GENERALIZED ZARISKI-VAN KAMPEN THEOREM AND ITS APPLICATION TO GRASSMANNIAN DUAL VARIETIES

ICHIRO SHIMADA

Dedicated to the memory of Professor Nguyen Huu Duc

Abstract. We formulate and prove a generalization of Zariski-van Kampen theorem on the topological fundamental groups of smooth complex algebraic varieties. As an application, we prove a hyperplane section theorem of Lefschetz-Zariski-van Kampen type for the fundamental groups of the complements to the Grassmannian dual varieties.

1. Introduction

We work over the complex number field $\mathbb{C}$. By a *variety*, we mean a reduced irreducible quasi-projective scheme. The fundamental group $\pi_1(V)$ of a variety V is the topological fundamental group of the analytic space underlying V. The conjunction of paths is read from left to right; that is, for paths $\alpha : I := [0,1] \to V$ and $\beta : I \to V$, we define $\alpha\beta : I \to V$ only when $\alpha(1) = \beta(0)$.

For a subset S of a group G, we denote by $\langle S \rangle$ the subgroup of G generated by the elements of S. Let a group Γ act on G from the right. Then the subgroup

$$N_\Gamma := \langle \, \{ g^{-1} g^{\gamma} \mid g \in G, \gamma \in \Gamma \} \, \rangle$$

of G is normal, because $h^{-1}(g^{-1}g^{\gamma})h = ((gh)^{-1}(gh)^{\gamma})(h^{-1}h^{\gamma})^{-1}$. We then put

$$G/\!/\Gamma := G/N_\Gamma,$$

and call $G/\!/\Gamma$ the *Zariski-van Kampen quotient* of G by Γ.

Let $f : X \to Y$ be a dominant morphism from a smooth variety X to a smooth variety Y with a connected general fiber. There exists a non-empty Zariski open subset $Y^\circ \subset Y$ such that f is locally trivial in the $\mathcal{C}^\infty$-category over Y°. We put $X^\circ := f^{-1}(Y^\circ)$, and denote by $f^\circ : X^\circ \to Y^\circ$ the restriction of f to X°. We choose a base point $b \in Y^\circ$, put $F_b := f^{-1}(b)$, and choose a base point $\tilde{b} \in F_b$.

We investigate the kernel of the homomorphism

$$\iota_* \, : \, \pi_1(F_b, \tilde{b}) \, \to \, \pi_1(X, \tilde{b})$$

induced by the inclusion $\iota : F_b \hookrightarrow X$. The classical Zariski-van Kampen theorem, which started from [29], describes $\mathrm{Ker}(\iota_*)$ in terms of the monodromy action of $\pi_1(Y^\circ, b)$ on $\pi_1(F_b, \tilde{b})$ *under the assumption that a cross-section of f passing through $\tilde{b}$ exists.* The cross-section plays a double role; one is to define the monodromy action of $\pi_1(Y^\circ, b)$ on $\pi_1(F_b, \tilde{b})$, and the other is to prevent $\pi_2(Y)$ from contributing

2000 *Mathematics Subject Classification.* 14F35, 14D05.

Partially supported by JSPS Core-to-Core Program 18005: "New Developments of Arithmetic Geometry, Motive, Galois Theory, and Their Practical Applications".

to $\mathrm{Ker}(\iota_*)$. However, the cross-section rarely exists in applications. If we do not have any cross-section, then the monodromy of $\pi_1(Y^\circ, b)$ on $\pi_1(F_b)$ is not well-defined, and moreover $\pi_2(Y)$ may contribute to $\mathrm{Ker}(\iota_*)$. (See Example 3.4.)

In this paper, we give a generalization of Zariski-van Kampen theorem (Theorem 3.20), which describes $\mathrm{Ker}(\iota_*)$ under weaker conditions on the existence of the cross-section. Informally, our theorem states that, if there exists a cross-section on a subspace of Y whose π_2 surjects to $\pi_2(Y)$, then, under additional assumptions on the singular fibers of f, $\mathrm{Ker}(\iota_*)$ is generated by the monodromy relations arising from the *lifted monodromy*, which is defined as follows.

Since $f^\circ : X^\circ \to Y^\circ$ is locally trivial, the groups $\pi_1(f^{-1}(f(x)), x)$ form a locally constant system on X° when x moves on X°, and hence $\pi_1(X^\circ, \tilde{b})$ acts on $\pi_1(F_b, \tilde{b})$ from the right in a natural way. We denote this action by

$$\mu \,:\, \pi_1(X^\circ, \tilde{b}) \;\to\; \mathrm{Aut}(\pi_1(F_b, \tilde{b})), \tag{1.1}$$

and call μ the *lifted monodromy.*

Combining our main result with Nori's lemma [14] (see Proposition 3.1), we obtain the following:

Corollary 1.1. *Suppose that the following three conditions are satisfied:*

(C1) *the locus* $\mathrm{Sing}(f)$ *of critical points of* f *is of codimension* ≥ 2 *in* X*,*

(C2) *there exists a Zariski closed subset* Ξ_0 *of* Y *with codimension* ≥ 2 *such that* $F_y := f^{-1}(y)$ *is non-empty and irreducible for any* $y \in Y \setminus \Xi_0$*, and*

(Z) *there exist a subspace* $Z \subset Y$ *containing* b *and a continuous cross-section* $s_Z : Z \to f^{-1}(Z)$ *of* f *over* Z *satisfying* $s_Z(Z) \cap \mathrm{Sing}(f) = \emptyset$ *and* $s_Z(b) = \tilde{b}$ *such that the inclusion* $Z \hookrightarrow Y$ *induces a surjection* $\pi_2(Z, b) \twoheadrightarrow \pi_2(Y, b)$*.*

Let $i_{X*} : \pi_1(X^\circ, \tilde{b}) \to \pi_1(X, \tilde{b})$ *be the homomorphism induced by the inclusion* $i_X : X^\circ \hookrightarrow X$*. Then* $\mathrm{Ker}(\iota_*)$ *is equal to*

$$\mathcal{R} := \langle\, \{ g^{-1} g^{\mu(\gamma)} \mid g \in \pi_1(F_b, \tilde{b}),\ \gamma \in \mathrm{Ker}(i_{X*}) \} \,\rangle, \tag{1.2}$$

and we have the exact sequence

$$1 \;\longrightarrow\; \pi_1(F_b, \tilde{b}) /\!/ \mathrm{Ker}(i_{X*}) \;\xrightarrow{\iota_*}\; \pi_1(X, \tilde{b}) \;\xrightarrow{f_*}\; \pi_1(Y, b) \;\longrightarrow\; 1.$$

Remark 1.2. The condition (Z) is trivially satisfied if $\pi_2(Y) = 0$; for example, when Y is an affine space $\mathbb{A}^N$, an abelian variety, or a Riemann surface of genus > 0.

In our previous papers [17], [23] and [24], we have given three different proofs to a special case of Theorem 3.20, where Y is an affine space $\mathbb{A}^N$. Even this special case has yielded many applications ([16, 18, 19, 20, 21, 22, 25]). Thus we can expect more applications of the generalized Zariski-van Kampen theorem of this paper.

As an easy application, we obtain the following:

Corollary 1.3. *Let* $f : X \to Y$ *be a morphism from a smooth variety* X *to a smooth variety* Y*. Suppose that* $\pi_2(Y) = 0$*, that* f *is projective with the general fiber* F_b *being connected, and that* $\mathrm{Sing}(f)$ *is of codimension* ≥ 3 *in* X*. Let* $\iota : F_b \hookrightarrow X$ *be the inclusion. Then the sequence*

$$1 \;\longrightarrow\; \pi_1(F_b) \;\xrightarrow{\iota_*}\; \pi_1(X) \;\xrightarrow{f_*}\; \pi_1(Y) \;\longrightarrow\; 1$$

is exact.

As the next application, we investigate the fundamental group of the complement of the *Grassmannian dual variety*, and prove a hyperplane section theorem of Zariski-Lefschetz-van Kampen type.

A Zariski closed subset of a projective space $\mathbb{P}^N$ is said to be *non-degenerate* if it is not contained in any hyperplane of $\mathbb{P}^N$. We denote by $\mathrm{Gr}^c(\mathbb{P}^N)$ the Grassmannian variety of $(N-c)$-dimensional linear subspaces of $\mathbb{P}^N$. For a point $t \in (\mathbb{P}^N)^\vee = \mathrm{Gr}^1(\mathbb{P}^N)$ of the dual projective space, let $H_t \subset \mathbb{P}^N$ denote the corresponding hyperplane.

Let W be a closed subscheme of $\mathbb{P}^N$ such that every irreducible component is of dimension n. For $c \le n$, the *Grassmannian dual variety of* W *in* $\mathrm{Gr}^c(\mathbb{P}^N)$ is defined to be the locus of $L \in \mathrm{Gr}^c(\mathbb{P}^N)$ such that the scheme-theoretic intersection of W and the linear subspace $L \subset \mathbb{P}^N$ *fails* to be smooth of dimension $n-c$. For a non-negative integer k, we denote by $U_k(W, \mathbb{P}^N)$ the complement of the Grassmannian dual variety of W in $\mathrm{Gr}^{n-k}(\mathbb{P}^N)$; that is, $U_k(W, \mathbb{P}^N) \subset \mathrm{Gr}^{n-k}(\mathbb{P}^N)$ is the Zariski open subset of all $L \in \mathrm{Gr}^{n-k}(\mathbb{P}^N)$ that intersect W along a smooth scheme of dimension k.

Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension $n \ge 2$. The fundamental group $\pi_1((\mathbb{P}^N)^\vee \setminus X^\vee) = \pi_1(U_{n-1}(X, \mathbb{P}^N))$ of the complement of the dual variety has been studied in several papers (for example, [3, 4]). However, there seem to be few studies on its generalization to Grassmannian varieties. We will investigate the fundamental groups $\pi_1(U_k(X, \mathbb{P}^N))$ for $k = 0, \dots, n-2$.

We choose a *general* line Λ in $(\mathbb{P}^N)^\vee$, and consider the corresponding pencil $\{H_t\}_{t \in \Lambda}$ of hyperplanes. Let $A := \bigcap H_t \cong \mathbb{P}^{N-2}$ denote the axis of the pencil. We put

$$Y_t := X \cap H_t \quad \text{and} \quad Z_\Lambda := X \cap A.$$

Let k be an integer such that $0 \le k \le n-2$. Regarding $\mathrm{Gr}^{c-1}(H_t)$ as a closed subvariety of $\mathrm{Gr}^c(\mathbb{P}^N)$, and $\mathrm{Gr}^{c-2}(A)$ as a closed subvariety of $\mathrm{Gr}^{c-1}(H_t)$, we have canonical inclusions

$$U_k(Z_\Lambda, A) \hookrightarrow U_k(Y_t, H_t) \hookrightarrow U_k(X, \mathbb{P}^N).$$

Since $k \le n-2$, the space $U_k(Z_\Lambda, A)$ is non-empty. (When $k = n-2$, the space $U_{n-2}(Z_\Lambda, A)$ is equal to the one-point set $\mathrm{Gr}^0(A) = \{A\}$.) We choose a base point

$$L_o \in U_k(Z_\Lambda, A),$$

which serves also as a base point of $U_k(X, \mathbb{P}^N)$ and of $U_k(Y_t, H_t)$ by the natural inclusions above. Consider the space

$$\mathcal{U}_k(X, \mathbb{P}^N, \Lambda) := \{ \, (L, t) \in U_k(X, \mathbb{P}^N) \times \Lambda \;\mid\; L \subset H_t \, \}$$

with the projection

$$f_\Lambda : \mathcal{U}_k(X, \mathbb{P}^N, \Lambda) \to \Lambda.$$

The fiber of f_Λ over $t \in \Lambda$ is canonically identified with $U_k(Y_t, H_t)$, and the point L_o furnishes us with a holomorphic section

$$s_o : \Lambda \to \mathcal{U}_k(X, \mathbb{P}^N, \Lambda)$$

of f_Λ. There exists a proper Zariski closed subset Σ_Λ of Λ such that f_Λ is locally trivial over $\Lambda \setminus \Sigma_\Lambda$ in the $\mathcal{C}^\infty$-category. We choose a base point $0 \in \Lambda \setminus \Sigma_\Lambda$. By the section s_o, the fundamental group $\pi_1(\Lambda \setminus \Sigma_\Lambda, 0)$ acts on $\pi_1(U_k(Y_0, H_0), L_o)$ in the classical (not lifted) monodromy.

Using the fact that $\Lambda \hookrightarrow (\mathbb{P}^N)^\vee$ induces an isomorphism $\pi_2(\Lambda) \cong \pi_2((\mathbb{P}^N)^\vee)$, we derive from Theorem 3.20 the following:

Theorem 1.4. *Consider the homomorphism*

$$\iota_* \,:\, \pi_1(U_k(Y_0, H_0), L_o) \;\to\; \pi_1(U_k(X, \mathbb{P}^N), L_o)$$

induced by the inclusion $\iota : U_k(Y_0, H_0) \hookrightarrow U_k(X, \mathbb{P}^N)$.

(1) *If* $k \le n-2$, *then* ι_* *is surjective and induces an isomorphism*

$$\pi_1(U_k(Y_0, H_0), L_o) /\!\!/ \pi_1(\Lambda \setminus \Sigma_\Lambda, 0) \;\;\xrightarrow{\sim}\;\; \pi_1(U_k(X, \mathbb{P}^N), L_o).$$

(2) *If* $k < n-2$, *the monodromy action of* $\pi_1(\Lambda \setminus \Sigma_\Lambda, 0)$ *on* $\pi_1(U_k(Y_0, H_0), L_o)$ *is trivial. In particular, the homomorphism* ι_* *is an isomorphism for* $k < n-2$.

Remark that this theorem resembles the classical Lefschetz hyperplane section theorem on the homotopy groups of smooth projective varieties: namely, the inclusion $Y_0 \hookrightarrow X$ induces surjective homomorphisms $\pi_k(Y_0) \twoheadrightarrow \pi_k(X)$ for $k \le n-1$, and isomorphisms $\pi_k(Y_0) \xrightarrow{\sim} \pi_k(X)$ for $k < n-1$.

The isomorphism in the assertion (2) of Theorem 1.4 seems to fail to hold for $k = n-2$, as can be seen from the argument in §6 of this paper.

As the third application, we study $\pi_1(U_k(X, \mathbb{P}^N), L_o)$ for $k = 0$. By Theorem 1.4, it is enough to investigate the case where $\dim X = 2$, and to study the monodromy action of $\pi_1(\Lambda \setminus \Sigma_\Lambda, 0)$ on $\pi_1(U_0(Y_0, H_0), L_o)$, where $Y_0 = X \cap H_0$ is a smooth compact Riemann surface.

First we define the simple braid group SB_g^d of d strings on a compact Riemann surface C of genus $g > 0$. We denote by $\mathrm{Div}^d(C)$ the variety of effective divisors of degree d on C, and by $\mathrm{rDiv}^d(C) \subset \mathrm{Div}^d(C)$ the Zariski open subset consisting of *reduced* divisors. We fix a base point

$$D_0 = p_1 + \cdots + p_d$$

of $\mathrm{rDiv}^d(C)$. The braid group $B_g^d = B(C, D_0)$ is defined to be the fundamental group $\pi_1(\mathrm{rDiv}^d(C), D_0)$. (See [2].)

Definition 1.5. The *simple braid group* $SB_g^d = SB(C, D_0)$ is defined to be the kernel of the homomorphism $B(C, D_0) \to \pi_1(\mathrm{Div}^d(C), D_0)$ induced by the inclusion $\mathrm{rDiv}^d(C) \hookrightarrow \mathrm{Div}^d(C)$.

Let $\mathcal{M}_g^d = \mathcal{M}(C, D_0)$ be the topological group of orientation-preserving diffeomorphisms γ of C acting from the right that satisfy ${p_i}^\gamma = p_i$ for each point p_i of D_0. We denote by

$$\mathit{\Gamma}_g^d = \mathit{\Gamma}(C, D_0) := \pi_0(\mathcal{M}(C, D_0))$$

the group of isotopy classes of diffeomorphisms in $\mathcal{M}_g^d = \mathcal{M}(C, D_0)$, which acts on $SB_g^d = SB(C, D_0)$ from the right in a natural way.

Let $C \subset \mathbb{P}^M$ be a smooth non-degenerate projective curve of degree d and genus $g > 0$, and let $D_0 \in \mathrm{rDiv}^d(C)$ be a general hyperplane section. We will investigate $\pi_1(U_0(C, \mathbb{P}^M), D_0)$; that is, the fundamental group of the complement of the *dual hypersurface* of C.

In [8] and [23], we studied this group under conditions that $d \ge 2g+2$ and that the invertible sheaf $\mathcal{O}_C(D_0)$ corresponds to a *general* point of the Picard variety $\mathrm{Pic}^d(C)$ of isomorphism classes of line bundles of degree d.

Using the fact that $\pi_2(\mathrm{Pic}^d(C)) = 0$, we derive from our main theorem (Theorem 3.20) the following result, which states the same result as in [8] and [23] under weaker conditions.

Definition 1.6. We say that $C \subset \mathbb{P}^M$ is *Plücker general* if the dual curve $\rho(C)^\vee \subset (\mathbb{P}^2)^\vee$ of the image $\rho(C) \subset \mathbb{P}^2$ of the general projection $\rho : C \to \mathbb{P}^2$ has only ordinary nodes and ordinary cusps as its singularities.

Theorem 1.7. *Suppose that $d \geq g+4$ and that C is Plücker general in $\mathbb{P}^M$. Then $\pi_1(U_0(C, \mathbb{P}^M), D_0)$ is isomorphic to $SB(C, D_0)$.*

Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective surface of degree d, and let $\{Y_t\}_{t\in\Lambda}$ be a pencil of hyperplane sections of X parameterized by a general line $\Lambda \subset (\mathbb{P}^N)^\vee$ with the base locus $Z_\Lambda := X \cap A$, where $A = \bigcap H_t$ is the axis of the pencil $\{H_t\}_{t\in\Lambda}$ of hyperplanes. Let

$$\varphi : \mathcal{Y} := \{\ (x,t) \in X \times \Lambda\ \mid\ x \in H_t\ \} \to \Lambda$$

be the fibration of the pencil. Then φ is locally trivial over $\Lambda \setminus \Sigma'_\Lambda$ in the $\mathcal{C}^\infty$-category, where Σ'_Λ is the set of critical values of φ. Let 0 be a general point of Λ. The corresponding member Y_0 is a compact Riemann surface of genus

$$g := (d + H_0 \cdot K_X)/2 + 1.$$

Note that $U_0(Z_\Lambda, A) = \{A\}$, and that each point of Z_Λ yields a holomorphic section of $\varphi : \mathcal{Y} \to \Lambda$. By the classical monodromy, we obtain a homomorphism

$$\pi_1(\Lambda \setminus \Sigma'_\Lambda, 0) \ \to\ \mathit{\Gamma}^d_g = \mathit{\Gamma}(Y_0, Z_\Lambda), \tag{1.3}$$

and hence $\pi_1(\Lambda \setminus \Sigma'_\Lambda, 0)$ acts on the simple braid group $SB^d_g = SB(Y_0, Z_\Lambda)$ from the right. We denote by

$$\Gamma_\Lambda \ \subset\ \mathit{\Gamma}^d_g = \mathit{\Gamma}(Y_0, Z_\Lambda)$$

the image of the monodromy homomorphism (1.3). Combining Theorems 1.4 and 1.7, we obtain the following:

Corollary 1.8. *Let X, $\{Y_t\}_{t\in\Lambda}$, $Z_\Lambda = X \cap A$ and Γ_Λ be as above. Suppose that $g > 0$, $d \geq g+4$, and that a general hyperplane section of X is Plücker general. Then $\pi_1(U_0(X, \mathbb{P}^N), A)$ is isomorphic to the Zariski-van Kampen quotient $SB(Y_0, Z_\Lambda)/\!/\Gamma_\Lambda$.*

A motivation of the study of the fundamental group $\pi_1(U_0(X, \mathbb{P}^N))$ for a surface $X \subset \mathbb{P}^N$ is the conjecture of Auroux, Donaldson, Katzarkov and Yotov [1] about the fundamental group $\pi_1(\mathbb{P}^2 \setminus B)$ of the complement of the branch curve $B \subset \mathbb{P}^2$ of the general projection $X \to \mathbb{P}^2$, which had been intensively studied by Moishezon, Teicher, Robb. The weakening of the conditions from our previous works ([8], [23]) to the present result (Theorem 1.7) is important with respect to this application. See Remark 6.4.

The plan of this paper is as follows. In §2, we state some elementary facts about Zariski-van Kampen quotients. In §3, we prove the generalized Zariski-van Kampen theorem (Theorem 3.20). We then prove its variant (Theorem 3.33), and deduce Corollaries 1.1 and 1.3. The main ingredient of the proof is the notion of *free loop pairs of monodromy relation type* (Definitions 3.23 and 3.24), and Proposition 3.29. Using these results, we prove Theorem 1.4 in §4, and Theorem 1.7 in §5. In the last section, we explain the relation between $\pi_1(U_0(X, \mathbb{P}^N))$ and the conjecture of Auroux, Donaldson, Katzarkov, Yotov.

Conventions and Notation

(1) The constant map to a point P is denoted by 1_P.
(2) We denote by $I \subset \mathbb{R}$ the interval $[0,1]$, by $\Delta \subset \mathbb{C}$ the open unit disc, and by $\bar{\Delta} \subset \mathbb{C}$ the closed unit disc.
(3) For a continuous map $\delta : \bar{\Delta} \to T$ to a topological space T, we denote by
$$\partial_\varepsilon \delta \;:\; I \;\to\; T$$
the loop given by $t \mapsto \delta(\exp(2\pi\sqrt{-1}t))$.

2. Zariski-van Kampen quotient

Definition 2.1. Let G be a group, and let S be a subset of G. We denote by $\langle S \rangle_G$ or simply by $\langle S \rangle$ the smallest subgroup of G containing S, and by $\langle\langle S \rangle\rangle_G$ or simply by $\langle\langle S \rangle\rangle$ the smallest *normal* subgroup of G containing S.

We let a group Γ act on a group G from the right. The following are easy:

Lemma 2.2. *For any $\gamma \in \Gamma$, the subgroup $\langle \{g^{-1}g^\gamma \mid g \in G\} \rangle_G$ of G is normal. Hence, for any subset $\Sigma \subset \Gamma$, the subgroup $\langle \{g^{-1}g^\sigma \mid g \in G, \sigma \in \Sigma\} \rangle_G$ is normal.*

Lemma 2.3. *Let S be a subset of G, and let Σ be a subset of Γ. If $G = \langle S \rangle_G$ and $\Gamma = \langle \Sigma \rangle_\Gamma$, then we have*
$$\langle\langle \{s^{-1}s^\sigma \mid s \in S, \sigma \in \Sigma\} \rangle\rangle_G = \langle \{g^{-1}g^\sigma \mid g \in G, \sigma \in \Sigma\} \rangle_G = \langle \{g^{-1}g^\gamma \mid g \in G, \gamma \in \Gamma\} \rangle_G.$$

Definition 2.4. We define $G \rtimes \Gamma$ to be the group with the underlying set $G \times \Gamma$ and with the product defined by
$$(g, \gamma)(h, \delta) := (g \cdot \left(h^{(\gamma^{-1})}\right), \gamma\delta).$$
We then define homomorphisms $i : G \to G \rtimes \Gamma$, $p : G \rtimes \Gamma \to \Gamma$ and $s : \Gamma \to G \rtimes \Gamma$ by $i(g) := (g, 1)$, $p(g, \gamma) := \gamma$ and $s(\gamma) := (1, \gamma)$. Then we obtain an exact sequence
$$1 \;\longrightarrow\; G \;\xrightarrow{\;i\;}\; G \rtimes \Gamma \;\xrightarrow{\;p\;}\; \Gamma \;\longrightarrow\; 1 \tag{2.1}$$
with the cross-section s of p, and the action $g \mapsto g^\gamma$ of $\gamma \in \Gamma$ on G coincides with the inner-automorphism $g \mapsto s(\gamma)^{-1}gs(\gamma)$ by $s(\gamma) \in G \rtimes \Gamma$ on the normal subgroup $G = i(G)$ of $G \rtimes \Gamma$.

The following two lemmas are elementary:

Lemma 2.5. *Let $\mathcal{G}$ be a group. Suppose that we are given an exact sequence*
$$1 \;\longrightarrow\; G \;\xrightarrow{\;i'\;}\; \mathcal{G} \;\xrightarrow{\;p'\;}\; \Gamma \;\longrightarrow\; 1 \tag{2.2}$$
with a cross-section $s' : \Gamma \to \mathcal{G}$ of p' that is a homomorphism of groups. Suppose also that the action of $\gamma \in \Gamma$ on $g \in G$ is equal to the inner-automorphism by $s'(\gamma)$; that is, we have $i'(g^\gamma) = s'(\gamma)^{-1}i'(g)s'(\gamma)$ for any $g \in G$ and $\gamma \in \Gamma$. Then there exists an isomorphism $\mathcal{G} \cong G \rtimes \Gamma$ such that the exact sequences (2.1) *and* (2.2) *coincide and the cross-section s corresponds to s' by this isomorphism.*

Lemma 2.6. *The composite homomorphism*
$$G \;\xrightarrow{\;i\;}\; G \rtimes \Gamma \;\longrightarrow\; (G \rtimes \Gamma)/\langle\langle s(\Gamma) \rangle\rangle_{G \rtimes \Gamma}$$
is surjective, and its kernel is equal to $\langle \{g^{-1}g^\gamma \mid g \in G, \gamma \in \Gamma\} \rangle$; that is, the Zariski-van Kampen quotient $G /\!/ \Gamma$ is isomorphic to $(G \rtimes \Gamma)/\langle\langle s(\Gamma) \rangle\rangle$.

3. Fundamental groups of algebraic fiber spaces

Let X and Y be smooth varieties, and let $f : X \to Y$ be a dominant morphism. We denote by $\mathrm{Sing}(f) \subset X$ the Zariski closed subset of the critical points of f. For a point $y \in Y$, we put

$$F_y := f^{-1}(y).$$

Let $\alpha : T \to Y$ be a continuous map from a topological space T. Then a continuous map $\tilde{\alpha} : T \to X$ is said to be a *lift of* α if $f \circ \tilde{\alpha} = \alpha$.

We fix, once and for all, a proper Zariski closed subset

$$\Sigma \subset Y$$

such that $f^\circ : X^\circ \to Y^\circ$ is locally trivial in the $\mathcal{C}^\infty$-category, where

$$Y^\circ := Y \setminus \Sigma, \quad X^\circ := f^{-1}(Y^\circ) \quad \text{and} \quad f^\circ := f|_{X^\circ} : X^\circ \to Y^\circ.$$

(In particular, $\mathrm{Sing}(f)$ is contained in $f^{-1}(\Sigma)$.) It follows from Hironaka's resolution of singularities that such a proper Zariski closed subset $\Sigma \subset Y$ exists. We then fix base points

$$b \in Y^\circ \quad \text{and} \quad \tilde{b} \in F_b \subset X^\circ,$$

and consider the homomorphisms

$$\iota_* : \pi_1(F_b, \tilde{b}) \to \pi_1(X, \tilde{b}) \quad \text{and} \quad f_* : \pi_1(X, \tilde{b}) \to \pi_1(Y, b)$$

induced by the inclusion $\iota : F_b \hookrightarrow X$ and the morphism $f : X \to Y$, respectively. The aim of Zariski-van Kampen theorem is to describe $\mathrm{Ker}(\iota_*)$.

The following result of Nori [14] will be used throughout this paper:

Proposition 3.1. *Suppose that F_b is connected, and that there exists a Zariski closed subset $\Xi' \subset Y$ of codimension ≥ 2 such that $F_y \setminus (F_y \cap \mathrm{Sing}(f)) \neq \emptyset$ for any $y \in Y \setminus \Xi'$. Then $f_* : \pi_1(X, \tilde{b}) \to \pi_1(Y, b)$ is surjective, and its kernel is equal to the image of $\iota_* : \pi_1(F_b, \tilde{b}) \to \pi_1(X, \tilde{b})$.*

Proof. See Nori [14, Lemma 1.5] and [23, Proposition 3.1]. □

Let $\tilde{\alpha} : I \to X^\circ$ be a path, and we put $\alpha := f^\circ \circ \tilde{\alpha}$. Then $\tilde{\alpha}$ induces an isomorphism $\pi_1(F_{\alpha(0)}, \tilde{\alpha}(0)) \xrightarrow{\sim} \pi_1(F_{\alpha(1)}, \tilde{\alpha}(1))$, which depends only on the homotopy class (relative to ∂I) of the path $\tilde{\alpha}$. Hence we can write this isomorphism as

$$[\tilde{\alpha}]_* : \pi_1(F_{\alpha(0)}, \tilde{\alpha}(0)) \xrightarrow{\sim} \pi_1(F_{\alpha(1)}, \tilde{\alpha}(1)).$$

The *lifted monodromy*

$$\mu : \pi_1(X^\circ, \tilde{b}) \to \mathrm{Aut}(\pi_1(F_b, \tilde{b}))$$

introduced in §1 (see (1.1)) is obtained by applying this construction to the loops in X° with the base point $\tilde{b}$. By the definition, we have the following:

Proposition 3.2. *For any $[\tilde{\alpha}] \in \pi_1(X^\circ, \tilde{b})$ and $g \in \pi_1(F_b, \tilde{b})$, we have*

$$\iota^\circ_*(g^{\mu([\tilde{\alpha}])}) = [\tilde{\alpha}]^{-1} \cdot \iota^\circ_*(g) \cdot [\tilde{\alpha}]$$

in $\pi_1(X^\circ, \tilde{b})$, where $\iota^\circ_* : \pi_1(F_b, \tilde{b}) \to \pi_1(X^\circ, \tilde{b})$ is the homomorphism induced by the inclusion $\iota^\circ : F_b \hookrightarrow X^\circ$.

First we prove the following:

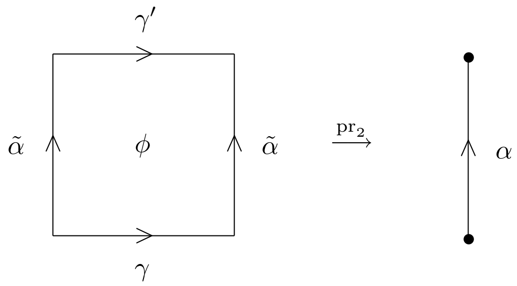


FIGURE 3.1. The extension ϕ

Proposition 3.3. *Suppose that a loop* $\tilde{\alpha} : (I, \partial I) \to (X^\circ, \tilde{b})$ *is null-homotopic in* $(X, \tilde{b})$. *Then* $g^{-1} g^{\mu([\tilde{\alpha}])} \in \mathrm{Ker}(\iota_*)$ *for any* $g \in \pi_1(F_b, \tilde{b})$.

Proof. We put $\alpha := f^\circ \circ \tilde{\alpha}$, and $\sqcup := (I \times \{0\}) \cup (\partial I \times I)$. Let $g \in \pi_1(F_b, \tilde{b})$ be represented by a loop $\gamma : (I, \partial I) \to (F_b, \tilde{b})$. We define $\phi_\sqcup : \sqcup \to X^\circ$ by

$$\phi_\sqcup(s, 0) := \gamma(s), \quad \phi_\sqcup(0, t) := \tilde{\alpha}(t), \quad \text{and} \quad \phi_\sqcup(1, t) := \tilde{\alpha}(t).$$

Then we have $f^\circ \circ \phi_\sqcup = (\alpha \circ \mathrm{pr}_2)|_\sqcup$, where $\mathrm{pr}_2 : I \times I \to I$ is the second projection. Since $\sqcup$ is a strong deformation retract of $I \times I$ and f° is locally trivial, the extension of $(\alpha \circ \mathrm{pr}_2)|_\sqcup : \sqcup \to Y^\circ$ to $\alpha \circ \mathrm{pr}_2 : I \times I \to Y^\circ$ lifts to an extension from $\phi_\sqcup : \sqcup \to X^\circ$ to a continuous map $\phi : I \times I \to X^\circ$ that satisfies $\phi|_\sqcup = \phi_\sqcup$ and $f^\circ \circ \phi = \alpha \circ \mathrm{pr}_2$. (See Figure 3.1.) Then the loop

$$\gamma' := \phi|_{I \times \{1\}} \; : \; (I, \partial I) \; \to \; (F_b, \tilde{b})$$

represents $g^{\mu([\tilde{\alpha}])}$. Since $\phi|_{\{0\} \times I} = \tilde{\alpha}$ and $\phi|_{\{1\} \times I} = \tilde{\alpha}$, we have

$$[\gamma]^{-1} [\tilde{\alpha}] [\gamma'] [\tilde{\alpha}]^{-1} = 1$$

in $\pi_1(X^\circ, \tilde{b})$. Since $[\tilde{\alpha}] = 1$ in $\pi_1(X, \tilde{b})$ by the assumption, we have $[\gamma]^{-1}[\gamma'] = 1$ in $\pi_1(X, \tilde{b})$. $\square$

By Proposition 3.3, the normal subgroup $\mathcal{R}$ defined by (1.2) is contained in $\mathrm{Ker}(\iota_*)$. However $\mathcal{R}$ is not equal to $\mathrm{Ker}(\iota_*)$ in general. We give two examples.

Example 3.4. Let $L \to \mathbb{P}^1$ be a line bundle of degree $d > 0$, and let $L^\times \subset L$ be the complement of the zero-section. Since the projection $f : X = L^\times \to Y = \mathbb{P}^1$ is locally trivial, we can put $\Sigma = \emptyset$, and hence $\mathcal{R} = \{1\}$. However, the kernel of

$$\iota_* \; : \; \pi_1(F_b) = \pi_1(\mathbb{C}^\times) \cong \mathbb{Z} \; \to \; \pi_1(L^\times) \cong \mathbb{Z}/d\mathbb{Z}$$

is non-trivial. Indeed, $\mathrm{Ker}(\iota_*)$ is equal to the image of the boundary homomorphism $\pi_2(\mathbb{P}^1) \to \pi_1(\mathbb{C}^\times)$ in the homotopy exact sequence.

Example 3.5. Consider the morphism

$$f \; : \; X = \mathbb{C}^2 \; \to \; Y = \mathbb{C}$$

given by $f(x, y) := xy$. We can put $\Sigma = \{0\}$, and hence the fundamental group of $X^\circ = \mathbb{C}^2 \setminus \{xy = 0\}$ is isomorphic to $\mathbb{Z}^2$. The general fiber F_b is isomorphic to $\mathbb{P}^1$ minus two points, and the lifted monodromy action of $\pi_1(X^\circ)$ on $\pi_1(F_b) \cong \mathbb{Z}$ is trivial. Therefore we have $\mathcal{R} = \{1\}$, while we have $\mathrm{Ker}(\iota_*) = \pi_1(F_b) \cong \mathbb{Z}$.

Our ultimate goal is to show that the three conditions in Corollary 1.1 is sufficient for $\mathcal{R} = \mathrm{Ker}(\iota_*)$ to hold.

From now on, we suppose that $f : X \to Y$ satisfies the first two of the three conditions in Corollary 1.1; namely, we assume the following:

(C1) $\mathrm{Sing}(f)$ is of codimension ≥ 2 in X, and
(C2) there exists a Zariski closed subset $\Xi_0 \subset Y$ of codimension ≥ 2 such that F_y is non-empty and irreducible for any $y \in Y \setminus \Xi_0$.

Remark 3.6. By the conditions (C1) and (C2), the following hold:

(C0) for $y \in Y^\circ$, the fiber F_y is connected, and
(C3) there exists a Zariski closed subset $\Xi_1 \subset Y$ of codimension ≥ 2 such that $F_y \setminus (F_y \cap \mathrm{Sing}(f))$ is non-empty and connected for every $y \in Y \setminus \Xi_1$.

In particular, we see that f_* is surjective and $\mathrm{Im}(\iota_*) = \mathrm{Ker}(f_*)$ holds by Nori's lemma (Proposition 3.1).

Let $\Sigma_1, \dots, \Sigma_N$ be the irreducible components of Σ with codimension 1 in Y. There exists a proper Zariski closed subset $\Xi \subset \Sigma$ with the following properties. We put

$$Y^\sharp := Y \setminus \Xi, \quad \Sigma_i^\sharp := \Sigma_i \setminus (\Sigma_i \cap \Xi) = \Sigma_i \cap Y^\sharp, \quad \Sigma^\sharp := \Sigma \setminus \Xi = \Sigma \cap Y^\sharp.$$

(Ξ0) The codimension of Ξ in Y is ≥ 2.
(Ξ1) The Zariski closed subsets $\Xi_0 \subset Y$ in the condition (C2) and $\Xi_1 \subset Y$ in the condition (C3) are contained in Ξ.
(Ξ2) Each $\Sigma_i^\sharp$ is a smooth hypersurface of $Y^\sharp$, and $\Sigma^\sharp$ is a disjoint union of $\Sigma_1^\sharp, \dots, \Sigma_N^\sharp$; that is, Ξ contains all the irreducible components of Σ with codimension ≥ 2 in Y and the singular locus of Σ.
(Ξ3) For each $y \in \Sigma_i^\sharp$, there exist an open neighborhood $U \subset Y^\sharp$ of y in $Y^\sharp$ and an analytic isomorphism

$$\phi : (U, U \cap \Sigma) \xrightarrow{\sim} \Delta^{m-1} \times (\Delta, 0), \qquad \text{where } m = \dim Y,$$

with the following properties. Let $\psi : U \to \Delta^{m-1}$ be the composite of $\phi : U \cong \Delta^{m-1} \times \Delta$ and the projection $\Delta^{m-1} \times \Delta \to \Delta^{m-1}$. Then

$$\Psi := \psi \circ f \,:\, f^{-1}(U) \,\to\, \Delta^{m-1}$$

is smooth, and the commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "f"] \arrow[dr, "\Psi"'] & & U \arrow[dl, "\psi"] \\ & \Delta^{m-1} & \end{tikzcd}$$

is a trivial family of $\mathcal{C}^\infty$-maps over Δ^{m-1} in the $\mathcal{C}^\infty$-category.

Because of the choice of Ξ, for *any* point $y \in \Sigma_i^\sharp$, there exists an open disc $\Delta \subset Y^\sharp$ with the following properties:

($\Delta^\sharp$1) $\Delta \cap \Sigma = \{y\}$, and Δ intersects $\Sigma_i^\sharp$ transversely at y,
($\Delta^\sharp$2) $f^{-1}(\Delta)$ is a complex manifold,
($\Delta^\sharp$3) $f|_{f^{-1}(\Delta)} : f^{-1}(\Delta) \to \Delta$ is a one-dimensional family of complex analytic spaces that is locally trivial in the $\mathcal{C}^\infty$-category over $\Delta \setminus \{y\}$, and
($\Delta^\sharp$4) the central fiber $F_y := f^{-1}(y)$ is an irreducible hypersurface of $f^{-1}(\Delta)$, and $F_y \setminus (F_y \cap \mathrm{Sing}(f))$ is non-empty and connected.

Moreover the diffeomorphism type of $f|_{f^{-1}(\Delta)} : f^{-1}(\Delta) \to \Delta$ depends only on the index i of Σ_i.

We put

$$X^\sharp := f^{-1}(Y^\sharp),\ \ f^\sharp := f|_{X^\sharp} : X^\sharp \to Y^\sharp,\ \ \Theta^\sharp_i := (f^\sharp)^{-1}(\Sigma^\sharp_i)\ \text{ and }\ \Theta^\sharp := (f^\sharp)^{-1}(\Sigma^\sharp).$$

Then each $\Theta^\sharp_i$ is an irreducible hypersurface of $X^\sharp$, and $\Theta^\sharp$ is a disjoint union of $\Theta^\sharp_1, \dots, \Theta^\sharp_N$. Note that we have $X^\circ = X^\sharp \setminus \Theta^\sharp$.

Remark 3.7. By the condition (C1), the Zariski closed subset $f^{-1}(\Xi)$ of X is also of codimension ≥ 2, and hence the inclusions induce isomorphisms $\pi_1(X^\sharp, \tilde{b}) \cong \pi_1(X, \tilde{b})$ and $\pi_1(Y^\sharp, b) \cong \pi_1(Y, b)$.

We introduce notions of *transversal discs*, *leashed discs* and *lassos*.

Definition 3.8. Let $H \subset M$ be a reduced hypersurface of a complex manifold M of dimension m, and let $H_1, \dots, H_l$ be the irreducible components of H. We fix a base point $b_M \in M \setminus H$.

(1) Let N be a real k-dimensional $\mathcal{C}^\infty$-manifold with $2 \leq k \leq 2m$ (possibly with boundaries and corners), and let $\phi : N \to M$ be a continuous map. Let p be a point of N that is not in the corner of N. If $k = 2$, we further assume that $p \notin \partial N$. We say that $\phi : N \to M$ *intersects* H *at* p *transversely* if the following hold:

(ϕ1) $\phi(p) \in H \setminus \mathrm{Sing}(H)$, and

(ϕ2) there exist local coordinates $(u_1, \dots, u_k)$ of N at p and local coordinates $(v_1, \dots, v_{2m})$ of the $\mathcal{C}^\infty$-manifold underlying M at $\phi(p)$ such that
- $p = (0, \dots, 0)$, $\phi(p) = (0, \dots, 0)$,
- if $p \in \partial N$, then N is given by $u_k \geq 0$ locally at p,
- H is locally defined by $v_1 = v_2 = 0$ in M, and
- ϕ is given by $(u_1, \dots, u_k) \mapsto (v_1, \dots, v_{2m}) = (u_1, \dots, u_k, 0, \dots, 0)$.

We say that $\phi : N \to M$ *intersects* H *transversely* if $\phi^{-1}(H)$ is disjoint from the corner of N (when $k = 2$, we assume that $\phi^{-1}(H) \cap \partial N = \emptyset$) and ϕ intersects H transversely at every point of $\phi^{-1}(H)$.

If ϕ intersects H transversely, then $\phi^{-1}(H)$ is a real $(k-2)$-dimensional submanifold of N. If $k > 2$, then the boundary of $\phi^{-1}(H)$ is equal to $\phi^{-1}(H) \cap \partial N$, while if $k = 2$, then $\phi^{-1}(H)$ is a finite set of points in the interior of N.

(2) A continuous map $\delta : \bar{\Delta} \to M$ is called a *transversal disc around* H_i if $\delta^{-1}(H) = \{0\}$, $\delta(0) \in H_i$ and δ intersects H transversely at 0. In this case, the *sign* of δ is the local intersection number ($+1$ or -1) of δ with H_i at $\delta(0)$.

(3) An *isotopy* between transversal discs δ and δ' around H_i is a continuous map

$$h : \bar{\Delta} \times I \to M$$

such that, for each $t \in I$, the restriction $\delta_t := h|_{\bar{\Delta} \times \{t\}} : \bar{\Delta} \to M$ of h to $\bar{\Delta} \times \{t\}$ is a transversal disc around H_i, and such that $\delta_0 = \delta$ and $\delta_1 = \delta'$ hold.

(4) A *leashed disc* around H_i with the base point b_M is a pair $\rho = (\delta, \eta)$ of a transversal disc $\delta : \bar{\Delta} \to M$ around H_i and a path $\eta : I \to M \setminus H$ from $\delta(1) = \partial_\varepsilon \delta(0) = \partial_\varepsilon \delta(1)$ to b_M. (Recall that $\partial_\varepsilon \delta$ is the loop given by $t \mapsto \delta(\exp(2\pi\sqrt{-1}t))$. See Convention (3).) The *sign* of a leashed disc $\rho = (\delta, \eta)$ is the sign of δ.

(5) The *lasso* $\lambda(\rho)$ associated with a leashed disc $\rho = (\delta, \eta)$ is the loop $\eta^{-1} \cdot (\partial_\varepsilon \delta) \cdot \eta$ in $M \setminus H$ with the base point b_M.

(6) An *isotopy* of leashed discs around H_i with the base point b_M is the pair of continuous maps

$$(h_{\bar{\Delta}}, h_I) : (\bar{\Delta}, I) \times I \to (M, M \setminus H)$$

such that, for each $t \in I$, the restriction of $(h_{\bar{\Delta}}, h_I)$ to $(\bar{\Delta}, I) \times \{t\}$ is a leashed disc around H_i with the base point b_M.

Remark 3.9. The isotopy class of a leashed disc ρ is denoted by $[\rho]$. If $[\rho] = [\rho']$, then $[\lambda(\rho)] = [\lambda(\rho')]$ holds in $\pi_1(M \setminus H, b_M)$.

The following is obvious:

Proposition 3.10. (1) *Any two transversal discs around H_i with the same sign are isotopic.*

(2) *The homotopy classes of lassos associated with all the leashed discs around H_i with a fixed sign form a conjugacy class in $\pi_1(M \setminus H, b_M)$.*

(3) *The kernel of the homomorphism $\pi_1(M \setminus H, b_M) \to \pi_1(M, b_M)$ induced by the inclusion is generated by the homotopy classes of all lassos around $H_1, \dots, H_l$.*

We apply these notions to the hypersurfaces

$$\Sigma^\sharp = \Sigma_1^\sharp \cup \dots \cup \Sigma_N^\sharp \text{ of } Y^\sharp, \quad \text{and} \quad \Theta^\sharp = \Theta_1^\sharp \cup \dots \cup \Theta_N^\sharp \text{ of } X^\sharp.$$

Definition 3.11. (1) A *transversal lift* of a transversal disc $\delta : \bar{\Delta} \to Y^\sharp$ around $\Sigma_i^\sharp$ is a lift $\tilde{\delta} : \bar{\Delta} \to X^\sharp$ of δ with $\tilde{\delta}(0) \notin \mathrm{Sing}(f)$ such that $\tilde{\delta}$ intersects the irreducible hypersurface $\Theta_i^\sharp$ transversely at 0.

(2) Let $\rho = (\delta, \eta)$ be a leashed disc around $\Sigma_i^\sharp$ with the base point b. A *transversal lift* of ρ is a pair $\tilde{\rho} = (\tilde{\delta}, \tilde{\eta})$ such that $\tilde{\delta} : \bar{\Delta} \to X^\sharp$ is a transversal lift of $\delta : \bar{\Delta} \to Y^\sharp$ and $\tilde{\eta} : I \to X^\circ$ is a lift of $\eta : I \to Y^\circ$ such that $\tilde{\eta}(0) = \tilde{\delta}(1)$ and $\tilde{\eta}(1) = \tilde{b}$.

Remark 3.12. Any transversal lift of a transversal disc (resp. a leashed disc) around $\Sigma_i^\sharp$ is a transversal disc (resp. a leashed disc) around $\Theta_i^\sharp$. Moreover the lifting does not change the sign.

Definition 3.13. (1) Let δ_0 and δ_1 be two transversal discs on $Y^\sharp$ around $\Sigma_i^\sharp$, and let $h : \bar{\Delta} \times I \to Y^\sharp$ be an isotopy of transversal discs from δ_0 to δ_1. A *lift* of the isotopy h is a continuous map

$$\tilde{h} : \bar{\Delta} \times I \to X^\sharp$$

such that, for each $t \in I$, the restriction $\tilde{\delta}_t := \tilde{h}|_{\bar{\Delta} \times \{t\}}$ is a transversal lift of the transversal disc $\delta_t := h|_{\bar{\Delta} \times \{t\}}$ on $Y^\sharp$. In particular, we have $f \circ \tilde{h} = h$ and $\tilde{h}(\bar{\Delta} \times I) \cap \mathrm{Sing}(f) = \emptyset$. Moreover $\tilde{h}$ is an isotopy of transversal discs around $\Theta_i^\sharp$ from $\tilde{\delta}_0$ to $\tilde{\delta}_1$. By abuse of notation, we sometimes say that the isotopy $\tilde{\delta}_t$ is the transversal lift of the isotopy δ_t, understanding that t is the homotopy parameter.

(2) Let ρ_0 and ρ_1 be two leashed discs on $Y^\sharp$ around to $\Sigma_i^\sharp$, and let $(h_{\bar{\Delta}}, h_I) : (\bar{\Delta}, I) \times I \to (Y^\sharp, Y^\circ)$ be an isotopy of leashed discs from ρ_0 to ρ_1. A *lift* of the isotopy $(h_{\bar{\Delta}}, h_I)$ is a pair of continuous maps

$$(\tilde{h}_{\bar{\Delta}}, \tilde{h}_I) : (\bar{\Delta}, I) \times I \to (X^\sharp, X^\circ)$$

such that, for each $t \in I$, the restriction $\tilde{\rho}_t := (\tilde{h}_{\bar{\Delta}}, \tilde{h}_I)|_{(\bar{\Delta}, I) \times \{t\}}$ is a transversal lift of the leashed disc $\rho_t := (h_{\bar{\Delta}}, h_I)|_{(\bar{\Delta}, I) \times \{t\}}$ on $Y^\sharp$.

The following are obvious from the condition $(\Delta^\sharp 4)$:

Proposition 3.14. *Every transversal disc around $\Sigma_i^\sharp$ has a transversal lift on $X^\sharp$. Moreover, every isotopy δ_t of transversal discs around $\Sigma_i^\sharp$ from δ_0 to δ_1 lifts to an isotopy $\tilde{\delta}_t$ from a given transversal lift $\tilde{\delta}_0$ of δ_0 to a given transversal lift $\tilde{\delta}_1$ of δ_1.*

Remark 3.15. Every leashed disc on $Y^\sharp$ around $\Sigma_i^\sharp$ has a transversal lift on $X^\sharp$. Moreover, every isotopy ρ_t of leashed discs on $Y^\sharp$ has a lift $\tilde{\rho}_t$ on $X^\sharp$ from a given transversal lift $\tilde{\rho}_0$ of ρ_0, but the ending lift $\tilde{\rho}_1$ cannot be arbitrarily given.

Definition 3.16. Let ρ be a leashed disc on $Y^\sharp$ around $\Sigma_i^\sharp$, and let $\tilde{\rho}$ be a transversal lift of ρ. Then we have the lasso $\lambda(\tilde{\rho})$, which is a loop in X° with the base point $\tilde{b}$. Recall that μ is the lifted monodromy. We put

$$N(\tilde{\rho}) := \langle\ \{g^{-1}g^{\mu([\lambda(\tilde{\rho})])}\mid g\in\pi_1(F_b,\tilde{b})\}\ \rangle_{\pi_1(F_b,\tilde{b})}.$$

Proposition-Definition 3.17. *Let ρ' be a leashed disc on $Y^\sharp$ isotopic to ρ, and let $\tilde{\rho}'$ be a transversal lift of ρ'. Then we have*

$$N(\tilde{\rho}) \;=\; N(\tilde{\rho}').$$

Therefore, for an isotopy class $[\rho]$ of leashed discs on $Y^\sharp$, we can define a normal subgroup $N^{[\rho]}$ of $\pi_1(F_b,\tilde{b})$ by choosing a transversal lift $\tilde{\rho}$ of a representative ρ of $[\rho]$, and putting

$$N^{[\rho]} := N(\tilde{\rho}).$$

Proof. By Remarks 3.9 and 3.15, the isotopy from ρ to ρ' lifts to an isotopy from $\tilde{\rho}$ to some lift $\tilde{\rho}_1'$ of ρ', and we have $[\lambda(\tilde{\rho})]=[\lambda(\tilde{\rho}_1')]$ in $\pi_1(X^\circ,\tilde{b})$. (However $[\lambda(\tilde{\rho}_1')]$ and $[\lambda(\tilde{\rho}')]$ may be distinct in general.) Therefore it is enough to show that $N(\tilde{\rho}^{(1)})=N(\tilde{\rho}^{(2)})$ holds for any two transversal lifts $\tilde{\rho}^{(1)}=(\tilde{\delta}^{(1)},\tilde{\eta}^{(1)})$ and $\tilde{\rho}^{(2)}=(\tilde{\delta}^{(2)},\tilde{\eta}^{(2)})$ of a single leashed disc $\rho=(\delta,\eta)$ on $Y^\sharp$. We can assume that the transversal disc $\delta:\bar{\Delta}\to Y^\sharp$ around $\Sigma_i^\sharp$ is an embedding of a complex manifold. We denote by $\bar{\Delta}_\rho$ the image of δ, and by Δ_ρ the interior of $\bar{\Delta}_\rho$. We can further assume that $\bar{\Delta}_\rho$ is sufficiently small, and that

$$E_\rho := f^{-1}(\Delta_\rho)$$

is a smooth complex manifold by the condition ($\Delta^\sharp 2$). We then put

$$\overline{E}_\rho\;=f^{-1}(\bar{\Delta}_\rho),\quad \overline{E}_\rho^{\times}\;=f^{-1}(\bar{\Delta}_\rho^{\times}),$$

where $\bar{\Delta}_\rho^\times := \bar{\Delta}_\rho\setminus\{\delta(0)\}=\bar{\Delta}_\rho\cap Y^\circ$. We also put $q:=\delta(1)=\eta(0)\in\partial\bar{\Delta}_\rho$ and

$$\tilde{q}^{(1)} := \tilde{\delta}^{(1)}(1)=\tilde{\eta}^{(1)}(0)\in F_q,\quad \tilde{q}^{(2)} := \tilde{\delta}^{(2)}(1)=\tilde{\eta}^{(2)}(0)\in F_q.$$

Since f is locally trivial over $\eta(I)\subset Y^\circ$ and $\sqcap=(\partial I\times I)\cup(I\times\{1\})$ is a strong deformation retract of $I\times I$, there exists a continuous map $\Omega: I\times I\to X^\circ$ such that the following hold for any $s,t\in I$:

$$f(\Omega(s,t))=\eta(t),\quad \Omega(s,1)=\tilde{b},\quad \Omega(0,t)=\tilde{\eta}^{(1)}(t),\quad \Omega(1,t)=\tilde{\eta}^{(2)}(t).$$

(See Figure 3.2.) Then, for each $t\in I$, the map $s\mapsto\Omega(s,t)$ is a path in $F_{\eta(t)}$ from $\tilde{\eta}^{(1)}(t)$ to $\tilde{\eta}^{(2)}(t)$. We denote by $\omega: I\to F_q$ the path in F_q from $\tilde{q}^{(1)}$ to $\tilde{q}^{(2)}$ defined by $\omega(s):=\Omega(s,0)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
\pi_1(F_b,\tilde{b}) & \underset{[\tilde{\eta}^{(1)}]_*}{\overset{\sim}{\longleftarrow}} & \pi_1(F_q,\tilde{q}^{(1)}) & \overset{i_{q*}}{\longrightarrow} & \pi_1(\overline{E}_\rho,\tilde{q}^{(1)}) \\
\| & & {\scriptstyle[\omega]_*}\downarrow\wr & & {\scriptstyle[\omega]_*}\downarrow\wr \\
\pi_1(F_b,\tilde{b}) & \underset{[\tilde{\eta}^{(2)}]_*}{\overset{\sim}{\longleftarrow}} & \pi_1(F_q,\tilde{q}^{(2)}) & \overset{i_{q*}}{\longrightarrow} & \pi_1(\overline{E}_\rho,\tilde{q}^{(2)}),
\end{array}$$

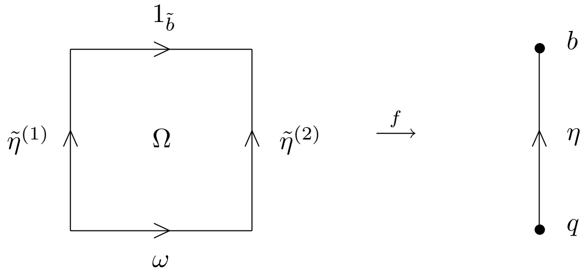


FIGURE 3.2. The map Ω

where $i_q : F_q \hookrightarrow \overline{E}_\rho$ is the inclusion. Hence, in order to prove $N(\tilde{\rho}^{(1)}) = N(\tilde{\rho}^{(2)})$, it is enough to show the following equality:

$$[\tilde{\eta}^{(1)}]_*^{-1}(N(\tilde{\rho}^{(1)})) = \operatorname{Ker}(i_{q*} : \pi_1(F_q, \tilde{q}^{(1)}) \to \pi_1(\overline{E}_\rho, \tilde{q}^{(1)})).$$

Since $f|_{\overline{E}_\rho} : \overline{E}_\rho \to \bar{\Delta}_\rho$ is locally trivial over $\bar{\Delta}_\rho^\times$ with the general fiber being connected by (C0), and since there exists a cross-section

$$^s\tilde{\delta}^{(1)} : \bar{\Delta}_\rho \to \overline{E}_\rho$$

of $f|_{\overline{E}_\rho}$ given by the transversal lift $\tilde{\delta}^{(1)}$ of δ, we have an exact sequence

$$1 \longrightarrow \pi_1(F_q, \tilde{q}^{(1)}) \xrightarrow{i_{q*}} \pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)}) \xrightarrow{(f|_{\overline{E}_\rho^\times})_*} \pi_1(\bar{\Delta}_\rho^\times, q) \longrightarrow 1$$

with the cross-section

$$s : \pi_1(\bar{\Delta}_\rho^\times, q) \to \pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)})$$

of $(f|_{\overline{E}_\rho^\times})_*$ that maps the positive generator $[\partial_\varepsilon \delta]$ of $\pi_1(\bar{\Delta}_\rho^\times, q) \cong \mathbb{Z}$ to $[\partial_\varepsilon \tilde{\delta}^{(1)}] \in \pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)})$. By the cross-section $^s\tilde{\delta}^{(1)}$ of $f|_{\overline{E}_\rho}$ over $\bar{\Delta}_\rho$, we have the classical monodromy action of $\pi_1(\bar{\Delta}_\rho^\times, q)$ on $\pi_1(F_q, \tilde{q}^{(1)})$. By the definition, the action of $[\partial_\varepsilon \delta] \in \pi_1(\bar{\Delta}_\rho^\times, q)$ is equal to

$$g \mapsto g^{\mu([\partial_\varepsilon \tilde{\delta}^{(1)}])} = [\partial_\varepsilon \tilde{\delta}^{(1)}]^{-1} \cdot g \cdot [\partial_\varepsilon \tilde{\delta}^{(1)}] \quad \text{for} \quad g \in \pi_1(F_q, \tilde{q}),$$

where the product is taken in $\pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)})$ and $\pi_1(F_q, \tilde{q}^{(1)})$ is regarded as a normal subgroup of $\pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)})$ by i_{q*}. Hence, by Lemma 2.5, $\pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)})$ is isomorphic to the semi-direct product $\pi_1(F_q, \tilde{q}^{(1)}) \rtimes \pi_1(\bar{\Delta}_\rho^\times, q)$ constructed by the monodromy action. On the other hand, by the condition $(\Delta^\sharp 4)$, the central fiber $F_{\delta(0)}$ of $\overline{E}_\rho \to \bar{\Delta}_\rho$ is an irreducible hypersurface of $\overline{E}_\rho$, and hence the kernel of

$$j_* : \pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)}) \to \pi_1(\overline{E}_\rho, \tilde{q}^{(1)})$$

induced by the inclusion $j : \overline{E}_\rho^\times \hookrightarrow \overline{E}_\rho$ is generated by the conjugacy class of lassos around $F_{\delta(0)}$. (See Proposition 3.10.) Since $\partial_\varepsilon \tilde{\delta}^{(1)} = \lambda(\tilde{\delta}^{(1)})$ is a lasso around $F_{\delta(0)}$, the kernel of j_* is equal to the normal subgroup $\langle\langle \{[\partial_\varepsilon \tilde{\delta}^{(1)}]\} \rangle\rangle = \langle\langle \operatorname{Im}(s) \rangle\rangle$. By Lemmas 2.3 and 2.6, the kernel of the composite

$$\pi_1(F_q, \tilde{q}^{(1)}) \xrightarrow{i_{q*}} \pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)}) \xrightarrow{j_*} \pi_1(\overline{E}_\rho, \tilde{q}^{(1)}) = \pi_1(\overline{E}_\rho^\times, \tilde{q}^{(1)}) / \langle\langle \operatorname{Im}(s) \rangle\rangle$$

is equal to

$$N' := \langle \{ \, g^{-1} g^{\mu([\partial_\varepsilon \tilde{\delta}^{(1)}])} \;\mid\; g \in \pi_1(F_q, \tilde{q}^{(1)}) \, \} \rangle.$$

Since $[\tilde{\eta}^{(1)}]_*(g^{\mu([\partial_\varepsilon \tilde{\delta}^{(1)}])}) = ([\tilde{\eta}^{(1)}]_*(g))^{\mu([\lambda(\tilde{\rho}^{(1)})])}$ for any $g \in \pi_1(F_q, \tilde{q}^{(1)})$, we see that $[\tilde{\eta}^{(1)}]_*$ induces an isomorphism $N' \xrightarrow{\sim} N(\tilde{\rho}^{(1)})$. □

Proposition 3.18. *Let $\tilde{\gamma} : (I, \partial I) \to (X^\circ, \tilde{b})$ be a loop, and we put $\gamma := f \circ \tilde{\gamma}$. Then, for any leashed disc $\rho = (\delta, \eta)$ on $Y^\sharp$ around $\Sigma_i^\sharp$, we have*

$$(N^{[\rho]})^{\mu([\tilde{\gamma}])} = N^{[(\delta, \eta\gamma)]}.$$

Proof. Let g be an element of $\pi_1(F_b, \tilde{b})$, and let h denote $g^{\mu([\tilde{\gamma}])}$. Then, for a transversal lift $\tilde{\rho} = (\tilde{\delta}, \tilde{\eta})$ of ρ, we have

$$(g^{-1} g^{\mu([\lambda(\tilde{\rho})])})^{\mu([\tilde{\gamma}])} = h^{-1} h^{\mu([\tilde{\gamma}]^{-1}[\lambda(\tilde{\rho})][\tilde{\gamma}])}.$$

Since $\tilde{\gamma}^{-1}\lambda(\tilde{\rho})\tilde{\gamma} = \tilde{\gamma}^{-1}\tilde{\eta}^{-1} \cdot \partial_\varepsilon \tilde{\delta} \cdot \tilde{\eta}\tilde{\gamma}$ is a lasso associated with the transversal lift $(\tilde{\delta}, \tilde{\eta}\tilde{\gamma})$ of the leashed disc $(\delta, \eta\gamma)$, we obtain the proof. □

Corollary 3.19. *If $N^{[\rho]} = 1$ holds for one leashed disc ρ around $\Sigma_i^\sharp$, then we have $N^{[\rho]} = 1$ for any leashed disc ρ around $\Sigma_i^\sharp$.*

We can now state the main result of this section.

Theorem 3.20. *Suppose that the conditions* (C1), (C2) *and the following condition* (Z) *are satisfied:*

- (Z) *There exists a continuous cross-section $s_Z : Z \to f^{-1}(Z)$ of f over a subspace $Z \subset Y$ satisfying $b \in Z$, $s_Z(b) = \tilde{b}$, $s_Z(Z) \cap \operatorname{Sing}(f) = \emptyset$ and such that the inclusion $Z \hookrightarrow Y$ induces a surjection $\pi_2(Z, b) \twoheadrightarrow \pi_2(Y, b)$.*

Let $\mathcal{L}$ be the set of isotopy classes of all leashed discs on $Y^\sharp$ around $\Sigma_1^\sharp, \dots, \Sigma_N^\sharp$. Then $\operatorname{Ker}(\iota_*)$ *is equal to*

$$\mathcal{N} := \langle \; \textstyle\bigcup_{[\rho] \in \mathcal{L}} N^{[\rho]} \; \rangle_{\pi_1(F_b, \tilde{b})}.$$

Remark 3.21. If $\pi_2(Y) = 0$, then the condition (Z) is always satisfied, because we can put $Z = \{b\}$ and $s_Z(b) = \tilde{b}$.

For the proof, we define the notion of *free loop pairs of monodromy relation type.* Let $\mathbb{S}^1$ denote the oriented circle.

Definition 3.22. Let T be a topological space. A *free loop* on T is a continuous map $\varphi : \mathbb{S}^1 \to T$. A *homotopy* from a free loop φ to a free loop φ' is a continuous map $\Phi : \mathbb{S}^1 \times I \to T$ such that $\Phi|_{\mathbb{S}^1 \times \{0\}} = \varphi$ and $\Phi|_{\mathbb{S}^1 \times \{1\}} = \varphi'$. The homotopy class of a free loop φ is denoted by $[\varphi]_{\mathrm{FL}}$.

Suppose that T is path-connected, and let b_T be a base point of T. Then the natural map $[\alpha] \mapsto [\alpha]_{\mathrm{FL}}$ induces a bijection from the set of conjugacy classes of $\pi_1(T, b_T)$ to the set of homotopy classes of free loops on T.

Let D be a topological space homeomorphic to $\bar{\Delta}$, let b_D be a point of D, and let ∂D be the boundary of D with an orientation.

Definition 3.23. A *free loop pair* is a pair

$$(\psi, (\psi|_{\partial D})^\sim) \, : \, (D, \partial D) \, \to \, (Y^\circ, X^\circ)$$

of a continuous map $\psi : D \to Y^\circ$ and a lift $(\psi|_{\partial D})^\sim : \partial D \to X^\circ$ of the restriction $\psi|_{\partial D} : \partial D \to Y^\circ$ of ψ to ∂D.

Let $(\psi, (\psi|_{\partial D})^\sim) : (D, \partial D) \to (Y^\circ, X^\circ)$ be a free loop pair. Consider the pull-back

$$\psi^*(f^\circ) \,:\, \psi^*(X^\circ) := X^\circ \times_{Y^\circ} D \,\to\, D$$

of the locally trivial map $f^\circ : X^\circ \to Y^\circ$ by ψ. Since D is contractible, we have a contraction $c : \psi^*(X^\circ) \to F_{\psi(b_D)}$, which is the homotopy inverse of the inclusion $F_{\psi(b_D)} \hookrightarrow \psi^*(X^\circ)$. Then the cross-section

$${}^s(\psi|_{\partial D})^\sim \,:\, \partial D \,\to\, \psi^*(X^\circ)$$

of $\psi^*(f^\circ)$ over ∂D obtained from $(\psi|_{\partial D})^\sim : \partial D \to X^\circ$ defines a homotopy class $[(\psi|_{\partial D})^\sim]_{\mathrm{FL}}$ of free loops on $F_{\psi(b_D)}$ via the contraction c, and hence a conjugacy class $\mathrm{C}(\psi, (\psi|_{\partial D})^\sim)$ of $\pi_1(F_{\psi(b_D)}, \tilde{b}')$, where $\tilde{b}' \in F_{\psi(b_D)}$ is an arbitrary base point. Remark that $\mathrm{C}(\psi, (\psi|_{\partial D})^\sim)$ does not depend on the choice of the contraction c.

Definition 3.24. We choose a path $\tilde{\alpha}$ in X° from $\tilde{b} \in F_b$ to $\tilde{b}' \in F_{\psi(b_D)}$. We say that the free loop pair

$$(\psi, (\psi|_{\partial D})^\sim) \,:\, (D, \partial D) \,\to\, (Y^\circ, X^\circ)$$

is *of monodromy relation type around* $\Sigma_i^\sharp$ if the pull-back of the conjugacy class $\mathrm{C}(\psi, (\psi|_{\partial D})^\sim) \subset \pi_1(F_{\psi(b_D)}, \tilde{b}')$ by the isomorphism $[\tilde{\alpha}]_* : \pi_1(F_b, \tilde{b}) \xrightarrow{\sim} \pi_1(F_{\psi(b_D)}, \tilde{b}')$ is contained in $N^{[\rho]}$ for some leashed disc ρ on $Y^\sharp$ around $\Sigma_i^\sharp$.

Remark 3.25. It is obvious that this definition does not depend on the choice of the orientation of ∂D. It also follows from Proposition 3.18 that this definition does not depend on the choice of the path $\tilde{\alpha}$ connecting $\tilde{b} \in F_b$ and $\tilde{b}' \in F_{\psi(b_D)}$.

Definition 3.26. A *homotopy of free loop pairs* is a pair of continuous maps

$$(h, (h|_{\partial D})^\sim) \,:\, (D, \partial D) \times I \,\to\, (Y^\circ, X^\circ)$$

such that, for each $u \in I$, the restriction of $(h, (h|_{\partial D})^\sim)$ to $(D, \partial D) \times \{u\}$ is a free loop pair.

Remark 3.27. Suppose that two free loop pairs are homotopic. If one is of monodromy relation type around $\Sigma_i^\sharp$, then so is the other.

Remark 3.28. Let $\psi_u : D \to Y^\circ$ be a homotopy of continuous maps from ψ_0 to ψ_1 parametrized by $u \in I$. Since f° is locally trivial, the homotopy $\psi_u|_{\partial D} : \partial D \to Y^\circ$ lifts to a homotopy $(\psi_u|_{\partial D})^\sim : \partial D \to X^\circ$ that starts from any given lift $(\psi_0|_{\partial D})^\sim$ of $\psi_0|_{\partial D}$ and hence we obtain a homotopy $(\psi_u, (\psi_u|_{\partial D})^\sim)$ of free loop pairs starting from a given $(\psi_0, (\psi_0|_{\partial D})^\sim)$. (The ending lift $(\psi_1|_{\partial D})^\sim$ cannot be arbitrarily given.)

Proposition 3.29. *Let δ_0 and δ_1 be two transversal discs on $Y^\sharp$ around $\Sigma_i^\sharp$, and let $h : \bar{\Delta} \times I \to Y^\sharp$ be an isotopy of transversal discs from $\delta_0 = h|_{\bar{\Delta}\times\{0\}}$ to $\delta_1 = h|_{\bar{\Delta}\times\{1\}}$. Let D be a closed subset of $\partial\bar{\Delta} \times (I \setminus \partial I)$ homeomorphic to $\bar{\Delta}$, and put*

$$T := \partial(\bar{\Delta} \times I) \setminus (D \setminus \partial D),$$

so that $\partial T = \partial D$. Suppose that we are given a lift

$$(h|_T)^\sim \,:\, T \,\to\, X^\sharp$$

of $h|_T : T \to Y^\sharp$ such that the restrictions

$$\tilde{\delta}_0 := (h|_T)^\sim|_{\bar{\Delta}\times\{0\}} : \bar{\Delta} \to X^\sharp \quad \textit{and} \quad \tilde{\delta}_1 := (h|_T)^\sim|_{\bar{\Delta}\times\{1\}} : \bar{\Delta} \to X^\sharp$$

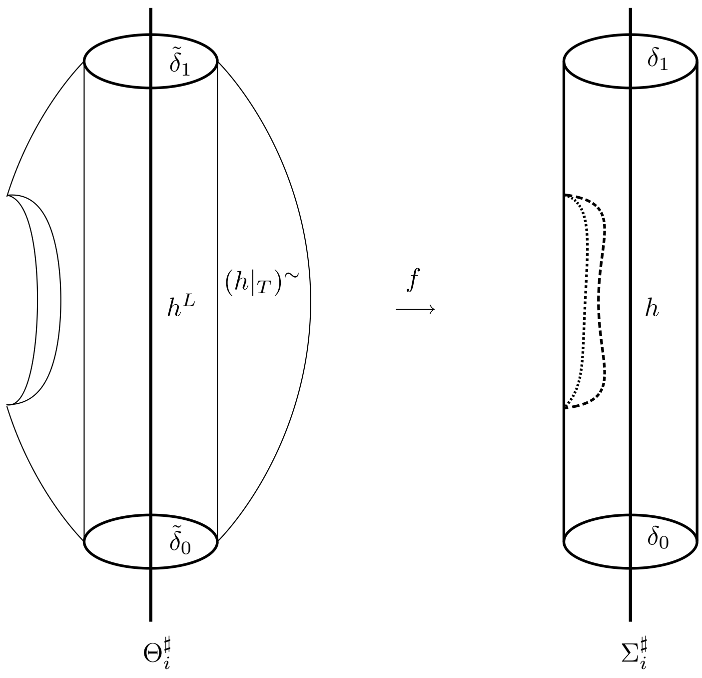


FIGURE 3.3. $(h|_T)^\sim$ and h^L

are transversal lifts of δ_0 *and* δ_1*, respectively. Then the free loop pair*

$$(h|_D, (h|_T)^\sim|_{\partial D}) \,:\, (D, \partial D) \,\to\, (Y^\circ, X^\circ)$$

is of monodromy relation type around $\Sigma_i^\sharp$*.*

Remark 3.30. In Figure 3.3, the closed subset D is the region surrounded by the dashed curve on the right tube $\bar{\Delta} \times I$.

Proof of Proposition 3.29. First note that, since h is an isotopy of transversal discs, the image of $\partial\bar{\Delta} \times I$ by h is contained in Y°, and hence we have $h|_D(D) \subset Y^\circ$.

By Remarks 3.27 and 3.28, we can assume that $D \cap (\{1\} \times I) = \emptyset$ by moving D by a homeomorphism of $\partial\bar{\Delta} \times I$ homotopic to the identity. We consider the continuous map

$$\tau \,:\, I^2 \,\to\, \partial\bar{\Delta} \times I$$

given by $\tau(s,t) := (\exp(2\pi\sqrt{-1}s), t)$. Then we have $D \subset \tau(I^2 \setminus \partial I^2)$ and $\tau(\partial I^2) \subset T$. Under a suitable homeomorphism between D and I^2, the inclusion $D \hookrightarrow \partial\bar{\Delta} \times I$ is homotopic to τ. We put

$$H_0 := h \circ \tau \,:\, I^2 \to Y^\circ$$

and define a lift $(H_0|_{\partial I^2})^\sim$ of $H_0|_{\partial I^2}$ by

$$(H_0|_{\partial I^2})^\sim := (h|_T)^\sim \circ (\tau|_{\partial I^2}) \,:\, \partial I^2 \to X^\circ.$$

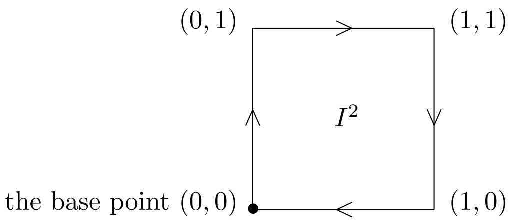


FIGURE 3.4. An orientation of ∂I^2

By Remarks 3.27 and 3.28 again, it is enough to prove that the free loop pair

$$(H_0, (H_0|_{\partial I^2})^\sim) \,:\, (I^2, \partial I^2) \,\to\, (Y^\circ, X^\circ)$$

is of monodromy relation type around $\Sigma_i^\sharp$. For simplicity, we put

$$\begin{aligned} q &:= \delta_0(1) = h(1,0) = H_0(0,0) = H_0(1,0), \quad \text{and} \\ \tilde{q} &:= \tilde{\delta}_0(1) = (h|_T)^\sim(1,0) = (H_0|_{\partial I^2})^\sim(0,0) = (H_0|_{\partial I^2})^\sim(1,0) \in F_q. \end{aligned}$$

By Proposition 3.14, we have an isotopy

$$h^L \,:\, \bar{\Delta} \times I \,\to\, X^\sharp$$

of transversal discs around $\Theta_i^\sharp$ from $\tilde{\delta}_0 = (h|_T)^\sim|_{\bar{\Delta}\times\{0\}}$ to $\tilde{\delta}_1 = (h|_T)^\sim|_{\bar{\Delta}\times\{1\}}$ that is a lift of the isotopy $h : \bar{\Delta} \times I \to Y^\sharp$;

$$f \circ h^L = h.$$

In Figure 3.3, the left tube is h^L, while the barrel with a hole is $(h|_T)^\sim$. We put

$$\delta_t := h|_{\bar{\Delta}\times\{t\}} \,:\, \bar{\Delta} \,\to\, Y^\sharp \quad \text{and} \quad \tilde{\delta}_t := h^L|_{\bar{\Delta}\times\{t\}} \,:\, \bar{\Delta} \,\to\, X^\sharp.$$

Then $\tilde{\delta}_t$ is a transversal lift of δ_t. Next we put

$$k_0 := h|_{\{1\}\times I} \,:\, I \,\to\, Y^\circ,$$

which is a path on Y° from $q = \delta_0(1)$ to $\delta_1(1)$, and

$$\tilde{k}_0 := (h|_T)^\sim|_{\{1\}\times I} = (H_0|_{\partial I^2})^\sim|_{\{0\}\times I} = (H_0|_{\partial I^2})^\sim|_{\{1\}\times I},$$

which is a lift of k_0 from $\tilde{q} = \tilde{\delta}_0(1)$ to $\tilde{\delta}_1(1)$. Note that, with the base point $(0,0)$ and the orientation of ∂I^2 given in Figure 3.4, the map $(H_0|_{\partial I^2})^\sim : \partial I^2 \to X^\circ$ is equal to

$$\tilde{k}_0 \cdot \partial_\varepsilon \tilde{\delta}_1 \cdot \tilde{k}_0^{-1} \cdot \partial_\varepsilon \tilde{\delta}_0^{-1}$$

as a loop with the base point $\tilde{q} = (H_0|_{\partial I^2})^\sim(0,0) \in F_q$. We define a homotopy

$$H_u \,:\, I^2 \,\to\, Y^\circ \qquad (u \in I)$$

with u being the homotopy parameter by $H_u(s,t) := H_0(s,(1-u)t)$, and will construct a homotopy $(H_u|_{\partial I^2})^\sim : \partial I^2 \to X^\circ$ that covers the homotopy $H_u|_{\partial I^2}$ and starts from $(H_0|_{\partial I^2})^\sim$ above. We define

$$K \,:\, I \times I \,\to\, Y^\circ$$

by $K(t,u) := k_0((1-u)t)$, and put $k_u := K|_{I\times\{u\}}$ for $u \in I$. Then k_u gives a homotopy with parameter $u \in I$ from k_0 to the constant map $k_1 = 1_q$. We then

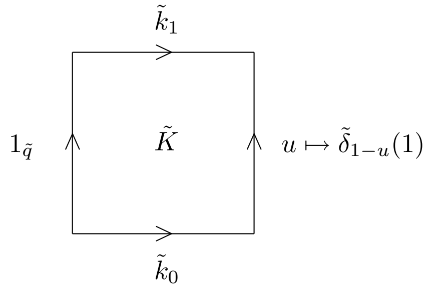


FIGURE 3.5. The map $\tilde{K}$

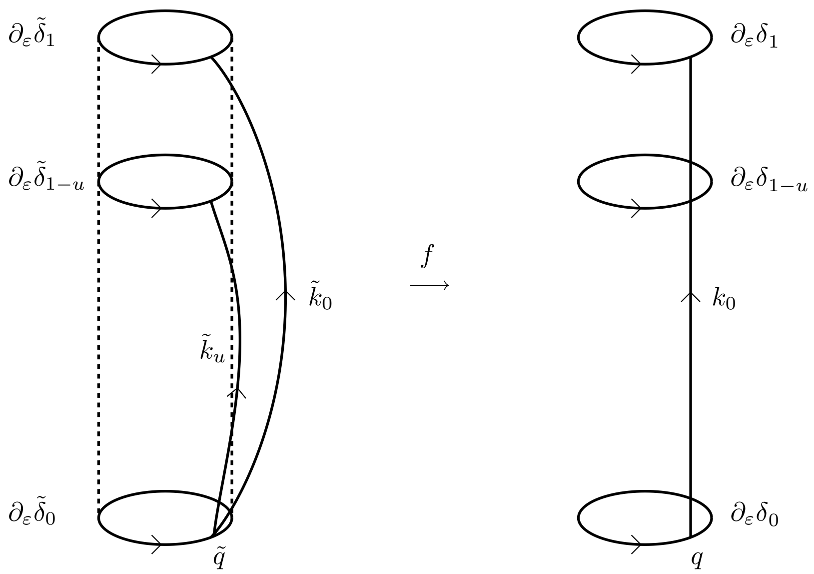


FIGURE 3.6. The loop $(H_u|_{\partial I^2})^{\sim}$

define a lift $(K|_{\sqcup})^{\sim} : \sqcup \to X^\circ$ of $K|_{\sqcup} : \sqcup \to Y^\circ$, where $\sqcup := (\partial I \times I) \cup (I \times \{0\})$, by the following:

$$(K|_{\sqcup})^{\sim}(t,u) := \begin{cases} \tilde{q} & \text{if } t = 0, \\ \tilde{k}_0(t) & \text{if } u = 0, \\ \tilde{\delta}_{1-u}(1) = h^L(1, 1-u) & \text{if } t = 1. \end{cases}$$

Since f° is locally trivial, the lift $(K|_{\sqcup})^{\sim}$ extends to a lift $\tilde{K} : I \times I \to X^\circ$ of K. (See Figure 3.5.) Then we obtain a lift

$$\tilde{k}_u := \tilde{K}|_{I \times \{u\}},$$

of k_u, which is a path from $\tilde{q} \in F_q$ to the point $\tilde{\delta}_{1-u}(1) = h^L(1, 1-u)$ of $F_{\delta_{1-u}(1)}$. (See Figure 3.6.) We then define a lift

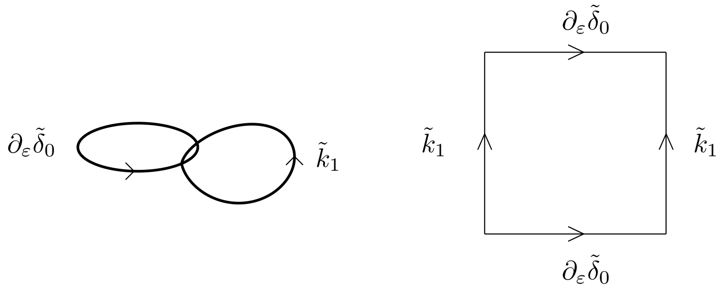


FIGURE 3.7. Two figures for $(H_1|_{\partial I^2})^\sim = \tilde{k}_1 \cdot \partial_\varepsilon\tilde{\delta}_0 \cdot \tilde{k}_1^{-1} \cdot \partial_\varepsilon\tilde{\delta}_0^{-1}$

$$(H_u|_{\partial I^2})^\sim \,:\, \partial I^2 \,\to\, X^\circ \qquad (u \in I)$$

of $H_u|_{\partial I^2}$ as a loop by

$$\tilde{k}_u \cdot \partial_\varepsilon\tilde{\delta}_{1-u} \cdot \tilde{k}_u^{-1} \cdot \partial_\varepsilon\tilde{\delta}_0^{-1},$$

where ∂I^2 is oriented and segmented as Figure 3.4 above. Then $(H_u, (H_u|_{\partial I^2})^\sim)$ is a homotopy of free loop pairs parametrized by $u \in I$. By Remarks 3.27 and 3.28 again, it is enough to prove that the free loop pair

$$(H_1, (H_1|_{\partial I^2})^\sim) \,:\, (I^2, \partial I^2) \,\to\, (Y^\circ, X^\circ)$$

is of monodromy relation type around $\Sigma_i^\sharp$. Note that

$$(H_1|_{\partial I^2})^\sim = \tilde{k}_1 \cdot \partial_\varepsilon\tilde{\delta}_0 \cdot \tilde{k}_1^{-1} \cdot \partial_\varepsilon\tilde{\delta}_0^{-1},$$

(see Figure 3.7), and that the lift $\tilde{k}_1$ of the constant map $k_1 = 1_q$ is a loop in F_q with the base point $\tilde{q}$. Since $H_1(s,t) = H_0(s,0) = \partial_\varepsilon\delta_0(s)$ for any t, the pull-back

$$H_1^*(f^\circ) \,:\, H_1^*(X^\circ) \,\to\, I^2$$

of $f^\circ : X^\circ \to Y^\circ$ by H_1 is the product of the pull-back

$$(\partial_\varepsilon\delta_0)^*(f^\circ) \,:\, (\partial_\varepsilon\delta_0)^*(X^\circ) \,\to\, I$$

of f° by $\partial_\varepsilon\delta_0 : I \to Y^\circ$ and the identity map of the second factor I. Let

$${}^s(H_1|_{\partial I^2})^\sim \,:\, \partial I^2 \,\to\, H_1^*(X^\circ) = (\partial_\varepsilon\delta_0)^*(X^\circ) \times I$$

be the cross-section of $H_1^*(f^\circ)$ over ∂I^2 obtained from $(H_1|_{\partial I^2})^\sim$. We will describe the image of the free loop ${}^s(H_1|_{\partial I^2})^\sim$ by a contraction

$$c' \,:\, H_1^*(X^\circ) \,\to\, F_q.$$

We construct the contraction c' as the composite of the projection

$$\mathrm{pr}_1 \,:\, (H_1|_{\partial I^2})^\sim \,\to\, (\partial_\varepsilon\delta_0)^*(X^\circ)$$

onto the first factor and a contraction $c : (\partial_\varepsilon\delta_0)^*(X^\circ) \to F_q$. Let

$$\sigma \,:\, \partial I^2 \,\to\, (\partial_\varepsilon\delta_0)^*(X^\circ)$$

be the composite of ${}^s(H_1|_{\partial I^2})^\sim$ with the projection pr_1. The fibers $F_q^{(0)}$ and $F_q^{(1)}$ of $(\partial_\varepsilon\delta_0)^*(f^\circ) : (\partial_\varepsilon\delta_0)^*(X^\circ) \to I$ over $0 \in I$ and $1 \in I$ are canonically identified with F_q. Let $\tilde{q}^{(0)} \in F_q^{(0)}$ and $\tilde{q}^{(1)} \in F_q^{(1)}$ be the points corresponding to $\tilde{q} \in F_q$. Then $(H_1|_{\partial I^2})^\sim|_{\{0\}\times I}$ (resp. $(H_1|_{\partial I^2})^\sim|_{\{1\}\times I}$) gives rise to a loop $\tilde{k}_1^{(0)}$ in $F_q^{(0)}$ with the base point $\tilde{q}^{(0)}$ (resp. a loop $\tilde{k}_1^{(1)}$ in $F_q^{(1)}$ with the base point $\tilde{q}^{(1)}$). Each of them

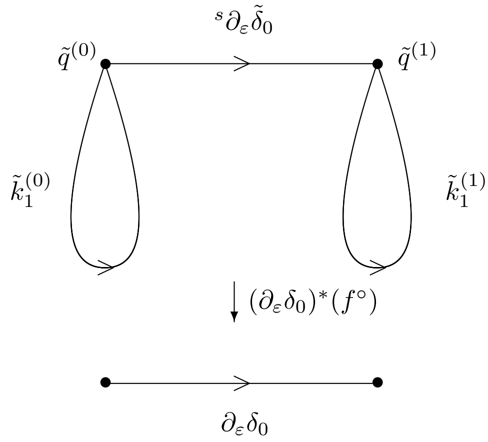


FIGURE 3.8. The loop $\sigma = (\tilde{k}_1^{(0)}) \cdot ({}^{s}\partial_\varepsilon\tilde{\delta}_0) \cdot (\tilde{k}_1^{(1)})^{-1} \cdot ({}^{s}\partial_\varepsilon\tilde{\delta}_0)^{-1}$

corresponds to the loop $\tilde{k}_1$ by the obvious identifications $(F_q, \tilde{q}) = (F_q^{(0)}, \tilde{q}^{(0)}) = (F_q^{(1)}, \tilde{q}^{(1)})$. On the other hand, the loop $\partial_\varepsilon\tilde{\delta}_0$ gives rise to a cross-section

$$ {}^{s}\partial_\varepsilon\tilde{\delta}_0 \,:\, I \,\to\, (\partial\delta_0)^*(X^\circ) $$

of $(\partial_\varepsilon\delta_0)^*(f^\circ)$ that connects $\tilde{q}^{(0)}$ and $\tilde{q}^{(1)}$. The loop σ on $(\partial_\varepsilon\delta_0)^*(X^\circ)$ is then equal to the conjunction

$$ (\tilde{k}_1^{(0)}) \cdot ({}^{s}\partial_\varepsilon\tilde{\delta}_0) \cdot (\tilde{k}_1^{(1)})^{-1} \cdot ({}^{s}\partial_\varepsilon\tilde{\delta}_0)^{-1}. $$

(See Figure 3.8.) We denote by $S \subset (\partial_\varepsilon\delta_0)^*(X^\circ)$ the image of the section ${}^{s}\partial_\varepsilon\tilde{\delta}_0$, and choose a contraction

$$ c \,:\, ((\partial_\varepsilon\delta_0)^*(X^\circ), S) \,\to\, (F_q^{(0)}, \tilde{q}^{(0)}) = (F_q, \tilde{q}) $$

to the fiber over $0 \in I$ that contracts the section S to the point $\tilde{q}$. We put

$$ \gamma := \mu([\partial_\varepsilon\tilde{\delta}_0]) \in \mathrm{Aut}(\pi_1(F_q, \tilde{q})). $$

By the definition of the lifted monodromy, the loop

$$ ({}^{s}\partial_\varepsilon\tilde{\delta}_0) \cdot (\tilde{k}_1^{(1)}) \cdot ({}^{s}\partial_\varepsilon\tilde{\delta}_0)^{-1} $$

on $\partial_\varepsilon\delta_0^*(X^\circ)$ is contracted by c to a loop in F_q that represents

$$ [\tilde{k}_1]^{(\gamma^{-1})} \in \pi_1(F_q, \tilde{q}), $$

while the loop $\tilde{k}_1^{(0)}$ on $F_q^{(0)}$ obviously represents $[\tilde{k}_1] \in \pi_1(F_q, \tilde{q})$. Therefore, by the contraction c, the loop σ on $(\partial_\varepsilon\delta_0)^*(X^\circ)$ is mapped to a loop that represents

$$ [\tilde{k}_1]([\tilde{k}_1]^{(\gamma^{-1})})^{-1} = (\kappa^{-1}\kappa^{\gamma})^{-1}, $$

where $\kappa := ([\tilde{k}_1]^{(\gamma^{-1})})^{-1}$. Hence the conjugacy class of $\pi_1(F_q, \tilde{q})$ corresponding to the free loop pair $(H_1, (H_1|_{\partial I^2})^{\sim})$ is contained in the normal subgroup $N(\partial_\varepsilon\tilde{\delta}_0) = N^{[\partial_\varepsilon\delta_0]}$ generated by the monodromy relations along $[\partial_\varepsilon\delta_0]$. $\square$

Corollary 3.31. *We put*

$$\begin{array}{rcll} \mathbb{T} & := & \{\ (x,y,z)\in\mathbb{R}^3 \ \mid\ x^2+y^2\le 1, z\in I\ \}, & \\ A_\zeta & := & \{\ (x,y,z)\in\mathbb{T}\ \mid\ z=\zeta\ \}, \quad \textit{and} & \\ \Upsilon & := & \{\ (x,y,z)\in\mathbb{T}\ \mid\ x^2+y^2=1\ \}\cup A_1 & = \ \partial\,\mathbb{T}\setminus A_0^\circ, \end{array}$$

where A_0° is the interior of the closed disc A_0. Let $\varphi:\mathbb{T}\to Y^\sharp$ be a continuous map such that $\varphi(\mathbb{T})\cap\Sigma^\sharp\subset\Sigma_i^\sharp$ and

$$\varphi^{-1}(\Sigma_i^\sharp)=\{\ (x,0,z)\in\mathbb{T}\ \mid\ x^2+(z-1)^2=1/2\ \}$$

hold, and such that $\varphi|_{A_1}: A_1\to Y^\sharp$ intersects $\Sigma^\sharp$ transversely at $(\pm 1/\sqrt{2},0,1)$. Suppose that we have a lift $(\varphi|_\Upsilon)^\sim:\Upsilon\to X^\sharp$ of $\varphi|_\Upsilon:\Upsilon\to Y^\sharp$ that intersects $\Theta_i^\sharp$ transversely at the two points $(\pm 1/\sqrt{2},0,1)$. Let $(\varphi|_\Upsilon)^\sim|_{\partial A_0}:\partial A_0\to X^\circ$ be the restriction of $(\varphi|_\Upsilon)^\sim$ to $\partial\Upsilon=\partial A_0$. Then the free loop pair

$$(\varphi|_{A_0},(\varphi|_\Upsilon)^\sim|_{\partial A_0})\,:\,(A_0,\partial A_0)\;\to\;(Y^\circ,X^\circ)$$

is of monodromy relation type around $\Sigma_i^\sharp$.

Corollary 3.32. *Let $\delta:\bar\Delta\to Y^\sharp$ be a transversal disc around $\Sigma_i^\sharp$, and let $\tilde\delta$ and $\tilde\delta'$ be two transversal lifts of δ. We put $q:=\delta(1)$ and $\tilde q:=\tilde\delta(1)\in F_q$, $\tilde q':=\tilde\delta'(1)\in F_q$. Suppose that we are given a path $\gamma_0:I\to F_q$ from $\tilde q$ to $\tilde q'$. Then we can deform γ_0 to a path γ_t on $F_{\partial_\varepsilon\delta(t)}$ from $\partial_\varepsilon\tilde\delta(t)$ to $\partial_\varepsilon\tilde\delta'(t)$; that is, we have a continuous map $\Gamma:I\times I\to X^\sharp$ such that*

$$f(\Gamma(s,t))=\partial_\varepsilon\delta(t),\quad \Gamma(s,0)=\gamma_0(s),\quad \Gamma(0,t)=\partial_\varepsilon\tilde\delta(t),\quad \Gamma(1,t)=\partial_\varepsilon\tilde\delta'(t),$$

and $\gamma_t:=\Gamma|_{I\times\{t\}}$. Consider the path γ_1 on F_q from $\tilde q$ to $\tilde q'$. The conjunction $\gamma_0\gamma_1^{-1}$ is a loop on F_q, which we write $\gamma_0\gamma_1^{-1}:D\to F_q$, where D is homeomorphic to $\bar\Delta$. Then the free loop pair

$$(1_q,\gamma_0\gamma_1^{-1})\,:\,(D,\partial D)\;\to\;(Y^\circ,X^\circ)$$

is of monodromy relation type around $\Sigma_i^\sharp$.

Now we start the proof of Theorem 3.20.

Proof of Theorem 3.20. By Proposition 3.3, we have $N^{[\rho]}\subset\operatorname{Ker}(\iota_*)$ for any $[\rho]\in\mathcal{L}$, because the lasso $\lambda(\tilde\rho)$ is null-homotopic in X for any transversal lift $\tilde\rho$ of ρ. Therefore $\mathcal{N}\subset\operatorname{Ker}(\iota_*)$ follows.

Let a loop $\gamma:(I,\partial I)\to(F_b,\tilde b)$ represent an element $[\gamma]$ of $\operatorname{Ker}(\iota_*)$. We will show that $[\gamma]\in\mathcal{N}$. There exists a homotopy

$$h\,:\,(I^2,\sqcap)\;\to\;(X,\tilde b)$$

from γ to $1_{\tilde b}$ in X stationary on ∂I; that is, $h|_{I\times\{0\}}=\gamma$ and $h|_\sqcap=1_{\tilde b}$, where $\sqcap:=(\partial I\times I)\cup(I\times\{1\})\subset I^2$. By the condition (C1), we can perturb h so that

$$h(I^2)\cap\operatorname{Sing}(f)=\emptyset \tag{3.1}$$

holds. Since $(f\circ h)|_{\partial I^2}=1_b$, the map $f\circ h: I^2\to Y$ represents an element of $\pi_2(Y,b)$. By the condition (Z), we have a continuous map

$$l\,:\,(I^2,\partial I^2)\;\to\;(Z,b)$$

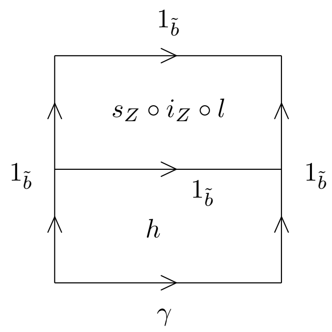


FIGURE 3.9. The map h'

such that $[f \circ h] + [i_Z \circ l] = 0$ holds in $\pi_2(Y, b)$, where $i_Z : Z \hookrightarrow Y$ is the inclusion. We then consider the continuous map $s_Z \circ i_Z \circ l : (I^2, \partial I^2) \to (X, \tilde{b})$. Replacing h with $h' : (I^2, \sqcap) \to (X, \tilde{b})$ defined by

$$h'(x, y) := \begin{cases} h(x, 2y) & \text{if } 2y \leq 1, \\ s_Z \circ i_Z \circ l(x, 2y - 1) & \text{if } 2y \geq 1, \end{cases}$$

we have

$$[f \circ h] = 0 \quad \text{in} \quad \pi_2(Y, b). \tag{3.2}$$

(See Figure 3.9.) Moreover, since $s_Z(Z) \cap \mathrm{Sing}(f) = \emptyset$ by the condition (Z), we still have (3.1). Then any small perturbation of $f \circ h$ can be lifted to a small perturbation of h. Since Ξ is of codimension ≥ 2 in Y, we can assume that $(f \circ h)(I^2) \cap \Sigma \subset \Sigma^\sharp$, and that $f \circ h$ intersects $\Sigma^\sharp$ transversely (see Definition 3.8). We put

$$(f \circ h)^{-1}(\Sigma^\sharp) = \{P_1, \dots, P_n\} \ \subset \ I^2 \setminus \partial I^2.$$

We will construct a continuous map

$$j \ : \ V := I^2 \setminus (D_1^\circ \cup \cdots \cup D_m^\circ) \ \to \ X^\sharp$$

with the following properties:

- (j1) $D_1, \dots, D_m$ are mutually disjoint closed discs in $I^2 \setminus (\partial I^2 \cup \{P_1, \dots, P_n\})$, and D_μ° is the interior of D_μ; in particular, V contains $P_1, \dots, P_n$ in its interior,
- (j2) $j(\partial I^2) = \{\tilde{b}\}$,
- (j3) $f \circ j = f \circ h|_V$ holds, and hence we have $j^{-1}(\Theta^\sharp) = \{P_1, \dots, P_n\}$,
- (j4) j intersects $\Theta^\sharp$ transversely at the points P_ν for $\nu = 1, \dots, n$, and
- (j5) for each D_μ, the free loop pair

$$((f \circ h)|_{D_\mu}, j|_{\partial D_\mu}) \ : \ (D_\mu, \partial D_\mu) \ \to \ (Y^\circ, X^\circ)$$

  is of monodromy relation type.

By (3.2), there exists a homotopy

$$H \ : \ (I^2 \times I, B) \ \to \ (Y, b)$$

from $f \circ h$ to 1_b that is stationary on ∂I^2; that is, $H|_{I^2\times\{0\}} = f \circ h$ and $H|_B = 1_b$, where

$$B := (\partial I^2 \times I) \cup (I^2 \times \{1\}) \ \subset\ I^2 \times I.$$

Since Ξ is of real codimension ≥ 4 in Y, we can perturb H and assume the following:

(H1) $H(I^2 \times I) \cap \Sigma$ is contained in $\Sigma^\sharp$,

(H2) H intersects $\Sigma^\sharp$ transversely (in the sense of Definition 3.8), so that

$$L := H^{-1}(\Sigma^\sharp)$$

is a disjoint union of smooth real curves, and

(H3) the projection $\mathrm{pr}_L : L \to I$ to the second factor of $I^2 \times I$ has only ordinary critical points in L; that is, pr_L is a Morse function on L.

We have

$$\partial L = L \cap (I^2 \times \{0\}) = (f \circ h)^{-1}(\Sigma^\sharp) = \{P_1, \ldots, P_n\}.$$

Let $L_1, \ldots, L_k$ be the connected components of L. Then each L_κ is a curve connecting two points of $\{P_1, \ldots, P_n\}$, or a curve without boundary. In particular, the cardinality n of the points $(f \circ h)^{-1}(\Sigma^\sharp)$ is even.

We denote by $p_1^+, \ldots, p_l^+$ (resp. $p_1^-, \ldots, p_m^-$) the critical points in $L \setminus \partial L$ of the projection $\mathrm{pr}_L : L \to I$ at which the Morse function pr_L attains a local maximum (resp. a local minimum), and call them the *positive (resp. negative) critical points of* pr_L. (See Figure 3.10, in which L is drawn in thick curve.)

Let $\mathbb{T}$ and A_ζ be as in Corollary 3.31. For each negative critical point p_μ^-, we can choose a continuous map

$$\tau_\mu \,:\, \mathbb{T} \,\to\, I^2 \times I$$

with the following properties:

(τ1) each τ_μ is a homeomorphism onto its image $T_\mu := \tau_\mu(\mathbb{T})$, and $T_1, \ldots, T_m$ are mutually disjoint,

(τ2) there exists a strictly increasing function $t_\mu : I \to I$ with $t_\mu(0) = 0$ that makes the following diagram commutative;

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\tau_\mu} & I^2 \times I \\ \downarrow & & \downarrow \\ I & \xrightarrow{t_\mu} & I, \end{array}$$

where the vertical arrows are the projections onto the last factors,

(τ3) $\tau_\mu^{-1}(\partial(I^2 \times I)) = A_0$ and $\tau_\mu(A_0) \subset (I^2 \setminus \partial I^2) \times \{0\}$,

(τ4) $\tau_\mu^{-1}(L) = \{(x, 0, z) \in T \mid x^2 + (z-1)^2 = 1/2\}$ and $\tau_\mu(1/2, 0, 1/2) = p_\mu^-$, so that p_μ^- is the only critical point of pr_L in $T_\mu \cap L$, and

(τ5) $H \circ (\tau_\mu|_{A_1}) : A_1 \to Y^\sharp$ intersects $\Sigma^\sharp$ transversely at $(\pm 1/\sqrt{2}, 0, 1) \in A_1$.

We put

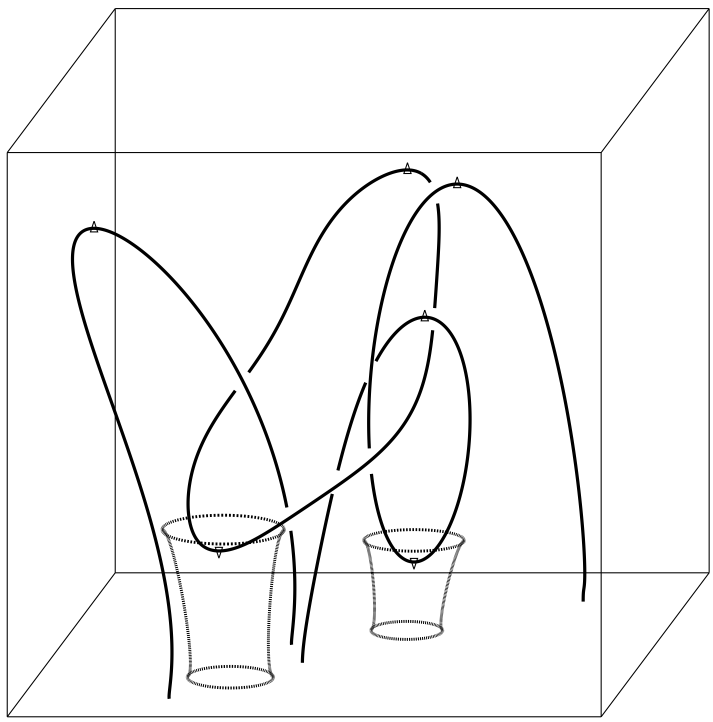
$$T := T_1 \cup \cdots \cup T_m.$$

(In Figure 3.10, each T_μ is depicted by dashed curves.) We also put

$$\mathbb{T}^\circ := \{(x, y, z) \in \mathbb{T} \mid x^2 + y^2 < 1, z < 1\}$$

(the union of the interior of $\mathbb{T}$ and the bottom open disc), and

$$T_\mu^\circ := \tau_\mu(\mathbb{T}^\circ), \quad T^\circ := T_1^\circ \cup \cdots \cup T_m^\circ \quad \text{and} \quad J := (I^2 \times I) \setminus T^\circ.$$



$\triangle$: the points p^+_λ, $\triangledown$: the points p^-_μ.

FIGURE 3.10. L and T

Note that J is the closure of $(I^2 \times I) \setminus T$. Then

$$L' := L \cap J$$

is a disjoint union of smooth real curves $L'_1, \dots, L'_l$, and each connected component L'_λ of L' contains exactly one positive critical point p^+_λ in $L'_\lambda \setminus \partial L'_\lambda$. Moreover, each L'_λ has two boundary points Q_λ and Q'_λ, each of which is either one point among $\{P_1, \dots, P_n\}$ or one of $\tau_\mu(\pm 1/\sqrt{2}, 0, 1)$ for some μ. If Q_λ is one of $P_1, \dots, P_n$, let $D(Q_\lambda)$ be a sufficiently small closed disc on $I^2 \times \{0\}$ with the center Q_λ. If Q_λ is one of $\tau_\mu(\pm 1/\sqrt{2}, 0, 1))$, let $D(Q_\lambda)$ be a sufficiently small closed disc on $\tau_\mu(A_1)$ with the center Q_λ. We choose a closed disc $D(Q'_\lambda)$ with the center Q'_λ in the same way. Note that $H|_{D(Q_\lambda)} : D(Q_\lambda) \to Y^\sharp$ and $H|_{D(Q'_\lambda)} : D(Q_\lambda) \to Y^\sharp$ are the transversal discs around the irreducible component $\Sigma^\sharp_{i(\lambda)}$ of $\Sigma^\sharp$ that contains $H(p^+_\lambda)$. Then, for each $\lambda = 1, \dots, l$, we have a tubular neighborhood

$$m_\lambda \, : \, \bar{\Delta} \times I \, \to \, J$$

of L'_λ in J with the following properties:

(m1) each m_λ is a homeomorphism onto its image M_λ, and $M_1, \dots, M_l$ are mutually disjoint,

(m2) $m_\lambda^{-1}(L') = \{0\} \times I$ and $m_\lambda(\{0\} \times I) = L'_\lambda$,

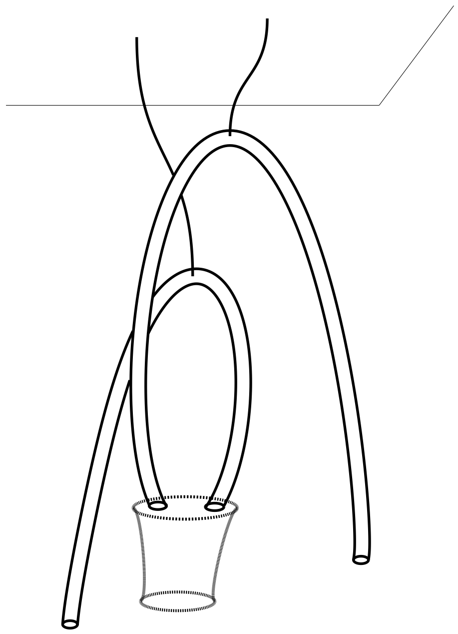


FIGURE 3.11. Two of $M_\lambda \cup W_\lambda$

(m3) m_λ is differentiable and locally a submersion at each point of $\{0\} \times I$, and
(m4) $m_\lambda^{-1}(\partial J) = \bar{\Delta} \times \partial I$ and $m_\lambda(\bar{\Delta} \times \{0\}) = D(Q_\lambda)$, $m_\lambda(\bar{\Delta} \times \{1\}) = D(Q'_\lambda)$.

Then the composite $H \circ m_\lambda : \bar{\Delta} \times I \to Y^\sharp$ is an isotopy between the transversal discs $H|_{D(Q_\lambda)}$ and $H|_{D(Q'_\lambda)}$. We put

$$M := M_1 \cup \cdots \cup M_l.$$

Let $c_\lambda \in I$ be the real number such that $m_\lambda(0, c_\lambda) = p_\lambda^+$. We choose a point $p_\lambda^{+\prime}$ on $m_\lambda(\partial\bar{\Delta} \times \{c_\lambda\}) \subset \partial M_\lambda$ and a path

$$w_\lambda : I \to J$$

from $p_\lambda^{+\prime}$ to a point $p_\lambda^{+\prime\prime}$ of $I^2 \times \{1\}$ with the following properties:

(w1) each w_λ is a homeomorphism onto its image W_λ, and $W_1, \dots, W_l$ are mutually disjoint,
(w2) $w_\lambda^{-1}(M) = \{0\}$, $w_\lambda^{-1}(\partial J) = \{1\}$, and
(w3) the composite $\mathrm{pr}_2 \circ w_\lambda : I \to I$ of w_λ with the second projection $I^2 \times I \to I$ is strictly increasing.

We put

$$W := W_1 \cup \cdots \cup W_l.$$

In Figure 3.11, two of $M_\lambda \cup W_\lambda$ are illustrated. The ceiling is $I^2 \times \{1\}$, from which W_λ are dangling, and the tubes are M_λ.

The following fact is the crucial point in the construction of $j : V \to X^\sharp$:

(3.3) $\qquad B \cup M \cup W$ *is a strong deformation retract of* J.

We choose transversal lifts $(H|_{D(Q_\lambda)})^\sim$ and $(H|_{D(Q'_\lambda)})^\sim$ of the transversal discs $H|_{D(Q_\lambda)}$ and $H|_{D(Q'_\lambda)}$ around $\Sigma^\sharp_{i(\lambda)}$, respectively. Then the isotopy $H \circ m_\lambda : \bar{\Delta} \to Y^\sharp$ between $H|_{D(Q_\lambda)}$ and $H|_{D(Q'_\lambda)}$ lifts to an isotopy between $(H|_{D(Q_\lambda)})^\sim$ and $(H|_{D(Q'_\lambda)})^\sim$, which yields a lift $(H|_{M_\lambda})^\sim$ of $H|_{M_\lambda}$. Hence we obtain a lift

$$(H|_M)^\sim \,:\, M \,\to\, X^\sharp$$

of $H|_M$. We define a lift $(H|_B)^\sim$ of $H|_B$ to be the constant map $1_{\tilde{b}}$. Then we can lift the path $H \circ w_\lambda$ to a path from $(H|_M)^\sim(p_\lambda^{+\prime})$ to $(H|_B)^\sim(p_\lambda^{+\prime\prime}) = \tilde{b}$, and thus we obtain a lift

$$(H|_W)^\sim \,:\, W \,\to\, X^\sharp$$

of $H|_W$. Joining these three lifts together, we obtain a lift

$$(H|_{B\cup M\cup W})^\sim \,:\, B \cup M \cup W \,\to\, X^\sharp$$

of $H|_{B\cup M\cup W}$. By the fact (3.3), we can extend the lift $(H|_{B\cup M\cup W})^\sim$ to a lift

$$(H|_J)^\sim \,:\, J \,\to\, X^\sharp$$

of $H|_J$, because the pull-back $(H|_J)^*(f^\sharp)$ of $f^\sharp : X^\sharp \to Y^\sharp$ by $H|_J : J \to Y^\sharp$ is locally trivial over the complement of the interior of M in J.

Recall that the floor $I^2 \times \{0\}$ of the source space $I^2 \times I$ of H is the source space I^2 of $f \circ h$. For $\mu = 1, \dots, m$, we put

$$D_\mu := \tau_\mu(A_0).$$

These $D_1, \dots, D_m$ satisfy the condition (j1). Then

$$V := I^2 \setminus (D_1^\circ \cup \dots \cup D_m^\circ)$$

is identified with $J \cap (I^2 \times \{0\})$. We put

$$j := (H|_J)^\sim|_V,$$

which is a lift of $f \circ h|_V = H|_V$. Hence j satisfies (j3). It is obvious that j satisfies (j1) and (j2). Since $(H|_M)^\sim$ is constructed as a union of isotopies of transversal discs around $\Theta^\sharp$, the continuous map

$$j|_{M\cap V} = (H|_M)^\sim|_{M\cap V} \,:\, M \cap V \,\to\, X^\sharp$$

intersects $\Theta^\sharp$ transversely at each P_ν. Therefore j satisfies (j4). By the properties ($\tau 4$) and ($\tau 5$) of τ_μ and Corollary 3.31, we see that j satisfies (j5). Thus the hoped-for continuous map $j : V \to X^\sharp$ is constructed.

For $\nu = 1, \dots, n$, we choose a sufficiently small closed disc $D_{m+\nu}$ with the center P_ν in $I^2 \setminus \partial I^2$ in such a way that the $m+n$ closed discs $D_1, \dots, D_{m+n}$ are mutually disjoint.

For each $\mu = 1, \dots, m+n$, we choose a path

$$\alpha_\mu \,:\, I \,\to\, I^2$$

from a point $R_\mu = (\rho_\mu, 0) \in I \times \{0\}$ to a point $S_\mu \in \partial D_\mu$ with the following properties:

($\alpha 1$) $0 < \rho_1 < \dots < \rho_{m+n} < 1$,

($\alpha 2$) each α_μ is injective and the images $\alpha_\mu(I)$ ($\mu = 1, \dots, m+n$) are mutually disjoint, and

($\alpha 3$) $\alpha_\mu^{-1}(\partial I^2) = \{0\}$, $\alpha_\mu^{-1}(D_\mu) = \{1\}$, and $\alpha_\mu^{-1}(D_{\mu'}) = \emptyset$ if $\mu \neq \mu'$.

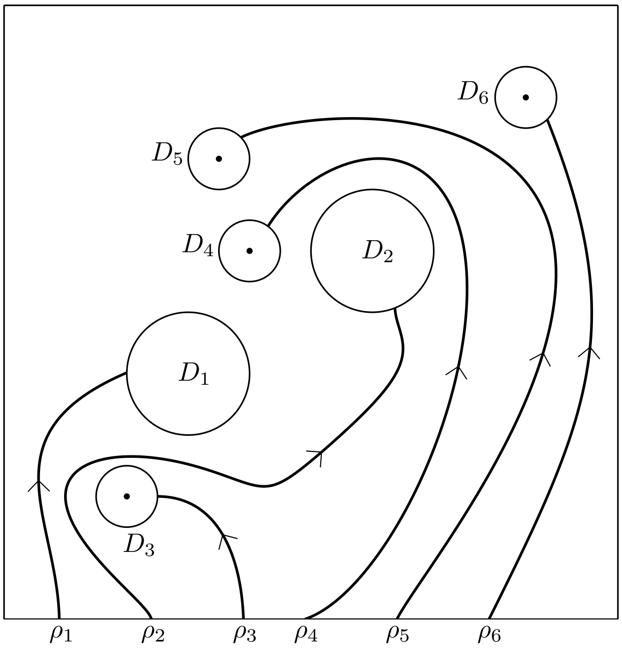


FIGURE 3.12. The paths α_μ

In Figure 3.12, the paths α_μ are illustrated by thick curves. Then there exists a continuous map

$$\ell : \mathbf{I}^2 \to I^2$$

with the following properties, where $\mathbf{I} := I = [0,1] \subset \mathbb{R}$. (We use the boldface $\mathbf{I}$ to distinguish the source plane $\mathbf{I}^2$ and the target plane I^2 of ℓ.)

(ℓ1) ℓ induces a homeomorphism from $\mathbf{I}^2 \setminus \partial \mathbf{I}^2$ to

$$I^2 \setminus \left(\partial I^2 \cup \bigcup_{\mu=1}^{m+n} (D_\mu \cup \alpha_\mu(I)) \right),$$

(ℓ2) if $(x,y) \in \sqcap := (\partial \mathbf{I} \times \mathbf{I}) \cup (\mathbf{I} \times \{1\})$, then $\ell(x,y) = (x,y)$, and

(ℓ3) there exist real numbers $c_\mu, d_\mu, d'_\mu, c'_\mu \in \mathbf{I}$ for $\mu = 1, \dots, m+n$ with

$$\begin{array}{ccccccccccc} 0 & < & c_1 & < & d_1 & < & d'_1 & < & c'_1 & < & \\ & < & c_2 & < & d_2 & < & d'_2 & < & c'_2 & < & \\ & & & & & \cdots & & & & & \\ & < & c_{m+n} & < & d_{m+n} & < & d'_{m+n} & < & c'_{m+n} & < & 1 \end{array}$$

such that the following hold:

– $\ell(c_\mu, 0) = \ell(c'_\mu, 0) = R_\mu \in I \times \{0\}$, $\ell(d'_\mu, 0) = \ell(d_\mu, 0) = S_\mu \in \partial D_\mu$,

– $\ell|_{[c_\mu, d_\mu] \times \{0\}}$ is equal to α_μ via a parameter change $[c_\mu, d_\mu] \cong I$, and $\ell|_{[d'_\mu, c'_\mu] \times \{0\}}$ is equal to α_μ^{-1} via a parameter change $[d'_\mu, c'_\mu] \cong I$,

– $\ell|_{[d_\mu, d'_\mu] \times \{0\}}$ is the loop that goes from S_μ to S_μ along ∂D_μ clockwise, and

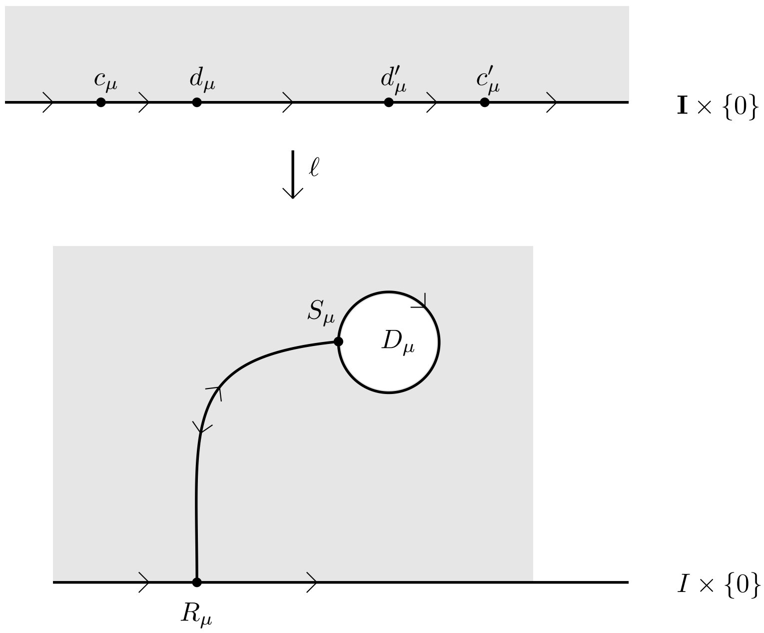


FIGURE 3.13. The map ℓ

– $\ell|_{[c'_{\mu-1}, c_\mu]\times\{0\}}$ is equal to the path $[\rho_{\mu-1}, \rho_\mu] \to I \times \{0\}$ given by $t \mapsto (t, 0)$ via a parameter change $[c'_{\mu-1}, c_\mu] \cong [\rho_{\mu-1}, \rho_\mu]$, where we put $\rho_0 := 0, c'_0 := 0$ and $\rho_{m+n+1} := 1, c_{m+n+1} := 1$.

(See Figure 3.13.) Since the image of ℓ is contained in V and is disjoint from $\{P_1, \ldots, P_n\}$, we have continuous maps

$$j \circ \ell : \mathbf{I}^2 \to X^\circ \quad \text{and} \quad h \circ \ell : \mathbf{I}^2 \to X^\circ$$

to X°. They satisfy

$$f^\circ \circ j \circ \ell = f^\circ \circ h \circ \ell$$

by the property (j3). By the properties (j2) and (ℓ2), they also satisfy

$$j \circ \ell|_\sqcap = 1_{\tilde{b}} \quad \text{and} \quad h \circ \ell|_\sqcap = 1_{\tilde{b}}.$$

We then define $G : \mathbf{I}^2 \times \mathbf{I} \to Y^\circ$ by the composition

$$G \ : \ \mathbf{I}^2 \times \mathbf{I} \ \xrightarrow{\mathrm{pr}_1} \ \mathbf{I}^2 \ \xrightarrow{f^\circ \circ j \circ \ell = f^\circ \circ h \circ \ell} \ Y^\circ,$$

where pr_1 is the first projection. We put

$$C := (\mathbf{I}^2 \times \partial\mathbf{I}) \cup (\sqcap \times \mathbf{I}) \ \subset \ \mathbf{I}^2 \times \mathbf{I},$$

and define a lift

$$(G|_C)^\sim \ : \ C \ \to \ X^\circ$$

of $G|_C : C \to Y^\circ$ by the following:

$$(G|_C)^\sim(x, y, z) := \begin{cases} h(\ell(x, y)) & \text{if } z = 0, \\ j(\ell(x, y)) & \text{if } z = 1, \\ \tilde{b} & \text{if } (x, y, z) \in \sqcap \times \mathbf{I}. \end{cases}$$

Since $f^\circ : X^\circ \to Y^\circ$ is locally trivial and C is a strong deformation retract of $\mathbf{I}^2 \times \mathbf{I}$, the map $(G|_C)^\sim$ extends to a lift

$$\tilde{G} \,:\, \mathbf{I}^2 \times \mathbf{I} \,\to\, X^\circ$$

of $G \,:\, \mathbf{I}^2 \times \mathbf{I} \,\to\, Y^\circ$. By construction, for $(x, y) \in \mathbf{I}^2$, the restriction of $\tilde{G}$ to $\{(x, y)\} \times \mathbf{I}$ is a path in the fiber

$$F_{f\circ h\circ \ell(x,y)} = F_{f\circ j\circ \ell(x,y)}$$

from the point $h \circ \ell(x, y)$ to the point $j \circ \ell(x, y)$. For $x \in \mathbf{I}$, we put

$$F_{[x]} := F_{f\circ h\circ \ell(x,0)} = F_{f\circ j\circ \ell(x,0)}, \quad \text{and} \quad \xi_{[x]} := \tilde{G}|_{\{(x,0)\}\times \mathbf{I}} \,:\, \mathbf{I} \,\to\, F_{[x]}.$$

Suppose that $x \notin \bigcup_{\mu=1}^{m+n}[c_\mu, c'_\mu]$, so that

$$(x', 0) := \ell(x, 0) \in I \times \{0\}.$$

By (j2), we see that $F_{[x]}$ is equal to F_b and $\xi_{[x]}$ is a path in F_b from $h(x', 0) = \gamma(x')$ to $j(x', 0) = \tilde{b}$. Moreover, we have $\xi_{[0]} = \xi_{[1]} = 1_{\tilde{b}}$ because $\tilde{G}|_{\sqcap\times\mathbf{I}} = 1_{\tilde{b}}$. Therefore, for $\mu = 0, 1, \dots, m + n$, the path

$$\gamma_\mu := \gamma|_{[\rho_\mu,\rho_{\mu+1}]} = h|_{[\rho_\mu,\rho_{\mu+1}]\times\{0\}} \,:\, [\rho_\mu, \rho_{\mu+1}] \,\to\, F_b$$

is homotopic to the path $\xi_{[c'_\mu]}\xi^{-1}_{[c_{\mu+1}]}$ in F_b, because the boundary of $\tilde{G}|_{[c'_\mu,c_{\mu+1}]\times\{0\}\times\mathbf{I}}$ is the loop $\xi_{[c'_\mu]}\cdot 1_{\tilde{b}}\cdot\xi^{-1}_{[c_{\mu+1}]}\cdot\gamma_\mu^{-1}$ in F_b, where $[c'_\mu, c_{\mu+1}] \times \{0\} \times \mathbf{I} \cong I^2$ is oriented and segmented as in Figure 3.4. Since γ is the conjunction $\gamma_0\gamma_1 \dots \gamma_{m+n}$, the homotopy class $[\gamma] \in \pi_1(F_b, \tilde{b})$ is equal to

$$[\, \xi_{[c'_0]}\xi^{-1}_{[c_1]}\xi_{[c'_1]}\xi^{-1}_{[c_2]} \cdots \xi_{[c'_{m+n}]}\xi^{-1}_{[c_{m+n+1}]} \,] \;=\; [\xi^{-1}_{[c_1]}\xi_{[c'_1]}]\cdot[\xi^{-1}_{[c_2]}\xi_{[c'_2]}]\cdot\dots\cdot[\xi^{-1}_{[c_{m+n}]}\xi_{[c'_{m+n}]}].$$

(See Figure 3.14.) Note that $\xi^{-1}_{[c_\mu]}\xi_{[c'_\mu]}$ is a loop in F_b with the base point $\tilde{b}$. It is enough to show that each $[\xi^{-1}_{[c_\mu]}\xi_{[c'_\mu]}] \in \pi_1(F_b, \tilde{b})$ is contained in $N^{[\rho]}$ for some transversal disc ρ around an irreducible component of $\Sigma^\sharp$.

Consider the path

$$\tilde{\alpha}_\mu := j \circ \alpha_\mu \,:\, I \,\to\, X^\circ$$

from $\tilde{b}$ to $\tilde{q}_\mu := j(S_\mu) \in F_{q_\mu}$, where $q_\mu := f(j(S_\mu)) = f(h(S_\mu))$, and the induced isomorphism

$$[\tilde{\alpha}_\mu]_* \,:\, \pi_1(F_b, \tilde{b}) \;\xrightarrow{\sim}\; \pi_1(F_{q_\mu}, \tilde{q}_\mu).$$

This isomorphism maps $[\xi^{-1}_{[c_\mu]}\xi_{[c'_\mu]}] \in \pi_1(F_b, \tilde{b})$ to

$$[\xi^{-1}_{[d_\mu]}\xi_{[d'_\mu]}] \in \pi_1(F_{q_\mu}, \tilde{q}_\mu).$$

(See Figure 3.15.) We consider $\xi^{-1}_{[d_\mu]}\xi_{[d'_\mu]}$ as a free loop $\partial\bar{\Delta} \to F_{q_\mu}$ in F_{q_μ}. It is enough to show that the free loop pair

$$(1_{q_\mu}, \xi^{-1}_{[d_\mu]}\xi_{[d'_\mu]}) \,:\, (\bar{\Delta}, \partial\bar{\Delta}) \,\to\, (Y^\circ, X^\circ)$$

is of monodromy relation type.

Suppose that $\mu > m$, so that D_μ is a disc with the center $P_{\mu-m} \in (f \circ h)^{-1}(\Sigma^\sharp)$. Then $(1_{q_\mu}, \xi^{-1}_{[d_\mu]}\xi_{[d'_\mu]})$ is of monodromy relation type by Corollary 3.32. Suppose

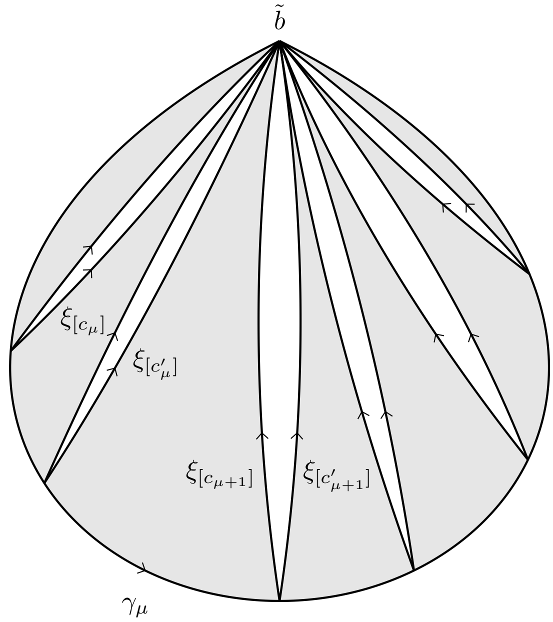


FIGURE 3.14. The paths γ_μ and $\xi_{[c_\mu]}, \xi_{[c'_\mu]}$

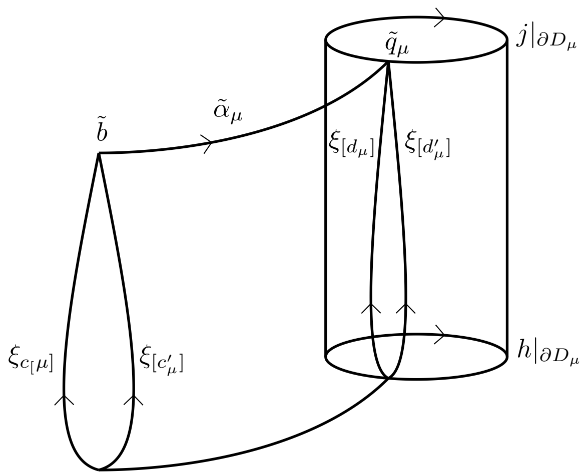


FIGURE 3.15. Deformation of the loop along $\tilde{\alpha}_\mu$

that $\mu \leq m$. By (j5), it is enough to show that the free loop pair $(1_{q_\mu}, \xi_{[d_\mu]}^{-1}\xi_{[d'_\mu]})$ is homotopic to the free loop pair

$$((f \circ h)|_{D_\mu}, j|_{\partial D_\mu}) : (D_\mu, \partial D_\mu) \to (Y^\circ, X^\circ)$$

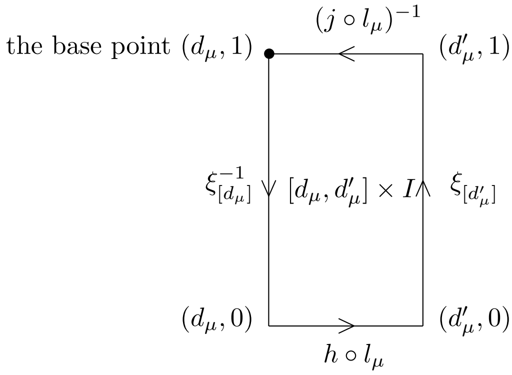


FIGURE 3.16. The orientation of $\partial([d_\mu, d'_\mu] \times I)$

under a suitable homeomorphism $\bar{\Delta} \cong D_\mu$. We put

$$l_\mu := \ell_{[d_\mu, d'_\mu] \times \{0\}} \;:\; [d_\mu, d'_\mu] \;\to\; \partial D_\mu.$$

Consider the continuous map

$$\zeta_\mu \;:\; [d_\mu, d'_\mu] \times I \;\to\; X^\circ$$

given by $\zeta_\mu(x, t) := \xi_{[x]}(t)$. With the base point and the orientation on the boundary of $[d_\mu, d'_\mu] \times I$ given in Figure 3.16, the boundary of ζ_μ is equal to the loop

$$\xi^{-1}_{[d_\mu]} \cdot (h \circ l_\mu) \cdot \xi_{[d'_\mu]} \cdot (j \circ l_\mu)^{-1}$$

with the base point $\tilde{q}_\mu$. Since the free loop $h \circ l_\mu$ is the boundary of $h|_{D_\mu}$, it is null-homotopic in X°. Hence the free loop $\xi^{-1}_{[d_\mu]} \cdot \xi_{[d'_\mu]}$ is homotopic to the free loop $j \circ l_\mu$ in X°. It can be easily seen that we can construct a homotopy of free loops from $j|_{\partial D_\mu} = j \circ l_\mu$ to $\xi^{-1}_{[d_\mu]} \cdot \xi_{[d'_\mu]}$ in X° as a lift of the restriction to ∂D_μ of a contraction from $f(h(D_\mu))$ to q_μ, because $f(h(D_\mu)) \subset Y^\circ$ holds for $\mu \le m$. Hence $(1_{q_\mu}, \xi^{-1}_{[d_\mu]} \xi_{[d'_\mu]})$ is homotopic to $((f \circ h)|_{D_\mu}, j|_{\partial D_\mu})$. □

The following is a semi-classical version of Theorem 3.20.

Theorem 3.33. *Suppose that the conditions* (C1) *and* (C2) *are satisfied. Suppose also that there exist a reduced connected curve C (possibly singular and/or reducible and not necessarily closed) on Y and a continuous cross-section*

$$s_C \;:\; C \;\to\; f^{-1}(C)$$

of f over C with the following properties:

- *$C^\circ := C \cap Y^\circ$ is non-empty and connected, and the inclusion $C^\circ \hookrightarrow Y^\circ$ induces a surjection $\pi_1(C^\circ, b) \twoheadrightarrow \pi_1(Y^\circ, b)$, where $b \in C^\circ$ is a base point,*
- *the inclusion $C \hookrightarrow Y$ induces a surjection $\pi_2(C, b) \twoheadrightarrow \pi_2(Y, b)$,*
- *$s_C(C) \cap \mathrm{Sing}(f) = \emptyset$, and*
- *for each irreducible component Σ_i of Σ with codimension 1 in Y, there exists a point $p_i \in C \cap \Sigma_i$ satisfying the following:*
  - *C and Σ are smooth at p_i, and C intersects Σ_i transversely at p_i,*
  - *the cross-section s_C is holomorphic at p_i.*

By the cross-section s_C, we have the classical monodromy action

$$\pi_1(C^\circ, b) \;\to\; \mathrm{Aut}(\pi_1(F_b, \tilde{b})), \quad \textit{where} \quad \tilde{b} := s_C(b) \in F_b := f^{-1}(b),$$

which we denote by $g \mapsto g^u$ for $u \in \pi_1(C^\circ, b)$. Then $\mathrm{Ker}(\iota_*)$ is equal to

$$N_K := \langle\ \{g^{-1}g^u \mid g \in \pi_1(F_b, \tilde{b}), u \in K\}\ \rangle,$$

where $K \subset \pi_1(C^\circ, b)$ is the kernel of $\pi_1(C^\circ, b) \to \pi_1(C, b)$ induced by the inclusion.

Proof. First of all, remark that the condition (Z) is satisfied with C and s_C being Z and s_Z in the condition (Z), and hence $\mathrm{Ker}(\iota_*)$ is equal to $\mathcal{N}$.

Let $\gamma : (I, \partial I) \to (C^\circ, b)$ be a loop that represents an element u of K. We have a homotopy (stationary on ∂I) h on C from γ to 1_b. Then $s_C \circ h$ is a homotopy on X from $s_C \circ \gamma$ to $1_{\tilde{b}}$. By definition, the classical monodromy action by u is equal to the lifted monodromy action by $[s_C \circ \gamma] \in \pi_1(X^\circ, \tilde{b})$. Since $s_C \circ \gamma$ is null-homotopic in X, we see that $g^{-1}g^u = g^{-1}g^{\mu([s_C \circ \gamma])}$ is contained in $\mathrm{Ker}(\iota_*)$ by Proposition 3.3. Thus $N_K \subset \mathrm{Ker}(\iota_*)$ is proved.

In order to prove $\mathcal{N} = \mathrm{Ker}(\iota_*) \subset N_K$, it is enough to show that, for any leashed disc $\rho = (\delta, \eta)$ around an irreducible component $\Sigma_i^\sharp$ of $\Sigma^\sharp$ in $Y^\sharp$, the normal subgroup $N^{[\rho]}$ is contained in N_K. We have a point p_i of $C \cap \Sigma_i$ at which C and Σ are smooth and intersect transversely. Let

$$\delta_{i,C} \;:\; \bar{\Delta} \;\hookrightarrow\; C$$

be a sufficiently small closed disc on C such that $\delta_{i,C}(0) = p_i$. Since s_C is holomorphic at p_i and $s_C(p_i) \notin \mathrm{Sing}(f)$ by the assumption, $\Theta := f^{-1}(\Sigma)$ is smooth at $s_C(p_i)$, and $s_C \circ \delta_{i,C}$ intersects Θ at $s_C(p_i)$ transversely. If $p_i \in \Xi$, then we perturb $\delta_{i,C}$ to a $\mathcal{C}^\infty$-map $\delta'_{i,C} : \bar{\Delta} \to Y^\sharp$ such that $\delta_{i,C}|_{\partial\bar{\Delta}} = \delta'_{i,C}|_{\partial\bar{\Delta}}$. If $p_i \notin \Xi$, then we put $\delta'_{i,C} := \delta_{i,C}$. Then $\delta'_{i,C}$ is a transversal disc around $\Sigma_i^\sharp$ such that $\delta'_{i,C}(\partial\bar{\Delta}) \subset C^\circ$. Since $s_C(p_i) \notin \mathrm{Sing}(f)$, we can lift the perturbation from $\delta_{i,C}$ to $\delta'_{i,C}$ to a perturbation from $s_C \circ \delta_{i,C}$ to

$$\tilde{\delta}'_{i,C} \;:\; \bar{\Delta} \;\hookrightarrow\; X^\sharp$$

in such a way that

$$\tilde{\delta}'_{i,C}|_{\partial\bar{\Delta}} = s_C \circ \delta'_{i,C}|_{\partial\bar{\Delta}} = s_C \circ \delta_{i,C}|_{\partial\bar{\Delta}},$$

and that $\tilde{\delta}'_{i,C}$ is a transversal lift of $\delta'_{i,C}$ around $\Theta_i^\sharp$. The transversal disc δ of the given leashed disc $\rho = (\delta, \eta)$ is isotopic to $\delta'_{i,C}$ (Proposition 3.10). Hence ρ is isotopic to a leashed disc

$$\rho' = (\delta'_{i,C}, \eta')$$

for some path η' on Y° from $\delta_{i,C}(1) = \delta'_{i,C}(1) \in C^\circ$ to b. Since C° is connected, there exists a path ζ on C° from b to $\eta'(0) = \delta_{i,C}(1)$. Then $\zeta\eta'$ is a loop on Y° with the base point b. Since the inclusion $C^\circ \hookrightarrow Y^\circ$ induces a surjection $\pi_1(C^\circ, b) \twoheadrightarrow \pi_1(Y^\circ, b)$, there exists a loop ξ on C° with the base point b that is homotopic to $\zeta\eta'$ in Y°. Then $\rho = (\delta, \eta)$ is isotopic to the leashed disc

$$\rho_C := (\delta'_{i,C}, \zeta^{-1}\xi).$$

Note that $\zeta^{-1}\xi$ is a path on C°. Since $\tilde{\delta}'_{i,C}(1) = s_C(\delta'_{i,C}(1))$, the pair

$$\tilde{\rho}_C := (\tilde{\delta}'_{i,C}, s_C \circ (\zeta^{-1}\xi))$$

is a leashed disc, which is a transversal lift of ρ_C. Hence $N^{[\rho]}$ is generated by the monodromy relations $g^{-1}g^{\mu([\lambda(\tilde{\rho}_C)])}$ along $[\lambda(\tilde{\rho}_C)]$. Note that the lasso $\lambda(\rho_C)$ is a loop on C° that is null-homotopic in C, so that we have $[\lambda(\rho_C)] \in K$. Because $s_C \circ \lambda(\rho_C) = \lambda(\tilde{\rho}_C)$, the generators $g^{-1}g^{\mu([\lambda(\tilde{\rho}_C)])}$ of $N^{[\rho]}$ are contained in N_K. $\square$

We give a sufficient condition under which $N^{[\rho]} = 1$ holds for one (and hence any) leashed disc ρ around $\Sigma_i^\sharp$. (See Corollary 3.19.)

Suppose that X is the complement to a reduced hypersurface W in a smooth variety $\overline{X}$, and that f is the restriction to X of a *projective* morphism $\bar{f} : \overline{X} \to Y$. For $y \in Y$, we put $\overline{F}_y := \bar{f}^{-1}(y)$, and denote by W_y the *scheme-theoretic* intersection of $\overline{F}_y$ with W. Let $\mathrm{Sing}(\bar{f}) \subset \overline{X}$ be the Zariski closed subset of critical points of $\bar{f}$.

Proposition 3.34. *We assume the conditions* (C1) *and* (C2)*. Suppose that, for a general point y of Σ_i, the intersection $\overline{F}_y \cap \mathrm{Sing}(\bar{f})$ is of codimension ≥ 2 in $\overline{F}_y$ and $W_y \setminus (W_y \cap \mathrm{Sing}(\bar{f}))$ is a reduced hypersurface of $\overline{F}_y \setminus (\overline{F}_y \cap \mathrm{Sing}(\bar{f}))$. Then $N^{[\rho]} = 1$ holds for a leashed disc ρ around $\Sigma_i^\sharp$.*

Proof. Let y_0 be a general point y_0 of Σ_i, and let $U \subset Y$ be a sufficiently small contractible neighborhood of y_0. Since $\bar{f}$ is projective, there exists an embedding over U of $\bar{f}^{-1}(U)$ into $\mathbb{P}^N \times U$;

$$\begin{array}{ccc} \bar{f}^{-1}(U) & \hookrightarrow & \mathbb{P}^N \times U \\ & \searrow \quad \swarrow & \\ & U. & \end{array}$$

By this embedding, we consider each $\overline{F}_y$ for $y \in U$ as a closed subscheme of $\mathbb{P}^N$ of dimension $\dim X - \dim Y$. We choose a general linear subspace $P \subset \mathbb{P}^N$ of codimension $\dim \overline{F}_y - 1$. By the assumption $\dim(\overline{F}_y \cap \mathrm{Sing}(\bar{f})) \leq \dim \overline{F}_y - 2$ for any $y \in U \cap \Sigma_i$, we have $(P \times U) \cap \mathrm{Sing}(\bar{f}) = \emptyset$ and we can assume that $P \cap \overline{F}_y$ is a smooth projective curve for any $y \in U$. By the assumption on W_y, we see that $P \cap W_y$ is a reduced divisor of $P \cap \overline{F}_y$ whose degree is independent of $y \in U$. Hence the family

$$P \cap F_y = P \cap (\overline{F}_y \setminus W_y) \qquad (y \in U)$$

of punctured Riemann surfaces is trivial (in the $\mathcal{C}^\infty$-category) over U. Let $\delta : \bar{\Delta} \to Y^\sharp$ be a transversal disc around $\Sigma_i^\sharp$ such that $\delta(\bar{\Delta}) \subset U$. Then we have a transversal lift $\tilde{\delta} : \bar{\Delta} \to X^\sharp$ of δ such that $\tilde{\delta}(z) \in P \cap F_{\delta(z)}$ holds for any $z \in \bar{\Delta}$. We put

$$q := \delta(1), \qquad \tilde{q} := \tilde{\delta}(1) \in P \cap F_q.$$

The lifted monodromy of $[\partial_\varepsilon \tilde{\delta}]$ on $\pi_1(P \cap F_q, \tilde{q})$ is trivial. On the other hand, the inclusion $P \cap F_q \hookrightarrow F_q$ induces a surjective homomorphism

$$\pi_1(P \cap F_q, \tilde{q}) \twoheadrightarrow \pi_1(F_q, \tilde{q})$$

by the Lefschetz-Zariski hyperplane section theorem. (See, for example, [5] or [6]). Hence the lifted monodromy of $[\partial_\varepsilon \tilde{\delta}]$ on $\pi_1(F_q, \tilde{q})$ is also trivial. $\square$

We prove the two corollaries stated in Introduction.

Proof of Corollary 1.1. Since the lasso of any transversal lift of a leashed disc on $Y^\sharp$ around $\Sigma_i^\sharp$ is null-homotopic in X, we have $\mathcal{N} \subset \mathcal{R}$. Hence Corollary 1.1

follows from Theorem 3.20, Proposition 3.3 and Nori's lemma (Proposition 3.1 and Remark 3.6). □

Proof of Corollary 1.3. It is enough to show that f satisfies the condition (C2), and that, for each Σ_i, $N^{[\rho]} = 1$ holds for a leashed disc ρ around $\Sigma_i^\sharp$.

Since f is projective and the general fiber is connected, every fiber of f is non-empty and connected. Suppose that F_y is reducible for a general point y of some irreducible hypersurface Σ' of Y. Let $\Delta \subset Y$ be a small open disc intersecting Σ' transversely at y such that $f^{-1}(\Delta)$ is smooth. Then F_y is a reducible hypersurface of $f^{-1}(\Delta)$. Since F_y is connected and projective, there exist distinct irreducible components F_y' and F_y'' of F_y that intersect. Since $F_y' \cap F_y''$ is of codimension 2 in $f^{-1}(\Delta)$, we obtain a contradiction to the assumption that $\operatorname{Sing}(f)$ is of codimension ≥ 3 in X. Thus the condition (C2) is satisfied.

Let y be a general point of Σ_i. By the assumption that $\operatorname{Sing}(f)$ is of codimension ≥ 3 in X, we see that $F_y \cap \operatorname{Sing}(f)$ is of codimension ≥ 2 in F_y. Applying Proposition 3.34 to the case where $W = \emptyset$ and $X = \overline{X}$, we obtain $N^{[\rho]} = 1$ for a leashed disc ρ around $\Sigma_i^\sharp$. □

4. Proof of Theorem 1.4

Proof of Theorem 1.4. We assume $k \leq n-2$, where n is the dimension of the smooth non-degenerate projective variety $X \subset \mathbb{P}^N$. We put

$$\mathcal{U}_k(X, \mathbb{P}^N, (\mathbb{P}^N)^\vee) := \{ \, (L, t) \in U_k(X, \mathbb{P}^N) \times (\mathbb{P}^N)^\vee \ \mid \ L \subset H_t \, \},$$

and consider the projection

$$f_{(\mathbb{P}^N)^\vee} : \mathcal{U}_k(X, \mathbb{P}^N, (\mathbb{P}^N)^\vee) \to (\mathbb{P}^N)^\vee.$$

Then the fiber of $f_{(\mathbb{P}^N)^\vee}$ over $t \in (\mathbb{P}^N)^\vee$ is canonically identified with $U_k(Y_t, H_t)$, where $Y_t = X \cap H_t$. The morphism

$$f_\Lambda : \mathcal{U}_k(X, \mathbb{P}^N, \Lambda) \to \Lambda$$

defined in Introduction is the pull-back of $f_{(\mathbb{P}^N)^\vee}$ by the inclusion $\Lambda \hookrightarrow (\mathbb{P}^N)^\vee$. Consider the following diagram:

$$\begin{array}{ccccc} \mathcal{U}_k(X, \mathbb{P}^N, \Lambda) & \hookrightarrow & \mathcal{U}_k(X, \mathbb{P}^N, (\mathbb{P}^N)^\vee) & \xrightarrow{\mathrm{pr}_1} & U_k(X, \mathbb{P}^N) \\ {\scriptstyle f_\Lambda} \downarrow & \square & \downarrow {\scriptstyle f_{(\mathbb{P}^N)^\vee}} & & \\ \Lambda & \hookrightarrow & (\mathbb{P}^N)^\vee, & & \end{array}$$

where pr_1 is the projection onto the first factor. The fiber of pr_1 over $L \in U_k(X, \mathbb{P}^N)$ is isomorphic to a linear subspace $\{t \in (\mathbb{P}^N)^\vee \mid L \subset H_t\}$ of $(\mathbb{P}^N)^\vee$, and hence pr_1 is smooth and proper (and thus locally trivial) with simply-connected fibers. Therefore $\mathcal{U}_k(X, \mathbb{P}^N, (\mathbb{P}^N)^\vee)$ is smooth and irreducible, and pr_1 induces an isomorphism

$$\pi_1(\mathcal{U}_k(X, \mathbb{P}^N, (\mathbb{P}^N)^\vee), s_o(0)) \cong \pi_1(U_k(X, \mathbb{P}^N), L_o). \tag{4.1}$$

The fiber of $f_{(\mathbb{P}^N)^\vee}$ over $t \in (\mathbb{P}^N)^\vee$ is a Zariski open subset of $\mathrm{Gr}^{n-1-k}(H_t)$. Hence $f_{(\mathbb{P}^N)^\vee}$ is smooth. There exists a Zariski closed subset Ξ'' of $(\mathbb{P}^N)^\vee$ of codimension ≥ 2 such that, if $t \in (\mathbb{P}^N)^\vee \setminus \Xi''$, then Y_t has only isolated singular points. (See [9], for example.) Then $U_k(Y_t, H_t)$ is non-empty and irreducible for $t \in (\mathbb{P}^N)^\vee \setminus \Xi''$. Therefore $f_{(\mathbb{P}^N)^\vee}$ satisfies the conditions (C1) and (C2). In particular, by Nori's lemma (Proposition 3.1), we see that the inclusion of the general fiber induces a surjective homomorphism

$$\iota_* : \pi_1(U_k(Y_0, H_0), L_o) \twoheadrightarrow \pi_1(\mathcal{U}_k(X, \mathbb{P}^N, (\mathbb{P}^N)^\vee), s_o(0)). \tag{4.2}$$

On the other hand, in virtue of the *general* line $\Lambda \subset (\mathbb{P}^N)^\vee$ and the holomorphic section s_o over Λ, we see that $f_{(\mathbb{P}^N)^\vee}$ satisfies the conditions of Theorem 3.33, and hence ι_* induces an injective homomorphism

$$\pi_1(U_k(Y_0, H_0), L_o) /\!/ \pi_1(\Lambda \setminus \Sigma_\Lambda, 0) \hookrightarrow \pi_1(\mathcal{U}_k(X, \mathbb{P}^N, (\mathbb{P}^N)^\vee), s_o(0)). \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we complete the proof of Theorem 1.4(1).

In particular, the inclusion $U_k(Y_0, H_0) \hookrightarrow U_k(X, \mathbb{P}^N)$ induces a surjective homomorphism on the fundamental groups. If $k < n-2$, then we can apply this result to the inclusion $U_k(Z_\Lambda, A) \hookrightarrow U_k(Y_0, H_0)$, and obtain a surjection

$$\pi_1(U_k(Z_\Lambda, A), L_o) \twoheadrightarrow \pi_1(U_k(Y_0, H_0), L_o).$$

By construction, this homomorphism is equivariant under the classical monodromy action of $\pi_1(\Lambda \setminus \Sigma_\Lambda, 0)$ given by the cross-section s_o. Since $\pi_1(\Lambda \setminus \Sigma_\Lambda, 0)$ acts on $\pi_1(U_k(Z_\Lambda, A), L_o)$ trivially, we obtain the proof of Theorem 1.4(2). □

5. The simple braid group

Let C be a compact Riemann surface of genus $g > 0$, and let $D_0 = p_1 + \cdots + p_d$ be a reduced effective divisor on C of degree d, which we use as a base point of the space $\mathrm{rDiv}^d(C)$ of reduced divisors of degree d on C. Let $\mathrm{Pic}^d(C)$ be the Picard variety of isomorphism classes $[L]$ of line bundles L of degree d on C. We denote by

$$\bar\lambda : \mathrm{Div}^d(C) \to \mathrm{Pic}^d(C)$$

the natural morphism, and consider the induced homomorphism

$$\bar\lambda_* : \pi_1(\mathrm{Div}^d(C), D_0) \to \pi_1(\mathrm{Pic}^d(C), \bar\lambda(D_0)) = H_1(C, \mathbb{Z}).$$

Proposition 5.1. *Suppose that* $d \geq g$. (1) *We have* $\mathrm{Sing}(\bar\lambda) = \bar\lambda^{-1}(\bar\lambda(\mathrm{Sing}(\bar\lambda)))$. (2) *If* $d \geq 2g-1$ *then* $\mathrm{Sing}(\bar\lambda) = \emptyset$. *If* $d \leq 2g-2$ *then* $\dim \mathrm{Sing}(\bar\lambda) \leq g-1$ *and* $\dim \bar\lambda(\mathrm{Sing}(\bar\lambda)) \leq 2g-2-d$.

Proof. Note that $\bar\lambda$ is surjective because $d \geq g$. For $D \in \mathrm{Div}^d(C)$, we have

$$\bar\lambda^{-1}(\bar\lambda(D)) = |\mathcal{O}_C(D)| \cong \mathbb{P}^{d-g+s(D)},$$

where $s(D) := h^0(C, K_C(-D))$. Hence $D \in \mathrm{Sing}(\bar\lambda)$ if and only if $s(D) > 0$, and therefore the assertion (1) follows, and moreover, we have

$$\dim \bar\lambda(\mathrm{Sing}(\bar\lambda)) \leq \dim \mathrm{Sing}(\bar\lambda) - (d-g+1).$$

On the other hand, we have $s(D) > 0$ if and only if D is a sub-divisor of a member of the $(g-1)$-dimensional linear system $|K_C|$. Since $\deg K_C = 2g-2$, we obtain the proof. □

Remark 5.2. Suppose $d \geq g$. Then $\mathrm{Sing}(\bar\lambda)$ is the locus of *special divisors* of degree d on C, and $\bar\lambda(\mathrm{Sing}(\bar\lambda))$ is the locus of *special line bundles* of degree d on C.

Proposition 5.3. *Suppose that* $d \geq g$. *Then* $\bar\lambda_*$ *is an isomorphism.*

Proof. The general fiber of $\bar\lambda$ is isomorphic to $\mathbb{P}^{d-g}$. By Proposition 5.1, the assumption $d \geq g$ implies that $\bar\lambda(\mathrm{Sing}(\bar\lambda)) \subset \mathrm{Pic}^d(C)$ is of codimension ≥ 2. Hence Proposition 5.3 follows from Nori's lemma (Proposition 3.1). □

Proposition 5.4. (1) *Suppose that $d \geq g+2$. Then there exists a Zariski closed subset $\Xi_1 \subset \mathrm{Pic}^d(C)$ of codimension ≥ 2 such that the complete linear system $|L|$ is base-point free for any $[L] \in \mathrm{Pic}^d(C) \setminus \Xi_1$.*

(2) *Suppose that $d \geq g+4$. Then there exists a Zariski closed subset $\Xi_2 \subset \mathrm{Pic}^d(C)$ of codimension ≥ 2 such that $|L|$ is very ample for any $[L] \in \mathrm{Pic}^d(C) \setminus \Xi_2$.*

Proof. Suppose that $d \geq g+2$, and let L be a line bundle of degree d. If $|L|$ has a base point p, then $L(-p)$ is a special line bundle, and hence $[L] \in \mathrm{Pic}^d(C)$ is contained in the image of the morphism

$$\bar{\lambda}'(\mathrm{Sing}(\bar{\lambda}')) \times C \;\;\to\;\; \mathrm{Pic}^d(C) \tag{5.1}$$

given by $([M], p) \mapsto [M(p)]$, where $\bar{\lambda}' : \mathrm{Div}^{d-1}(C) \to \mathrm{Pic}^{d-1}(C)$ is the natural morphism. Since $\dim \bar{\lambda}'(\mathrm{Sing}(\bar{\lambda}')) \leq 2g-d-1$ by Proposition 5.1, the image of (5.1) is of codimension ≥ 2.

Suppose that $d \geq g+4$. If a base-point free line bundle L of degree d is not very ample, then there exist points p, q of C such that $h^0(L(-p-q)) = h^0(L(-p))$ holds, and hence $L(-p-q)$ is a special line bundle of degree $d-2$. We complete the proof by the same argument as above. □

We denote by

$$\lambda \,:\, \mathrm{rDiv}^d(C) \;\to\; \mathrm{Pic}^d(C)$$

the restriction of $\bar{\lambda}$ to $\mathrm{rDiv}^d(C)$, and consider the homomorphism

$$\lambda_* \,:\, B(C,d) := \pi_1(\mathrm{rDiv}^d(C), D_0) \;\to\; H_1(C,\mathbb{Z}) = \pi_1(\mathrm{Pic}^d(C))$$

induced by λ. From Proposition 5.3, we obtain the following:

Corollary 5.5. *Suppose that $d \geq g$. Then the simple braid group $SB(C, D_0)$ defined in Definition 1.5 is equal to the kernel of the homomorphism λ_*.*

Let $\sigma : (I, \partial I) \to (\mathrm{rDiv}^d(C), D_0)$ be a loop. Then there exist paths $\sigma_i : I \to C$ for $i = 1, \dots, d$ such that $\sigma_i(0) = p_i$ and such that $\sigma(t) = \sigma_1(t) + \cdots + \sigma_d(t)$ for all $t \in I$. The homology class $\lambda_*([\sigma]) \in H_1(C,\mathbb{Z})$ is represented by the 1-cycle obtained as the conjunction of the paths $\sigma_1, \dots, \sigma_d$.

Let $\Gamma^d(C) \subset \mathrm{Div}^d(C)$ be the big diagonal in $\mathrm{Div}^d(C) = C^d/\mathfrak{S}_d$, where $\mathfrak{S}_d$ is the symmetric group acting on the Cartesian product C^d of d copies of C by permutation of the components. We have

$$\mathrm{rDiv}^d(C) = \mathrm{Div}^d(C) \setminus \Gamma^d(C).$$

For $[L] \in \mathrm{Pic}^d(C)$, we put

$$\Gamma(L) := \Gamma^d(C) \cap \bar{\lambda}^{-1}([L]) \quad \text{and} \quad |L|^{\mathrm{red}} := \lambda^{-1}([L]) = |L| \setminus \Gamma(L),$$

where $\bar{\lambda}^{-1}([L])$ is identified with $|L|$.

Remark 5.6. Suppose that L is very ample, and let $C_L \subset \mathbb{P}^{d-g+s(L)}$ denote the image of the embedding of C by $|L|$. Then, under the identification $|L| \cong (\mathbb{P}^{d-g+s(L)})^\vee$, $\Gamma(L)$ is equal to the dual hypersurface $C_L^\vee$ of C_L, and hence it is of degree

$$d^\vee := 2(d+g-1).$$

Proposition 5.7. *Suppose that $d \ge g+4$. If $[L] \in \mathrm{Pic}^d(C)$ is general, then the inclusion $|L|^{\mathrm{red}} \hookrightarrow \mathrm{rDiv}^d(C)$ induces as isomorphism*

$$\pi_1(|L|^{\mathrm{red}}, D_0) \;\cong\; SB(C, D_0),$$

where D_0 is a point of $|L|^{\mathrm{red}}$.

Proof. We put $\Xi := \bar{\lambda}(\mathrm{Sing}(\bar{\lambda})) \cup \Xi_2$, where Ξ_2 is the Zariski closed subset in Proposition 5.4. Then Ξ is a Zariski closed subset of codimension ≥ 2 in $\mathrm{Pic}^d(C)$ and $\bar{\lambda}^{-1}(\Xi)$ is of codimension ≥ 2 in $\mathrm{Div}^d(C)$ by Proposition 5.1. Moreover $\bar{\lambda}^{-1}(\Xi)$ contains $\mathrm{Sing}(\bar{\lambda})$, and L' is very ample if $[L'] \notin \Xi$. We consider the restriction

$$f \,:\, X := \mathrm{rDiv}^d(C) \setminus \lambda^{-1}(\Xi) \;\to\; Y := \mathrm{Pic}^d(C) \setminus \Xi$$

of λ to $X = \mathrm{rDiv}^d(C) \setminus \lambda^{-1}(\Xi)$. We have

$$\begin{aligned} \pi_1(Y, [L]) \;&=\; \pi_1(\mathrm{Pic}^d(C), [L]) \;=\; H_1(C, \mathbb{Z}), \\ \pi_1(X, D_0) \;&=\; \pi_1(\mathrm{rDiv}^d(C), D_0) \;=\; B(C, D_0), \\ \pi_2(Y) \;&=\; \pi_2(\mathrm{Pic}^d(C)) \;=\; 0. \end{aligned}$$

By the last equality, the morphism f satisfies (Z). Since f is smooth with every fiber being non-empty Zariski open subsets of $\mathbb{P}^{d-g}$, the conditions (C1) and (C2) are also satisfied. Therefore we can apply Theorem 3.20. Using Proposition 3.34 and Remark 5.6, the lifted monodromy action of $\pi_1(X^\circ, D_0)$ on $\pi_1(|L|^{\mathrm{red}}, D_0)$ is trivial. Combining this result with Corollary 1.1, we see that $\pi_1(|L|^{\mathrm{red}}, D_0)$ is equal to the kernel of the homomorphism $B(C, D_0) \to H_1(C, \mathbb{Z})$ induced by f, which is $SB(C, D_0)$ by Corollary 5.5. □

Now we prove our third main result.

Proof of Theorem 1.7. We denote by L the line bundle on $C \subset \mathbb{P}^M$ corresponding to the hyperplane section, and let $C_L \subset \mathbb{P}^N$ be the image of the embedding of C by $|L|$. Then $C \subset \mathbb{P}^M$ is the image of a projection $C_L \to \mathbb{P}^M$ with the center being disjoint from $C_L \subset \mathbb{P}^N$. Let $\rho : C \to \mathbb{P}^2$ be a general projection. By this sequence of the linear projections $\mathbb{P}^N \dashrightarrow \mathbb{P}^M \dashrightarrow \mathbb{P}^2$, we have the canonical embeddings of linear subspaces

$$(\mathbb{P}^2)^\vee \hookrightarrow (\mathbb{P}^M)^\vee \hookrightarrow (\mathbb{P}^N)^\vee.$$

Let $\rho(C)^\vee \subset (\mathbb{P}^2)^\vee$, $C^\vee \subset (\mathbb{P}^M)^\vee$ and $(C_L)^\vee \subset (\mathbb{P}^N)^\vee$ be the dual hypersurfaces of $\rho(C) \subset \mathbb{P}^2$, $C \subset \mathbb{P}^M$ and $C_L \subset \mathbb{P}^N$, respectively. Then we have

$$\rho(C)^\vee = (\mathbb{P}^2)^\vee \cap C^\vee = (\mathbb{P}^2)^\vee \cap (C_L)^\vee, \quad C^\vee = (\mathbb{P}^M)^\vee \cap (C_L)^\vee.$$

We will consider the homomorphisms

$$\pi_1((\mathbb{P}^2)^\vee \setminus \rho(C)^\vee) \;\to\; \pi_1((\mathbb{P}^M)^\vee \setminus C^\vee) \;\to\; \pi_1((\mathbb{P}^N)^\vee \setminus (C_L)^\vee)$$

induced by the inclusions. Since $C \subset \mathbb{P}^M$ is Plücker general by the assumption, the degree $d^\vee$ of $\rho(C)^\vee$, the number $\delta^\vee$ of ordinary nodes on $\rho(C)^\vee$ and the number $\kappa^\vee$ of ordinary cusps on $\rho(C)^\vee$ are given by the Plücker formula;

$$d^\vee = 2d + 2g - 2, \quad \delta^\vee = 2d^2 + 4dg + 2g^2 - 10d - 14g + 12, \quad \kappa^\vee = 3d + 6g - 6.$$

(See [30, Chap. 7], for example.) In particular, the section $\rho(C)^\vee$ of $(C_L)^\vee$ by $(\mathbb{P}^2)^\vee \subset (\mathbb{P}^N)^\vee$ is equisingular to the *general* plane section of $(C_L)^\vee$. By the classical Zariski hyperplane section theorem ([5], [6], [31]), we see that the inclusion induces an isomorphism

$$\pi_1((\mathbb{P}^2)^\vee \setminus \rho(C)^\vee) \;\cong\; \pi_1((\mathbb{P}^N)^\vee \setminus (C_L)^\vee).$$

On the other hand, the scheme-theoretic intersection of $(C_L)^\vee$ and $(\mathbb{P}^2)^\vee$ in $(\mathbb{P}^N)^\vee$ is reduced, and hence the scheme-theoretic intersection of $C^\vee$ and $(\mathbb{P}^2)^\vee$ in $(\mathbb{P}^M)^\vee$ is also reduced, and thus the inclusion induces a surjective homomorphism

$$\pi_1((\mathbb{P}^2)^\vee \setminus \rho(C)^\vee) \twoheadrightarrow \pi_1((\mathbb{P}^M)^\vee \setminus C^\vee).$$

Therefore we conclude that the inclusions induce isomorphisms

$$\pi_1((\mathbb{P}^2)^\vee \setminus \rho(C)^\vee) \;\cong\; \pi_1((\mathbb{P}^M)^\vee \setminus C^\vee) \;\cong\; \pi_1((\mathbb{P}^N)^\vee \setminus (C_L)^\vee).$$

Remark that $(\mathbb{P}^M)^\vee \setminus C^\vee$ is equal to $U_0(C, \mathbb{P}^M)$, and $(\mathbb{P}^N)^\vee \setminus (C_L)^\vee$ is equal to $|L|^{\mathrm{red}}$. Therefore it is enough to show that $\pi_1(|L|^{\mathrm{red}})$ or $\pi_1((\mathbb{P}^2)^\vee \setminus \rho(C)^\vee)$ is isomorphic to the simple braid group SB^g_d. Note that, since $[L]$ is not necessarily a general point of $\mathrm{Pic}^d(C)$, we cannot apply Proposition 5.7. We overcome this difficulty using Harris' theorem [7].

Note that $\rho(C)$ is a plane curve of degree d with $\delta := (d-1)(d-2)/2 - g$ ordinary nodes and no other singularities. Let $\mathbb{P}_*(H^0(\mathbb{P}^2, \mathcal{O}(d)))$ be the space of all plane curves of degree d, and let $\mathcal{S}_{d,\delta} \subset \mathbb{P}_*(H^0(\mathbb{P}^2, \mathcal{O}(d)))$ be the locus of reduced plane curves $\Gamma \subset \mathbb{P}^2$ of degree d such that $\operatorname{Sing}\Gamma$ consists of only δ ordinary nodes. In [7], Harris gave an affirmative answer to the Severi problem, in virtue of which we know that $\mathcal{S}_{d,\delta}$ is irreducible. We then denote by $\mathcal{S}^\circ_{d,\delta} \subset \mathcal{S}_{d,\delta}$ the locus of $\Gamma \in \mathcal{S}_{d,\delta}$ such that the dual curve $\Gamma^\vee$ has only ordinary nodes and ordinary cusps as its singularities. Then $\mathcal{S}^\circ_{d,\delta}$ is a Zariski open subset of $\mathcal{S}_{d,\delta}$ containing $\rho(C)$.

Let C' be an arbitrary compact Riemann surface of genus g, and let $[L']$ be a *general* point of $\mathrm{Pic}^d(C')$. Since $d \ge g+4$, we see from Proposition 5.4 that $|L'|$ is very ample of dimension $d-g$. We denote by $C'_{L'} \subset \mathbb{P}^{d-g}$ the image of the embedding $C' \hookrightarrow \mathbb{P}^{d-g}$ by $|L'|$, and consider the general projection $\rho' : C'_{L'} \to \mathbb{P}^2$. Then $\rho'(C'_{L'})$ is a point of $\mathcal{S}_{d,\delta}$. Since $\mathcal{S}_{d,\delta}$ is irreducible, we can connect the two points $\rho(C) \in \mathcal{S}_{d,\delta}$ and $\rho'(C'_{L'}) \in \mathcal{S}_{d,\delta}$ by an irreducible closed curve $T \subset \mathcal{S}_{d,\delta}$. We put $T^0 := T \cap \mathcal{S}^\circ_{d,\delta}$, which is a Zariski open subset of T containing $\rho(C)$. When Γ moves on $\mathcal{S}^\circ_{d,\delta}$ the dual curves $\Gamma^\vee$ form an equisingular family of plane curves. Therefore we have

$$\pi_1((\mathbb{P}^2)^\vee \setminus \rho(C)^\vee) \;\cong\; \pi_1((\mathbb{P}^2)^\vee \setminus \Gamma^\vee) \quad \text{for any } \Gamma \in T^0. \tag{5.2}$$

On the other hand, by Propositions 5.4 and 5.7, there exists a Zariski open dense subset $T^1 \subset T$ containing $\rho'(C'_{L'})$ such that the complete linear system $|\mathcal{O}_\Gamma(1)|$ of a hyperplane section of $\Gamma \subset \mathbb{P}^2$ is very ample on the normalization $\Gamma^\sim$ of Γ for any $\Gamma \in T^1$, that $\dim|\mathcal{O}_\Gamma(1)| = d-g$ for any $\Gamma \in T^1$, and that

$$\pi_1((\mathbb{P}^2)^\vee \setminus \Gamma^\vee) \;\cong\; \pi_1(|\mathcal{O}_\Gamma(1)|^{\mathrm{red}}) \;\cong\; SB^d_g \quad \text{for any } \Gamma \in T^1. \tag{5.3}$$

Here we have used the classical Zariski hyperplane section theorem again. Since $T^0 \cap T^1 \neq \emptyset$, we complete the proof of Theorem 1.7 by combining the isomorphisms (5.2), (5.3). $\square$

6. The conjecture of Auroux, Donaldson, Katzarkov and Yotov

Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective surface of degree d, and let $B \subset \mathbb{P}^2$ be the branch curve of a general projection $X \to \mathbb{P}^2$. The fundamental group $\pi_1(\mathbb{P}^2 \setminus B)$ has been studied intensively by Moishezon, Teicher and Robb ([10], [11], [12], [13], [28], [27], [15], ……). In many examples, it has turned out that $\pi_1(\mathbb{P}^2 \setminus B)$ is rather "small". In [1, Conjectures 1.3 and 1.6], Auroux, Donaldson, Katzarkov and Yotov formulated the following conjecture (not only for algebraic surfaces but also for symplectic 4-manifolds), and confirmed it for some new examples.

Note that there exist natural homomorphisms

$$\pi_1(\mathbb{P}^2 \setminus B) \;\to\; \mathfrak{S}_d \quad \text{and} \quad \pi_1(\mathbb{P}^2 \setminus B) \;\to\; H_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/\deg(B)\mathbb{Z}.$$

For a smooth projective surface X and a line bundle L on X, we denote by

$$\lambda_{(X,L)} \,:\, H^2(X,\mathbb{Z}) \;\to\; \mathbb{Z}^2$$

the homomorphism given by $\lambda_{(X,L)}(\alpha) := (\alpha \cup c_1(L), \alpha \cup c_1(K_S + 3L))$, where $\cup$ denotes the cup-product.

Conjecture 6.1. *Let L be an ample line bundle of a smooth projective surface S, and let $X_m \subset \mathbb{P}^{N(m)}$ be the image of the embedding of S by the complete linear system $|L^{\otimes m}|$. We denote by $B_m \subset \mathbb{P}^2$ the branch curve of a general projection $X_m \to \mathbb{P}^2$. Let G_m^0 be the kernel of the natural homomorphism*

$$\pi_1(\mathbb{P}^2 \setminus B_m) \;\to\; \mathfrak{S}_d \times \mathbb{Z}/\deg(B_m)\mathbb{Z}.$$

Suppose that S is simply-connected and that m is large enough. Then the abelianization of G_m^0 is isomorphic to $(\mathbb{Z}^2/\operatorname{Im}(\lambda_{(X,mL)}))^{d-1}$, and the commutator subgroup $[G_m^0, G_m^0]$ is a quotient of $(\mathbb{Z}/2\mathbb{Z})^2$.

For a smooth non-degenerate projective surface $X \subset \mathbb{P}^N$, the fundamental groups $\pi_1(U_0(X,\mathbb{P}^N))$ and $\pi_1(\mathbb{P}^2 \setminus B)$ are related as follows. Note that the target space $\mathbb{P}^2$ of the general projection $X \to \mathbb{P}^2$ is identified with the closed subvariety

$$\{\, L \in \mathrm{Gr}^2(\mathbb{P}^N) \;\mid\; L \text{ contains the center of the projection} \,\}$$

of $\mathrm{Gr}^2(\mathbb{P}^N)$, and $\mathbb{P}^2 \setminus B$ is identified with the pull-back of $U_0(X,\mathbb{P}^N)$ by this embedding $\mathbb{P}^2 \hookrightarrow \mathrm{Gr}^2(\mathbb{P}^N)$.

Proposition 6.2. *The inclusion $\mathbb{P}^2 \setminus B \hookrightarrow U_0(X,\mathbb{P}^N)$ induces a surjective homomorphism $\pi_1(\mathbb{P}^2 \setminus B) \twoheadrightarrow \pi_1(U_0(X,\mathbb{P}^N))$.*

Proof. Consider the incidence variety

$$\begin{array}{ccc} \{\, (L,M) \in \mathrm{Gr}^2(\mathbb{P}^N) \times \mathrm{Gr}^3(\mathbb{P}^N) \;\mid\; L \supset M \,\} & \xrightarrow{\mathrm{pr}_1} & \mathrm{Gr}^2(\mathbb{P}^N) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \\ \mathrm{Gr}^3(\mathbb{P}^N), & & \end{array}$$

where pr_1 and pr_2 are the natural projections, and put

$$\mathcal{U} := \mathrm{pr}_1^{-1}(U_0(X,\mathbb{P}^N)).$$

Since pr_1 is smooth with every fiber being isomorphic to $\mathbb{P}^{N-2}$, we see that $\mathcal{U}$ is smooth, irreducible, and that $\mathrm{pr}_1|_{\mathcal{U}}$ induces an isomorphism $\pi_1(\mathcal{U}) \cong \pi_1(U_0(X, \mathbb{P}^N))$. For $M \in \mathrm{Gr}^3(\mathbb{P}^N)$, the target space Π_M of the projection

$$\rho_M : X \to \Pi_M$$

with the center M is the fiber of pr_2 over M, and we have

$$\Pi_M \setminus B_M \cong (\mathrm{pr}_2|_{\mathcal{U}})^{-1}(M) = \mathrm{pr}_2^{-1}(M) \cap \mathcal{U},$$

where $B_M \subset \Pi_M$ is the branch curve of ρ_M. Hence it is enough to show that the inclusion of the general fiber of $\mathrm{pr}_2|_{\mathcal{U}}$ over M induces a surjective homomorphism

$$\pi_1((\mathrm{pr}_2|_{\mathcal{U}})^{-1}(M)) \twoheadrightarrow \pi_1(\mathcal{U}). \tag{6.1}$$

Since pr_2 is smooth, so is $\mathrm{pr}_2|_{\mathcal{U}}$. Moreover the locus of all $M \in \mathrm{Gr}^3(\mathbb{P}^N)$ such that $(\mathrm{pr}_2|_{\mathcal{U}})^{-1}(M) = \emptyset$ is contained in a Zariski closed subset of codimension ≥ 2 in $\mathrm{Gr}^3(\mathbb{P}^N)$. Hence Nori's lemma (Proposition 3.1) implies the surjectivity (6.1). $\square$

Thus we see that the group $\pi_1(U_0(X, \mathbb{P}^N))$ is "smaller" than $\pi_1(\mathbb{P}^2 \setminus B)$. In view of Corollary 1.8 and Conjecture 6.1, we expect that the image Γ_Λ of the monodromy (1.3) should be "large".

The group Γ_Λ is generated by the Dehn twists associated with the ordinary nodes of the singular members of the pencil $\{Y_t\}_{t\in\Lambda}$. Hence the group Γ_Λ and its action on $SB(Y_0, Z_\Lambda)$ can be visualized by drawing on Y_0 the reduced divisor Z_Λ and the vanishing cycles for the singular members of the pencil.

As for the largeness of Γ_Λ, we have the following result of Smith [26, Theorem 1.3 and Corollary 4.3].

Theorem 6.3 (Smith)**.** *The vanishing cycles of the Lefschetz fibration $\mathcal{Y} \to \Lambda$ fill up the fiber Y_0; that is, their complement is a bunch of discs. Moreover distinct points of Z_Λ are on distinct discs.*

The second assertion follows from the argument in the proof of [26, Theorem 5.1], and the fact that the homology classes of the sections of $\mathcal{Y} \to \Lambda$ corresponding to the points of Z_Λ are distinct.

Remark 6.4. In the calculation of $\pi_1(U_0(X_m, \mathbb{P}^{N(m)}))$ by means of Corollary 1.8, the assumption $d \geq g + 4$ is satisfied when m is large enough. Indeed, the degree d of X_m is given by $d = m^2L^2$, while the genus g of the general hyperplane section Y_0 of X_m is given by $g = (m^2L^2 + mL \cdot K_X)/2 + 1$.

References

[1] D. Auroux, S. K. Donaldson, L. Katzarkov, and M. Yotov. Fundamental groups of complements of plane curves and symplectic invariants. *Topology*, 43(6):1285–1318, 2004.

[2] J. S. Birman. *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J., 1974. Annals of Mathematics Studies, No. 82.

[3] J. A. Carlson and D. Toledo. Discriminant complements and kernels of monodromy representations. *Duke Math. J.*, 97(3):621–648, 1999.

[4] I. Dolgachev and A. Libgober. On the fundamental group of the complement to a discriminant variety. In *Algebraic geometry (Chicago, Ill., 1980)*, volume 862 of *Lecture Notes in Math.*, pages 1–25. Springer, Berlin, 1981.

[5] M. Goresky and R. MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3).* Springer-Verlag, Berlin, 1988.

[6] H. A. Hamm and Lê Dũng Tráng. Lefschetz theorems on quasiprojective varieties. *Bull. Soc. Math. France*, 113(2):123–142, 1985.

[7] J. Harris. On the Severi problem. *Invent. Math.*, 84(3):445–461, 1986.
[8] Vik. S. Kulikov and I. Shimada. On the fundamental groups of complements to the dual hypersurfaces of projective curves. preprint, MPI 96-32.
[9] K. Lamotke. The topology of complex projective varieties after S. Lefschetz. *Topology*, 20(1):15–51, 1981.
[10] B. Moishezon. Stable branch curves and braid monodromies. In *Algebraic geometry (Chicago, Ill., 1980)*, volume 862 of *Lecture Notes in Math.*, pages 107–192. Springer, Berlin, 1981.
[11] B. Moishezon and M.Teicher. Simply-connected algebraic surfaces of positive index. *Invent. Math.*, 89(3):601–643, 1987.
[12] B. Moishezon. On cuspidal branch curves. *J. Algebraic Geom.*, 2(2):309–384, 1993.
[13] B. Moishezon and M.Teicher. Fundamental groups of complements of branch curves as solvable groups. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, pages 329–345, Ramat Gan, 1996. Bar-Ilan Univ.
[14] M. V. Nori. Zariski's conjecture and related problems. *Ann. Sci. École Norm. Sup. (4)*, 16(2):305–344, 1983.
[15] A. Robb. On branch curves of algebraic surfaces. In *Singularities and complex geometry (Beijing, 1994)*, volume 5 of *AMS/IP Stud. Adv. Math.*, pages 193–221. Amer. Math. Soc., Providence, RI, 1997.
[16] I. Shimada. Remarks on fundamental groups of complements of divisors on algebraic varieties. *Kodai Math. J.*, 17(2):311–319, 1994.
[17] I. Shimada. Fundamental groups of open algebraic varieties. *Topology*, 34(3):509–531, 1995.
[18] I. Shimada. A generalization of Lefschetz-Zariski theorem on fundamental groups of algebraic varieties. *Internat. J. Math.*, 6(6):921–932, 1995.
[19] I. Shimada. A note on Zariski pairs. *Compositio Math.*, 104(2):125–133, 1996.
[20] I. Shimada. Fundamental groups of complements to singular plane curves. *Amer. J. Math.*, 119(1):127–157, 1997.
[21] I. Shimada. On the commutativity of fundamental groups of complements to plane curves. *Math. Proc. Cambridge Philos. Soc.*, 123(1):49–52, 1998.
[22] I. Shimada. Equisingular families of plane curves with many connected components. *Vietnam J. Math.*, 31(2):193–205, 2003.
[23] I. Shimada. Fundamental groups of algebraic fiber spaces. *Comment. Math. Helv.*, 78(2):335–362, 2003.
[24] I. Shimada. On the Zariski-van Kampen theorem. *Canad. J. Math.*, 55(1):133–156, 2003.
[25] I. Shimada. Zariski hyperplane section theorem for Grassmannian varieties. *Canad. J. Math.*, 55(1):157–180, 2003.
[26] I. Smith. Geometric monodromy and the hyperbolic disc. *Q. J. Math.*, 52(2):217–228, 2001.
[27] M. Teicher. Braid groups, algebraic surfaces and fundamental groups of complements of branch curves. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 127–150. Amer. Math. Soc., Providence, RI, 1997.
[28] M. Teicher. The fundamental group of a $\mathbf{CP}^2$ complement of a branch curve as an extension of a solvable group by a symmetric group. *Math. Ann.*, 314(1):19–38, 1999.
[29] E. R. van Kampen. On the fundamental group of an algebraic curve. *Amer. J. Math.*, 55:255–260, 1933.
[30] C. T. C. Wall. *Singular points of plane curves*, volume 63 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.
[31] O. Zariski. A theorem on the Poincaré group of an algebraic hypersurface. *Ann. of Math. (2)*, 38(1):131–141, 1937.

Department of Mathematics, Graduate School of Science, Hiroshima University, Kagamiyama, Higashi-Hiroshima, 739-8526, JAPAN

E-mail address: `shimada@math.sci.hiroshima-u.ac.jp`