INVOLUTIONS IN THE AFFINE CONWAY GROUP: COMPUTATIONAL DATA

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In the files

`affineConwayCompdata.txt` (about 700KB),

we present the computational data used in the preprint

Ichiro Shimada. Involutions in the affine Conway group. Preprint.

This document explains the structure and contents of the data.

Remarks

- In the following, we freely use the notation introduced in the preprint above.
- We use the format of `GAP`. In particular, some parts of the data are given in the `Record` format of `GAP`.
- Elements of a vector space are given by *row* vectors. A linear map is expressed by the multiplication of a matrix from the *right*.

1. Contents of `affineConwayCompdata.txt`

In the following, we arrange the elements of $\Omega = \mathbb{P}^1(\mathbb{F}_{23})$ in order as

$$\infty, 0, 1, 2, \dots, 22,$$

and we fix isomorphisms $\mathbb{F}_2^{\Omega} \cong \mathbb{F}_2^{24}$ and $\mathbb{Z}^{\Omega} \cong \mathbb{Z}^{24}$ accordingly. For example, the vector $v_\infty \in \mathbb{Z}^{\Omega}$ is $[1, 0, \dots, 0]$ and $v_{22} \in \mathbb{Z}^{\Omega}$ is $[0, \dots, 0, 1]$.

- `theOmega` is the list $[$`infinity`$, 0, 1, 2, \dots, 22]$.
- `GolayWords` is the extended binary Golay code $\mathcal{C}_{24} \subset \mathbb{F}_2^{24}$. This data is the list of $2^{12} = 4096$ codewords of $\mathcal{C}_{24}$, each of which is expressed as a subset of `theOmega`.
- `LeechBasis` gives a basis of the Leech lattice $\Lambda \subset \mathbb{Z}^{\Omega}$. This data is a list of 24 vectors in $\mathbb{Z}^{\Omega}$.
- `GramLeech` is the Gram matrix of Λ with respect to `LeechBasis`.

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- `erecs` is the list of 4 records, each of which describes the involution $\varepsilon_n = \varepsilon(C_n) \in \cdot 0$ for $n = 8, 12, 16, 24$. Each record `erec` has the following structure:

```
erecs
   n
   word
   invol
   plusrec
      basis
      Gram
      OGrec
         size
         gens
      Kerqrec
         size
         gens
   minusrec
      basis
      Gram
      OGrec
         size
         gens
      Kerqrec
         size
         gens
```

  - `n` is $n \in \{8, 12, 16, 24\}$.
  - `word` is the element C_n of `GolayWords` such that $\varepsilon_n = \varepsilon(C_n)$.
  - `invol` is the matrix representation of the involusion $\varepsilon_n \in \cdot 0$ with respect to the basis `LeechBasis` of Λ.
  - `plusrec` is a record that describes the invariant part $\Lambda(\varepsilon_n, +)$ of ε_n. When $n = 24$, this is a null record, because $\Lambda(\varepsilon_{24}, +)$ is of rank 0. It contains the following items:
    - `basis` is a basis of $\Lambda(\varepsilon_n, +)$, which is a list of vectors in Λ.
    - `Gram` is the Gram matrix of $\Lambda(\varepsilon_n, +)$ with respect to the `basis` above.
    - `OGrec` is a record that describes the group finite $\mathrm{O}(\Lambda(\varepsilon_n, +))$. It contains the following items:
      - `size` is the size of $\mathrm{O}(\Lambda(\varepsilon_n, +))$.
      - `gens` is a finite set of generators of $\mathrm{O}(\Lambda(\varepsilon_n, +))$, which is a set of matrix representations of isometries of $\Lambda(\varepsilon_n, +)$ with espect to the `basis` above.

  - ∗ `Kerqrec` is a record that describes the kernel of the natural homomorphism $\eta_+ \colon \mathrm{O}(\Lambda(\varepsilon_n, +)) \to \mathrm{O}(q_+)$ to the automorphism group of the discriminant form q_+ of $\Lambda(\varepsilon_n, +)$. It contains the following items:
    - · `size` is the size of $\operatorname{Ker} \eta_+$.
    - · `gens` is a finite set of generators of $\operatorname{Ker} \eta_+$, which is a set of matrix representations of isometries of $\Lambda(\varepsilon_n, +)$ with espect to the `basis` above.
  - `minusrec` is a record that describes the anti-invariant part $\Lambda(\varepsilon_n, -)$ of ε_n. The items in this record are defined in the same way as those of `plusrec` with all "+" being replaced by "−". When $n = 24$, the item `minusrec.Kerqrec` is a null record, because the discriminant group of $\Lambda(\varepsilon_{24}, -) = \Lambda$ is trivial.

- `GramL26` is the Gram matrix of $L_{26} = U \oplus \Lambda(-1)$ with respect to the basis $\boldsymbol{w}_0$, $\boldsymbol{w}_1$ of U and the `LeechBasis` of $\Lambda(-1)$. Hence `GramL26` is the block diagonal matrix with diagonal entries $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and `GramLeech`. In the following, we fix this basis of $L_{26} = U \oplus \Lambda(-1)$ and refer it as `L26Basis`.
- `gammarecs` is the list of 9 records, each of which describes the involution $\gamma = (\varepsilon_n, \tau) \in \cdot\infty$. Each record `gammarec` has the following structure:

```
gammarecs
   n
   epsilon
   tau
   invol
   plusrec
      basis
      Gram
      Xitau
   minusrec
      basis
      Gram
      OGrec
         size
         gens
      Kerqrec
         size
         gens
```

  - `n` is the n in $\gamma = (\varepsilon_n, \tau)$, which is equal to the rank of $L_{26}(\gamma, -)$.

- `epsilon` is the matrix representation of $\varepsilon_n \in \cdot 0$ with respect to `LeechBasis`.
- `tau` is the vector representation of $\tau \in \Lambda$ with respect to `LeechBasis`.
- `invol` is the matrix representation of $\gamma \in \cdot\infty$ with respect to the basis `L26Basis` of L_{26}.
- `plusrec` is a record that describes the invariant part $L_{26}(\gamma, +)$ of γ. It contains the following items:
  - `basis` is a basis of $L_{26}(\gamma, +)$, which is a list of vectors. Each element of `basis` is written as a vector of length 26 with respect to `L26Basis`.
  - `Gram` is the Gram matrix of $L_{26}(\gamma, +)$ with respect to the `basis` above.
  - `Xitau` is the set Ξ_τ when $n \neq 24$, and is `"not available"` when $n = 24$. For $n \neq 24$, each element of `Xitau` is written as a vector of length 24 with respect to `LeechBasis`. (The set Ξ_τ for $n = 24$ is not used in the preprint.)
- `minusrec` is a record that describes the anti-invariant part $L_{26}(\gamma, -)$ of γ. It contains the same items as `erec.plusrec` and `erec.minusrec`, except that, for the case $n = 24$, the records `minusrec.OGrec` and `minusrec.Kerqrec` are null records. (The data for $n = 24$ are not used in the preprint.) Remark that, whereas `erec.minusrec` describes a positive-definite lattice, the record `gammarec.minusrec` describes a a *negative*-definite lattice $L_{26}(\gamma, -)$.

References

[1] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.

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