is the laminated lattice of rank n. See [8, Chapter 6] for the definition. In particular, we have $\Lambda = \Lambda_{24}$, and Λ_{16} is isomorphic to the Barnes–Wall lattice BW_{16}.
- E_8 is the root lattice of type E_8, that is, a positive-definite even unimodular lattice of rank 8, which is unique up to isomorphism. We have $E_8(2) \cong \Lambda_8$.
- Σ_{12} is a positive-definite *odd* unimodular lattice of rank 12 without vectors of norm 1. Note that Σ_{12} is unique up to isomorphism. See [8, Table 16.7].
- $\mathrm{I}_{1,9}$ is an odd unimodular hyperbolic lattice of rank 10, which is unique up to isomorphism, and is isomorphic to the orthogonal direct sum of a copy of $[1]$ and nine copies of $[-1]$.
- $\mathrm{II}_{1,9}$ is an even unimodular hyperbolic lattice of rank 10, which is unique up to isomorphism, and is isomorphic to $U \oplus E_8(-1)$.
- V_{2m} is a lattice of rank 2 with a Gram matrix

$$\begin{bmatrix} 0 & 2 \\ 2 & -2m \end{bmatrix}. \tag{1.3}$$

  The lattices V_{2m} and $V_{2m'}$ are isomorphic if and only if $m \equiv m' \bmod 2$.

Conway [8, Chapter 10] also calculated the centralizers $\mathrm{Cen}(\cdot 0, g)$ of the involutions g in $\cdot 0$. We extend this result to the calculation of the centralizers $\mathrm{Cen}(\cdot\infty, \gamma)$ of the involutions γ in $\cdot\infty$. See Proposition 3.2.

Using the computation of $\mathrm{Cen}(\cdot\infty, \gamma)$ and the Borcherds method [1, 2], we can investigate the hyperbolic lattice $L_{26}(\gamma, +)$. In this Introduction, we present only the results concerning the lattices $L_{26}(\gamma, +) = \mathrm{I}_{1,9}$ (corresponding to the involution $\gamma_{16,120}$) and $L_{26}(\gamma, +) = \mathrm{II}_{1,9}$ (corresponding to the involution $\gamma_{16,135}$). Let L be a hyperbolic lattice, and let $\mathcal{P}$ be a positive cone of L. Let $\mathcal{F}$ be a locally finite family of hyperplanes in $\mathcal{P}$. An *$\mathcal{F}$-chamber* is a closed polyhedral cone in $\mathcal{P}$ bounded by members of $\mathcal{F}$ whose interior is disjoint from every member of $\mathcal{F}$.

Theorem 1.3. *Let L be either $\mathrm{I}_{1,9}$ or $\mathrm{II}_{1,9}$. Then there exist a subgroup Γ of $\mathrm{O}(L,\mathcal{P})$ and a locally finite family $\mathcal{F}$ of hyperplanes in $\mathcal{P}$ such that the following holds:*

(i) *The index of Γ in $\mathrm{O}(L,\mathcal{P})$ is*

$$[\mathrm{O}(L,\mathcal{P}):\Gamma] = \begin{cases} 240 & \text{if } L = \mathrm{I}_{1,9}, \\ 71145 & \text{if } L = \mathrm{II}_{1,9}. \end{cases}$$

(ii) *The family $\mathcal{F}$ is invariant under the action of Γ, and Γ acts transitively on the set of $\mathcal{F}$-chambers.*

(iii) *Fixing an $\mathcal{F}$-chamber D_0, we see that the stabilizer $\mathrm{Stab}_\Gamma(D_0)$ of D_0 in Γ acts transitively on the set of walls of D_0.*

(iv) *The group $\mathrm{Stab}_\Gamma(D_0)$ is an extension of a finite group by a free abelian group of rank 8.*

Thus we obtained a Γ-invariant tessellation of the cone $\mathcal{P}$ by polyhedral cones.

Corollary 1.4. *Let $g \in \Gamma$ be an element that maps D_0 to an $\mathcal{F}$-chamber adjacent to D_0. Then Γ is generated by $\mathrm{Stab}_\Gamma(D_0)$ together with g.*

As mentioned above, we use the Borcherds method for the proof. So far, this method has been applied only in situations where each $\mathcal{F}$-chamber has finitely many walls. The novelty of our computation is that the $\mathcal{F}$-chambers have infinitely many walls.

1.2. **Motivation from algebraic geometry.** The Borcherds method [1, 2] is applied to the computation of automorphism groups of K3 surfaces and Enriques surfaces. The first geometric application of this method was given by Kondo [11] to Jacobian Kummer surfaces. See [18, 19] for generalization of the method and computational details, and [3, 4] for applications to Enriques surfaces. The Borcherds method is based on the result of Conway [6] on the hyperbolic lattice L_{26}. We can interpret Conway's result as a statement about the nef-and-big cone of a *virtual* K3 surface $\mathbb{X}$ whose Néron–Severi lattice $S_{\mathbb{X}}$ is isomorphic to L_{26}. In fact, in [20], we regard the Leech lattice as the Mordell–Weil lattice [23] of an elliptic fibration of this *non-existing* $\mathbb{X}$, and obtained a new construction of the Leech lattice by considering other elliptic fibrations on $\mathbb{X}$. In this context, if we ignore the question of what the period of $\mathbb{X}$ should be, we can consider $\cdot\infty$ as the automorphism group of $\mathbb{X}$, and our result in this paper can be regarded as the classification of involutions of $\mathbb{X}$. Then the conjugacy class of the involution $\epsilon := \gamma_{16,135}$ can be regarded as the conjugacy class of Enriques involutions of $\mathbb{X}$, because of the following result of Keum [10].

Proposition 1.5. *Let X be a complex $K3$ surface with the Néron–Severi lattice S_X. Then an involution ι of X is an Enriques involution if and only if the invariant sublattice $S_X(\iota,+)$ of the action of ι on S_X is isomorphic to $\mathrm{II}_{1,9}(2)$, and its orthogonal complement $S_X(\iota,-)$ contains no (-2)-vectors.* □

Let $\mathbb{Y} = \mathbb{X}/\langle\epsilon\rangle$ denote the *virtual* Enriques surface corresponding to the Enriques involution ϵ. The inclusion

$$L_{26}(\epsilon, +) = S_{\mathbb{Y}}(2) \cong \mathrm{II}_{1,9}(2) \quad\hookrightarrow\quad S_{\mathbb{X}} = L_{26}$$

is the embedding of the Néron–Severi lattices induced by the étale double covering $\mathbb{X} \to \mathbb{Y}$. This embedding has already appeared in [4] as one of the 17 embeddings of $\mathrm{II}_{1,9}(2)$ into L_{26}. The embeddings in [4] other than this embedding have applications in the computation of the automorphism groups of various Enriques surfaces in [3]. We hope that this new embedding $L_{26}(\epsilon, +) \hookrightarrow L_{26}$ has some new geometric and/or lattice theoretic applications.

Since the seminal work of Nikulin [15], involutions of K3 surfaces have been extensively studied. The problem of enumerating all Enriques involutions (up to conjugacy) of a given K3 surface was studied by several authors. For example, Ohashi [16], building on the work of Kondo [11], classified all Enriques involutions on Jacobian Kummer surfaces. See also [22].

See Remark 6.9 for a further exploration of this heuristic analogy.

1.3. **Plan of the paper.** In Section 2, we fix notation and terminology concerning lattices, review the theory of discriminant forms due to Nikulin [14], and explain some computational methods. In Section 3, we review Conway's results [8, Chapter 10] on $\cdot 0$ and present the nine conjugacy classes of involutions in $\cdot\infty$. Theorem 1.1 is proved in this section. In Section 4, we investigate the action of involutions of $\cdot\infty$ on L_{26}, and prove Theorem 1.2. In Section 5, we apply the Borcherds method to the invariant sublattices of the involutions of $\cdot\infty$ in L_{26}. In Section 6, we present the detailed computational results obtained by the Borcherds method. Theorem 1.3 and similar results for other invariant lattices are proved in this section.

We have used GAP [9] for the proof of our main results. Computational data are available from [21].

2. Preliminaries

In Section 2.1, we fix notation and terminology about lattices. In Section 2.2, we apply the theory of discriminant forms to involutions in the orthogonal group of a lattice. In Section 2.3, we explain some computational methods for finite groups. Since this section is preliminary and technical, the reader may skip it on a first reading and return to it later when necessary.

2.1. **Notation and terminology about lattices.** For a lattice L and a subset A of $L \otimes \mathbb{R}$, we put

$$\mathrm{O}(L, A) := \{\, g \in \mathrm{O}(L) \mid A^g = A \,\}.$$

As was already noted, we say that a lattice of rank $n > 1$ is *hyperbolic* if the signature of the associated real quadratic space is $(1, n-1)$. For a hyperbolic lattice L with a positive cone $\mathcal{P}$, the group $\mathrm{O}(L, \mathcal{P})$ is a subgroup of the group of isometries of the associated hyperbolic space.

A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r\rangle = -2$. It defines a reflection $s_r \in \mathrm{O}(L)$ by $x^{s_r} := x + \langle x, r\rangle r$. For an even hyperbolic lattice L with a positive cone $\mathcal{P}$, the *Weyl group* $W(L)$ is the subgroup of $\mathrm{O}(L,\mathcal{P})$ generated by all reflections s_r with respect to (-2)-vectors r. For a vector $v \in L\otimes\mathbb{Q}$ with $\langle v,v\rangle<0$, we put

$$(v)^{\perp} := \{\, x\in\mathcal{P} \mid \langle x,v\rangle = 0 \,\}.$$

If r is a (-2)-vector, then $(r)^{\perp}$ is the mirror of s_r.

Definition 2.1. A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \ \setminus \ \bigcup (r)^{\perp},$$

where the union is taken over all (-2)-vectors r.

We have $\mathrm{O}(L,\mathcal{P}) = W(L) \rtimes \mathrm{O}(L,N)$ for any standard fundamental domain N of the action of $W(L)$ on $\mathcal{P}$.

2.2. **Discriminant forms and involutions.** Let M be an even lattice, and let $M^{\vee} := \mathrm{Hom}(M,\mathbb{Z})$ denote its *dual lattice*. Identifying $M^{\vee}$ as a subset of $M\otimes\mathbb{Q}$, we let $\mathrm{O}(M)$ act on $M^{\vee}$ from the right (*not* contragrediently). The intersection form $M\times M\to\mathbb{Z}$ extends to an intersection form $M^{\vee}\times M^{\vee}\to\mathbb{Q}$. We define

$$q_M \ : \ M^{\vee}/M \to \mathbb{Q}/2\mathbb{Z}$$

by $q_M(v \bmod M) := \langle v,v\rangle \bmod 2\mathbb{Z}$ for $v\in M^{\vee}$. We call $M^{\vee}/M$ the *discriminant group* of M, and q_M the *discriminant form* of M. We denote by $\mathrm{O}(q_M)$ the automorphism group of the finite quadratic form q_M. Then we have a natural homomorphism

$$\eta_M \ : \ \mathrm{O}(M) \to \mathrm{O}(q_M).$$

In this paper, we encounter many finite quadratic forms such that the underlying groups are direct sums of finite copies of $\mathbb{Z}/2\mathbb{Z}$. We use the following notation:

$$\begin{aligned} &u := \left((\mathbb{Z}/2\mathbb{Z})^2,\ \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \right), \quad v := \left((\mathbb{Z}/2\mathbb{Z})^2,\ \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \right), \\ &a := (\mathbb{Z}/2\mathbb{Z},\ [1/2])\,, \quad b := (\mathbb{Z}/2\mathbb{Z},\ [3/2])\,. \end{aligned} \tag{2.1}$$

Remark 2.2. We put $q_{[\mu,\nu,\alpha,\beta]} := u^{\mu}\oplus v^{\nu}\oplus a^{\alpha}\oplus b^{\beta}$. Then we have the following isomorphisms of finite quadratic forms:

$$\begin{aligned} q_{[2,0,0,0]} &\cong q_{[0,2,0,0]}, \quad q_{[0,0,4,0]} \cong q_{[0,0,0,4]}, \\ q_{[1,0,1,0]} &\cong q_{[0,0,2,1]}, \quad q_{[1,0,0,1]} \cong q_{[0,0,1,2]}, \\ q_{[0,1,1,0]} &\cong q_{[0,0,0,3]}, \quad q_{[0,1,0,1]} \cong q_{[0,0,3,0]}. \end{aligned}$$

We apply Nikulin's theory of discriminant forms [14] to the study of involutions in $\mathrm{O}(M)$. Let $\varepsilon\in\mathrm{O}(M)$ be an involution, and let $M_+ := M(\varepsilon,+)$ and $M_- := M(\varepsilon,-)$ be the sublattices defined by (1.2). Then we have an embedding

$$\mathrm{Cen}(\mathrm{O}(M),\varepsilon) \ \hookrightarrow \ \mathrm{O}(M_+)\times\mathrm{O}(M_-) \tag{2.2}$$

by the restrictions $g\mapsto (g|M_+, g|M_-)$. Assume that M is unimodular. Then the submodule

$$M/(M_+\oplus M_-) \ \subset \ (M_+^{\vee}/M_+)\times(M_-^{\vee}/M_-)$$

is the graph of an anti-isomorphism $q_{M_+} \cong -q_{M_-}$ of finite quadratic forms, which induces an isomorphism

$$j_M \;:\; \mathrm{O}(q_{M_+}) \cong \mathrm{O}(q_{M_-}).$$

Consequently, the embedding (2.2) identifies $\mathrm{Cen}(\mathrm{O}(M), \varepsilon)$ with

$$\{\, (g_+, g_-) \in \mathrm{O}(M_+) \times \mathrm{O}(M_-) \mid j_M(\eta_{M_+}(g_+)) = \eta_{M_-}(g_-) \,\},$$

that is, we have the following fiber-product diagram:

$$\begin{array}{ccc} \mathrm{Cen}(\mathrm{O}(M), \varepsilon) & \to & \mathrm{O}(M_-) \\ \downarrow & \square & \downarrow \\ \mathrm{O}(M_+) & \to & \mathrm{O}(q_{M_+}) \cong \mathrm{O}(q_{M_-}). \end{array} \tag{2.3}$$

2.3. **Computational method.** In some arguments, we need to determine, for a positive- or negative-definite lattice M, the sizes of $\mathrm{O}(M)$, $\mathrm{O}(q_M)$, and/or the image of the natural homomorphism $\eta_M \colon \mathrm{O}(M) \to \mathrm{O}(q_M)$. In this section, we explain the method used in these computations.

Let M be a positive-definite lattice of rank m. A finite generating set and the size of $\mathrm{O}(M)$ are obtained by the famous Plesken-Souvignier algorithm [17]. A generating set and the size of the automorphism group $\mathrm{O}(q)$ of a finite quadratic form q can also be computed using a similar stabilizer-chain method.

Remark 2.3. A closed formula for the size of $\mathrm{O}(q)$ is given in [5, Theorem 5.2].

The size of the image of the natural homomorphism $\eta_M \colon \mathrm{O}(M) \to \mathrm{O}(q_M)$ is calculated as follows. Let $A := M^\vee/M$ denote the discriminant group. We first compute a generating set $\{g_1, \dots, g_N\}$ of $\mathrm{O}(M)$, together with their images $\eta_M(g_i) \in \mathrm{O}(q_M)$ under η_M. Each $\eta_M(g_i)$ induces a permutation of the finite set A. The size of the image of η_M is then equal to the size of the subgroup of the symmetric group $\mathfrak{S}(A)$ generated by these permutations, which can be computed using the Schreier–Sims algorithm.

Remark 2.4. Since the implementation of the Schreier–Sims algorithm in `GAP` [9] is enhanced by randomized methods, this computation by `GAP` is generally very fast. We take full advantage of this high performance of `GAP` in our study.

We can calculate $\mathrm{Ker}\,\eta_M$ as follows. We fix a basis $b_1, \dots, b_m$ of $M^\vee$, and identify elements g of $\mathrm{O}(M^\vee) = \mathrm{O}(M)$ with tuples $[b'_1, \dots, b'_m]$ of vectors $b'_i \in M^\vee$ satisfying

$$\langle b_i, b_j \rangle = \langle b'_i, b'_j \rangle \;\; (i, j = 1, \dots, m)$$

via $b_i^g = b'_i$ for $i = 1, \dots, m$. Then the subgroup $\mathrm{Ker}\,\eta_M$ of $\mathrm{O}(M)$ corresponds to the set of all tuples $[b'_1, \dots, b'_m]$ such that $b_i - b'_i \in M$ holds for $i = 1, \dots, m$. Enumerating such tuples $[b'_1, \dots, b'_m]$ by the stabilizer chain method as in the Plesken-Souvignier algorithm, we obtain the size of $\mathrm{Ker}\,\eta_M$ and a finite generating set of $\mathrm{Ker}\,\eta_M$.

3. Proof of Theorem 1.1

In Section 3.1, we investigate the relationship between involutions of $\cdot\infty$ and those of $\cdot 0$. We also describe the centralizer $\mathrm{Cen}(\cdot\infty, \gamma)$ for an involution $\gamma \in \cdot\infty$. In Section 3.2, we review Conway's result [8, Chapter 10] on $\cdot 0$, and present the four involutions ε_n ($n = 8, 12, 16, 24$) in $\cdot 0$. In Sections 3.3–3.6, we study the conjugacy class $[\varepsilon_n]$ in $\cdot 0$ for each n and determine the corresponding conjugacy classes in $\cdot\infty$. This completes the proof of Theorem 1.1.

3.1. **Involutions of $\cdot 0$ and of $\cdot\infty$.** We use the following notation. For a group G, let $\mathcal{I}(G)$ denote the set of conjugacy classes of involutions in G. For an involution $g \in G$, let $[g] \in \mathcal{I}(G)$ denote the conjugacy class of g.

The product in $\cdot\infty$ is given by

$$(g_1, \tau_1)(g_2, \tau_2) = (g_1 g_2, \tau_1^{g_2} + \tau_2).$$

Hence, for $(g, \tau) \in \cdot\infty$, we have

$$(g, \tau) \text{ is an involution } \iff g \in \cdot 0 \text{ is an involution and } \tau \in \Lambda(g, -). \tag{3.1}$$

Moreover, we have

$$(h, \sigma)(g, \tau)(h, \sigma)^{-1} = (hgh^{-1}, (\tau + \sigma^g - \sigma)^{h^{-1}}). \tag{3.2}$$

Hence two involutions (g, τ) and (g', τ') of $\cdot\infty$ are conjugate if and only if

$$\exists\, h \in \cdot 0\ \ \exists\, \sigma \in \Lambda \ \text{ such that }\ g' = hgh^{-1} \text{ and } \tau'^h = \tau + \sigma^g - \sigma. \tag{3.3}$$

Therefore the homomorphism $\cdot\infty \to \cdot 0$ given by $(g, \tau) \mapsto g$ induces a surjective mapping

$$\Pi : \mathcal{I}(\cdot\infty) \twoheadrightarrow \mathcal{I}(\cdot 0).$$

Let $g \in \cdot 0$ be an involution. The fiber of Π over $[g] \in \mathcal{I}(\cdot 0)$ is described as follows. We denote by

$$\pi_g : \Lambda \to \Lambda(g, -)$$

the homomorphism defined by $\lambda \mapsto \lambda - \lambda^g$. Note that π_g is $\mathrm{Cen}(\cdot 0, g)$-equivariant, and hence $\mathrm{Cen}(\cdot 0, g)$ acts on the cokernel of π_g. For $\tau \in \Lambda(g, -)$, we denote by $[\tau]$ the element $\tau + \mathrm{Im}(\pi_g)$ of $\mathrm{Coker}(\pi_g)$, and by $\langle\tau\rangle \in \mathrm{Coker}(\pi_g)/\mathrm{Cen}(\cdot 0, g)$ the orbit of $[\tau]$ under the action of $\mathrm{Cen}(\cdot 0, g)$ on $\mathrm{Coker}(\pi_g)$.

The following proposition has already been proved in the more general setting in [12, Lemma 4.13]. For the convenience of the reader, however, we provide a proof in our simpler setting.

Proposition 3.1. *Suppose that $[(g_1, \tau_1)] \in \Pi^{-1}([g])$, so that there exists an element $h_1 \in \cdot 0$ such that $g_1 h_1 = h_1 g$ holds. Note that $\tau_1^{h_1} \in \Lambda(g, -)$. Then the map $[(g_1, \tau_1)] \mapsto \langle\tau_1^{h_1}\rangle$ is well defined, and yields a bijection*

$$f \ :\ \Pi^{-1}([g]) \ \cong\ \mathrm{Coker}(\pi_g)/\mathrm{Cen}(\cdot 0, g). \tag{3.4}$$

Proof. Suppose that $[(g_2, \tau_2)] = [(g_1, \tau_1)]$. Then, by (3.3), we have $h_{21} \in \cdot 0$ and $\sigma_{21} \in \Lambda$ such that $g_2 h_{21} = h_{21} g_1$ and $\tau_2^{h_{21}} = \tau_1 + \sigma_{21}^{g_1} - \sigma_{21}$. Let $h_2 \in \cdot 0$ be an

0	∞	1	11	2	22
19	3	20	4	10	18
15	6	14	16	17	8
5	9	21	13	7	12

FIGURE 3.1. The standard MOG labeling

element such that $g_2h_2 = h_2g$. It is enough to show $\langle\tau_2^{h_2}\rangle = \langle\tau_1^{h_1}\rangle$ holds. Note that $k := h_2^{-1}h_{21}h_1$ belongs to $\mathrm{Cen}(\cdot 0, g)$. Then we have

$$\tau_2^{h_2} = \tau_2^{h_{21}h_1k^{-1}} = (\tau_1 + \sigma_{21}^{g_1} - \sigma_{21})^{h_1k^{-1}} = (\tau_1^{h_1} + (\sigma_{21}^{h_1g} - \sigma_{21}^{h_1}))^{k^{-1}}.$$

Thus f is well-defined.

Since $f([(g,\tau)]) = \langle\tau\rangle$ for any $\tau \in \Lambda(g,-)$, the surjectivity of f is obvious. Note that, if $[(g',\tau')] \in \Pi^{-1}([g])$, then there exists $\tau_0 \in \mathrm{Coker}(\pi_g)$ such that $[(g',\tau')] = [(g,\tau_0)]$, and hence $f([(g',\tau')]) = \langle\tau_0\rangle$. Therefore, to show the injectivity of f, it is enough to show that, for $\tau,\tau' \in \mathrm{Coker}(\pi_g)$, the equality $\langle\tau\rangle = \langle\tau'\rangle$ implies $[(g,\tau)] = [(g,\tau')]$. If $\langle\tau\rangle = \langle\tau'\rangle$ holds, then we have $k \in \mathrm{Cen}(\cdot 0, g)$ and $\sigma \in \Lambda$ such that $\tau' = \tau^k + \sigma - \sigma^g$. Then $(k,\sigma)(g,\tau')(k,\sigma)^{-1} = (g,\tau)$ holds. □

Let (g,τ) be an involution in $\cdot\infty$. We put

$$\Sigma_g([\tau]) := \text{ the stabilizer of } [\tau] \in \mathrm{Coker}(\pi_g) \text{ in } \mathrm{Cen}(\cdot 0, g).$$

Then we have the following:

Proposition 3.2. *We have an exact sequence*

$$0 \longrightarrow \Lambda(g,+) \longrightarrow \mathrm{Cen}(\cdot\infty,(g,\tau)) \longrightarrow \Sigma_g([\tau]) \longrightarrow 1. \tag{3.5}$$

Proof. By (3.2), it follows that

$$(h,\sigma) \in \mathrm{Cen}(\cdot\infty,(g,\tau)) \iff h \in \mathrm{Cen}(\cdot 0, g) \text{ and } \tau - \tau^h = \pi_g(\sigma). \tag{3.6}$$

Therefore we have the projection $\mathrm{Cen}(\cdot\infty,(g,\tau)) \to \mathrm{Cen}(\cdot 0, g)$ given by $(h,\sigma) \mapsto h$. The kernel of this projection is equal to

$$\{\, (\mathrm{id},\lambda) \mid \lambda \in \Lambda(g,+) \,\} \;\cong\; \Lambda(g,+).$$

On the other hand, by (3.6), an element h of $\mathrm{Cen}(\cdot 0, g)$ belongs to the image of this projection if and only if there exists a vector $\sigma \in \Lambda$ such that $\tau - \tau^h = \pi_g(\sigma)$, or equivalently, if and only if $h \in \Sigma_g([\tau])$. Thus we obtain the exact sequence (3.5). □

3.2. **Conway's result on involutions of** $\cdot 0$. Conway investigated involutions of $\cdot 0$ and computed their centralizers in [8, Chapter 10, Section 3.7]. In this section, we review Conway's result.

Let Ω be the set $\mathbb{P}^1(\mathbb{F}_{23}) = \{\infty, 0, 1, \dots, 22\}$, and let $\mathcal{C}_{24} \subset 2^\Omega$ be the extended binary Golay code, that is, the extended quadratic residue code over $\mathbb{F}_{23}$. We also use the Miracle Octad Generator (MOG) to denote elements of 2^Ω. The elements of Ω and the MOG positions are identified using the standard labeling [8, Figure 11.7], which we reproduce in Figure 3.1. For $n \geq 0$, we denote by $\mathcal{C}_{24}(n)$ the set of codewords $w \in \mathcal{C}_{24}$ of size n. Note that $\mathcal{C}_{24}(n) \neq \emptyset$ if and only if $n \in \{0, 8, 12, 16, 24\}$. For $S \in 2^{\Omega}$, let $v_S \in \mathbb{Z}^{\Omega}$ denote the vector

$$v_S(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

When $S = \{i\}$ for $i \in \Omega$, we write v_i instead of $v_{\{i\}}$. Let Λ be the submodule of $\mathbb{Z}^{\Omega}$ generated by the vectors $2v_C$, where C runs through $\mathcal{C}_{24}(8)$, and the vector $v_{\Omega} - 4v_{\infty}$. We equip $\mathbb{Z}^{\Omega}$ with the inner product $\langle\ ,\ \rangle$ given by

$$\langle x, y \rangle := \frac{1}{8} \sum_{i \in \Omega} x(i) y(i).$$

Then Λ with $\langle\ ,\ \rangle$ is the Leech lattice. For a non-zero codeword $C \in \mathcal{C}_{24}$, we define $\varepsilon(C) \in \mathrm{O}(\mathbb{Z}^{\Omega})$ by

$$v_i^{\varepsilon(C)} := \begin{cases} -v_i & \text{if } i \in C, \\ v_i & \text{if } i \notin C. \end{cases}$$

Then $\varepsilon(C)$ preserves $\Lambda \subset \mathbb{Z}^{\Omega}$, and hence gives an involution $\varepsilon(C) \in \cdot 0$. Since the Mathieu group $M_{24} \subset \mathfrak{S}(\Omega)$ acts transitively on $\mathcal{C}_{24}(n)$ for each n, and M_{24} is naturally embedded into $\cdot 0$, the conjugacy class in $\cdot 0$ of $\varepsilon(C)$ for $C \in \mathcal{C}_{24}(n)$ depends only on n. We choose a codeword $C_n \in \mathcal{C}_{24}(n)$ and put

$$\varepsilon_n := \varepsilon(C_n).$$

Note that $\varepsilon_{24} = -\mathrm{id}$. Then Conway [8, Chapter 10, Section 3.7] proved the following:

Theorem 3.3 (Conway)**.** *The set $\mathcal{I}(\cdot 0)$ of conjugacy classes of involutions in $\cdot 0$ consists of 4 elements $[\varepsilon_8]$, $[\varepsilon_{12}]$, $[\varepsilon_{16}]$, $[\varepsilon_{24}]$.* □

For $C \in \mathcal{C}_{24}$, we put $\overline{C} := \Omega \setminus C \in \mathcal{C}_{24}$, and let $\mathbb{Z}^C \subset \mathbb{Z}^{\Omega}$ denote the submodule generated by the vectors v_i $(i \in C)$. Then we have

$$\Lambda(\varepsilon(C), -) = \Lambda \cap \mathbb{Z}^C, \quad \Lambda(\varepsilon(C), +) = \Lambda \cap \mathbb{Z}^{\overline{C}}. \tag{3.7}$$

For each ε_n, we compute Gram matrices of $\Lambda(\varepsilon_n, \pm)$ with respect to certain bases and the matrix representation of the homomorphism

$$\pi_n := \pi_{\varepsilon_n} \colon \Lambda \to \Lambda(\varepsilon_n, -); \quad \lambda \mapsto \lambda - \lambda^{\varepsilon_n}$$

with respect to these bases. By these computations, we obtain the following. See Table 3.1 for a summary. Recall that $\mathit{\Lambda}_n$ is the laminated lattice of rank n (see [8, Figure 6.2]). Note that $\mathit{\Lambda}_8$ is isomorphic to $E_8(2)$, and $\mathit{\Lambda}_{16}$ is isomorphic to the Barnes–Wall lattice BW_{16}. Recall also that $\mathit{\Sigma}_{12}$ is an *odd* unimodular positive-definite lattice of rank 12 that contains no vectors of norm 1. Note that $\mathit{\Sigma}_{12}$ is unique up to isomorphism (see [8, Table 16.7]).

- $n = 8$. We choose as C_8 the codeword $\{0, \infty, 19, 3, 15, 6, 5, 9\}$, which corresponds to the positions with $*$ in Figure 3.2 of MOG. Then we obtain

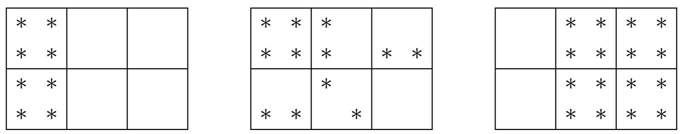
$$\Lambda(\varepsilon_8, -) \cong \mathit{\Lambda}_8, \quad \Lambda(\varepsilon_8, +) \cong \mathit{\Lambda}_{16}.$$

  We confirm that π_8 is surjective.

n	$\Lambda(\varepsilon_n,+)$	$\Lambda(\varepsilon_n,-)$	$\mathrm{disc}(\Lambda(\varepsilon_n,-))$	$\mathrm{Coker}(\pi_n)$
8	$\mathit{\Lambda}_{16}$	$\mathit{\Lambda}_8$	$(\mathbb{Z}/2\mathbb{Z})^8$	0
12	$\mathit{\Sigma}_{12}(2)$	$\mathit{\Sigma}_{12}(2)$	$(\mathbb{Z}/2\mathbb{Z})^{12}$	0
16	$\mathit{\Lambda}_8$	$\mathit{\Lambda}_{16}$	$(\mathbb{Z}/2\mathbb{Z})^8$	$(\mathbb{Z}/2\mathbb{Z})^8$
24	0	Λ	0	$(\mathbb{Z}/2\mathbb{Z})^{24}$

$\mathrm{disc}(L)$ means the discriminant group $L^\vee/L$.

TABLE 3.1. $\Lambda(\varepsilon_n,+)$ and $\Lambda(\varepsilon_n,-)$



C_8 C_{12} C_{16}

FIGURE 3.2. Codewords

- $n=12$. We choose as C_{12} the codeword $\{0,\infty,1,19,3,20,10,18,14,5,9,13\}$. See also Figure 3.2. Then we obtain

$$\Lambda(\varepsilon_{12},-)\cong \mathit{\Sigma}_{12}(2),\quad \Lambda(\varepsilon_{12},+)\cong \mathit{\Sigma}_{12}(2).$$

  The homomorphism π_{12} is surjective.
- $n=16$. We choose as C_{16} the complement $\Omega\setminus C_8$ of C_8. Then we have $\varepsilon_{16}=-\varepsilon_8$, and hence

$$\Lambda(\varepsilon_{16},-)=\Lambda(\varepsilon_8,+)\cong \mathit{\Lambda}_{16},\quad \Lambda(\varepsilon_{16},+)=\Lambda(\varepsilon_8,-)\cong \mathit{\Lambda}_8.$$

  The cokernel of π_{16} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^8$.
- $n=24$. We have $\varepsilon_{24}=-\mathrm{id}$ and hence $\Lambda(\varepsilon_{24},-)=\Lambda$. The cokernel of π_{24} is $\Lambda/2\Lambda\cong(\mathbb{Z}/2\mathbb{Z})^{24}$.

Our next task is to calculate $\mathrm{Cen}(\cdot 0,\varepsilon_n)$ and its action on the cokernel of π_n. Although these centralizers were already computed in [8, Chapter 10], we rederive them using discriminant forms (see Section 2.2).

3.3. **Involution** ε_8**.** We put

$$G_8:=\mathrm{O}_8^+(2),$$

which is a simple group of order 174182400. See [7, page 85]. We put

$$q_8:=q_{\mathit{\Lambda}_8}=q_{E_8(2)},\quad q_{16}:=q_{\mathit{\Lambda}_{16}}.$$

Then q_8 is isomorphic to $u^{\oplus 4}$, where u is given in (2.1). Since Λ is unimodular, we have $q_{16}\cong -q_8$. Since $u=-u$, we also have $q_{16}\cong u^{\oplus 4}$. By [7, page 85], we have

$$\mathrm{O}(q_8)\cong G_8.2,\quad \mathrm{O}(\mathit{\Lambda}_8)=W(E_8)\cong 2.G_8.2.$$

Remark 3.4. The homomorphism $\mathrm{O}(q_8)\to\{\pm1\}$ with the kernel G_8 is obtained by the permutation representation on the set of 4-dimensional totally isotropic subspaces of q_8. See [25, Section 3.8]. The structure $W(E_8) \cong 2.G_8.2$ is given by the inclusion $\{\pm\mathrm{id}\} \subset W(E_8)$ and $\det\colon W(E_8) \to \{\pm 1\}$.

The group $\mathrm{O}(\Lambda_8) = W(E_8)$ is generated by the 8 reflections corresponding to the standard fundamental roots of E_8. Computing their actions on q_8, we confirm that the natural homomorphism $\mathrm{O}(\Lambda_8) \to \mathrm{O}(q_8)$ is surjective with the kernel $\{\pm\mathrm{id}\}$. We compute the size $|\mathrm{O}(\Lambda_{16})|$ and a finite set of generators of $\mathrm{O}(\Lambda_{16})$ by the method explained in Section 2.3. The size of $\mathrm{O}(\Lambda_{16})$ is $512 \cdot |G_8|$. Since $\Lambda_{16} \cong \mathrm{BW}_{16}$, this result is consistent with $\mathrm{O}(\mathrm{BW}_{16}) \cong 2^{1+8}.G_8$ given in [8, Chapter 4, Section 10]. Using the method in Section 2.3 again and noting that G_8 is simple, we see that

(3.8) the image of the natural homomorphism $\eta_{16}\colon \mathrm{O}(\Lambda_{16}) \to \mathrm{O}(q_{16})$ is of size $|G_8|$, and hence is equal to G_8 in $\mathrm{O}(q_{16}) \cong G_8.2$.

We also calculate

$$K_{16} := \mathrm{Ker}\ \eta_{16}$$

by the method described in Section 2.3. It turns out that K_{16} is an extraspecial 2-group of plus type, namely $(2^{1+8})_+$.

Combining all these computations and using the diagram (2.3), we obtain

$$\mathrm{Cen}(\cdot 0, \varepsilon_8) \cong \{\pm\mathrm{id}\} \times \mathrm{O}(\Lambda_{16}).$$

Since $\pi_8\colon \Lambda \to \Lambda(\varepsilon_8, -)$ is surjective, Proposition 3.1 implies that the subset $\Pi^{-1}([\varepsilon_8])$ of $\mathcal{I}(\cdot\infty)$ consists of a single element, which is the conjugacy class represented by an involution

$$\gamma_8 := (\varepsilon_8, 0).$$

3.4. **Involution ε_{12}.** We slightly simplify the argument of Conway [8, Chapter 10, Section 3.7] by explicitly constructing the lattice Σ_{12}. Let Σ_{12} denote the submodule of $\mathbb{Z}^{12}$ generated by vectors $\pm 2e_i \pm 2e_j$ $(i \neq j)$, and $e_1 + \dots + e_{12}$, where $e_1, \dots, e_{12}$ are the standard basis of $\mathbb{Z}^{12}$. Then, by the intersection form

$$\langle x, y\rangle = \frac{1}{4}\sum_{i=1}^{12} x_i y_i$$

on $\mathbb{Z}^{12}$, the submodule Σ_{12} becomes the odd unimodular positive-definite lattice with no vectors v of norm $\langle v, v\rangle = 1$.

Lemma 3.5. *The action of* $\mathrm{O}(\Sigma_{12})$ *on* Σ_{12} *preserves the* frame

$$F := \{\, \pm 4e_i \mid i = 1, \dots, 12 \,\} \subset \Sigma_{12}.$$

Proof. Note that the set $(\Sigma_{12})_2$ of vectors $u \in \Sigma_{12}$ with $\langle u, u\rangle = 2$ consists of the 264 vectors of the form $\pm 2e_i \pm 2e_j$ $(i \neq j)$. We say that $u, v, w \in (\Sigma_{12})_2$ form a *regular triple* if $\langle u, v\rangle = \langle v, w\rangle = \langle w, u\rangle = 1$. If $\{u, v, w\}$ is a regular triple, then the number of vectors $x \in (\Sigma_{12})_2$ such that $\langle u, x\rangle = \langle v, x\rangle = \langle w, x\rangle = 1$ is either 0 or 16. We say that the regular triple $\{u, v, w\}$ is a *triangle* if there exists no such $x \in (\Sigma_{12})_2$. See Figure 3.3, where $a, b, c, d \in \{2, -2\}$. There exist exactly 1760 triangles, and the action of $\mathrm{O}(\Sigma_{12})$ preserves the set of triangles. Since the frame F is equal to

$$\{\, u + v - w \mid \{u, v, w\} \text{ is a triangle} \,\},$$

$$\begin{array}{cccccccc} u: & a & b & 0 & 0 & 0 & \dots \\ v: & a & 0 & c & 0 & 0 & \dots \\ w: & 0 & b & c & 0 & 0 & \dots \\ & & & \text{triangle} & & & \end{array} \qquad \begin{array}{cccccccc} u: & a & b & 0 & 0 & 0 & \dots \\ v: & a & 0 & c & 0 & 0 & \dots \\ w: & a & 0 & 0 & d & 0 & \dots \\ & & & \text{non-triangle} & & & \end{array}$$

FIGURE 3.3. Regular triples

we conclude that $\mathrm{O}(\Sigma_{12})$ preserves F. □

Corollary 3.6. *We have* $\mathrm{O}(\Sigma_{12}) = A \rtimes \mathfrak{S}_{12}$, *where*

$$A := \{\, \alpha = (\alpha_1, \dots, \alpha_{12}) \in \{\pm 1\}^{12} \mid \prod \alpha_i = 1 \,\} \;\;\cong\;\; (\mathbb{Z}/2\mathbb{Z})^{11}$$

acts on Σ_{12} *by* $e_i^\alpha = \alpha_i e_i$ *for* $i = 1, \dots, 12$, *and the symmetric group* $\mathfrak{S}_{12}$ *acts on* Σ_{12} *by permuting the coordinates.* □

We fix a codeword $C_{12} \in \mathcal{C}_{24}(12)$ and consider the involution $\varepsilon_{12} := \varepsilon(C_{12})$. By (3.7), the lattice $\Lambda(\varepsilon_{12}, -)$ (resp. $\Lambda(\varepsilon_{12}, +)$) is isomorphic to $\Sigma_{12}(2)$ under the isomorphism $\mathbb{Z}^{C_{12}} \cong \mathbb{Z}^{12}$ (resp. $\mathbb{Z}^{\overline{C}_{12}} \cong \mathbb{Z}^{12}$) induced by some (and hence any) bijection $C_{12} \cong \{1, \dots, 12\}$ (resp. $\overline{C}_{12} \cong \{1, \dots, 12\}$), and the frame $F \subset \Sigma_{12}$ corresponds to

$$\{\, \pm 8v_i \mid i \in C_{12} \,\} \subset \Lambda(\varepsilon_{12}, -), \qquad (\text{resp. } \{\, \pm 8v_j \mid j \in \overline{C}_{12} \,\} \subset \Lambda(\varepsilon_{12}, +)),$$

where v_i ($i \in \Omega$) are the standard basis of $\mathbb{Z}^\Omega$. Lemma 3.5 implies that the action of $\mathrm{Cen}(\cdot 0, \varepsilon_{12})$ on Λ preserves $\{\, \pm 8v_i \mid i \in \Omega \,\}$, and hence, by Theorem 26 of [8, Chapter 10, Section 3.3], the group $\mathrm{Cen}(\cdot 0, \varepsilon_{12})$ is contained in the subgroup

$$\langle \varepsilon(C) \mid C \in \mathcal{C}_{24} \rangle \rtimes M_{24}$$

of $\cdot 0$, where $M_{24} \subset \mathfrak{S}(\Omega)$ is the automorphism group of $\mathcal{C}_{24}$. Note that, for any $C \in \mathcal{C}_{24}$, the sizes of $C_{12} \cap C$ and $\overline{C}_{12} \cap C$ are even, and hence $\varepsilon(C_{12} \cap C)$ and $\varepsilon(\overline{C}_{12} \cap C)$ belong to A via $\mathrm{O}(\Lambda(\varepsilon_{12}, -)) \cong \mathrm{O}(\Sigma_{12})$ and $\mathrm{O}(\Lambda(\varepsilon_{12}, +)) \cong \mathrm{O}(\Sigma_{12})$, respectively, where A is introduced in Corollary 3.6. Thus we obtain

$$\mathrm{Cen}(\cdot 0, \varepsilon_{12}) = \langle \varepsilon(C) \mid C \in \mathcal{C}_{24} \rangle \rtimes \{\, g \in M_{24} \mid C_{12}^g = C_{12} \,\} = (\mathbb{Z}/2\mathbb{Z})^{12} \rtimes M_{12}. \tag{3.9}$$

Since $\pi_{12} \colon \Lambda \to \Lambda(\varepsilon_{12}, -)$ is surjective, Proposition 3.1 implies that the fiber $\Pi^{-1}([\varepsilon_{12}]) \subset \mathcal{I}(\cdot\infty)$ consists of a single element, which is represented by an involution

$$\gamma_{12} := (\varepsilon_{12}, 0).$$

Remark 3.7. The discriminant form of $\Sigma_{12}(2)$ is isomorphic to $q_{12} := a^{\oplus 12}$, and the size of $\mathrm{O}(q_{12})$ is 51231497335603200. By the method explained in Section 2.3, we confirm that the kernel of the natural homomorphism from $\mathrm{O}(\Sigma_{12}(2)) = \mathrm{O}(\Sigma_{12})$ to $\mathrm{O}(q_{12})$ is $\{\pm \mathrm{id}\}$.

3.5. **Involution ε_{16}.** Recall that we have $\mathrm{Cen}(\cdot 0, \varepsilon_{16}) = \mathrm{Cen}(\cdot 0, \varepsilon_8)$ by the choice of $C_{16} = \Omega \setminus C_8$. We study the action of $\mathrm{Cen}(\cdot 0, \varepsilon_{16})$ on the cokernel of the homomorphism $\pi_{16} \colon \Lambda \to \Lambda(\varepsilon_{16}, -) = \mathit{\Lambda}_{16}$. Since $\mathit{\Lambda}_{16}^{\vee}/\mathit{\Lambda}_{16}$ is 2-elementary, we have

$$\mathit{\Lambda}_{16}^{\vee} \supset \mathit{\Lambda}_{16} \supset 2\mathit{\Lambda}_{16}^{\vee}. \tag{3.10}$$

By direct computation, we see that the image of π_{16} is equal to $2\mathit{\Lambda}_{16}^{\vee}$. Since the discriminant form q_{16} of $\mathit{\Lambda}_{16}$ is isomorphic to $u^{\oplus 4}$ and takes values in $\mathbb{Z}/2\mathbb{Z}$, we have $\langle x, x \rangle \in \mathbb{Z}$ for any $x \in \mathit{\Lambda}_{16}^{\vee}$. Therefore the map

$$v + \mathrm{Im}(\pi_{16}) \ \mapsto \ \frac{1}{2}\langle v, v \rangle \mod 2\mathbb{Z}$$

gives rise to a quadratic form

$$\tilde{q}_{16} : \ \mathrm{Coker}(\pi_{16}) \ \to \ \mathbb{Q}/2\mathbb{Z}.$$

A direct computation shows that $\tilde{q}_{16}$ is isomorphic to $u^{\oplus 4}$. The action of $\mathrm{Cen}(\cdot 0, \varepsilon_{16})$ on $\mathrm{Coker}(\pi_{16})$ factors through the projection $\mathrm{Cen}(\cdot 0, \varepsilon_{16}) \to \mathrm{O}(\mathit{\Lambda}_{16})$, which is surjective by the diagram (2.3) and the surjectivity of $W(E_8) \to \mathrm{O}(q_8)$. Moreover, this action preserves $\tilde{q}_{16}$. Since we have calculated a generating set of $\mathrm{O}(\mathit{\Lambda}_{16})$, we can explicitly compute the orbit decomposition of $\mathrm{Coker}(\pi_{16})$ by $\mathrm{O}(\mathit{\Lambda}_{16})$; it consists of the following three orbits

$$\{0\}, \quad \{\, v \in \mathrm{Coker}(\pi_{16}) \mid \tilde{q}_{16}(v) = 1 \,\}, \quad \{\, v \in \mathrm{Coker}(\pi_{16}) \mid v \neq 0, \ \ \tilde{q}_{16}(v) = 0 \,\}.$$

Their sizes are 1, 120, and 135, respectively. Hence, by Proposition 3.1, it follows that the fiber $\Pi^{-1}([\varepsilon_{16}]) \subset \mathcal{I}(\cdot\infty)$ consists of three conjugacy classes, which are represented by involutions

$$\gamma_{16,1} := (\varepsilon_{16}, 0), \quad \gamma_{16,120} := (\varepsilon_{16}, \tau_{120}), \quad \gamma_{16,135} := (\varepsilon_{16}, \tau_{135}),$$

where τ_{120} (resp. τ_{135}) is a non-zero vector of $\Lambda(\varepsilon_{16}, -)$ not in the image of π_{16} such that $\langle \tau_{120}, \tau_{120} \rangle \equiv 2 \bmod 4$ (resp. $\langle \tau_{135}, \tau_{135} \rangle \equiv 0 \bmod 4$). We choose from $\Lambda(\varepsilon_{16}, -) = \mathit{\Lambda}_{16}$ a vector of norm 6 as τ_{120}, and a vector of norm 4 as τ_{135}. (There exist exactly 4320 vectors of norm 4 and 61440 vectors of norm 6 in $\mathit{\Lambda}_{16}$. None of them belongs to $2\mathit{\Lambda}_{16}^{\vee} = \mathrm{Im}(\pi_{16})$.)

Remark 3.8. By direct computation, we find that the kernel K_{16} of the natural homomorphism $\eta_{16} \colon \mathrm{O}(\mathit{\Lambda}_{16}) \to \mathrm{O}(q_{16})$ is in fact *equal* to the kernel of the natural homomorphism $\tilde{\eta}_{16} \colon \mathrm{O}(\mathit{\Lambda}_{16}) \to \mathrm{O}(\tilde{q}_{16})$. Suppose that $g \in K_{16}$. Then the action of g on the filtration (3.10) of $\mathit{\Lambda}_{16}^{\vee}$ induces the identities on $\mathit{\Lambda}_{16}^{\vee}/\mathit{\Lambda}_{16}$ and on $\mathrm{Coker}(\pi_{16}) = \mathit{\Lambda}_{16}/2\mathit{\Lambda}_{16}^{\vee}$, and hence gives rise to a homomorphism $\mathit{\Lambda}_{16}^{\vee}/\mathit{\Lambda}_{16} \to \mathrm{Coker}(\pi_{16})$ by

$$x \bmod \mathit{\Lambda}_{16} \ \ \mapsto \ \ x^g - x \bmod \mathrm{Im}(\pi_{16})$$

for $x \in \mathit{\Lambda}_{16}^{\vee}$. Thus we obtain an embedding

$$K_{16}/\{\pm \mathrm{id}\} \hookrightarrow \mathrm{Hom}(\mathit{\Lambda}_{16}^{\vee}/\mathit{\Lambda}_{16}, \mathrm{Coker}(\pi_{16})),$$

which gives an isomorphism $K_{16} \cong (2^{1+8})_+$.

3.6. **Involution ε_{24}.** We have $\varepsilon_{24} = -\mathrm{id}$ and hence $\Lambda(\varepsilon_{24}, -) = \Lambda$. The orbit decomposition of $\mathrm{Coker}(\pi_{24}) = \Lambda/2\Lambda$ by $\mathrm{Cen}(\cdot 0, \varepsilon_{24}) = \cdot 0$ was determined by Conway [8, Chapter 10, Section 3.3]. It consists of 4 orbits $\{0\}, \overline{\Lambda}(2), \overline{\Lambda}(3), \overline{\Lambda}(4)$, where

$$\overline{\Lambda}(n) := \{\, \lambda \bmod 2\Lambda \mid \langle \lambda, \lambda \rangle = 2n \,\}.$$

Their sizes are 1, 98280, 8386560, and 8292375, respectively. Hence, by Proposition 3.1, we see that $\Pi^{-1}([\varepsilon_{24}]) \subset \mathcal{I}(\cdot\infty)$ consists of four conjugacy classes, which are represented by involutions

$$\gamma_{24,0} := (-\mathrm{id}, 0), \quad \gamma_{24,4} := (-\mathrm{id}, t_2), \quad \gamma_{24,6} := (-\mathrm{id}, t_3), \quad \gamma_{24,8} := (-\mathrm{id}, t_4),$$

where $t_m \in \Lambda$ is a vector such that $\langle t_m, t_m \rangle = 2m$.

Combining the results of Sections 3.3, 3.4, 3.5, 3.6, we obtain a proof of Theorem 1.1. Using Proposition 3.2, we also obtain descriptions of $\mathrm{Cen}(\cdot\infty, \gamma)$ for involutions γ in $\cdot\infty$.

4. Action of $\cdot\infty$ on L_{26}

In Section 4.1, we present a general theory on the action of affine isometries of an even positive-definite lattice M on the hyperbolic lattice $L = U \oplus M(-1)$. Applying this theory, we prove Theorem 1.2 in Section 4.2.

4.1. **Action of affine isometries on a hyperbolic lattice.** Let M be an even positive-definite lattice. We denote by

$$\mathrm{AO}(M) := \mathrm{O}(M) \ltimes M$$

the group of affine isometries of M acting on M as $x^{(g,\tau)} = x^g + \tau$ for $g \in \mathrm{O}(M)$ and $\tau \in M$. We consider the even hyperbolic lattice

$$L := U \oplus M(-1),$$

where U is the hyperbolic plane with the basis $\boldsymbol{w}_0, \boldsymbol{w}_1$ satisfying (1.1). To avoid confusion, we write $\langle \quad \rangle_M$ for the intersection form of M (*not* of $M(-1)$), and $\langle \quad \rangle_L$ for the intersection form of L. An element of L is written as

$$(a, b, x) := a\boldsymbol{w}_0 + b\boldsymbol{w}_1 + x,$$

where $a, b \in \mathbb{Z}$ and $x \in M$. Then we have

$$\langle\, (a, b, x),\ (a', b', x')\, \rangle_L \;=\; ab' + a'b - \langle x, x' \rangle_M.$$

Let $\mathcal{P}_L$ be the positive cone containing $\boldsymbol{w}_0 = (1, 0, 0) \in L$ in its closure. It is easy to verify the following:

Proposition 4.1. *The group* $\mathrm{AO}(M)$ *acts on the lattice* $L = U \oplus M(-1)$ *by*

$$\begin{aligned}
\boldsymbol{w}_0^{(g,\tau)} &= \boldsymbol{w}_0, \\
\boldsymbol{w}_1^{(g,\tau)} &= \frac{\langle \tau, \tau \rangle_M}{2}\, \boldsymbol{w}_0 + \boldsymbol{w}_1 + (0, 0, \tau), \\
(0, 0, x)^{(g,\tau)} &= \langle \tau, x^g \rangle_M\, \boldsymbol{w}_0 + (0, 0, x^g).
\end{aligned}$$

Hence we have an injection $\mathrm{AO}(M) \hookrightarrow \mathrm{O}(L, \mathcal{P}_L)$. *The image of this embedding is equal to the stabilizer* $\mathrm{O}(L, \boldsymbol{w}_0)$ *of* $\boldsymbol{w}_0$ *in* $\mathrm{O}(L, \mathcal{P}_L)$. □

Remark 4.2. When $\tau = 0$, the action of $(g, 0) \in \mathrm{AO}(M)$ on L is equal to $\mathrm{id}_U \oplus g$.

Suppose that $(g, \tau) \in \mathrm{AO}(M)$ is an involution, that is, $g \neq \mathrm{id}_M$, $g^2 = \mathrm{id}_M$, and $\tau \in M(g, -)$. We consider the sublattice

$$M(g, -, \tau) := \{\, x \in M(g, -) \mid \langle x, \tau\rangle_M \equiv 0 \bmod 2 \,\}$$

of $M(g, -)$. Then the following is easy to confirm:

Proposition 4.3. *For an involution* $\gamma := (g, \tau) \in \mathrm{AO}(M)$, *we have*

$$L(\gamma, -) = \left\{ \left(\frac{\langle x, \tau\rangle_M}{2}, 0, x \right) \;\middle|\; x \in M(g, -, \tau) \right\},$$

which is isomorphic to $M(g, -, \tau)(-1)$. *In particular,* $L(\gamma, -)$ *is negative-definite of rank* $\leq \operatorname{rank} M$, *and hence its orthogonal complement* $L(\gamma, +)$ *is hyperbolic.* □

Remark 4.4. The inclusion $M(-1) \hookrightarrow L$ induces an anti-isomorphism $-q_M \cong q_L$. The action of $\mathrm{AO}(M) \subset \mathrm{O}(L, \mathcal{P})$ on $q_L \cong -q_M$ is the composite of the natural homomorphisms $\mathrm{AO}(M) \to \mathrm{O}(M) \to \mathrm{O}(q_M)$.

4.2. **Proof of Theorem 1.2.** We apply the general results in the previous section to the case where $M = \Lambda$ and $L = L_{26}$, and obtain an isomorphism

$$\cdot\infty \cong \mathrm{O}(L_{26}, \boldsymbol{w}_0).$$

For the 9 conjugacy classes $[\gamma] = [(g, \tau)]$ of involutions of $\cdot\infty$, we explicitly calculate the sublattices $L_{26}(\gamma, +)$ and $L_{26}(\gamma, -)$, and prove Theorem 1.2.

Remark 4.5. For involutions represented by $\gamma = (\varepsilon, 0)$, the computation is trivial by Remark 4.2. For involutions $\gamma = (\varepsilon_{16}, \tau_{120})$ and $\gamma = (\varepsilon_{16}, \tau_{135})$, we make use of the uniqueness of the unimodular lattices $\mathrm{I}_{1,9}$ and $\mathrm{II}_{1,9}$, respectively.

Remark 4.6. The lattice $L_{26}(\gamma, +)$ for $\gamma = \gamma_{24,2m} = (-\mathrm{id}, t_m)$ with $2m \in \{4, 6, 8\}$ has a basis $\{\boldsymbol{w}_0, 2\boldsymbol{w}_1 + \tau\}$. under which the Gram matrix is equal to (1.3). The isomorphism classes of the lattices $L_{26}(\gamma, -)$ for $\gamma = \gamma_{24,6}$ and $\gamma = \gamma_{24,8}$ are distinguished by their discriminant forms, which are anti-isomorphic to $q_{V_6} \cong a \oplus b$ and $q_{V_8} \cong u$, respectively.

5. Borcherds method for the invariant sublattices of involutions

The Borcherds method is a method for computing a generating set of a subgroup of $\mathrm{O}(S, \mathcal{P}_S)$, where S is a hyperbolic primitive sublattice of $L_{26} = U \oplus \Lambda(-1)$. In Sections 5.1 and 5.2, we fix notions and terminology for the Borcherds method. In Section 5.3, we fix an involution $\gamma = (\varepsilon, \tau) \in \cdot\infty$, and apply the Borcherds method to the primitive hyperbolic sublattice $L_+ := L_{26}(\gamma, +)$. We define a subgroup $\mathit{\Gamma}$ of $\mathrm{O}(L_+, \mathcal{P}_+)$ with finite index, and establish some facts for the computation of a generating set of $\mathit{\Gamma}$, which will be carried out in the next section.

5.1. **Conway chamber.** Let $\mathcal{P}_{26}$ be the positive cone of L_{26} containing $\boldsymbol{w}_0 \in U$ in its closure. We call standard fundamental domains of the action of $W(L_{26})$ on $\mathcal{P}_{26}$ *Conway chambers.* A *Weyl vector* is a non-zero primitive vector $\boldsymbol{w} \in L_{26}$ with $\langle \boldsymbol{w}, \boldsymbol{w} \rangle = 0$ contained in the closure of $\mathcal{P}_{26}$ such that the negative-definite lattice $[\boldsymbol{w}]^\perp/[\boldsymbol{w}]$ is isomorphic to $\Lambda(-1)$, where $[\boldsymbol{w}]^\perp := \{\, v \in L_{26} \mid \langle \boldsymbol{w}, v \rangle = 0 \,\}$. For example, the vector $\boldsymbol{w}_0 \in L_{26}$ is a Weyl vector.

Let $\boldsymbol{w}$ be a Weyl vector. A (-2)-vector r of L_{26} is called a *Leech root with respect to* $\boldsymbol{w}$ if $\langle r, \boldsymbol{w} \rangle = 1$. The set of all Leech roots with respect to $\boldsymbol{w}$ is denoted by $\mathcal{L}(\boldsymbol{w})$. We then put

$$C(\boldsymbol{w}) := \{\, x \in \mathcal{P}_{26} \mid \langle x, r \rangle \geq 0 \ \text{ for all } \ r \in \mathcal{L}(\boldsymbol{w}) \,\}.$$

Conway [6] proved the following:

Theorem 5.1 (Conway)**.** *The mapping $\boldsymbol{w} \mapsto C(\boldsymbol{w})$ is a bijection from the set of Weyl vectors of L_{26} to the set of Conway chambers.* □

Corollary 5.2. *The embedding $\cdot\infty \hookrightarrow \mathrm{O}(L_{26}, \mathcal{P}_{26})$ given in Section 4.1 gives rise to an isomorphism $\cdot\infty \cong \mathrm{O}(L_{26}, C(\boldsymbol{w}_0))$.* □

We have

$$\mathcal{L}(\boldsymbol{w}_0) = \{\, r(\lambda) \mid \lambda \in \Lambda \,\}, \quad \text{where} \quad r(\lambda) := \left(\frac{\langle \lambda, \lambda \rangle_\Lambda - 2}{2},\ 1,\ \lambda \right). \tag{5.1}$$

5.2. **$\mathcal{F}$-chambers and Γ-simpleness.** Let S be a hyperbolic lattice and $\mathcal{P}_S$ a positive cone of S. Let $\mathcal{F}$ be a locally finite family of hyperplanes in $\mathcal{P}_S$. An *$\mathcal{F}$-chamber* is the closure in $\mathcal{P}_S$ of a connected component of the complement

$$\mathcal{P}_S \ \setminus \ \bigcup_{H \in \mathcal{F}} H$$

of the union of all hyperplanes in $\mathcal{F}$. Let D be an $\mathcal{F}$-chamber. A *wall* of D is a closed subset of D of the form $D \cap H$, where H is a member of $\mathcal{F}$ such that $D \cap H$ contains a non-empty open subset of H. We denote by $\mathcal{W}(D)$ the set of walls of D. A vector $v \in S \otimes \mathbb{Q}$ is a *defining vector* of a wall $w \in \mathcal{W}(D)$ if $w = D \cap (v)^\perp$ and $\langle v, a \rangle > 0$ for some (and hence any) interior point a of D. For a wall $D \cap H$ of D, there exists a unique $\mathcal{F}$-chamber D' such that $D \neq D'$ and that $D \cap H = D' \cap H$. We call this D' the *$\mathcal{F}$-chamber adjacent to D across the wall $D \cap H$.*

Let Γ be a subgroup of the group

$$\mathrm{O}(S, \mathcal{F}) := \{\, g \in \mathrm{O}(S, \mathcal{P}_S) \mid \text{the action of } g \text{ on } \mathcal{P}_S \text{ preserves } \mathcal{F} \,\}.$$

Then Γ acts on the set of $\mathcal{F}$-chambers. We say that the set of $\mathcal{F}$-chambers is *Γ-simple* if Γ acts transitively on the set of $\mathcal{F}$-chambers. We fix an $\mathcal{F}$-chamber D_0, and put

$$\mathrm{Stab}_\Gamma(D_0) := \{\, g \in \Gamma \mid D_0^g = D_0 \,\}.$$

Then $\mathrm{Stab}_\Gamma(D_0)$ acts on the set $\mathcal{W}(D_0)$ of walls of D_0. Let $\mathcal{W}(D_0)/\mathrm{Stab}_\Gamma(D_0)$ denote the set of $\mathrm{Stab}_\Gamma(D_0)$-orbits in $\mathcal{W}(D_0)$.

Example 5.3. Consider the case where $S = L_{26}$ and

$$\mathcal{F} = \mathcal{F}_{26} := \{\, (r)^{\perp} \mid r \in L_{26},\ \langle r, r\rangle = -2 \,\}.$$

The Conway chambers are precisely the $\mathcal{F}_{26}$-chambers, and for a Weyl vector $\boldsymbol{w}$, the set $\mathcal{L}(\boldsymbol{w})$ is a set of defining vectors of all walls of $C(\boldsymbol{w})$. By definition of $\mathcal{F}_{26}$, we have $\mathrm{O}(L_{26}, \mathcal{F}_{26}) = \mathrm{O}(L_{26}, \mathcal{P}_{26})$, and for $\Gamma = \mathrm{O}(L_{26}, \mathcal{P}_{26})$, the set of Conway chambers is Γ-simple. Moreover, we have $\mathrm{Stab}_{\Gamma}(C(\boldsymbol{w}_0)) = \cdot\infty$, and $\cdot\infty$ permutes the walls of $C(\boldsymbol{w}_0)$ by $r(\lambda)^{(g,\tau)} = r(\lambda^g + \tau)$, where $r(\lambda) \in \mathcal{L}(\boldsymbol{w}_0)$ is defined in (5.1).

The Borcherds method [1, 2] is based on the following observation:

Proposition 5.4. *Suppose that the set of $\mathcal{F}$-chambers is Γ-simple. For each orbit o of the action of $\mathrm{Stab}_{\Gamma}(D_0)$ on $\mathcal{W}(D_0)$, we choose an element $g_o \in \Gamma$ such that $D_0^{g_o}$ is adjacent to D_0 across a wall belonging to o. Then the group Γ is generated by the union of $\mathrm{Stab}_{\Gamma}(D_0)$ and $\{\, g_o \mid o \in \mathcal{W}(D_0)/\,\mathrm{Stab}_{\Gamma}(D_0) \,\}$.*

Proof. Let φ be an arbitrary element of Γ. We consider the $\mathcal{F}$-chamber D_0^{φ}. Since $\mathcal{P}_S$ is connected, we have a sequence $D_0, D_1, \dots, D_m = D_0^{\varphi}$ of $\mathcal{F}$-chambers such that D_{i-1} and D_i are adjacent across a wall for $i = 1, \dots, m$. We prove by induction on i that there exists a product $g(i)$ of elements of $\mathrm{Stab}_{\Gamma}(D_0)$ and elements g_o for $o \in \mathcal{W}(D_0)/\,\mathrm{Stab}_{\Gamma}(D_0)$ such that $D_i = D_0^{g(i)}$. Then we have $\varphi = hg(m)$ for some $h \in \mathrm{Stab}_{\Gamma}(D_0)$. □

Remark 5.5. We give several remarks on applications of the Borcherds method to K3 and Enriques surfaces.

(1) In these applications, the lattice S is the Néron–Severi lattice of a surface X, and Γ is the image of $\mathrm{Aut}(X)$ in $\mathrm{O}(S, \mathcal{P}_S)$. In many cases, the inclusion $\Gamma \subset \mathrm{O}(S, \mathcal{F})$ is guaranteed by the *period condition* for X. See [19, Section 5.2].

(2) In some applications, the assumption of Γ-simpleness fails. See [19, Section 5.1] for an algorithm that takes the place of Proposition 5.4 in these cases.

(3) In all geometric applications studied so far, the number of walls of an $\mathcal{F}$-chamber is finite. In the situations treated in this paper, the $\mathcal{F}$-chambers have infinitely many walls, as will be explained in the next section.

5.3. **Invariant sublattice of an involution in L_{26}.** We fix an involution

$$\gamma = (\varepsilon, \tau) \in \cdot\infty$$

acting on L_{26}, and apply the Borcherds method to the primitive hyperbolic sublattice

$$L_+ := L_{26}(\gamma, +)$$

of L_{26} with the positive cone $\mathcal{P}_+ := \mathcal{P}_{26} \cap (L_+ \otimes \mathbb{R})$. Let

$$\mathrm{pr}_+ \ :\ L_{26} \otimes \mathbb{R} \to L_+ \otimes \mathbb{R}$$

denote the orthogonal projection. A hyperplane $(r)^{\perp} \in \mathcal{F}_{26}$ of $\mathcal{P}_{26}$ defined by a (-2)-vector $r \in L_{26}$ intersects $\mathcal{P}_+$ if and only if $\langle \mathrm{pr}_+(r), \mathrm{pr}_+(r)\rangle < 0$, and in this case, we have $\mathcal{P}_+ \cap (r)^{\perp} = (\mathrm{pr}_+(r))^{\perp}$. We put

$$\mathcal{F}_+ := \{\, (\mathrm{pr}_+(r))^{\perp} \mid r \in L_{26} \text{ is a } (-2)\text{-vector satisfying } \langle \mathrm{pr}_+(r), \mathrm{pr}_+(r)\rangle < 0 \,\},$$

which is a locally finite family of hyperplanes in $\mathcal{P}_+$. Then a closed subset D of $\mathcal{P}_+$ is an $\mathcal{F}_+$-chamber if and only if the following hold:

(a) D contains a non-empty open subset of $\mathcal{P}_+$, and
(b) there exists a Conway chamber $C(\boldsymbol{w})$ such that $D = \mathcal{P}_+ \cap C(\boldsymbol{w})$.

We consider the following groups:

$$
\begin{aligned}
\widetilde{\Gamma} &:= \{\, \tilde{g} \in \mathrm{O}(L_{26}, \mathcal{P}_{26}) \mid L_+^{\tilde{g}} = L_+ \,\} = \mathrm{Cen}(\mathrm{O}(L_{26}, \mathcal{P}_{26}), \gamma), \\
\Gamma &:= \text{the image in } \mathrm{O}(L_+, \mathcal{P}_+) \text{ of } \widetilde{\Gamma} \text{ under the restriction } \tilde{g} \mapsto \tilde{g}|L_+.
\end{aligned}
\tag{5.2}
$$

Since the action of $\widetilde{\Gamma}$ on $\mathcal{P}_{26}$ preserves $\mathcal{F}_{26}$, we have

$$\Gamma \ \subset \ \mathrm{O}(L_+, \mathcal{F}_+).$$

We will prove that the subgroup Γ of $\mathrm{O}(L_+, \mathcal{P}_+)$ is of finite index, and that we can apply Proposition 5.4 to the set of $\mathcal{F}_+$-chambers to compute Γ.

We put

$$L_- := L_{26}(\gamma, -),$$

and let q_+ and q_- be the discriminant forms of L_+ and L_-, respectively, with the natural homomorphisms $\eta_\pm \colon \mathrm{O}(L_\pm) \to \mathrm{O}(q_\pm)$. Since $L_{26} \subset L_+^\vee \times L_-^\vee$ is unimodular, it induces an isomorphism $q_+ \cong q_-$ and hence gives rise to $\mathrm{O}(q_+) \cong \mathrm{O}(q_-)$. Then we have a fiber-product diagram

$$
\begin{array}{ccc}
\widetilde{\Gamma} & \to & \mathrm{O}(L_-) \\
\downarrow & \square & \downarrow {\scriptstyle \eta_-} \\
\mathrm{O}(L_+, \mathcal{P}_+) & \underset{\eta_+}{\longrightarrow} & \mathrm{O}(q_+) \cong \mathrm{O}(q_-),
\end{array}
\tag{5.3}
$$

which is just a hyperbolic version of the diagram (2.3). Since $\mathrm{O}(q_\pm)$ is finite, the index of Γ in $\mathrm{O}(L_+, \mathcal{P}_+)$ is finite. Moreover, we also have the following:

Proposition 5.6. *Suppose that $\eta_+ \colon \mathrm{O}(L_+, \mathcal{P}_+) \to \mathrm{O}(q_+)$ is surjective. Then the index of Γ in $\mathrm{O}(L_+, \mathcal{P}_+)$ is equal to the index of the image of $\eta_- \colon \mathrm{O}(L_-) \to \mathrm{O}(q_-)$ in $\mathrm{O}(q_-)$.* □

Proposition 5.7. *Every $\mathcal{F}_+$-chamber $D = \mathcal{P}_+ \cap C(\boldsymbol{w})$ contains an interior point of the Conway chamber $C(\boldsymbol{w})$. In particular, for every $\mathcal{F}_+$-chamber D, the Conway chamber $C(\boldsymbol{w}')$ such that $D = \mathcal{P}_+ \cap C(\boldsymbol{w}')$ is unique.*

Proof. Recall from Theorem 1.2 that the orthogonal complement L_- of L_+ in L_{26} is isomorphic to a sublattice of $\Lambda(-1)$, and hence does not contain any (-2)-vector. Therefore no hyperplane $(r)^\perp \in \mathcal{F}_{26}$ of $\mathcal{P}_{26}$ defined by a (-2)-vector r contains the subspace $\mathcal{P}_+$. □

Proposition 5.8. *The closed subset*

$$D_0 := \mathcal{P}_+ \cap C(\boldsymbol{w}_0)$$

of $\mathcal{P}_+$ is an $\mathcal{F}_+$-chamber.

Proof. It is enough to show that D_0 contains a non-empty open subset of $\mathcal{P}_+$. To avoid confusion, we use $\langle\quad\rangle_L$ to denote the intersection form of L_{26} and $\langle\quad\rangle_\Lambda$ to denote that of Λ. For $x \in \Lambda\otimes\mathbb{R}$, we write x^2 for $\langle x, x\rangle_\Lambda$. Let t be a real number with $t > 2$. We put

$$p := t\boldsymbol{w}_0 + \boldsymbol{w}_1 + \boldsymbol{w}_1^\gamma = \left(\ t + \frac{\tau^2}{2},\ \ 2,\ \ \tau\ \right).$$

Then we have $p^\gamma = p$, $\langle p, \boldsymbol{w}_0\rangle_L > 0$, and $\langle p, p\rangle_L > 0$, and hence $p \in \mathcal{P}_+$. We have

$$\langle r(\lambda), p\rangle_L = t + \frac{\lambda^2}{2} + \frac{(\tau-\lambda)^2}{2} - 2 \geq t - 2$$

for any $\lambda \in \Lambda$. Therefore the point p is an interior point of D_0 relative to $\mathcal{P}_+$. □

Proposition 5.9. *The restriction homomorphism $\tilde{g} \mapsto \tilde{g}|L_+$ gives rise to a surjective homomorphism* $\mathrm{Cen}(\cdot\infty, \gamma) \twoheadrightarrow \mathrm{Stab}_\mathit{\Gamma}(D_0)$.

Proof. Recall that $\widetilde{\mathit{\Gamma}} = \mathrm{Cen}(\mathrm{O}(L_{26}, \mathcal{P}_{26}), \gamma)$. Suppose that $\tilde{g} \in \mathrm{Cen}(\cdot\infty, \gamma)$ and we put $g = \tilde{g}|L_+$. Then we have $g \in \mathit{\Gamma}$. Since $\tilde{g}$ stabilizes $C(\boldsymbol{w}_0)$, we have $D_0^g = D_0$. Therefore $g \in \mathrm{Stab}_\mathit{\Gamma}(D_0)$ holds. To show the surjectivity, let g be an element of $\mathrm{Stab}_\mathit{\Gamma}(D_0)$. Since $g \in \mathit{\Gamma}$, we have an element $\tilde{g} \in \widetilde{\mathit{\Gamma}}$ such that $g = \tilde{g}|L_+$. Proposition 5.7 implies that $\tilde{g}$ stabilizes $C(\boldsymbol{w}_0)$, and hence $\tilde{g}$ belongs to $\mathrm{Cen}(\cdot\infty, \gamma)$. □

Combining Proposition 5.9 with the exact sequence (3.5), we see that $\mathrm{Stab}_\mathit{\Gamma}(D_0)$ is finitely generated, and that a generating set can be computed.

Proposition 5.10. *The set of $\mathcal{F}_+$-chambers in $\mathcal{P}_+$ is $\mathit{\Gamma}$-simple.*

Proof. Let $D = \mathcal{P}_+ \cap C(\boldsymbol{w})$ be an $\mathcal{F}_+$-chamber. We will find an element φ of $\mathit{\Gamma}$ that maps D to D_0. There exists an element g of $W(L_{26})$ such that $C(\boldsymbol{w})^g = C(\boldsymbol{w}_0)$. Since ${L_+}^g = L_{26}(g^{-1}\gamma g, +)$, we have $D^g \subset L_{26}(g^{-1}\gamma g, +)\otimes\mathbb{R}$. Since D contains an interior point of $C(\boldsymbol{w})$ by Proposition 5.7, the space $L_{26}(g^{-1}\gamma g, +)\otimes\mathbb{R}$ contains an interior point of $C(\boldsymbol{w}_0)$. Therefore $g^{-1}\gamma g$ stabilizes $C(\boldsymbol{w}_0)$, and hence $g^{-1}\gamma g \in \cdot\infty$. The lattices $L_{26}(g^{-1}\gamma g, +)$ and $L_{26}(g^{-1}\gamma g, -) = {L_-}^g$ are isomorphic to L_+ and L_-, respectively. Hence, by Theorem 1.2, the involutions $g^{-1}\gamma g$ and γ are conjugate in $\cdot\infty$. Thus we have an element $h \in \cdot\infty$ such that $\gamma g h = g h \gamma$. It follows that gh belongs to $\widetilde{\mathit{\Gamma}}$ and hence preserves L_+ and $\mathcal{P}_+$. We then put

$$\varphi := gh|L_+ \in \mathit{\Gamma}.$$

Since $C(\boldsymbol{w})^{gh} = C(\boldsymbol{w}_0)$, we obtain $D^\varphi = D_0$. □

6. Walls of the $\mathcal{F}_+$-chamber D_0

In this section, we fix an involution $\gamma = (\varepsilon, \tau)$ of $\cdot\infty$ as in Section 5.3, and investigate the hyperbolic lattice $L_+ = L_{26}(\gamma, +)$ more closely. Here ε is one of $\varepsilon_8, \varepsilon_{12}, \varepsilon_{16}$. (The case $\varepsilon = \varepsilon_{24}$ is not interesting as $\mathrm{rank}\, L_+ = 2$.) In Section 6.1, we show that the action of $\mathrm{Stab}_\mathit{\Gamma}(D_0)$ on the set $\mathcal{W}(D_0)$ of walls of D_0 has only finitely many orbits. In Sections 6.2, 6.3, 6.4, we compute the index of $\mathit{\Gamma}$ in $\mathrm{O}(L_+, \mathcal{P}_+)$, and describe the walls of D_0.

In this section, we continue to use $\langle\quad\rangle_L$ and $\langle\quad\rangle_\Lambda$ to denote the intersection form of L_{26} and of Λ, respectively. For $x \in \Lambda\otimes\mathbb{R}$, we write x^2 for $\langle x, x\rangle_\Lambda$, as in the proof of Proposition 5.8.

6.1. **Finiteness of the number of orbits in $\mathcal{W}(D_0)$.** The following theorem combined with the results proved in the previous section enables us to compute a finite generating set of $\mathit{\Gamma}$ by applying Proposition 5.4 to $D_0 = \mathcal{P}_+ \cap C(\boldsymbol{w}_0)$. In the proof, we present a method to enumerate the orbits explicitly.

Theorem 6.1. *The action of* $\mathrm{Stab}_{\mathit{\Gamma}}(D_0)$ *on* $\mathcal{W}(D_0)$ *has only finitely many orbits.*

Proof. We put

$$L_+ := L_{26}(\gamma,+),\quad L_- := L_{26}(\gamma,-),\quad \Lambda_+ := \Lambda(\varepsilon,+),\quad \Lambda_- := \Lambda(\varepsilon,-),$$

and denote the orthogonal projections by

$$\mathrm{pr}^L_+ \colon L_{26}\to L_+^\vee,\quad \mathrm{pr}^L_- \colon L_{26}\to L_-^\vee,\qquad \mathrm{pr}^\Lambda_+ \colon \Lambda\to \Lambda_+^\vee,\quad \mathrm{pr}^\Lambda_- \colon \Lambda\to \Lambda_-^\vee.$$

Since L_{26} and Λ are unimodular, these projections are surjective. To simplify the notation, for $\lambda\in\Lambda$, we put

$$\lambda_+ := \mathrm{pr}^\Lambda_+(\lambda),\qquad \lambda_- := \mathrm{pr}^\Lambda_-(\lambda).$$

Since Λ_+ and Λ_- are orthogonal, and the component τ of the fixed involution $\gamma = (\varepsilon,\tau)$ belongs to Λ_-, we have

$$\lambda^2 = \lambda_+^2 + \lambda_-^2,\quad \langle\lambda,\tau\rangle_\Lambda = \langle\lambda_-,\tau\rangle_\Lambda,\quad \left(\lambda_+ + \frac{\tau}{2}\right)^2 = \lambda_+^2 + \frac{\tau^2}{4}.$$

For $\lambda\in\Lambda$, we put

$$\begin{aligned} m(\lambda) &:= \mathrm{pr}^L_+(r(\lambda)) = \frac{1}{2}(r(\lambda) + r(\lambda^\varepsilon+\tau)) \\ &= \left(\frac{\lambda^2}{2} - \frac{\langle\lambda,\tau\rangle_\Lambda}{2} + \frac{\tau^2}{4} - 1,\ \ 1,\ \ \lambda_+ + \frac{\tau}{2}\right) \\ &= \left(\frac{\lambda_+^2}{2} + \frac{1}{2}\left(\lambda_- - \frac{\tau}{2}\right)^2 + \frac{\tau^2}{8} - 1,\ \ 1,\ \ \lambda_+ + \frac{\tau}{2}\right). \end{aligned} \tag{6.1}$$

For $v\in\Lambda_-^\vee$, we put

$$n(v) := \left(v - \frac{\tau}{2}\right)^2.$$

Then we have

$$\langle m(\lambda), m(\lambda')\rangle_L = \frac{1}{2}\left((\lambda_+ - \lambda'_+)^2 + n(\lambda_-) + n(\lambda'_-)\ -\ 4\right). \tag{6.2}$$

In particular, we have

$$\langle m(\lambda), m(\lambda)\rangle_L = n(\lambda_-) - 2.$$

We also put

$$\begin{aligned} \widetilde{\mathcal{W}}(D_0) &:= \{\,\lambda\in\Lambda \mid (r(\lambda))^\perp\cap D_0 \text{ is a wall of } D_0\,\}, \\ \mathcal{M}(D_0) &:= \{\,\lambda\in\Lambda \mid \langle m(\lambda), m(\lambda)\rangle_L < 0\,\} = \{\,\lambda\in\Lambda \mid n(\lambda_-) < 2\,\}. \end{aligned}$$

Then we have

$$\widetilde{\mathcal{W}}(D_0) \subset \mathcal{M}(D_0). \tag{6.3}$$

Note that the stabilizer $\mathrm{Cen}(\cdot\infty, \gamma)$ of the Conway chamber $C(\boldsymbol{w}_0)$ acts on $\widetilde{\mathcal{W}}(D_0)$ and $\mathcal{M}(D_0)$. Since the restriction homomorphism $\tilde{g} \mapsto \tilde{g}|_{L_+}$ maps $\mathrm{Cen}(\cdot\infty, \gamma)$ to $\mathrm{Stab}_{\Gamma}(D_0)$, it is enough to show that the number of orbits of the action of $\mathrm{Cen}(\cdot\infty, \gamma)$ on $\mathcal{M}(D_0)$ is finite.

We put

$$\Xi_\tau := \left\{\ v \in \Lambda_-^\vee\ \mid n(v) < 2\ \right\}, \tag{6.4}$$

which is a finite subset of $\Lambda_-^\vee$. Since $\mathrm{pr}_-^\Lambda \colon \Lambda \to \Lambda_-^\vee$ is surjective, the projection $\mathcal{M}(D_0) \to \Xi_\tau$ is also surjective. Moreover, by translation, the group Λ_+ acts on each fiber of $\mathcal{M}(D_0) \to \Xi_\tau$ transitively and freely. Therefore we have

$$\mathcal{M}(D_0)/\Lambda_+ \ \cong\ \Xi_\tau.$$

Recall from (3.5) that we have an exact sequence

$$0 \ \longrightarrow\ \Lambda_+ \ \longrightarrow\ \mathrm{Cen}(\cdot\infty, \gamma) \ \longrightarrow\ \Sigma_\varepsilon([\tau]) \ \longrightarrow\ 1, \tag{6.5}$$

where $\Sigma_\varepsilon([\tau])$ is the subgroup of $\mathrm{Cen}(\cdot 0, \varepsilon)$ consisting of $h \in \mathrm{Cen}(\cdot 0, \varepsilon)$ such that there exists an element $\sigma \in \Lambda$ such that $\tau - \tau^h = \sigma - \sigma^\varepsilon$. Therefore the action of $\mathrm{Cen}(\cdot\infty, \gamma)$ on $\mathcal{M}(D_0)$ induces an action of $\Sigma_\varepsilon([\tau])$ on Ξ_τ. This induced action is described easily by introducing

$$\widetilde{\Xi}_\tau := \Xi_\tau - \frac{\tau}{2} = \left\{\ u \in \Lambda_-^\vee \otimes \mathbb{Q}\ \middle|\ u^2 < 2,\ \ u + \frac{\tau}{2} \in \Lambda_-^\vee\ \right\}. \tag{6.6}$$

We show that, via the natural action of $\mathrm{Cen}(\cdot 0, \varepsilon)$ on Λ_-, the subgroup $\Sigma_\varepsilon([\tau])$ of $\mathrm{Cen}(\cdot 0, \varepsilon)$ preserves the subset $\widetilde{\Xi}_\tau$ of $\Lambda_-^\vee \otimes \mathbb{Q}$, and that the induced action of $\Sigma_\varepsilon([\tau])$ on Ξ_τ is equal to the natural action of $\Sigma_\varepsilon([\tau])$ on $\widetilde{\Xi}_\tau$ via the bijection $\widetilde{\Xi}_\tau \cong \Xi_\tau$ given by $x \mapsto x + \tau/2$. Suppose that $h \in \Sigma_\varepsilon([\tau])$ and $v \in \Xi_\tau$. We choose a vector $\sigma \in \Lambda$ such that $\tau - \tau^h = \sigma - \sigma^\varepsilon = 2\sigma_-$. Then $(h, \sigma) \in \mathrm{Cen}(\cdot\infty, \gamma)$ is a lift of h in (6.5). We also choose a lift $\lambda \in \Lambda$ of v, that is, we have $v = \lambda_-$. Then the induced action of h on Ξ_τ maps v to $(\lambda^h + \sigma)_-$. This action is compatible with the natural action $x \mapsto x^h$ on $\widetilde{\Xi}_\tau$, because we have

$$\left(v - \frac{\tau}{2}\right)^h = (\lambda^h + \sigma)_- - \frac{\tau}{2} \ \ \in\ \widetilde{\Xi}_\tau.$$

Therefore we can explicitly calculate the set

$$\mathcal{M}(D_0)/\mathrm{Cen}(\cdot\infty, \gamma) = \mathcal{M}(D_0)/\Lambda_+/\Sigma_g([\tau]) \ \cong\ \Xi_\tau/\Sigma_g([\tau]). \tag{6.7}$$

Since Ξ_τ is finite, we see that $\widetilde{\mathcal{W}}(D_0)/\mathrm{Cen}(\cdot\infty, \gamma)$ is finite. Hence $\mathcal{W}(D_0)/\mathrm{Stab}_{\Gamma}(D_0)$ is also finite. □

We consider the problems when the equality $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$ holds and what are the fibers of the natural mapping $\widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$.

Lemma 6.2. *For $\lambda, \lambda' \in \mathcal{M}(D_0)$, we have*

$$\begin{aligned} & (m(\lambda))^\perp = (m(\lambda'))^\perp \\ \iff \quad & m(\lambda) = m(\lambda') \\ \iff \quad & \lambda_+ = \lambda'_+ \ \text{ and } \ n(\lambda_-) = n(\lambda'_-). \end{aligned}$$

Proof. These follow from the definition (6.1). □

First we recall elementary results in hyperbolic geometry. Let L be a hyperbolic lattice, and $\mathcal{P}$ a positive cone of L. For a vector $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle < 0$, let $(v)^\perp$ denote the hyperplane of $\mathcal{P}$ defined by $\langle x, v \rangle = 0$. Let $v \in L \otimes \mathbb{R}$ be a vector with $\langle v, v \rangle < 0$, and p a point of $\mathcal{P}$. The *foot of the perpendicular from p to the hyperplane* $(v)^\perp$ is given by

$$q := p - \frac{\langle p, v \rangle}{\langle v, v \rangle}\, v.$$

Then we have $\langle p, q \rangle > 0$, and by the Lorentzian Cauchy–Schwarz inequality, we have $\langle q, q \rangle > 0$. Therefore we have $q \in \mathcal{P}$. Let $v' \in L \otimes \mathbb{R}$ be another vector with $\langle v', v' \rangle < 0$. We assume that $\langle p, v \rangle > 0$ and $\langle p, v' \rangle > 0$. Then we have

$$\langle v, v' \rangle \geq 0 \implies \langle q, v' \rangle = \langle p, v' \rangle - \frac{\langle p, v \rangle}{\langle v, v \rangle} \langle v, v' \rangle > 0, \tag{6.8}$$

that is, if $\langle v, v' \rangle \geq 0$, then the foot $q \in (v)^\perp$ is located in the same half-space as p with respect to the hyperplane $(v')^\perp$.

Lemma 6.3. *Suppose that we have $\langle m(\lambda), m(\lambda') \rangle_L \geq 0$ for every pair $\lambda, \lambda' \in \mathcal{M}(D_0)$ satisfying $m(\lambda) \neq m(\lambda')$. Then we have $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$.*

Proof. Since $\mathcal{M}(D_0) \supset \widetilde{\mathcal{W}}(D_0)$, we have

$$D_0 = \{\, x \in \mathcal{P}_+ \mid \langle x, m(\lambda) \rangle_L \geq 0 \ \text{ for any } \ \lambda \in \mathcal{M}(D_0) \,\}.$$

Hence a vector $\lambda \in \mathcal{M}(D_0)$ belongs to $\widetilde{\mathcal{W}}(D_0)$ if and only if $(m(\lambda))^\perp \cap D_0$ contains a non-empty open subset of $(m(\lambda))^\perp$, or equivalently, if and only if there exists a point $q \in (m(\lambda))^\perp$ such that $\langle q, m(\lambda') \rangle_L > 0$ holds for any $\lambda' \in \mathcal{M}(D_0)$ with $m(\lambda) \neq m(\lambda')$. Considering the foot q of the perpendicular from an interior point p of D_0 to the hyperplane $(m(\lambda))^\perp$, we obtain the proof from (6.8). □

For a pair $\xi, \xi' \in \Xi_\tau$, we put

$$\Delta(\xi, \xi') := \{\, l \in \Lambda_+^\vee \mid l + \xi - \xi' \in \Lambda \,\}, \tag{6.9}$$

on which $\Lambda_+ = \Lambda_+^\vee \cap \Lambda$ acts freely and transitively. We then put

$$\ell(\xi, \xi') := \min\{\, l^2 \mid l \in \Delta(\xi, \xi') \,\}. \tag{6.10}$$

Then we obtain the following:

Lemma 6.4. *Suppose that, for any pair $\xi, \xi' \in \Xi_\tau$, we have*

$$(\ n(\xi) \neq n(\xi') \text{ or } \ \ell(\xi, \xi') \neq 0\ ) \implies \ell(\xi, \xi') + n(\xi) + n(\xi') \geq 4. \tag{6.11}$$

Then $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$ holds.

Proof. Assume that $\mathcal{M}(D_0) \supsetneqq \widetilde{\mathcal{W}}(D_0)$. By Lemma 6.3, there exists a pair λ, λ' of elements of $\mathcal{M}(D_0)$ such that $m(\lambda) \neq m(\lambda')$ and $\langle m(\lambda), m(\lambda')\rangle_L < 0$. We will show that the negation of (6.11) holds for $\xi := \lambda_-$ and $\xi' := \lambda'_-$. We put $l := \lambda_+ - \lambda'_+ \in \Lambda_+^\vee$, which is an element of $\Delta(\xi, \xi')$, and hence $\ell(\xi, \xi') \leq l^2$. By (6.2), the assumption $\langle m(\lambda), m(\lambda')\rangle_L < 0$ implies

$$l^2 + n(\xi) + n(\xi') < 4.$$

Thus the right-hand side of $\Longrightarrow$ in (6.11) is false. Moreover, we have $l^2 < 4$. Suppose that $n(\xi) = n(\xi')$. We show that $\ell(\xi, \xi') \neq 0$. By Lemma 6.2, the assumption $m(\lambda) \neq m(\lambda')$ implies $l \neq 0$. If $\ell(\xi, \xi') = 0$, then $\xi - \xi' \in \Lambda$ holds, and hence $\Delta(\xi, \xi') \subset \Lambda$. Then $l \in \Delta(\xi, \xi')$ and $l \neq 0$ imply $l^2 \geq 4$, a contradiction. Thus the left-hand side of $\Longrightarrow$ in (6.11) is true. $\square$

Remark 6.5. Since Ξ_τ is finite, checking the condition (6.11) for all pairs ξ, ξ' can be carried out by a brute-force method. More precisely, choosing an element $\delta \in \Delta(\xi, \xi')$ and fixing a bijection $\Delta(\xi, \xi') \cong \Lambda_+$ by $x \mapsto x - \delta$, the value $\ell(\xi, \xi')$ is calculated as the minimum of an inhomogeneous quadratic form on the free $\mathbb{Z}$-module Λ_+.

We write the natural mapping $\widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$ by μ, that is,

$$\mu(\lambda) := (m(\lambda))^\perp \cap D_0$$

for $\lambda \in \widetilde{\mathcal{W}}(D_0)$. Note that the fiber of μ over a wall w of D_0 is equal to the set of walls $(r(\lambda))^\perp \cap C(\boldsymbol{w}_0)$ of $C(\boldsymbol{w}_0)$ whose intersection with $\mathcal{P}_+$ is w. By Lemma 6.2, we have the following:

Lemma 6.6. *Suppose that $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$ holds. Then the fiber $\mu^{-1}(\mu(\lambda))$ of $\mu\colon \widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$ over a wall $(m(\lambda))^\perp \cap D_0$ of D_0 is equal to*

$$\{\, \lambda' \in \Lambda \mid \lambda'_+ = \lambda_+,\ \ \lambda'_- \in \Xi_\tau,\ \ n(\lambda'_-) = n(\lambda_-) \,\},$$

which is in one-to-one correspondence with the subset

$$\{\, \xi' \in \Xi_\tau \mid \xi' - \lambda_- \in \Lambda,\ \ n(\xi') = n(\lambda_-) \,\}.$$

of Ξ_τ via $\lambda' \mapsto \xi' = \lambda'_-$. $\square$

In the next three sections, we apply these lemmas to the cases $\gamma = (\varepsilon_8, 0), (\varepsilon_{12}, 0)$, and $\gamma = (\varepsilon_{16}, \tau)$ for $\tau = 0, \tau_{120}, \tau_{135}$. More precisely, we carry out the following computations:

- We calculate the index $[\mathrm{O}(L_+, \mathcal{P}_+) : \mathit{\Gamma}]$ by Proposition 5.6, where $\mathit{\Gamma}$ is the group defined in (5.2). For this purpose, we compute the index of the image of $\eta_-\colon \mathrm{O}(L_-) \to \mathrm{O}(q_-)$, and show that $\eta_+\colon \mathrm{O}(L_+, \mathcal{P}_+) \to \mathrm{O}(q_+)$ is surjective. Since L_- is negative-definite, we can use the method described in Section 2.3 for the computation of the index of $\operatorname{Im} \eta_-$. For the surjectivity of η_+, the general theory in [13, Chapter VIII] can be applied. However, we instead give simpler arguments for each case.

γ	L_+	L_-	$\lvert\mathrm{O}(L_-)\rvert$
$(\varepsilon_8, 0)$	$U \oplus \Lambda_{16}(-1)$	$E_8(-2)$	696729600
$(\varepsilon_{12}, 0)$	$U \oplus \Sigma_{12}(-2)$	$\Sigma_{12}(-2)$	980995276800
$(\varepsilon_{16}, 0)$	$U \oplus E_8(-2)$	$\Lambda_{16}(-1)$	89181388800
$(\varepsilon_{16}, \tau_{120})$	$\mathrm{I}_{1,9}(2)$	See Proposition 4.3	743178240
$(\varepsilon_{16}, \tau_{135})$	$\mathrm{II}_{1,9}(2)$	See Proposition 4.3	21139292160

γ	$q_+ \cong q_-$	$\lvert\mathrm{O}(q_+)\rvert = \lvert\mathrm{O}(q_-)\rvert$	index of $\mathrm{Im}\,\eta_-$
$(\varepsilon_8, 0)$	$u^{\oplus 4}$	348364800	1
$(\varepsilon_{12}, 0)$	$a^{\oplus 12}$	51231497335603200	104448
$(\varepsilon_{16}, 0)$	$u^{\oplus 4}$	348364800	2
$(\varepsilon_{16}, \tau_{120})$	$(a \oplus b)^{\oplus 5}$	89181388800	240
(τ_{135})	$u^{\oplus 5}$	46998591897600	71145

TABLE 6.1. Discriminant form and sizes of groups

- We compute the finite set Ξ_τ defined in (6.6). We then verify that $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$ holds by Lemma 6.4. See Remark 6.5. Next we calculate a finite generating set of the subgroup $\Sigma_\varepsilon([\tau])$ of $\mathrm{Cen}(\cdot 0, \varepsilon)$, and compute the orbit decomposition of Ξ_τ under the action of $\Sigma_\varepsilon([\tau])$. Thus we obtain the $\mathrm{Stab}_\Gamma(D_0)$-orbits in $\mathcal{W}(D_0)$.
- For each $\mathrm{Stab}_\Gamma(D_0)$-orbit in $\mathcal{W}(D_0)$, we compute by Lemma 6.6 the fiber $\mu^{-1}(\mu(\lambda))$ of $\mu\colon \widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$ over a representative wall $\mu(\lambda)$ of the orbit.

The results are summarized in Tables 6.1 and 6.2.

6.2. **Involution** $\gamma = (\varepsilon_8, 0)$**.** In this case, we have

$$\begin{aligned} \Lambda_+ &\cong \Lambda_{16}, & \Lambda_- &\cong \Lambda_8 \cong E_8(2),\\ L_+ &\cong U \oplus \Lambda_{16}(-1), & L_- &\cong E_8(-2), \end{aligned}$$

and the discriminant forms $q_+ \cong q_-$ of these lattices are isomorphic to $q_8 = u^{\oplus 4}$. We have proved the surjectivity of η_- in Section 3.3. Hence, by (5.3), we obtain

$$[\mathrm{O}(L_+, \mathcal{P}_+) : \Gamma] = 1.$$

Since $\tau = 0$, we have $\Xi_\tau = \widetilde{\Xi}_\tau$ and $\Sigma_\varepsilon([\tau]) = \mathrm{Cen}(\cdot 0, \varepsilon_8)$. Since E_8 is unimodular, we have $\Lambda_-^\vee = (1/2)E_8(2)$. Consequently, we obtain

$$\Xi_\tau \;=\; \widetilde{\Xi}_\tau \;=\; \{0\} \;\cup\; \{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 1 \,\},$$

which consists of $1 + 240$ vectors. We prove (6.11) holds for all pairs $\xi, \xi' \in \Xi_\tau$ by the method in Remark 6.5. Hence $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$ holds. The action of $\Sigma_\varepsilon([\tau])$ on Ξ_τ is given by the projection $\mathrm{Cen}(\cdot 0, \varepsilon_8) \to \mathrm{O}(\Lambda_-)$. By the diagram (2.3), and results in Section 3.3 (in particular, the fact (3.8)), the image of the projection

γ	index of $\varGamma$	orbits in Ξ_τ	ξ^2 for $\xi \in \widetilde{\Xi}_\tau$	$\mu^{-1}(\mu(\lambda))$
$(\varepsilon_8, 0)$	1	$241 = 1$	0	f_0
		$+240$	1	F_0
$(\varepsilon_{12}, 0)$	104448	$2313 = 1$	0	f_0
		$+264$	1	F_0
		$+2048$	$3/2$	F_1
$(\varepsilon_{16}, 0)$	2	1	0	f_0
$(\varepsilon_{16}, \tau_{120})$	240	512	$3/2$	F_1
$(\varepsilon_{16}, \tau_{135})$	71145	32	1	F_0

Explanation of the column labelled $\mu^{-1}(\mu(\lambda))$:
f_0 means that $\mu^{-1}(\mu(\lambda))$ is a singleton $\{\lambda\}$,
F_0 means that $\mu^{-1}(\mu(\lambda)) = \{\lambda, \lambda'\}$ with $\langle r(\lambda), r(\lambda')\rangle = 0$,
F_1 means that $\mu^{-1}(\mu(\lambda)) = \{\lambda, \lambda'\}$ with $\langle r(\lambda), r(\lambda')\rangle = 1$.

TABLE 6.2. Index $[\mathrm{O}(L_+, \mathcal{P}_+) : \varGamma]$ and orbit decomposition of walls

$\mathrm{Cen}(\cdot 0, \varepsilon_8) \to \mathrm{O}(\Lambda_-)$ is the pullback of $\mathrm{O}_8^+(2) \subset \mathrm{O}(q_8)$ by the natural homomorphism $\eta_8 \colon W(E_8) \to \mathrm{O}(q_8)$. Thus we have

$$\mathrm{Im}(\mathrm{Cen}(\cdot 0, \varepsilon_8) \to \mathrm{O}(\Lambda_-)) = \{\, g \in W(E_8) \mid \det g = 1 \,\} = W(E_8)',$$

where $W(E_8)'$ is the derived group of $W(E_8)$. The action of $W(E_8)'$ on Ξ_τ has exactly two orbits: one is $\{0\}$ and the other is $\{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 1 \,\}$. Hence the action of $\mathrm{Stab}_\varGamma(D_0)$ on $\mathcal{W}(D_0)$ has also two orbits.

The orbit o_1 corresponding to $\{0\}$ consists of walls $\mu(\lambda)$, where λ runs through Λ_+, and the fiber of $\mu \colon \widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$ over $\mu(\lambda) \in o_1$ is a singleton $\{\lambda\}$. The orbit o_{240} corresponding to $\{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 1 \,\}$ consists of walls $\mu(\lambda)$, where $\lambda \in \Lambda$ runs through vectors of Λ satisfying $\lambda_-^2 = 1$. The fiber of $\mu \colon \widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$ over $\mu(\lambda) \in o_{240}$ consists of two vectors $\lambda = \lambda_+ + \lambda_-$ and $\lambda' = \lambda_+ - \lambda_-$. The corresponding pair of Leech roots $r(\lambda)$ and $r(\lambda')$ satisfies $\langle r(\lambda), r(\lambda')\rangle_L = 0$.

6.3. **Involution $\gamma = (\varepsilon_{12}, 0)$.** In this case, we have

$$\begin{aligned} \Lambda_+ &\cong \varSigma_{12}(2), & \Lambda_- &\cong \varSigma_{12}(2), \\ L_+ &\cong U \oplus \varSigma_{12}(-2), & L_- &\cong \varSigma_{12}(-2), \end{aligned}$$

and their discriminant forms $q_+ \cong q_-$ are all isomorphic to $q_{12} := a^{\oplus 12}$.

Proposition 6.7. *The natural homomorphism $\eta_+ \colon \mathrm{O}(L_+, \mathcal{P}_+) \to \mathrm{O}(q_+)$ is surjective. Hence the index $[\mathrm{O}(L_+, \mathcal{P}_+) : \varGamma]$ is equal to* 104448*.*

Proof. We have an inclusion $\mathrm{O}(\Sigma_{12}) \hookrightarrow \mathrm{O}(L_+, \mathcal{P}_+)$ given by $g \mapsto \mathrm{id}_U \oplus g$, and, by the method in Section 2.3, we observe that the image $\eta_+(\mathrm{O}(\Sigma_{12}))$ of this finite subgroup by η_+ is of size $490497638400 = 2^{10} \cdot 12! = |\mathrm{O}(\Sigma_{12})|/2$.

We use the construction of Σ_{12} in Section 3.4. Let v be the vector $2e_1 + \cdots + 2e_6$ of Σ_{12} with norm 6. We consider the vectors

$$\boldsymbol{u} := 2\boldsymbol{w}_0 + 3\boldsymbol{w}_1 + v, \quad \boldsymbol{u}' := 3\boldsymbol{w}_0 + 2\boldsymbol{w}_1 + v,$$

of $L_+ = U \oplus \Sigma_{12}(-2)$ with norm 0, where $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ are the chosen basis of U. Since $\langle \boldsymbol{u}, \boldsymbol{u}' \rangle = 1$, these two vectors generate a hyperbolic plane U' in L_+. The orthogonal complement $(U')^\perp$ of U' in L_+ is of the form $\Sigma'(-2)$, where Σ' is a positive-definite odd unimodular lattice with no vectors of norm 1. Therefore Σ' is isomorphic to Σ_{12}. We calculate the frame of Σ' by the method given in the proof of Lemma 3.5, and find an isomorphism $g' \colon \Sigma_{12} \cong \Sigma'$. Combining $\boldsymbol{w}_0 \mapsto \boldsymbol{u}$, $\boldsymbol{w}_1 \mapsto \boldsymbol{u}'$, and $g' \colon \Sigma_{12} \cong \Sigma'$, we obtain an automorphism $g \colon L_+ \to L_+$. Since $\boldsymbol{w}_0{}^g = \boldsymbol{u}$, the action of g preserves $\mathcal{P}_+$. Then $g \in \mathrm{O}(L_+, \mathcal{P}_+)$ is not contained in the subgroup $\mathrm{O}(\Sigma_{12})$. We calculate the size of the subgroup of $\mathrm{O}(q_+)$ generated by $\eta_+(\mathrm{O}(\Sigma_{12}))$ and $\eta_+(g)$ by the method given in Section 2.3, and confirm that this size is equal to $|\mathrm{O}(q_+)| = 51231497335603200$. Therefore η_+ is surjective.

The computation concerning η_- has already been done in Section 3.4, and we know that the image of the natural homomorphism $\eta_- \colon \mathrm{O}(L_-) \to \mathrm{O}(q_-)$ is of size $|\mathrm{O}(L_-)|/2 = 490497638400$. By Proposition 5.6, we obtain

$$[\mathrm{O}(L_+, \mathcal{P}_+) : \Gamma] = \frac{|\mathrm{O}(q_{12})|}{|\operatorname{Im}(\eta_-)|} = \frac{51231497335603200}{490497638400} = 104448.$$

This proves the proposition. $\square$

Since $\tau = 0$, we have $\Xi_\tau = \widetilde{\Xi}_\tau$ and $\Sigma_\varepsilon([\tau]) = \mathrm{Cen}(\cdot 0, \varepsilon_{12})$. Since Σ_{12} is unimodular, we have $\Lambda_-^\vee = (1/2)\Sigma_{12}(2)$. Therefore we have

$$\Xi_\tau = \widetilde{\Xi}_\tau = \{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 0, \text{ or } \xi^2 = 1, \text{ or } \xi^2 = 3/2 \,\},$$

which is of size $1 + 264 + 2048$. Using the construction of Σ_{12} in Section 3.4 and identifying $\Lambda_-^\vee$ with $(1/2)\Sigma_{12}(2)$, we see that

$$\begin{aligned} \{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 1 \,\} &= \{\, \pm e_i \pm e_j \mid i \neq j \,\}, \\ \{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 3/2 \,\} &= \{\, (\textstyle\sum s_i e_i)/2 \mid s_i \in \{1, -1\} \text{ and } \prod s_i = 1 \,\}. \end{aligned}$$

We prove (6.11) holds for all pairs $\xi, \xi' \in \Xi_\tau$ by the method in Remark 6.5. Hence $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$ holds.

Recall that $C_{12} \in \mathcal{C}_{24}$ is the codeword such that $\varepsilon(C_{12}) = \varepsilon_{12}$. From a generating set of M_{24} given in [8, Chapter 10, Section 2], we compute a generating set of the stabilizer M_{12} of C_{12} in M_{24}. Hence, by (3.9), we can compute the action of $\mathrm{Cen}(\cdot 0, \varepsilon_{12})$ on Λ explicitly. We confirm that this action decomposes $\Xi_\tau = \widetilde{\Xi}_\tau$ into three orbits of size 1, 264, 2048. Hence the action of $\mathrm{Stab}_\Gamma(D_0)$ on $\mathcal{W}(D_0)$ has also three orbits, which are described as follows.

(o_1) The orbit o_1 corresponding to $\{0\}$ consists of walls $\mu(\lambda)$, where λ runs through Λ_+. The fiber of $\mu\colon \widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$ over a wall $\mu(\lambda) \in o_1$ is a singleton $\{\lambda\}$.

(o_{264}) The orbit o_{264} corresponding to $\{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 1 \,\}$ consists of $\mu(\lambda)$, where $\lambda \in \Lambda$ runs through vectors of Λ satisfying $\lambda_-^2 = 1$. The fiber $\mu^{-1}(\mu(\lambda))$ of μ over $\mu(\lambda) \in o_{264}$ consists of two vectors $\lambda = \lambda_+ + \lambda_-$ and $\lambda' = \lambda_+ - \lambda_-$. The corresponding pair of Leech roots $r(\lambda)$, $r(\lambda')$ satisfies $\langle r(\lambda), r(\lambda')\rangle_L = 0$.

(o_{2048}) The orbit o_{2048} corresponding to $\{\, \xi \in \Lambda_-^\vee \mid \xi^2 = 3/2 \,\}$ consists of $\mu(\lambda)$, where $\lambda \in \Lambda$ runs through vectors of Λ satisfying $\lambda_-^2 = 3/2$. For a wall $\mu(\lambda) \in o_{2048}$, the fiber $\mu^{-1}(\mu(\lambda))$ consists of $\lambda = \lambda_+ + \lambda_-$ and $\lambda' = \lambda_+ - \lambda_-$, and the corresponding pair of Leech roots satisfies $\langle r(\lambda), r(\lambda')\rangle_L = 1$.

6.4. **Involutions $\gamma = (\varepsilon_{16}, \tau)$.** We have

$$\Lambda_+ \cong \mathit{\Lambda}_8 \cong E_8(2), \quad \Lambda_- \cong \mathit{\Lambda}_{16}.$$

The isomorphism classes of the lattices $L_\pm$ and the finite quadratic forms $q_\pm$ are described in Table 6.1. The indices of the images of η_- are also given there.

Proposition 6.8. *The natural homomorphism $\eta_+\colon \mathrm{O}(L_+, \mathcal{P}_+) \to \mathrm{O}(q_+)$ is surjective. Hence the index of $\mathit{\Gamma}$ in $\mathrm{O}(L_+, \mathcal{P}_+)$ is*

$$[\mathrm{O}(L_+, \mathcal{P}_+) : \mathit{\Gamma}] = \begin{cases} 2 & \text{if } \tau = 0, \\ 240 & \text{if } \tau = \tau_{120}, \\ 71145 & \text{if } \tau = \tau_{135}. \end{cases}$$

Proof. By the method described in Section 2.3, we calculate the values of $|\mathrm{O}(q_+)|$, which are given in Table 6.1. It is enough to find a finite subset S of $\mathrm{O}(L_+, \mathcal{P}_+)$ such that the subgroup $\langle \eta_+(S)\rangle$ of $\mathrm{O}(q_+)$ generated by $\eta_+(S)$ is of size equal to $|\mathrm{O}(q_+)|$.

Suppose that $\tau = 0$. In this case, we have $L_+ \cong U \oplus E_8(-2)$. We have a natural embedding $\mathrm{O}(E_8) = W(E_8) \hookrightarrow \mathrm{O}(L_+, \mathcal{P}_+)$ given by $g \mapsto \mathrm{id}_U \oplus g$. Then the set S consisting of the standard 8 reflections in $W(E_8)$ satisfies $|\langle \eta_+(S)\rangle| = |\mathrm{O}(q_+)|$.

Suppose that $\tau = \tau_{120}$. In this case, we have $L_+ \cong \mathrm{I}_{1,9}(2)$. By Vinberg's algorithm [24], we obtain a basis of the odd unimodular hyperbolic lattice $\mathrm{I}_{1,9}$ whose Coxeter–Dynkin diagram is shown in the upper part of Figure 6.1. (In this diagram, the black node represents a (-1)-vector, and the dashed edge indicates that $\langle \bullet, \circ\rangle = -1$.) The reflections with respect to these 10 vectors belongs to $\mathrm{O}(L_+, \mathcal{P}_+)$, and the set S consisting of these reflections satisfies $|\langle \eta_+(S)\rangle| = |\mathrm{O}(q_+)|$.

Suppose that $\tau = \tau_{135}$. In this case, we have $L_+ \cong \mathrm{II}_{1,9}(2)$, which has a basis whose Coxeter–Dynkin diagram is shown in the lower part of Figure 6.1. Then the set S of the 10 reflections corresponding to the nodes in this diagram satisfies $|\langle \eta_+(S)\rangle| = |\mathrm{O}(q_+)|$.

By Proposition 5.6, the index $[\mathrm{O}(L_+, \mathcal{P}_+) : \mathit{\Gamma}]$ is equal to the index of the image of the natural homomorphism $\eta_-\colon \mathrm{O}(L_-) \to \mathrm{O}(q_-)$ in $\mathrm{O}(q_-)$. The latter is computed by the method in Section 2.3, and is indicated in Table 6.1. This completes the proof. □

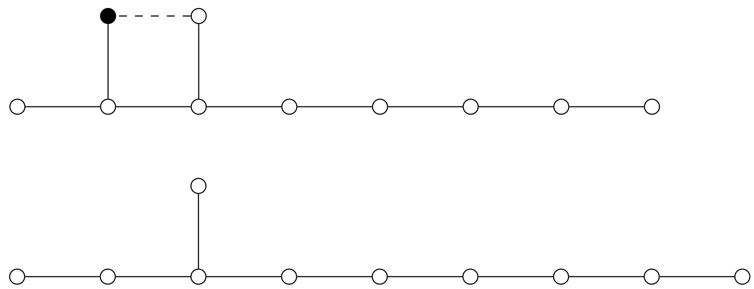


FIGURE 6.1. Coxeter-Dynkin diagrams for $\mathrm{I}_{1,9}$ and $\mathrm{II}_{1,9}$

Next we investigate the walls of D_0. Recall that, by the diagram (2.3) and the surjectivity of the natural homomorphism $\mathrm{O}(\Lambda_+) \cong W(E_8) \to \mathrm{O}(q_+) \cong \mathrm{O}(u^{\oplus 4})$, the projection $\mathrm{Cen}(\cdot 0, \varepsilon_{16}) \to \mathrm{O}(\Lambda_-)$ is surjective. Let $\overline{\Sigma}_\varepsilon([\tau])$ denote the stabilizer of $[\tau] \in \mathrm{Coker}(\pi_{16})$ in $\mathrm{O}(\Lambda_-)$. Then $\overline{\Sigma}_\varepsilon([\tau])$ is the image of $\Sigma_\varepsilon([\tau]) \subset \mathrm{Cen}(\cdot 0, \varepsilon_{16})$ by the projection, and the action of $\Sigma_\varepsilon([\tau])$ on $\widetilde{\widetilde{\Xi}}_\tau$ factors through the surjection $\Sigma_\varepsilon([\tau]) \to \overline{\Sigma}_\varepsilon([\tau])$. Note that, by Remark 3.8, the group K_{16} of order 512 is contained in $\overline{\Sigma}_\varepsilon([\tau])$.

6.4.1. *Case* $\tau = 0$. In this case, we have $\Xi_\tau = \widetilde{\widetilde{\Xi}}_\tau$, and it is a singleton $\{0\}$. We obviously have $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$, and the group $\overline{\Sigma}_\varepsilon([\tau])$ acts transitively $\widetilde{\widetilde{\Xi}}_\tau$. Hence $\mathrm{Stab}_\Gamma(D_0)$ acts transitively on $\mathcal{W}(D_0)$. The fiber of $\mu\colon \widetilde{\mathcal{W}}(D_0) \to \mathcal{W}(D_0)$ over a wall $\mu(\lambda)$ of D_0 is a singleton $\{\lambda\}$.

6.4.2. *Case* $\tau = \tau_{120}$. In this case, we have $|\Xi_\tau| = |\widetilde{\widetilde{\Xi}}_\tau| = 512$, and all vectors of $\widetilde{\widetilde{\Xi}}_\tau$ are of norm $3/2$. We have $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$. We confirm by direct computation that the subgroup K_{16} of $\overline{\Sigma}_\varepsilon([\tau])$ acts on $\widetilde{\widetilde{\Xi}}_\tau$ transitively. Hence, a fortiori, so does $\overline{\Sigma}_\varepsilon([\tau])$, and therefore $\mathrm{Stab}_\Gamma(D_0)$ acts transitively on $\mathcal{W}(D_0)$. The fiber of μ over $\lambda \in \mathcal{W}(D_0)$ consists of two vectors $\lambda = \lambda_+ + \lambda_-$ and $\lambda' = \lambda_+ + \tau - \lambda_-$, and the corresponding pair of Leech roots satisfies $\langle r(\lambda), r(\lambda')\rangle_L = 1$.

6.4.3. *Case* $\tau = \tau_{135}$. In this case, we have $|\Xi_\tau| = |\widetilde{\widetilde{\Xi}}_\tau| = 32$, and all vectors of $\widetilde{\widetilde{\Xi}}_\tau$ are of norm 1. We have $\mathcal{M}(D_0) = \widetilde{\mathcal{W}}(D_0)$. The group K_{16} acts on $\widetilde{\widetilde{\Xi}}_\tau$ transitively, and hence so does $\overline{\Sigma}_\varepsilon([\tau])$. Hence the action of $\mathrm{Stab}_\Gamma(D_0)$ on $\mathcal{W}(D_0)$ is transitive. The fiber $\mu^{-1}(\mu(\lambda))$ consists of two vectors $\lambda = \lambda_+ + \lambda_-$ and $\lambda' = \lambda_+ + \tau - \lambda_-$, and the corresponding pair of Leech roots satisfies $\langle r(\lambda), r(\lambda')\rangle_L = 0$.

Remark 6.9. Let us pursue the heuristic analogy given in Section 1.2 further. We consider $\gamma = (\varepsilon_{16}, \tau_{135})$ as an Enriques involution of the virtual K3 surface $\mathbb{X}$, and let $\mathbb{Y}$ be the corresponding virtual Enriques surface. We consider $C(\boldsymbol{w}_0)$ as the nef-and-big cone of $\mathbb{X}$. Then $D_0 = \mathcal{P}_+ \cap C(\boldsymbol{w}_0)$ is the nef-and-big cone of $\mathbb{Y}$. Each wall of D_0 corresponds to a smooth rational curve on $\mathbb{Y}$, and its pullback splits into a disjoint pair of smooth rational curves on $\mathbb{X}$. Moreover, if we ignore the period condition, we can regard $\cdot\infty$ as the automorphism group of $\mathbb{X}$ and $\mathrm{Stab}_\Gamma(D_0)$ as the automorphism group of $\mathbb{Y}$. Unfortunately, the kernel of the natural homomorphism

$\mathrm{Cen}(\cdot\infty, \gamma) \to \mathrm{Stab}_{\Gamma}(D_0)$ is not equal to $\langle\gamma\rangle$ but of size

$$32 = |\operatorname{Ker}(\mathrm{O}(L_-) \to \mathrm{O}(q_-)|$$

by the diagram (5.3). Therefore, it seems that, for the analogy to be perfect, the period condition should be appropriately considered.

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