NON-HOMEOMORPHIC CONJUGATE COMPLEX VARIETIES

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Abstract. We present a method to produce examples of non-homeomorphic conjugate complex varieties based on the genus theory of lattices. As an application, we give examples of arithmetic Zariski pairs.

1. Introduction

We denote by $\mathrm{Emb}(\mathbb{C})$ the set of embeddings $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ of the complex number field $\mathbb{C}$ into itself. A *complex variety* is a reduced irreducible quasi-projective scheme over $\mathbb{C}$ with the classical topology. For a complex variety X and $\sigma \in \mathrm{Emb}(\mathbb{C})$, we define a complex variety X^σ by the following diagram of the fiber product:

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}\, \mathbb{C} & \xrightarrow{\sigma^*} & \mathrm{Spec}\, \mathbb{C}. \end{array}$$

Two complex varieties X and X' are said to be *conjugate* if there exists $\sigma \in \mathrm{Emb}(\mathbb{C})$ such that X^σ is isomorphic to X' over $\mathbb{C}$. It is easy to see that the relation of being conjugate is an equivalence relation.

The purpose of this note is to give a simple method to produce many explicit examples of non-homeomorphic conjugate complex varieties. This method is based on a topological idea in [27], and the arithmetic theory of transcendental lattices of singular $K3$ surfaces in [28], which has been generalized by Schütt [24].

We give a brief outline of the history of examples of non-homeomorphic conjugate complex varieties. In 1964, Serre [25] gave the first example. He constructed a pair of conjugate complex smooth projective varieties that have non-isomorphic fundamental groups. In 1974, Abelson [1] presented a pair of conjugate complex smooth projective varieties that have isomorphic (finite) fundamental groups but are not homeomorphic. On the other hand, Grothendieck's "Esquisse d'un Programme" [19] appeared in 1984, in which the faithful action of the absolute Galois group of $\mathbb{Q}$ on the set of topological types of finite coverings of $\mathbb{P}^1$ branching only at 0, 1 and ∞ is discussed. In [5] and [6], Artal, Carmona and Cogolludo constructed an example of arithmetic Zariski pairs of plane curves in degree 12 by means of braid monodromies. Recently, after the first version of the manuscript of this paper appeared on the e-print archive (`arXiv:math/0701115`) in January 2007, many examples of non-homeomorphic or non-deformation-equivalent conjugate complex varieties have been constructed by various methods ([10], [11], [14], [16]).

I would like to thank the referee for comments and suggestions.

2000 *Mathematics Subject Classification.* 14F45 (Primary); 14K22, 14J28, 14H50 (Secondary).

2. A topological invariant

For a $\mathbb{Z}$-module A, we denote by A_{tor} the torsion part of A, and by A^{tf} the torsion-free quotient A/A_{tor}. Note that a symmetric bilinear form $A \times A \to \mathbb{Z}$ naturally induces a symmetric bilinear form $A^{\mathrm{tf}} \times A^{\mathrm{tf}} \to \mathbb{Z}$. A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a *non-degenerate* symmetric bilinear form $L \times L \to \mathbb{Z}$. For a topological space Z, let $H_k(Z)$ denote the homology group $H_k(Z, \mathbb{Z})$.

Let U be an oriented topological manifold of dimension $4n$. We denote by

$$\iota_U \;:\; H_{2n}(U) \times H_{2n}(U) \;\to\; \mathbb{Z}$$

the intersection pairing. We put

$$J_\infty(U) \;:=\; \textstyle\bigcap_K \operatorname{Im}(H_{2n}(U \setminus K) \to H_{2n}(U)),$$

where K runs through the set of compact subsets of U, and $H_{2n}(U \setminus K) \to H_{2n}(U)$ is the natural homomorphism induced by the inclusion. We then put

$$\widetilde{B}_U := H_{2n}(U)/J_\infty(U) \quad \text{and} \quad B_U := (\widetilde{B}_U)^{\mathrm{tf}}.$$

Since any topological cycle is compact, the intersection pairing ι_U induces symmetric bilinear forms

$$\tilde{\beta}_U \;:\; \widetilde{B}_U \times \widetilde{B}_U \;\to\; \mathbb{Z} \quad \text{and} \quad \beta_U \;:\; B_U \times B_U \;\to\; \mathbb{Z}.$$

It is obvious that, if U and U' are homeomorphic, then there exists an isomorphism $(B_U, \beta_U) \cong (B_{U'}, \beta_{U'})$.

Let X be a smooth complex projective variety of dimension $2n$. Then $H_{2n}(X)^{\mathrm{tf}}$ is a lattice by the intersection pairing ι_X. Let $Y_1, \dots, Y_m$ be irreducible subvarieties of X with codimension n. We put

$$Y := Y_1 \cup \dots \cup Y_m, \quad U := X \setminus Y,$$

and investigate the topological invariant (B_U, β_U) of the smooth complex variety U.

Remark 2.1. In this case, the submodule $J_\infty(U) \subset H_{2n}(U)$ and hence the topological invariant (B_U, β_U) can be calculated effectively by choosing a tubular neighborhood $\mathcal{T} \subset X$ of Y. Indeed, $J_\infty(U)$ is equal to the image of the homomorphism $i_U : H_{2n}(\mathcal{T} \cap U) \to H_{2n}(U)$ induced by the inclusion.

We denote by $\widetilde{\Sigma}_{(X,Y)}$ the submodule of $H_{2n}(X)$ generated by the homology classes $[Y_i] \in H_{2n}(X)$, and put $\Sigma_{(X,Y)} := (\widetilde{\Sigma}_{(X,Y)})^{\mathrm{tf}}$. We then put

$$\widetilde{\Lambda}_{(X,Y)} := \{\; x \in H_{2n}(X) \;\mid\; \iota_X(x, y) = 0 \;\text{ for any }\; y \in \widetilde{\Sigma}_{(X,Y)} \;\},$$

and $\Lambda_{(X,Y)} := (\widetilde{\Lambda}_{(X,Y)})^{\mathrm{tf}}$. Finally, we denote by

$$\tilde{\sigma}_{(X,Y)} : \widetilde{\Sigma}_{(X,Y)} \times \widetilde{\Sigma}_{(X,Y)} \;\to\; \mathbb{Z}, \qquad \sigma_{(X,Y)} : \Sigma_{(X,Y)} \times \Sigma_{(X,Y)} \;\to\; \mathbb{Z},$$
$$\tilde{\lambda}_{(X,Y)} : \widetilde{\Lambda}_{(X,Y)} \times \widetilde{\Lambda}_{(X,Y)} \;\to\; \mathbb{Z}, \qquad \lambda_{(X,Y)} : \Lambda_{(X,Y)} \times \Lambda_{(X,Y)} \;\to\; \mathbb{Z},$$

the symmetric bilinear forms induced by ι_X.

The proof of the following theorem is almost the same as the argument in the proof of [27, Theorem 2.4]. We present a proof for the sake of completeness.

Theorem 2.2. *Let X, Y and U be as above. Suppose that $\sigma_{(X,Y)}$ is non-degenerate. Then (B_U, β_U) is isomorphic to $(\Lambda_{(X,Y)}, \lambda_{(X,Y)})$.*

Proof. We choose a tubular neighborhood $\mathcal{T} \subset X$ of Y as in Remark 2.1, and put $\mathcal{T}^\times := \mathcal{T} \setminus Y = \mathcal{T} \cap U$. We then denote by

$$\begin{aligned} i_T &: H_{2n}(\mathcal{T}^\times) \to H_{2n}(\mathcal{T}), \quad & i_U &: H_{2n}(\mathcal{T}^\times) \to H_{2n}(U), \\ j_T &: H_{2n}(\mathcal{T}) \to H_{2n}(X), \quad & j_U &: H_{2n}(U) \to H_{2n}(X), \end{aligned}$$

the homomorphisms induced by the inclusions. We first show that

$$\mathrm{Im}(j_U) = \widetilde{\Lambda}_{(X,Y)}. \tag{2.1}$$

It is obvious that $\mathrm{Im}(j_U) \subseteq \widetilde{\Lambda}_{(X,Y)}$. Let $[W] \in \widetilde{\Lambda}_{(X,Y)}$ be represented by a real $2n$-dimensional topological cycle W. We can assume that $W \cap Y$ consists of a finite number of points in $Y \setminus \mathrm{Sing}(Y)$, and that, locally around each intersection point P, the topological cycle W is a differentiable manifold intersecting Y transversely at P. Let $P_{i,1}, \dots, P_{i,k(i)}$ (resp. $Q_{i,1}, \dots, Q_{i,l(i)}$) be the intersection points of W and Y_i with local intersection number 1 (resp. -1). Since $\iota_X([W],[Y_i]) = 0$, we have $k(i) = l(i)$. For each $j = 1, \dots, k(i)$, we choose a path

$$\xi_{i,j} : I \to Y_i \setminus \mathrm{Sing}(Y)$$

from $P_{i,j}$ to $Q_{i,j}$, where $I := [0,1] \subset \mathbb{R}$ is the closed interval. Let $\mathcal{B}$ denote a real $2n$-dimensional closed ball with the center O. We can *thicken* the path $\xi_{i,j}$ to a continuous map

$$\tilde{\xi}_{i,j} : \mathcal{B} \times I \to X$$

in such a way that $\tilde{\xi}_{i,j}^{-1}(Y)$ is equal to $\{O\} \times I$, that the restriction of $\tilde{\xi}_{i,j}$ to $\{O\} \times I$ is equal to $\xi_{i,j}$, and that the restriction of $\tilde{\xi}_{i,j}$ to $\mathcal{B} \times \{0\}$ (resp. to $\mathcal{B} \times \{1\}$) induces a homeomorphism from $\mathcal{B}$ to a closed neighborhood $\Delta^+_{i,j}$ of $P_{i,j}$ (resp. $\Delta^-_{i,j}$ of $Q_{i,j}$) in W. We then put

$$W' := \big(W \setminus \textstyle\bigcup_{i,j} (\Delta^+_{i,j} \cup \Delta^-_{i,j})\big) \cup \textstyle\bigcup_{i,j} \tilde{\xi}_{i,j}(\partial\mathcal{B} \times I).$$

We can give an orientation to each $\tilde{\xi}_{i,j}(\mathcal{B} \times I)$ in such a way that W' becomes a topological cycle. Then we have $[W] = [W']$ in $H_{2n}(X)$ and $W' \cap Y = \emptyset$. Therefore $[W] = [W']$ is contained in $\mathrm{Im}(j_U)$, and hence (2.1) is proved.

Next we show that

$$\mathrm{Ker}(j_U) \subseteq J_\infty(U). \tag{2.2}$$

Consider the Mayer-Vietoris sequence

$$\cdots \longrightarrow H_{2n}(\mathcal{T}^\times) \xrightarrow{\;i\;} H_{2n}(\mathcal{T}) \oplus H_{2n}(U) \xrightarrow{\;j\;} H_{2n}(X) \longrightarrow \cdots,$$

where $i(x) = (i_T(x), i_U(x))$ and $j(y,z) = j_T(y) - j_U(z)$. If $j_U(z) = 0$, then $(0,z) \in \mathrm{Ker}(j) = \mathrm{Im}(i)$, and hence $z \in \mathrm{Im}(i_U)$. On the other hand, we have $\mathrm{Im}(i_U) = J_\infty(U)$, because $\mathcal{T}$ is a tubular neighborhood of Y. Hence (2.2) is proved.

Since $(H_{2n}(X)^{\mathrm{tf}}, \iota_X)$ is a lattice and $(\Sigma_{(X,Y)}, \sigma_{(X,Y)})$ is a sublattice by the assumption, the orthogonal complement $(\Lambda_{(X,Y)}, \lambda_{(X,Y)})$ is also a lattice. By (2.1) and (2.2), we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(j_U) & \longrightarrow & H_{2n}(U) & \xrightarrow{j_U} & \widetilde{\Lambda}_{(X,Y)} & \longrightarrow & 0 \quad \text{(exact)} \\ & & \downarrow & & \| & & \downarrow \tilde{v} & & \\ 0 & \longrightarrow & J_\infty(U) & \longrightarrow & H_{2n}(U) & \longrightarrow & \widetilde{B}_U & \longrightarrow & 0 \quad \text{(exact)}, \end{array} \tag{2.3}$$

where the surjectivity of the third vertical arrow $\tilde{v}$ follows from the injectivity of the first vertical arrow (2.2). By the definition of the intersection pairing, we have

$\iota_U(z, z') = \iota_X(j_U(z), j_U(z'))$ for any $z, z' \in H_{2n}(U)$. Therefore the homomorphism $\tilde{v}$ in (2.3) satisfies

$$\widetilde{\lambda}_{(X,Y)}(\zeta, \zeta') = \tilde{\beta}_U(\tilde{v}(\zeta), \tilde{v}(\zeta'))$$

for any $\zeta, \zeta' \in \widetilde{\Lambda}_{(X,Y)}$. If $\tilde{v}(\zeta) \in (\widetilde{B}_U)_{\mathrm{tor}}$, then $\zeta \in (\widetilde{\Lambda}_{(X,Y)})_{\mathrm{tor}}$ holds, because $\lambda_{(X,Y)}$ is non-degenerate. Hence $\tilde{v}^{-1}((\widetilde{B}_U)_{\mathrm{tor}}) = (\widetilde{\Lambda}_{(X,Y)})_{\mathrm{tor}}$ holds. Therefore $\tilde{v}$ induces an isomorphism $(\Lambda_{(X,Y)}, \lambda_{(X,Y)}) \cong (B_U, \beta_U)$. □

3. Transcendental lattices

A submodule L' of a free $\mathbb{Z}$-module L is said to be *primitive* if $(L/L')_{\mathrm{tor}} = 0$.

Let X be a smooth complex projective variety of dimension $2n$. Then we have a natural isomorphism $H_{2n}(X, \mathbb{Z})^{\mathrm{tf}} \cong H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$ that transforms ι_X to the cup-product $(\ ,\ )_X$. Let $S_X \subset H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$ be the submodule generated by the classes $[Y] \in H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$ of irreducible subvarieties Y of X with codimension n, and let

$$s_X \,:\, S_X \times S_X \,\to\, \mathbb{Z}$$

be the restriction of the cup-product to S_X. Note that s_X is non-degenerate by Lefschetz decomposition and Hodge-Riemann bilinear relations. We consider the following condition:

(P) S_X is primitive in $H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$.

Remark 3.1. The condition (P) is satisfied if $\dim X = 2$, because $S_X = H^2(X, \mathbb{Z})^{\mathrm{tf}} \cap H^{1,1}(X)$ holds for a surface X. For the case where $\dim X > 2$, see Atiyah-Hirzebruch [9] and Totaro [32].

Let σ be an element of $\mathrm{Emb}(\mathbb{C})$, and consider the conjugate complex variety X^σ.

Proposition 3.2. *The map* $[Y] \mapsto [Y^\sigma]$ *induces an isomorphism* $(S_X, s_X) \cong (S_{X^\sigma}, s_{X^\sigma})$.

Proof. Let $\mathcal{Z}_X$ be the free $\mathbb{Z}$-module generated by irreducible subvarieties Y of codimension n in X, and let

$$\zeta_X \,:\, \mathcal{Z}_X \times \mathcal{Z}_X \,\to\, \mathbb{Z}$$

be the intersection pairing. Then S_X is the image of the cycle map $Z \mapsto [Z]$ from $\mathcal{Z}_X$ to $H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$. We put

$$\mathcal{B}_X := \{\ Z \in \mathcal{Z}_X \ \mid\ \zeta_X(Z, W) = 0 \ \text{ for any }\ W \in \mathcal{Z}_X\ \},$$

and consider the *numerical Néron-Severi lattice* $\mathrm{NS}_X := \mathcal{Z}_X/\mathcal{B}_X$ with the symmetric bilinear form $\bar{\zeta}_X : \mathrm{NS}_X \times \mathrm{NS}_X \to \mathbb{Z}$ induced by ζ_X. Since s_X is non-degenerate, the kernel of the cycle map $\mathcal{Z}_X \to H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$ coincides with $\mathcal{B}_X$, and hence (S_X, s_X) is isomorphic to $(\mathrm{NS}_X, \bar{\zeta}_X)$. In the same way, we see that $(S_{X^\sigma}, s_{X^\sigma})$ is isomorphic to $(\mathrm{NS}_{X^\sigma}, \bar{\zeta}_{X^\sigma})$. On the other hand, since the intersection pairing ζ_X is defined algebraically (see Fulton [18]), the map $Y \mapsto Y^\sigma$ induces an isomorphism $(\mathcal{Z}_X, \zeta_X) \cong (\mathcal{Z}_{X^\sigma}, \zeta_{X^\sigma})$, and hence it induces $(\mathrm{NS}_X, \bar{\zeta}_X) \cong (\mathrm{NS}_{X^\sigma}, \bar{\zeta}_{X^\sigma})$. □

Definition 3.3. We define the *transcendental lattice* T_X of X by

$$T_X := \{\ x \in H^{2n}(X, \mathbb{Z})^{\mathrm{tf}} \ \mid\ (x, y)_X = 0 \ \text{ for any }\ y \in S_X\ \}.$$

Theorem 3.4. *Let* $Y_1, \dots, Y_m$ *be irreducible subvarieties of* X *with codimension* n *whose classes* $[Y_i] \in H^{2n}(X, \mathbb{Q})$ *span* $S_X \otimes \mathbb{Q}$. *We put* $Y := \cup_{i=1}^m Y_i$ *and* $U := X \setminus Y$. *If* T_{X^σ} *is not isomorphic to* T_X, *then* U^σ *is not homeomorphic to* U.

Proof. Note that the classes $[{Y_i}^\sigma] \in H^{2n}(X^\sigma, \mathbb{Q})$ span $S_{X^\sigma} \otimes \mathbb{Q}$. Theorem 2.2 implies that (B_U, β_U) is isomorphic to T_X, and $(B_{U^\sigma}, \beta_{U^\sigma})$ is isomorphic to T_{X^σ}. Since (B_U, β_U) is a topological invariant of U, we obtain the result. □

We fix some terminologies about lattices. A lattice L is called *even* if $(v, v) \in 2\mathbb{Z}$ holds for any $v \in L$. For a prime integer p, let $\mathbb{Z}_p$ denote the ring of p-adic integers. We put $\mathbb{Z}_\infty := \mathbb{R}$. We say that two lattices L and L' *are in the same genus* if $L \otimes \mathbb{Z}_p$ and $L' \otimes \mathbb{Z}_p$ are isomorphic (as $\mathbb{Z}_p$-modules with $\mathbb{Z}_p$-valued symmetric bilinear forms) for all p (including ∞).

Proposition 3.5. *Suppose that $H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$ and $H^{2n}(X^\sigma, \mathbb{Z})^{\mathrm{tf}}$ are even. Suppose also that* (P) *holds for both of X and X^σ. Then T_{X^σ} is contained in the same genus as T_X.*

Proof. By the GAGA principle, the Hodge numbers $h^{p,q}(X) = \dim H^q(X, \Omega^p)$ of a smooth projective variety X are invariant under the conjugation. Since the signature of the cup-product on $H^{2n}(X, \mathbb{R})$ is given by these $h^{p,q}(X)$ (see, for example, [33, Theorem 6.33]), the cup-products on $H^{2n}(X, \mathbb{R})$ and on $H^{2n}(X^\sigma, \mathbb{R})$ have the same signature.

By the condition (P) for X, we see that S_X is a primitive sublattice of $H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$. Since $H^{2n}(X, \mathbb{Z})^{\mathrm{tf}}$ is unimodular, the discriminant form of T_X is isomorphic to the discriminant form of S_X multiplied by -1 by Nikulin's result [21, Corollary 1.6.2]. In the same way, we see that the discriminant form of T_{X^σ} is isomorphic to -1 times the discriminant form of S_{X^σ}. Since S_X and S_{X^σ} are isomorphic by Proposition 3.2, the discriminant forms of T_X and T_{X^σ} are isomorphic. Moreover, the signatures of T_X and T_{X^σ} are the same by the argument in the previous paragraph. Hence T_X and T_{X^σ} are contained in the same genus by [21, Corollary 1.9.4]. □

4. Arithmetic of singular abelian and $K3$ surfaces

Let X be a complex abelian surface (resp. a complex algebraic $K3$ surface). We say that X is *singular* if $\mathrm{rank}(S_X)$ attains the possible maximum 4 (resp. 20). Suppose that X is singular in this sense. Then we have

$$T_X = H^2(X, \mathbb{Z}) \cap (H^{2,0}(X) \oplus H^{0,2}(X)),$$

and T_X is an even positive-definite lattice of rank 2. Moreover, T_X has a canonical orientation given as follows; an ordered basis (e_1, e_2) of T_X is *positive* if the imaginary part of $(e_1, \omega)_X / (e_2, \omega)_X \in \mathbb{C}$ is positive, where ω is a basis of $H^{2,0}(X)$. We write $\widetilde{T}_X$ for the *oriented* transcendental lattice of X.

Theorem 4.1 (Shioda-Mitani [31])**.** *The map $A \mapsto \widetilde{T}_A$ induces a bijection from the set of isomorphism classes of complex singular abelian surfaces A to the set of isomorphism classes of even positive-definite oriented-lattices of rank* 2.

For complex singular abelian surfaces, we have a converse of Proposition 3.5.

Proposition 4.2. *Let A be a complex singular abelian surface, and let $\widetilde{T}'$ be an even positive-definite oriented-lattice of rank* 2 *such that the underlying lattice T' is contained in the same genus as T_A. Then there exists $\sigma \in \mathrm{Emb}(\mathbb{C})$ such that $\widetilde{T}_{A^\sigma} \cong \widetilde{T}'$.*

In [28], we proved Proposition 4.2 under additional conditions. Then Schütt [24] proved Proposition 4.2 in this full generality. We present a proof that does not make use of the idéle groups.

For the proof, we fix notation and prepare theorems in the theory of complex multiplications ([12], [20], [29]). For $a, b, c \in \mathbb{Z}$, we put

$$Q[a,b,c] := \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

For a negative integer d, we denote by $\mathcal{Q}_d$ the set of matrices $Q[a,b,c]$ such that $a, b, c \in \mathbb{Z}$, $a > 0$, $c > 0$ and $d = b^2 - 4ac$. The group $GL_2(\mathbb{Z})$ acts on $\mathcal{Q}_d$ by $Q \mapsto {}^t gQg$, where $Q \in \mathcal{Q}_d$ and $g \in GL_2(\mathbb{Z})$. Then the set of orbits $\mathcal{Q}_d/GL_2(\mathbb{Z})$ (resp. $\mathcal{Q}_d/SL_2(\mathbb{Z})$) is identified with the set of isomorphism classes of even positive-definite lattices (resp. oriented-lattices) of rank 2 with discriminant $-d$. For an $SL_2(\mathbb{Z})$-orbit $\Lambda \in \mathcal{Q}_d/\,SL_2(\mathbb{Z})$ and a positive integer m, we put

$$\langle m\rangle\Lambda := \{\ Q[ma, mb, mc] \ \mid \ Q[a,b,c] \in \Lambda\ \} \quad \in \quad \mathcal{Q}_{dm^2}/\,SL_2(\mathbb{Z}).$$

We denote by $\mathcal{Q}^*_d$ the subset of $\mathcal{Q}_d$ consisting of matrices $Q[a,b,c] \in \mathcal{Q}_d$ with $\gcd(a,b,c) = 1$. Then $\mathcal{Q}^*_d$ is stable under the action of $GL_2(\mathbb{Z})$.

Let $K \subset \mathbb{C}$ be an imaginary quadratic field. We denote by $\mathbb{Z}_K$ the ring of integers of K, and by D_K the discriminant of $\mathbb{Z}_K$. For a positive integer f, let $\mathcal{O}_f \subseteq \mathbb{Z}_K$ denote the order of conductor f. By a *grid*, we mean a $\mathbb{Z}$-submodule of K with rank 2. For grids L and L', we write $[L] = [L']$ if $L = \lambda L'$ holds for some $\lambda \in K^\times$. We put

$$\mathcal{O}(L) := \{\ \lambda \in K \ \mid \ \lambda L \subseteq L\ \}, \quad f(L) := [\mathbb{Z}_K : \mathcal{O}(L)], \quad d(L) := D_K f(L)^2.$$

By definition, we have $\mathcal{O}(L) = \mathcal{O}_{f(L)}$. Moreover, if $[L] = [L']$, then $f(L) = f(L')$. We equip L with a symmetric bilinear form defined by

$$(x,y)_L := \frac{n^2}{[\mathcal{O}_{f(L)} : nL]}\,\mathrm{Tr}_{K/\mathbb{Q}}(x\bar{y}),$$

where n is a positive integer such that $nL \subset \mathcal{O}_{f(L)}$. It turns out that $(\ ,\ )_L$ takes values in $\mathbb{Z}$. An ordered basis $[\alpha, \beta]$ of L is defined to be *positive* if the imaginary part of $\alpha/\beta \in \mathbb{C}$ is positive. With $(\ ,\ )_L$ and this orientation, L becomes an even positive-definite oriented-lattice of discriminant $-d(L)$. We denote by $\tilde{\lambda}(L) \in \mathcal{Q}_{d(L)}/\,SL_2(\mathbb{Z})$ the isomorphism class of the oriented-lattice L, and by $\lambda(L) \in \mathcal{Q}_{d(L)}/GL_2(\mathbb{Z})$ the isomorphism class of the underlying lattice. It is obvious that, if $[L] = [L']$, then $\tilde{\lambda}(L) = \tilde{\lambda}(L')$ holds.

Let L and M be grids. We denote by LM the grid generated by xy, where $x \in L$ and $y \in M$. Then $[L][M] := [LM]$ is well-defined. It is easy to prove that

$$f(LM) = \gcd(f(L), f(M)).$$

Let f be a positive integer. We put $d := D_K f^2$, and consider the set

$$Cl_d := \{\ [L] \ \mid \ f(L) = f\ \},$$

which is a finite abelian group by the product $[L][M] = [LM]$ with the unit element $[\mathcal{O}_f]$ and the inversion $[L]^{-1} = [L^{-1}]$, where $L^{-1} := \{\lambda \in K \mid \lambda L \subseteq \mathcal{O}_f\}$.

Proposition 4.3 (§3 and §7 in [12]). (1) *The map* $L \mapsto \tilde{\lambda}(L)$ *induces a bijection* $Cl_d \cong \mathcal{Q}_d^* / SL_2(\mathbb{Z})$ *with the inverse map being induced from* $Q \mapsto [L_Q]$, *where*

$$L_Q := \mathbb{Z} + \mathbb{Z}\left(\frac{-b+\sqrt{d}}{2a}\right) \quad \text{for} \quad Q := Q[a,b,c] \in \mathcal{Q}_d^*. \tag{4.1}$$

(2) *For grids* L *and* M *with* $f(L) = f(M) = f$, *the lattices* $\lambda(L)$ *and* $\lambda(M)$ *are contained in the same genus if and only if* $[L][M]^{-1} \in (Cl_d)^2$ *holds.*

Let $I \subseteq \mathcal{O}_f$ be an $\mathcal{O}_f$-ideal. Then $f(I)$ divides f. We say that I is a *proper* $\mathcal{O}_f$-ideal if $f = f(I)$ holds. For a non-zero integer μ, we say that I is *prime to* μ if $I + \mu\mathcal{O}_f = \mathcal{O}_f$ holds.

Proposition 4.4 (Chapter 8 of [20]). (1) *Any* $\mathcal{O}_f$*-ideal prime to* f *is proper. The map* $J \mapsto I = J \cap \mathcal{O}_f$ *is a bijection from the set of* $\mathbb{Z}_K$*-ideals* J *prime to* f *to the set of* $\mathcal{O}_f$*-ideals* I *prime to* f. *The inverse map is given by* $I \mapsto J = I\mathbb{Z}_K$.

(2) *Let* μ *be a non-zero integer. For any* $[M] \in Cl_d$, *there exists a proper* $\mathcal{O}_f$*-ideal* I *prime to* μ *such that* $[I] = [M]$.

For a grid L, we denote by $j(L) \in \mathbb{C}$ the j-invariant of the complex elliptic curve $\mathbb{C}/L$. It is obvious that $j(L) = j(L')$ holds if and only if $[L] = [L']$. We then put

$$H_d := K(j(\mathcal{O}_f)), \quad \text{where } d := D_K f^2.$$

The set $\{j(L) \,|\, f(L) = f\}$ is contained in H_d, and coincides with the set of conjugates of $j(\mathcal{O}_f)$ over K. Hence H_d/K is a finite Galois extension, which is called the *ring class field of* $\mathcal{O}_f$.

Theorem 4.5 (Chapter 10 of [20]). (1) *We define* $\varphi_d : Cl_d \to \mathrm{Gal}(H_d/K)$ *by*

$$j(\mathcal{O}_f)^{\varphi_d([M])} := j(M^{-1}) \text{ for } [M] \in Cl_d.$$

Then we have $j(L)^{\varphi_d([M])} = j(M^{-1}L)$ *for any* $[M], [L] \in Cl_d$, *and* φ_d *induces an isomorphism* $Cl_d \cong \mathrm{Gal}(H_d/K)$.

(2) *If a prime* $\mathfrak{p} \subset \mathbb{Z}_K$ *of* K *ramifies in* H_d, *then* $\mathfrak{p}$ *divides* $f\mathbb{Z}_K$. *For a* $\mathbb{Z}_K$*-ideal* J *prime to* f, *the Artin automorphism* $(J, H_d/K) \in \mathrm{Gal}(H_d/K)$ *is equal to* $\varphi_d([J \cap \mathcal{O}_f])$.

Shioda and Mitani [31] proved the following. Let $\widetilde{T}$ be an even positive-definite oriented-lattice of rank 2 given by $Q[a,b,c] \in \mathcal{Q}_d$. We put

$$K := \mathbb{Q}(\sqrt{d}), \;\; m := \gcd(a,b,c), \;\; d_0 := d/m^2, \;\; Q_0 := Q[a/m,\, b/m,\, c/m] \in \mathcal{Q}_{d_0}^*.$$

There exists a positive integer f such that $d = D_K(mf)^2$. We consider the grid $L_0 := L_{Q_0}$ of K associated with $Q_0 \in \mathcal{Q}_{d_0}^*$ by (4.1). Then we have $f(L_0) = f$ and $d(L_0) = d_0$. Note that $\langle m \rangle \tilde{\lambda}(L_0) \in \mathcal{Q}_d / SL_2(\mathbb{Z})$ is the isomorphism class containing $\widetilde{T}$. Note also that we have $\mathcal{O}_{mf} = \mathbb{Z} + \mathbb{Z}\,(b + \sqrt{d}\,)/2$. For grids L and M of K, we denote by $A(L,M)$ the complex abelian surface $\mathbb{C}/L \times \mathbb{C}/M$. It is well-known that the elliptic curve $\mathbb{C}/L$ is defined over the subfield $\mathbb{Q}(j(L))$ of $\mathbb{C}$. Hence $A(L,M)$ is defined over $\mathbb{Q}(j(L), j(M))$

Theorem 4.6 ([31]). (1) *The oriented transcendental lattice of* $A(L_0, \mathcal{O}_{mf})$ *is isomorphic to* $\widetilde{T}$. *In other words,* $\widetilde{T}_{A(L_0,\mathcal{O}_{mf})}$ *is contained in* $\langle m \rangle \tilde{\lambda}(L_0)$.

(2) *Let* L_1 *and* L_2 *be grids of* K. *Then* $A(L_1, L_2)$ *is isomorphic to* $A(L_0, \mathcal{O}_{mf})$ *if and only if* $[L_1 L_2] = [L_0]$ *and* $f(L_1)f(L_2) = mf^2$ *hold.*

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Since T_A and T' are in the same genus, they have the same discriminant, which we denote by $-d$. Let $\widetilde{T}_A$ and $\widetilde{T}'$ be represented by

$$Q_A = Q[a, b, c] \in \mathcal{Q}_d \quad \text{and} \quad Q' = Q[a', b', c'] \in \mathcal{Q}_d,$$

respectively. Since T_A and T' are in the same genus, we have

$$m := \gcd(a, b, c) = \gcd(a', b', c').$$

As above, we put $K := \mathbb{Q}(\sqrt{d})$, $d_0 := d/m^2$, and let f be the positive integer such that $d = D_K(mf)^2$. Let us consider the grids

$$L_0 := L_{Q[a/m,\, b/m,\, c/m]} \quad \text{and} \quad L'_0 := L_{Q[a'/m,\, b'/m,\, c'/m]}$$

associated with $(1/m)Q_A \in \mathcal{Q}^*_{d_0}$ and $(1/m)Q' \in \mathcal{Q}^*_{d_0}$ by (4.1). We have $f(L_0) = f(L'_0) = f$. The oriented-lattices $\widetilde{T}_A$ and $\widetilde{T}'$ are contained in the isomorphism classes $\langle m\rangle\tilde{\lambda}(L_0) \in \mathcal{Q}_d/\, SL_2(\mathbb{Z})$ and $\langle m\rangle\tilde{\lambda}(L'_0) \in \mathcal{Q}_d/\, SL_2(\mathbb{Z})$, respectively.

By Theorems 4.1 and 4.6, we have an isomorphism $A \cong A(L_0, \mathcal{O}_{mf})$ over $\mathbb{C}$. Note that $A(L_0, \mathcal{O}_{mf})$ is defined over $H_d \subset \mathbb{C}$, because we have $j(L_0) \in H_{d_0} \subseteq H_d$ by [20, Theorem 6 in Chapter 10].

Since T_A and T' are in the same genus, the lattices $\lambda(L_0), \lambda(L'_0) \in \mathcal{Q}^*_{d_0}/\, GL_2(\mathbb{Z})$ are also in the same genus. By Propositions 4.3 and 4.4, there exists a proper $\mathcal{O}_f$-ideal I_f prime to mf such that $[L'_0] = [I_f]^2[L_0]$ holds in Cl_{d_0}. We put $J := I_f\mathbb{Z}_K$ and $I_{mf} := J \cap \mathcal{O}_{mf}$. Then J is a $\mathbb{Z}_K$-ideal prime to mf satisfying $J \cap \mathcal{O}_f = I_f$, and I_{mf} is a proper $\mathcal{O}_{mf}$-ideal prime to mf. The Artin automorphism

$$\tau := (J, H_d/K) \;\in\; \mathrm{Gal}(H_d/K)$$

is equal to $\varphi_d([I_{mf}])$, and its restriction to H_{d_0} is equal to $\varphi_{d_0}([I_f]) \in \mathrm{Gal}(H_{d_0}/K)$. We extend $\tau^{-1} \in \mathrm{Gal}(H_d/K)$ to $\sigma \in \mathrm{Emb}(\mathbb{C})$. Then we have $j(L_0)^\sigma = j(I_f L_0)$ and $j(\mathcal{O}_{mf})^\sigma = j(I_{mf})$. Hence $A(L_0, \mathcal{O}_{mf})^\sigma$ is isomorphic to $A(I_f L_0, I_{mf})$, which is then isomorphic to $A(I_{mf} I_f L_0, \mathcal{O}_{mf})$ by Theorem 4.6. We have

$$[I_{mf} I_f L_0] = [I_f]^2[L_0] = [L'_0]. \tag{4.2}$$

To prove this, it is enough to show that $I_{mf}\mathcal{O}_f = I_f$. The inclusion $I_{mf}\mathcal{O}_f \subseteq I_f$ is obvious. Since I_{mf} is prime to mf, we have

$$I_f = I_f\mathcal{O}_{mf} = I_f(I_{mf} + mf\mathcal{O}_{mf}) \subseteq I_{mf}\mathcal{O}_f + mI_f.$$

Since $mI_f \subseteq m\mathcal{O}_f \subseteq \mathcal{O}_{mf}$, we have $mI_f \subseteq I_f \cap \mathcal{O}_{mf} = I_{mf}$. Therefore $I_f \subseteq I_{mf}\mathcal{O}_f$ holds, and (4.2) is proved. Consequently, the oriented transcendental lattice of

$$A^\sigma \;\cong\; A(L_0, \mathcal{O}_{mf})^\sigma \;\cong\; A(I_{mf} I_f L_0, \mathcal{O}_{mf}) \;\cong\; A(L'_0, \mathcal{O}_{mf})$$

is contained in $\langle m\rangle\tilde{\lambda}(L'_0)$ by Theorem 4.6, and hence is isomorphic to $\widetilde{T}'$. □

For complex singular $K3$ surfaces, we have the following result:

Theorem 4.7 (Shioda-Inose [30])**.** *The map $Y \mapsto \widetilde{T}_Y$ induces a bijection from the set of isomorphism classes of complex singular $K3$ surfaces Y to the set of isomorphism classes of even positive-definite oriented-lattices of rank 2.*

Let Y be a complex singular $K3$ surface, and A the complex singular abelian surface such that $\widetilde{T}_Y \cong \widetilde{T}_A$. Then Y is obtained from A by Shioda-Inose-Kummer construction (see [28, §6]). By [28, Proposition 6.4], we have $\widetilde{T}_{Y^\sigma} \cong \widetilde{T}_{A^\sigma}$ for any $\sigma \in \mathrm{Emb}(\mathbb{C})$. Combining this fact with Proposition 4.2, we obtain the following:

Proposition 4.8. *Let Y be a complex singular $K3$ surface, and let $\widetilde{T}'$ be an even positive-definite oriented-lattice of rank 2 such that the underlying lattice T' is contained in the same genus as T_Y. Then there exists $\sigma \in \mathrm{Emb}(\mathbb{C})$ such that $\widetilde{T}_{Y^\sigma} \cong \widetilde{T}'$.*

5. Applications

In this section, we consider only lattices without orientation. For a lattice T, let $g(T)$ denote the number of isomorphism classes of lattices in the genus of T. By Theorem 3.4 and Propositions 4.2, 4.8, we obtain the following:

Corollary 5.1. *Let X be a complex singular abelian surface or a complex singular $K3$ surface, and let $D \subset X$ be a reduced effective divisor such that the classes of irreducible components of D span $S_X \otimes \mathbb{Q}$. Then the set of the homeomorphism types of $(X \setminus D)^\sigma$ ($\sigma \in \mathrm{Emb}(\mathbb{C})$) contains at least $g(T_X)$ distinct elements.*

Another application is as follows. By a *plane curve*, we mean a complex reduced (possibly reducible) projective plane curve.

Definition 5.2. ([3], [5], [6]) A pair (C, C') of plane curves is said to be an *arithmetic Zariski pair* if the following hold:

(i) Let F be a homogeneous polynomial defining C. Then there exists $\sigma \in \mathrm{Emb}(\mathbb{C})$ such that C' is isomorphic (as a plane curve) to $C^\sigma := \{F^\sigma = 0\}$.

(ii) There exist tubular neighborhoods $\mathcal{T} \subset \mathbb{P}^2$ of C and $\mathcal{T}' \subset \mathbb{P}^2$ of C' such that $(\mathcal{T}, C)$ and $(\mathcal{T}', C')$ are diffeomorphic.

(iii) $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are *not* homeomorphic.

Remark 5.3. In general, the diffeomorphism type of a singular point of a plane curve changes under the conjugation. Consider the singularity of the curve

$$(y - a_1x)(y - a_2x)(y - a_3x)(y - a_4x) = 0$$

at the origin. The diffeomorphism type depends on the cross-ratio of (a_1, a_2, a_3, a_4), which varies under the action of $\sigma \in \mathrm{Emb}(\mathbb{C})$.

Remark 5.4. The first example of an arithmetic Zariski pair was discovered by Artal-Carmona-Cogolludo ([5], [6]) in degree 12.

Definition 5.5. ([23]) A plane curve C of degree 6 is called a *maximizing sextic* if C has only simple singularities and the total Milnor number of C attains the possible maximum 19.

Let C be a maximizing sextic. Then, for any $\sigma \in \mathrm{Emb}(\mathbb{C})$, the conjugate plane curve C^σ is also a maximizing sextic, and (C, C^σ) satisfies the condition (ii) in Definition 5.2, *because the Dynkin type of a simple singularity of a plane curve is defined algebraically* [7, 8]. We denote by $W_C \to \mathbb{P}^2$ the double covering branched along C, and by $Y_C \to W_C$ the minimal resolution. Then Y_C is a complex singular $K3$ surface. We denote by $T[C]$ the transcendental lattice of Y_C. Let $D_C \subset Y_C$ be the total inverse image of C by the composite morphism $Y_C \to W_C \to \mathbb{P}^2$, and we put $U_C := Y_C \setminus D_C$. Since the classes of irreducible components of D_C span $S_{Y_C} \otimes \mathbb{Q}$, the topological invariant (B_{U_C}, β_{U_C}) of U_C is isomorphic to $T[C]$. In particular, if $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C^\sigma)$ are homeomorphic, then U_C and U_{C^σ} are also homeomorphic, and hence $T[C]$ and $T[C^\sigma]$ are isomorphic. On the other hand, the set of isomorphism classes of the lattices $T[C^\sigma]$, where $\sigma \in \mathrm{Emb}(\mathbb{C})$, coincides with the genus of $T[C]$. Hence, if $g(T[C]) > 1$, then there exists $\sigma \in \mathrm{Emb}(\mathbb{C})$ such that

No.	R	$T[C]$ and	$T[C']$
1	$E_8 + A_{10} + A_1$	$L[6, 2, 8]$,	$L[2, 0, 22]$
2	$E_8 + A_6 + A_4 + A_1$	$L[8, 2, 18]$,	$L[2, 0, 70]$
3	$E_6 + D_5 + A_6 + A_2$	$L[12, 0, 42]$,	$L[6, 0, 84]$
4	$E_6 + A_{10} + A_3$	$L[12, 0, 22]$,	$L[4, 0, 66]$
5	$E_6 + A_{10} + A_2 + A_1$	$L[18, 6, 24]$,	$L[6, 0, 66]$
6	$E_6 + A_7 + A_4 + A_2$	$L[24, 0, 30]$,	$L[6, 0, 120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$L[30, 0, 42]$,	$L[18, 6, 72]$
8	$D_8 + A_{10} + A_1$	$L[6, 2, 8]$,	$L[2, 0, 22]$
9	$D_8 + A_6 + A_4 + A_1$	$L[8, 2, 18]$,	$L[2, 0, 70]$
10	$D_7 + A_{12}$	$L[6, 2, 18]$,	$L[2, 0, 52]$
11	$D_7 + A_8 + A_4$	$L[18, 0, 20]$,	$L[2, 0, 180]$
12	$D_5 + A_{10} + A_4$	$L[20, 0, 22]$,	$L[12, 4, 38]$
13	$D_5 + A_6 + A_5 + A_2 + A_1$	$L[12, 0, 42]$,	$L[6, 0, 84]$
14	$D_5 + A_6 + 2A_4$	$L[20, 0, 70]$,	$L[10, 0, 140]$
15	$A_{18} + A_1$	$L[8, 2, 10]$,	$L[2, 0, 38]$
16	$A_{16} + A_3$	$L[4, 0, 34]$,	$L[2, 0, 68]$
17	$A_{16} + A_2 + A_1$	$L[10, 4, 22]$,	$L[6, 0, 34]$
18	$A_{13} + A_4 + 2A_1$	$L[8, 2, 18]$,	$L[2, 0, 70]$
19	$A_{12} + A_6 + A_1$	$L[8, 2, 46]$,	$L[2, 0, 182]$
20	$A_{12} + A_5 + 2A_1$	$L[12, 6, 16]$,	$L[4, 2, 40]$
21	$A_{12} + A_4 + A_2 + A_1$	$L[24, 6, 34]$,	$L[6, 0, 130]$
22	$A_{10} + A_9$	$L[10, 0, 22]$,	$L[2, 0, 110]$
23	$A_{10} + A_9$	$L[8, 3, 8]$,	$L[2, 1, 28]$
24	$A_{10} + A_8 + A_1$	$L[18, 0, 22]$,	$L[10, 2, 40]$
25	$A_{10} + A_7 + A_2$	$L[22, 0, 24]$,	$L[6, 0, 88]$
26	$A_{10} + A_7 + 2A_1$	$L[10, 2, 18]$,	$L[2, 0, 88]$
27	$A_{10} + A_6 + A_2 + A_1$	$L[22, 0, 42]$,	$L[16, 2, 58]$
28	$A_{10} + A_5 + A_3 + A_1$	$L[12, 0, 22]$,	$L[4, 0, 66]$
29	$A_{10} + 2A_4 + A_1$	$L[30, 10, 40]$,	$L[10, 0, 110]$
30	$A_{10} + A_4 + 2A_2 + A_1$	$L[30, 0, 66]$,	$L[6, 0, 330]$
31	$A_8 + A_6 + A_4 + A_1$	$L[22, 4, 58]$,	$L[18, 0, 70]$
32	$A_7 + A_6 + A_4 + A_2$	$L[24, 0, 70]$,	$L[6, 0, 280]$
33	$A_7 + A_6 + A_4 + 2A_1$	$L[18, 4, 32]$,	$L[2, 0, 280]$
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$L[24, 0, 30]$,	$L[6, 0, 120]$

TABLE 5.1. Examples of arithmetic Zariski pairs of maximizing sextics

(C, C^σ) is an arithmetic Zariski pair. Using Yang's algorithm [34], we obtain the following theorem by computer-aided calculation:

Theorem 5.6. *There exist arithmetic Zariski pairs (C, C') of maximizing sextics with simple singularities of Dynkin type R for each R in Table 5.1.*

The lattices $T[C]$ and $T[C']$ are also presented in Table 5.1. We denote by $L[2a, b, 2c]$ the lattice of rank 2 represented by the matrix $Q[a, b, c]$.

Remark 5.7. In the previous paper [27], we have obtained a part of Theorem 5.6 by heavily using results of Artal-Carmona-Cogolludo [4] and Degtyarev [13]. Table 5.1 has been obtained during the calculation for the results in [26].

Remark 5.8. Table 5.1 is complete in a sense that, if (C, C') is an arithmetic Zariski pair of maximizing sextics such that $T[C] \not\cong T[C']$, then the data of (C, C') appear in Table 5.1. A detailed account of the algorithm for the calculation of Table 5.1 is given in [27].

Remark 5.9. Table 5.1 is *not* complete in a sense that there may be an arithmetic Zariski pair (C, C') of maximizing sextics such that $T[C] \cong T[C']$. A candidate of such an arithmetic Zariski pair is the pair (C, C^σ) of conjugate maximizing sextics of Dynkin type $2E_6 + A_5 + A_2$ defined over $\mathbb{Q}(\sqrt{3})$, which was discovered by Oka and Pho [22, nt99], where σ is the non-trivial element of $\mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$. The fundamental groups $\pi_1(\mathbb{P}^2 \setminus C)$ and $\pi_1(\mathbb{P}^2 \setminus C^\sigma)$ are calculated by Eyral and Oka in [17]. They conjectured that these two groups are not isomorphic (even though their pro-finite completions are isomorphic as a matter of course). In [15], Degtyarev suggested a method to distinguish $\pi_1(\mathbb{P}^2 \setminus C)$ and $\pi_1(\mathbb{P}^2 \setminus C^\sigma)$, but the conjecture is still open. On the other hand, the transcendental lattice does not distinguish $(\mathbb{P}^2, C)$ from $(\mathbb{P}^2, C^\sigma)$. Indeed we have

$$T[C] \cong T[C^\sigma] \cong L[6, 0, 6].$$

Remark 5.10. A method to calculate the lattice $T[C]$ from a defining equation of the maximizing sextic $C \subset \mathbb{P}^2$ is presented in [2].

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