MODULI CURVES OF SUPERSINGULAR $K3$ SURFACES IN CHARACTERISTIC 2 WITH ARTIN INVARIANT 2

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Dedicated to Professor Igor V. Dolgachev for his 60th birthday

Abstract. We construct explicitly moduli curves of polarized supersingular $K3$ surfaces in characteristic 2 with Artin invariant 2. As an application, we detect a "jump" phenomenon in a family of automorphism groups of supersingular $K3$ surfaces with a constant Néron-Severi lattice.

1. Introduction

A $K3$ surface is called *supersingular* if its numerical Néron-Severi lattice is of rank 22. Supersingular $K3$ surfaces exist only in positive characteristics. Artin showed in [1] that, in characteristic $p > 0$, the discriminant of the numerical Néron-Severi lattice of a supersingular $K3$ surface X is of the form $-p^{2\sigma(X)}$, where $\sigma(X)$ is a positive integer ≤ 10. This integer $\sigma(X)$ is called the *Artin invariant* of X.

We work over an algebraically closed field k of characteristic 2.

Definition 1.1. Let X be a supersingular $K3$ surface, and let $\mathcal{L}$ be a line bundle on X with $\mathcal{L}^2 = 2$. We say that $\mathcal{L}$ is a *polarization of type* ($\sharp$) if the following conditions are satisfied:

- the complete linear system $|\mathcal{L}|$ has no fixed components, and
- the set of curves contracted by the morphism $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ defined by $|\mathcal{L}|$ consists of 21 disjoint (-2)-curves.

In [10], we have shown that every supersingular $K3$ surface X in characteristic 2 has a polarization of type ($\sharp$), and that, if $\mathcal{L}$ is a polarization of type ($\sharp$) on X, then the morphism $\Phi_{|\mathcal{L}|}$ is purely inseparable. In [11], we have constructed a 9-dimensional moduli space $\mathfrak{M}$ of polarized supersingular $K3$ surfaces of type ($\sharp$). In this paper, we investigate the locus $\mathfrak{M}_2$ of $\mathfrak{M}$ corresponding to supersingular $K3$ surfaces with Artin invariant 2. As Artin [1] showed, this locus is of dimension 1. We will show that the curve $\mathfrak{M}_2$ is a disjoint union of three affine lines punctured at the origin. We will also construct explicitly the universal family of polarized supersingular $K3$ surfaces over certain finite covers of these punctured affine lines. The construction involves investigations of configurations of lines and conics on the projective plane in characteristic 2. These configurations are encoded by certain binary codes. In order to construct the moduli curve, we have to determine the automorphism groups of these codes. The automorphism group of the polarized $K3$ surface is also obtained from the automorphism group of the corresponding code.

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Let us briefly review the construction of the moduli space $\mathfrak{M}$ in [11]. For a non-zero homogeneous polynomial $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ of degree 6, we denote by

$$\pi_G : Y_G \to \mathbb{P}^2$$

the purely inseparable double cover of $\mathbb{P}^2$ defined by $W^2 = G(X, Y, Z)$.

Definition 1.2. Let $\mathcal{U}$ denote the locus of all non-zero homogeneous polynomials $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$ such that the surface Y_G has 21 ordinary nodes as its only singularities.

The locus $\mathcal{U}$ is Zariski open dense in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$. Indeed, in characteristic 2, the differential dG of G can be defined as a global section of $\Omega^1_{\mathbb{P}^2}(6)$ for any homogeneous polynomial $G \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$, because, by the isomorphism $\mathcal{O}_{\mathbb{P}^2}(6) \cong \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2}$, we can assume that the transition functions of the line bundle corresponding to $\mathcal{O}_{\mathbb{P}^2}(6)$ are all squares. Since $c_2(\Omega^1_{\mathbb{P}^2}(6)) = 21$, the subscheme $Z(dG)$ defined by $dG = 0$ is reduced of dimension 0 if and only if it consists of 21 points. The singular locus $\mathrm{Sing}(Y_G)$ of Y_G is equal to $\pi_G^{-1}(Z(dG))$, and the singular point of Y_G lying over a reduced point of $Z(dG)$ is an ordinary node. Hence the condition that G be a point of $\mathcal{U}$ is equivalent to the open condition that $Z(dG)$ be reduced of dimension 0.

Let $(X, \mathcal{L})$ be a polarized supersingular $K3$ surface of type ($\sharp$). Then there exists a homogeneous polynomial $G \in \mathcal{U}$ such that the Stein factorization of $\Phi_{|\mathcal{L}|}$ is written as

$$X \xrightarrow{\rho_G} Y_G \xrightarrow{\pi_G} \mathbb{P}^2.$$

Conversely, suppose that we are given $G \in \mathcal{U}$. Let $\rho_G : X_G \to Y_G$ be the minimal resolution of the surface Y_G. Then X_G is a supersingular $K3$ surface, and the invertible sheaf

$$\mathcal{L}_G := (\pi_G \circ \rho_G)^* \mathcal{O}_{\mathbb{P}^2}(1)$$

on X_G is a polarization of type ($\sharp$).

We put

$$\mathcal{V} := H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)).$$

Because we have $d(G + H^2) = dG$ for any $H \in \mathcal{V}$, the additive group $\mathcal{V}$ acts on the space $\mathcal{U}$ by

$$(G, H) \in \mathcal{U} \times \mathcal{V} \mapsto G + H^2 \in \mathcal{U}.$$

Proposition 1.3. *Let G and G' be homogeneous polynomials in $\mathcal{U}$. Then the following conditions are equivalent:*

(i) *Y_G and $Y_{G'}$ are isomorphic over $\mathbb{P}^2$,*
(ii) *$Z(dG) = Z(dG')$, and*
(iii) *there exist $c \in k^\times$ and $H \in \mathcal{V}$ such that $G' = c\,G + H^2$.*

See §2 for the proof.

Therefore the moduli space $\mathfrak{M}$ of polarized supersingular $K3$ surfaces of type ($\sharp$) is constructed by

$$\mathfrak{M} = PGL(3, k) \backslash \mathbb{P}_*(\mathcal{U}/\mathcal{V}).$$

For $G \in \mathcal{U}$, let $[G]$ denote the point of $\mathfrak{M}$ corresponding to G, which corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_G, \mathcal{L}_G)$ of type ($\sharp$). By Proposition 1.3, the automorphism group $\mathrm{Aut}(X_G, \mathcal{L}_G)$ of the polarized supersingular $K3$ surface is canonically identified with

$$\{\, g \in PGL(3, k) \mid g(Z(dG)) = Z(dG) \,\}.$$

The moduli space $\mathfrak{M}$ is stratified by the Artin invariant $\sigma(X_G)$ of X_G. We put

$$\mathfrak{M}_\sigma := \{ \,[G] \in \mathfrak{M} \;\mid\; \sigma(X_G) = \sigma\, \} \quad \text{and} \quad \mathfrak{M}_{\le\sigma} := \{ \,[G] \in \mathfrak{M} \;\mid\; \sigma(X_G) \le \sigma\, \}.$$

As was shown in [11], the locus $\mathfrak{M}_{\le 1} = \mathfrak{M}_1$ consists of a single point $[G_{\mathrm{DK}}]$, where

$$G_{\mathrm{DK}} := XYZ(X^3 + Y^3 + Z^3)$$

is the homogeneous polynomial discovered by Dolgachev and Kondo in [5]. The points $Z(dG_{\mathrm{DK}})$ coincide with the $\mathbb{F}_4$-rational points of $\mathbb{P}^2$, and hence the group $\mathrm{Aut}(X_{G_{\mathrm{DK}}}, \mathcal{L}_{G_{\mathrm{DK}}})$ is equal to $PGL(3, \mathbb{F}_4)$. We call $[G_{\mathrm{DK}}]$ the *Dolgachev-Kondo point.*

Now we can state our main results.

Theorem 1.4. *The locus $\mathfrak{M}_{\le 2}$ is a union of three irreducible curves $\overline{\mathfrak{M}}_A$, $\overline{\mathfrak{M}}_B$ and $\overline{\mathfrak{M}}_C$. In $\mathfrak{M}$, they are situated in such a way that, set-theoretically,*

$$\overline{\mathfrak{M}}_A \cap \overline{\mathfrak{M}}_B = \overline{\mathfrak{M}}_B \cap \overline{\mathfrak{M}}_C = \overline{\mathfrak{M}}_C \cap \overline{\mathfrak{M}}_A = \{[G_{\mathrm{DK}}]\}.$$

For $T = A, B$ and C, we put

$$\mathfrak{M}_T := \overline{\mathfrak{M}}_T \setminus \{[G_{\mathrm{DK}}]\}.$$

Hence $\mathfrak{M}_2$ is the disjoint union of $\mathfrak{M}_A$, $\mathfrak{M}_B$ and $\mathfrak{M}_C$.

Theorem 1.5. *For $T = A, B$ and C, the curve $\mathfrak{M}_T$ is isomorphic to an affine line punctured at the origin.*

We will describe the curves $\mathfrak{M}_T$ more explicitly. Let $\omega \in \mathbb{F}_4$ be a primitive third root of unity, and let $\bar\omega$ be $\omega + 1 = \omega^2$.

Theorem 1.6. *Let Γ_A be the group*

$$\left\{ \lambda, \lambda+1, \frac{1}{\lambda}, \frac{1}{\lambda+1}, \frac{\lambda}{\lambda+1}, \frac{\lambda+1}{\lambda} \right\}$$

acting on the punctured λ-line $\mathbb{A}^1 \setminus \{0, 1\} = \mathrm{Spec}\, k[\lambda, 1/\lambda(\lambda+1)]$. We put

$$J_A := \frac{(\lambda^2 + \lambda + 1)^3}{\lambda^2\,(\lambda+1)^2}$$

so that $k\,[\lambda, 1/\lambda(\lambda+1)]^{\Gamma_A} = k[J_A]$ holds. We also put

$$GA[\lambda] := XYZ\,(X+Y+Z)\left(X^2 + Y^2 + \left(\lambda^2 + \lambda\right) Z^2 + XY + YZ + ZX\right).$$

Then there exists an isomorphism

$$\mathfrak{M}_A \cong \mathrm{Spec}\, k[J_A, 1/J_A]$$

such that the family $W^2 = GA[\lambda]$ of sextic double planes over the finite Galois cover $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar\omega\} = \mathrm{Spec}\, k[\lambda, 1/(\lambda^4 + \lambda)]$ of the moduli curve $\mathfrak{M}_A$ yields the universal family of polarized supersingular $K3$ surfaces. The points $Z(dGA[\lambda])$ are given in Table 4.7. The origin $J_A = 0$ corresponds to the Dolgachev-Kondo point.

For $\alpha \in k \setminus \{0, 1, \omega, \bar\omega\}$, $\mathrm{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ is equal to the group

$$\left\{ \left[\begin{array}{c|c} \;A\; & \begin{matrix} a \\ b \end{matrix} \\ \hline 0\;0 & 1 \end{array}\right] \in PGL(3,k) \;\middle|\; \begin{array}{l} A \in GL(2, \mathbb{F}_2), \\ a, b \in \{0, 1, \alpha, \alpha+1\} \end{array} \right\} \tag{1.1}$$

of order 96.

Theorem 1.7. *We put*

$$Q_\lambda := (\bar{\omega}\lambda + \omega)\, X^2 + \bar{\omega}\, Y^2 + \omega\lambda\, Z^2 + (\lambda+1)\, XY + (\bar{\omega}\lambda+\omega) YZ + (\lambda+1)\, ZX,$$

and

$$GB[\lambda] := XYZ\,(X+Y+Z)\,Q_\lambda.$$

Let Γ_B *be the group*

$$\Big\{ \lambda,\ \omega\,\lambda+1,\ \frac{1}{\lambda+1},\ \frac{\lambda+\bar{\omega}}{\lambda+1},\ \frac{\bar{\omega}\lambda+\omega}{\lambda},\ \frac{\bar{\omega}}{\lambda},\ \frac{\omega}{\lambda+\bar{\omega}},\qquad \frac{\bar{\omega}\,(\lambda+1)}{\lambda+\bar{\omega}},\ \frac{\bar{\omega}\lambda}{\lambda+1},\ \frac{\lambda}{\lambda+\bar{\omega}},\ \frac{\lambda+1}{\lambda},\ \bar{\omega}(\lambda+1) \Big\}$$

acting on the punctured λ*-line* $\mathbb{A}^1 \setminus \{0,1,\bar{\omega}\} = \operatorname{Spec} k[\lambda, 1/\lambda(\lambda+1)(\lambda+\bar{\omega})]$. *We put*

$$J_B := \frac{(\lambda+\omega)^{12}}{\lambda^3(\lambda+1)^3(\lambda+\bar{\omega})^3}$$

so that $k\,[\lambda, 1/\lambda(\lambda+1)(\lambda+\bar{\omega})]^{\Gamma_B} = k[J_B]$ *holds. Then there exists an isomorphism*

$$\mathfrak{M}_B \cong \operatorname{Spec} k[J_B, 1/J_B]$$

such that the family $W^2 = GB[\lambda]$ *of sextic double planes over the finite Galois cover* $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\} = \operatorname{Spec} k[\lambda, 1/(\lambda^4+\lambda)]$ *of the moduli curve* $\mathfrak{M}_B$ *yields the universal family of polarized supersingular* $K3$ *surfaces. The points* $Z(dGB[\lambda])$ *are given in Table 5.5. The origin* $J_B = 0$ *corresponds to the Dolgachev-Kondo point.*

For any $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$, $\operatorname{Aut}(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$ *is equal to the subgroup of* $PGL(3,k)$ *generated by*

$$\begin{bmatrix} 0 & \omega & 0 \\ \bar{\omega} & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \bar{\omega} \\ 1 & 1 & 1 \\ \omega & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ \bar{\omega} & 1 & 0 \\ \omega & 0 & 1 \end{bmatrix}. \tag{1.2}$$

In particular, $\operatorname{Aut}(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$ *is isomorphic to the extended Heisenberg group of order* 18.

Theorem 1.8. *Let* Γ_C *be the group*

$$\{\ \alpha\lambda+\beta \ \mid\ \alpha \in \mathbb{F}_4^\times, \beta \in \mathbb{F}_4\ \}$$

of order 12 *acting on the* λ*-line* $\mathbb{A}^1 = k[\lambda]$. *We put*

$$J_C := (\lambda^4+\lambda)^3$$

so that $k[\lambda]^{\Gamma_C} = k[J_C]$ *holds. We also put*

$$GC[\lambda] := XYZ\left(X^3+Y^3+Z^3\right) + (\lambda^4+\lambda)X^3Y^3.$$

Then there exists an isomorphism

$$\mathfrak{M}_C \cong \operatorname{Spec} k[J_C, 1/J_C]$$

such that the family $W^2 = GC[\lambda]$ *of sextic double planes over the finite Galois cover* $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\} = \operatorname{Spec} k[\lambda, 1/(\lambda^4+\lambda)]$ *of the moduli curve* $\mathfrak{M}_C$ *yields the universal family of polarized supersingular* $K3$ *surfaces. The points* $Z(dGC[\lambda])$ *are given in Table 6.4. The origin* $J_C = 0$ *corresponds to the Dolgachev-Kondo point.*

For $\alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}$*,* $\mathrm{Aut}(X_{GC[\alpha]}, \mathcal{L}_{GC[\alpha]})$ *is equal to*

$$\left\{ \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ a^2c^2\alpha + e & b^2d^2\alpha + f & 1 \end{bmatrix} \in PGL(3, k) \;\middle|\; \begin{array}{l} a, b, c, d, e, f \in \mathbb{F}_4, \\ ad + bc = 1 \end{array} \right\} \tag{1.3}$$

of order 960*.*

Next we consider the isomorphism classes of *non-polarized* supersingular $K3$ surfaces with Artin invariant 2.

Definition 1.9. A reduced (possibly reducible) curve D in $\mathfrak{M}_T \times \mathfrak{M}_{T'}$ is called a *correspondence between* $\mathfrak{M}_T$ *and* $\mathfrak{M}_{T'}$. For a correspondence $D \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}$, let ${}^tD$ denote the correspondence in $\mathfrak{M}_{T'} \times \mathfrak{M}_T$ obtained from D by interchanging the first and the second factors. When D is a union of two curves D_1 and D_2 without common irreducible components, we write $D = D_1 + D_2$ and $D_2 = D - D_1$. Let $D_1 \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}$ and $D_2 \subset \mathfrak{M}_{T'} \times \mathfrak{M}_{T''}$ be correspondences. The *composite* $D_1 * D_2 \subset \mathfrak{M}_T \times \mathfrak{M}_{T''}$ of D_1 and D_2 is defined to be the image of

$$(D_1 \times \mathfrak{M}_{T''}) \cap (\mathfrak{M}_T \times D_2) \quad \subset \quad \mathfrak{M}_T \times \mathfrak{M}_{T'} \times \mathfrak{M}_{T''}$$

by the natural projection to $\mathfrak{M}_T \times \mathfrak{M}_{T''}$.

Definition 1.10. A correspondence D in $\mathfrak{M}_T \times \mathfrak{M}_{T'}$ is called an *isomorphism correspondence* if, for every point $([G], [G'])$ of D, the supersingular $K3$ surfaces X_G and $X_{G'}$ (without polarization) are isomorphic. An isomorphism correspondence $D \subset \mathfrak{M}_T \times \mathfrak{M}_{T'}$ is said to be *trivial* if T is equal to T' and D is the diagonal Δ_T of $\mathfrak{M}_T \times \mathfrak{M}_T$.

Using Cremona transformations by quintic curves, which played a central role in the study of $\mathrm{Aut}(X_{G_{\mathrm{DK}}})$ in [5], we have obtained examples of non-trivial isomorphism correspondences.

Definition 1.11. Let G be a homogeneous polynomial in $\mathcal{U}$. We say that a subset $\Sigma \subset Z(dG)$ of cardinality 6 is a *center of Cremona transformation for* $(X_G, \mathcal{L}_G)$ or *for* G if Σ satisfies the following conditions:

- no three points of Σ are collinear, and
- for each $p_i \in \Sigma$, there exists a conic curve $N'_i \subset \mathbb{P}^2$ such that $N'_i \cap Z(dG) = \Sigma \setminus \{p_i\}$.

Note that the conic curve N'_i is necessarily nonsingular.

Let $\Sigma = \{p_1, \dots, p_6\}$ be a center of Cremona transformation for $(X_G, \mathcal{L}_G)$. Consider the linear system $|\mathcal{I}^2_\Sigma(5)| \subset |\mathcal{O}_{\mathbb{P}^2}(5)|$ of quintic curves that pass through all the points of Σ and are singular at each point of Σ. Then $|\mathcal{I}^2_\Sigma(5)|$ is of dimension 2, and defines a birational map

$$\mathrm{CT}_\Sigma \;:\; \mathbb{P}^2 \;\cdots\!\to\; \mathbb{P}^2.$$

The birational map CT_Σ is the composite of the blowing up $\beta : S \to \mathbb{P}^2$ of the points of Σ and the blowing down $\beta' : S \to \mathbb{P}^2$ of the strict transforms N_i of the conic curves N'_i. We denote by p'_i the image of N_i by β'. Note that, if $p \in \mathbb{P}^2 \setminus \Sigma$, then the point $\mathrm{CT}_\Sigma(p) \in \mathbb{P}^2$ is well-defined.

Proposition 1.12 (Dolgachev-Kondo [5])**.** *We put*

$$Z' \;:=\; \{\ \mathrm{CT}_\Sigma(p) \;\mid\; p \in Z(dG) \setminus \Sigma\ \} \;\cup\; \{p'_1, \dots, p'_6\}.$$

Then there exists a homogeneous polynomial $G' \in \mathcal{U}$ such that $Z' = Z(dG')$. The birational map CT_Σ of $\mathbb{P}^2$ lifts to an isomorphism

$$\widetilde{\mathrm{CT}}_\Sigma \;:\; X_G \xrightarrow{\sim} X_{G'}$$

of supersingular $K3$ surfaces.

See also §8 of this paper for the proof of Proposition 1.12. Note that the polynomial G' is not uniquely determined, but the point $[G'] \in \mathfrak{M}$ is uniquely determined by G and Σ. We call $\widetilde{\mathrm{CT}}_\Sigma$ the *Cremona transformation of X_G with center Σ.*

Let T be A, B or C. As Tables 4.7, 5.5 and 6.4 show, the family

$$\{\ (p,\lambda)\ \mid\ p \in Z(dGT[\lambda])\ \}\quad\subset\quad \mathbb{P}^2 \times (\mathbb{A}^1 \setminus \{0,1,\omega,\bar\omega\})$$

of the points $Z(dGT[\lambda])$ consists of 21 connected components, each of which is étale of degree 1 over the punctured λ-line $\mathbb{A}^1 \setminus \{0,1,\omega,\bar\omega\}$. Therefore it makes sense to talk about a family $\Sigma[\lambda]$ of subsets of $Z(dGT[\lambda])$ that depends on λ continuously. It can be shown that, if $\Sigma[\alpha]$ is a center of Cremona transformation for $GT[\alpha]$ at one $\alpha \in k \setminus \{0,1,\omega,\bar\omega\}$, then so is $\Sigma[\alpha]$ at every $\alpha \in k \setminus \{0,1,\omega,\bar\omega\}$. In this case, we say that $\Sigma[\lambda]$ is a center of Cremona transformation for $GT[\lambda]$ or for $(X_{GT[\lambda]}, \mathcal{L}_{GT[\lambda]})$.

Suppose that $\Sigma[\lambda]$ is a center of Cremona transformation for $GT[\lambda]$. Then there exist a family $G'[\lambda]$ of homogeneous polynomials in $\mathcal{U}$ and a family of isomorphisms

$$\widetilde{\mathrm{CT}}_{\Sigma[\lambda]} \;:\; X_{GT[\lambda]} \;\xrightarrow{\sim}\; X_{G'[\lambda]}$$

depending on the parameter λ. The points $[G'[\lambda]]$ are of course contained in $\mathfrak{M}_2 = \mathfrak{M}_A \sqcup \mathfrak{M}_B \sqcup \mathfrak{M}_C$. Suppose that $[G'[\lambda]] \in \mathfrak{M}_{T'}$. Then the curve

$$\{\ ([GT[\lambda]], [G'[\lambda]]) \in \mathfrak{M}_T \times \mathfrak{M}_{T'}\ \mid\ \lambda \in \mathbb{A}^1 \setminus \{0,1,\omega,\bar\omega\}\ \}$$

is an irreducible isomorphism correspondence between $\mathfrak{M}_T$ and $\mathfrak{M}_{T'}$.

Theorem 1.13. (1) *There exist* 1644 *centers of Cremona transformation for the family $(X_{GA[\lambda]}, \mathcal{L}_{GA[\lambda]})$. They yield the following isomorphism correspondences:*

- 156 *of them give the trivial correspondence Δ_A,*
- 144 *of them give the correspondence*

  $$D_{A,A,1}\quad:\quad 1 + J_A\, J'_A + {J_A}^2 {J'_A}^2 + {J_A}^2 {J'_A}^3 + {J_A}^3 {J'_A}^2 = 0$$

  in $\mathfrak{M}_A \times \mathfrak{M}_A$,
- 720 *of them give the correspondence*

  $$D_{A,A,2} := D_{A,A,1} * D_{A,A,1} - \Delta_A \quad\subset\quad \mathfrak{M}_A \times \mathfrak{M}_A,$$
- 576 *of them give the correspondence*

  $$D_{A,B,1}\quad:\quad J_B + J_A\, J_B + J_A\, {J_B}^2 + {J_A}^2 J_B + {J_A}^4 = 0$$

  in $\mathfrak{M}_A \times \mathfrak{M}_B$,
- 48 *of them give the correspondence*

  $$D_{A,C,1}\quad:\quad J_C + J_A + J_A\, J_C + {J_A}^2 J_C + {J_A}^4 {J_C}^2 = 0$$

  in $\mathfrak{M}_A \times \mathfrak{M}_C$.

(2) *There exist* 1374 *centers of Cremona transformation for $(X_{GB[\lambda]}, \mathcal{L}_{GB[\lambda]})$. They yield the following isomorphism correspondences:*

- 798 *of them give the trivial correspondence Δ_B,*

T	T'	name	the equation
A	A	$D_{A,A,1}$	$J_A{}^3 J_A'{}^2 + J_A{}^2 J_A'{}^3 + J_A{}^2 J_A'{}^2 + J_A\, J_A' + 1$
A	A	$D_{A,A,2}$	$J_A{}^6 J_A'{}^2 + J_A{}^4 J_A'{}^4 + J_A{}^2 J_A'{}^6 + J_A{}^4 J_A'{}^3 + J_A{}^3 J_A'{}^4 + J_A{}^4 J_A'{}^2 + J_A{}^3 J_A'{}^3 + J_A{}^2 J_A'{}^4 + J_A{}^4 J_A' + J_A\, J_A'{}^4 + J_A{}^3 J_A' + J_A{}^2 J_A'{}^2 + J_A\, J_A'{}^3 + J_A{}^3 + J_A{}^2 J_A' + J_A\, J_A'{}^2 + J_A'{}^3$
B	B	$D_{B,B,1}$	$J_B{}^4 J_B' + J_B{}^3 J_B'{}^2 + J_B{}^2 J_B'{}^3 + J_B\, J_B'{}^4 + J_B{}^3 J_B' + J_B{}^2 J_B'{}^2 + J_B\, J_B'{}^3 + 1$
C	C	$D_{C,C,1}$	$J_C{}^4 J_C'{}^4 + J_C{}^3 J_C' + J_C{}^2 J_C'{}^2 + J_C\, J_C'{}^3 + J_C{}^3 + J_C{}^2 J_C' + J_C\, J_C'{}^2 + J_C'{}^3$
A	B	$D_{A,B,1}$	$J_A{}^4 + J_A{}^2 J_B + J_A\, J_B{}^2 + J_A\, J_B + J_B$
A	B	$D_{A,B,2}$	$J_A{}^6 J_B + J_A{}^5 J_B + J_A{}^4 J_B{}^2 + J_A{}^3 J_B{}^3 + J_A{}^4 J_B + J_A{}^2 J_B{}^2 + J_A{}^2 J_B + J_A\, J_B + 1$
B	C	$D_{B,C,1}$	$J_B\, J_C + 1$
B	C	$D_{B,C,2}$	$J_B{}^4 J_C{}^3 + J_B{}^3 J_C{}^3 + J_B{}^3 J_C{}^2 + J_B{}^2 J_C{}^2 + J_C{}^4 + J_B{}^2 J_C + J_B\, J_C + J_B$
C	A	$D_{C,A,1}$	$J_C{}^2 J_A{}^4 + J_C\, J_A{}^2 + J_C\, J_A + J_C + J_A$
C	A	$D_{C,A,2}$	$J_C{}^2 J_A{}^6 + J_C{}^2 J_A{}^5 + J_C{}^2 J_A{}^4 + J_C\, J_A{}^4 + J_C{}^2 J_A{}^2 + J_C{}^3 + J_C{}^2 J_A + J_C\, J_A{}^2 + J_A{}^3$

Table 1.1. Non-trivial irreducible isomorphism correspondences

- 216 *of them give the correspondence*
$$D_{B,A,1} := {}^tD_{A,B,1} \quad \subset \quad \mathfrak{M}_B \times \mathfrak{M}_A,$$
- 360 *of them give the correspondence*
$$D_{B,B,1} := D_{B,A,1} * D_{A,B,1} - \Delta_B \quad \subset \quad \mathfrak{M}_B \times \mathfrak{M}_B.$$

(3) *There exist* 2224 *centers of Cremona transformation for* $(X_{GC[\lambda]}, \mathcal{L}_{GC[\lambda]})$. *They yield the following isomorphism correspondences:*

- 1200 *of them give the trivial correspondence* Δ_C*,*
- 960 *of them give the correspondence*
$$D_{C,A,1} := {}^tD_{A,C,1} \quad \subset \quad \mathfrak{M}_C \times \mathfrak{M}_A,$$
- 64 *of them give the correspondence*
$$D_{C,C,1} := D_{C,A,1} * D_{A,C,1} - \Delta_C \quad \subset \quad \mathfrak{M}_C \times \mathfrak{M}_C.$$

Starting from the isomorphism correspondences by Cremona transformation above, making transposes and composites, and taking irreducible components, we obtain non-trivial irreducible isomorphism correspondences given in Table 1.1. When $T \neq T'$, we denote by $D_{T',T,\nu}$ the correspondence ${}^tD_{T,T',\nu}$ for $\nu = 1$ and 2. They have the relations in Table 8.5 at the end of §8.

Question 1.14. Are there any non-trivial irreducible isomorphism correspondences other than the ones in Table 1.1 and their transposes?

The Cremona transformations that yield the trivial isomorphism correspondence are also interesting, because they give automorphisms of the supersingular $K3$ surface X that may not be contained in $\mathrm{Aut}(X, \mathcal{L})$. See Remark 7.12.

Observation 1.15. Consider a Cremona transformation $\widetilde{\mathrm{CT}}_\Sigma$ on $(X_{GA[\lambda]}, \mathcal{L}_{GA[\lambda]})$ that yields the non-trivial isomorphism correspondence $D_{A,A,1}$. The curve $D_{A,A,1}$ intersects the diagonal Δ_A at two points $(J_A, J'_A) = (\omega, \omega)$ and $(\bar\omega, \bar\omega)$. Let η be an element of k such that the J_A-invariant of $(X_{GA[\eta]}, \mathcal{L}_{GA[\eta]})$ is ω or $\bar\omega$; that is, η is a root of

$$(\lambda^4 + \lambda^3 + 1)(\lambda^4 + \lambda + 1)(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1) = 0.$$

The Cremona transformation $\widetilde{\mathrm{CT}}_\Sigma$ gives rise to an automorphism of $X_{GA[\eta]}$, which cannot be deformed to any automorphisms of $X_{GA[\lambda]}$ for a generic λ. In other words, the automorphism group $\mathrm{Aut}(X_{GA[\lambda]})$ of the non-polarized supersingular $K3$ surface $X_{GA[\lambda]}$ *jumps* at $\lambda = \eta$, even though the numerical Néron-Severi lattice of $X_{GA[\lambda]}$ is constant around $\lambda = \eta$. Note that the automorphism group of a supersingular $K3$ surface is always embedded into the orthogonal group of its numerical Néron-Severi lattice ([8, §8, Proposition 3]).

The plan of this paper is as follows. In §2, we recall from [11] the definition of the binary code associated with a polarized supersingular $K3$ surface of type ($\sharp$). We stratify the moduli space $\mathfrak{M}$ according to the isomorphism classes $[\mathbf{C}]$ of the codes, and give a method to construct the stratum $\mathfrak{M}_{[\mathbf{C}]}$ from the code $\mathbf{C}$. In §3, we present three isomorphism classes $[\mathbf{C}_A]$, $[\mathbf{C}_B]$ and $[\mathbf{C}_C]$ of codes that are associated with polarized supersingular $K3$ surfaces of type ($\sharp$) with Artin invariant 2. In §4, §5 and §6, we carry out the method of the construction of $\mathfrak{M}_{[\mathbf{C}]}$ for $\mathbf{C} = \mathbf{C}_A, \mathbf{C}_B$ and $\mathbf{C}_C$, and prove Theorems 1.6, 1.7 and 1.8, respectively. In §7, we review from [5] the theory of Cremona transformations by quintic curves. In §8, we explain the algorithm to calculate the isomorphism correspondences given by Cremona transformations, and prove Theorem 1.13.

The isomorphism classes of codes associated with polarized supersingular $K3$ surfaces of Artin invariant $\sigma \geq 3$ are also given in [11]. For $\sigma = 3$, there are 13 isomorphism classes, and for $\sigma = 4$, there are 41 isomorphism classes. It would be a challenging problem in computational algebraic geometry to construct explicitly the moduli spaces of dimension $\sigma - 1$ corresponding to these isomorphism classes of codes, and to investigate the relations between them.

In [7], Rudakov and Shafarevich gave explicitly families of supersingular $K3$ surfaces in characteristic 2 for Artin invariants $\sigma = 1, \dots, 10$. The equation of the family for $\sigma = 2$ is

$$y^2 = x^3 + \mu\, t^6 x + t^5 (t+1)^4,$$

where μ is the "modulus". We would like to know the relation between μ and our moduli J_A, J_B and J_C.

The polarized supersingular $K3$ surface of type ($\sharp$) is an example of *Zariski surfaces.* A general theory of Zariski surfaces has been developed in [2].

Notation and terminologies.

(1) Let A be a commutative ring, and S a set. We denote by A^S the A-module of all maps from S to A.

(2) Let S be a finite set. The full symmetric group of S is denoted by $\mathfrak{S}(S)$, which acts on S from left. We denote by $\mathrm{Pow}(S)$ the power set of S. A canonical identification between $\mathrm{Pow}(S)$ and $\mathbb{F}_2^S$ is given by $f \in \mathbb{F}_2^S \mapsto f^{-1}(1) \subset S$. Hence $\mathrm{Pow}(S)$ has a structure of the $\mathbb{F}_2$-vector space by the symmetric difference

$$T_1 + T_2 = (T_1 \cup T_2) \setminus (T_1 \cap T_2) \qquad (T_1, T_2 \subset S).$$

A linear subspace of $\mathbb{F}_2^S = \mathrm{Pow}(S)$ is called a *code*, and an element of a code is called a *word*. A word is expressed either as a vector of dimension $|S|$ with coefficients in $\mathbb{F}_2$, or as a subset of S. The cardinality $|A|$ of a word $A \subset S$ is called the *weight* of A. The automorphism group $\mathrm{Aut}(\mathbf{C})$ of a code $\mathbf{C} \subset \mathrm{Pow}(S)$ is the subgroup of $\mathfrak{S}(S)$ consisting of all permutations preserving $\mathbf{C}$. Two codes $\mathbf{C}$ and $\mathbf{C}'$ in $\mathrm{Pow}(S)$ are said to be *isomorphic* if there exists a permutation $\sigma \in \mathfrak{S}(S)$ such that $\sigma(\mathbf{C}) = \mathbf{C}'$. The isomorphism class of codes represented by a code $\mathbf{C}$ is denoted by $[\mathbf{C}]$.

(3) A *lattice* is a free $\mathbb{Z}$-module Λ of finite rank equipped with a non-degenerate symmetric bilinear form $\Lambda \times \Lambda \to \mathbb{Z}$. A lattice is called *even* if $v^2 \in 2\mathbb{Z}$ holds for every $v \in \Lambda$. A lattice is called *hyperbolic* if the signature of the symmetric bilinear form on $\Lambda \otimes \mathbb{R}$ is $(1, r-1)$, where r is the rank of Λ. The *dual lattice* $\Lambda^\vee$ of Λ is the $\mathbb{Z}$-module $\mathrm{Hom}(\Lambda, \mathbb{Z})$. There exists a canonical embedding $\Lambda \hookrightarrow \Lambda^\vee$ of finite cokernel. Hence $\Lambda^\vee$ can be regarded as a submodule of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We have a natural $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda^\vee$ that extends the $\mathbb{Z}$-valued bilinear form on Λ. An *overlattice* of Λ is a submodule Λ' of $\Lambda^\vee$ containing Λ such that the canonical $\mathbb{Q}$-valued symmetric bilinear form on $\Lambda^\vee$ takes values in $\mathbb{Z}$ on Λ'.

2. The codes associated with the supersingular $K3$ surfaces

First we give a proof of Proposition 1.3.

Proof of Proposition 1.3. The equivalence of (i) and (iii) follows from the structure of the graded ring $\oplus_{m\geq 0} H^0(X, \mathcal{L}^{\otimes m})$, where X is a $K3$ surface and $\mathcal{L}$ is a line bundle of degree 2. (See [11, §7].) By [11, Theorem 2.1], $Z(dG) = Z(dG')$ holds if and only if $dG = c \cdot dG'$ for some $c \in k^\times$. Since the kernel of $G \mapsto dG$ is equal to $\{H^2 | H \in \mathcal{V}\}$, the equivalence of (ii) and (iii) follows. $\square$

2.1. Definition of the code $\mathcal{C}(X, \mathcal{L}, \gamma)$.

Let us fix a finite set

$$\mathcal{P} := \{P_1, \dots, P_{21}\}$$

consisting of 21 elements, on which the full symmetric group $\mathfrak{S}(\mathcal{P})$ acts from left.

Definition 2.1. We denote by $\mathcal{G}$ the space of all injective maps $\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$ such that there exists a homogeneous polynomial $G \in \mathcal{U}$ satisfying $\gamma(\mathcal{P}) = Z(dG)$.

The space $\mathcal{G}$ are constructed as follows. For $G \in \mathcal{U}$, let $\langle G \rangle \in \mathbb{P}_*(\mathcal{U}/\mathcal{V})$ denote the point corresponding to G. We denote by

$$\mathcal{Z} := \{\ (p, \langle G \rangle) \in \mathbb{P}^2 \times \mathbb{P}_*(\mathcal{U}/\mathcal{V})\ \mid\ p \in Z(dG)\ \} \quad\to\quad \mathbb{P}_*(\mathcal{U}/\mathcal{V})$$

the family of $Z(dG)$, which is finite and étale of degree 21 over $\mathbb{P}_*(\mathcal{U}/\mathcal{V})$. We prepare 21 copies of $\mathcal{Z}$ and make the fiber-product $\mathcal{Z}^{(21)}$ of them over $\mathbb{P}_*(\mathcal{U}/\mathcal{V})$. Then $\mathcal{G}$ is the union of irreducible components of $\mathcal{Z}^{(21)}$ that do not intersect the big diagonal.

Remark 2.2. We fix a base point $\langle G_0 \rangle \in \mathbb{P}_*(\mathcal{U}/\mathcal{V})$, and consider the monodromy action

$$\mu\ :\ \pi_1(\mathbb{P}_*(\mathcal{U}/\mathcal{V}), \langle G_0 \rangle) \ \to\ \mathfrak{S}(Z(dG_0))$$

of the algebraic fundamental group of $\mathbb{P}_*(\mathcal{U}/\mathcal{V})$ on $Z(dG_0)$. Then the number of irreducible components of $\mathcal{G}$ is equal to the index of the image of μ in $\mathfrak{S}(Z(dG_0))$. It was shown in [2, Chapter 4, Appendix 2] that the monodromy group on the singular points of a generic Zariski surface in characteristic ≥ 5 is equal to the full-symmetric group.

The group $\mathfrak{S}(\mathcal{P})$ acts on $\mathcal{G}$ from right, and $PGL(3,k)$ acts on $\mathcal{G}$ from left. By Proposition 1.3, we have

$$\mathfrak{M} = PGL(3,k)\backslash\mathcal{G}/\mathfrak{S}(\mathcal{P}).$$

Let

$$N_0 := \mathbb{Z}^{\mathcal{P}} \oplus \mathbb{Z}h = \bigoplus_{i=1}^{21} \mathbb{Z}e_i \oplus \mathbb{Z}h$$

be a free $\mathbb{Z}$-module of rank 22 generated by vectors $e_1, \dots, e_{21}$ corresponding to $P_1, \dots, P_{21} \in \mathcal{P}$ and a vector h. We equip N_0 with a structure of the even hyperbolic lattice by

$$e_i^2 = -2, \quad h^2 = 2, \quad e_i e_j = 0 \ \ (i \neq j), \quad he_i = 0.$$

The dual lattice

$$N_0^{\vee} = \operatorname{Hom}(N_0, \mathbb{Z}) \ \ \subset \ \ N_0 \otimes_{\mathbb{Z}} \mathbb{Q}$$

is generated by $e_i/2$ $(i = 1, \dots, 21)$ and $h/2$. Thus we have a canonical isomorphism

$$N_0^{\vee}/N_0 \ \cong \ \mathbb{F}_2^{\mathcal{P}} \oplus \mathbb{F}_2 \ = \ \operatorname{Pow}(\mathcal{P}) \oplus \mathbb{F}_2.$$

Hence we can write an element of $N_0^{\vee}/N_0$ in the form (A, α), where A is a subset of $\mathcal{P}$ and $\alpha \in \mathbb{F}_2$. We denote by

$$\mathrm{pr} \ : \ N_0^{\vee} \ \to \ N_0^{\vee}/N_0 = \operatorname{Pow}(\mathcal{P}) \oplus \mathbb{F}_2$$

the natural projection. We also denote by

$$\rho \ : \ N_0^{\vee}/N_0 = \operatorname{Pow}(\mathcal{P}) \oplus \mathbb{F}_2 \ \to \ \operatorname{Pow}(\mathcal{P})$$

the natural projection onto the first factor. The following is obvious:

Lemma 2.3. *Let $\widetilde{\mathcal{C}}$ be a subspace of the $\mathbb{F}_2$-vector space $\operatorname{Pow}(\mathcal{P}) \oplus \mathbb{F}_2$. Then the submodule $\mathrm{pr}^{-1}(\widetilde{\mathcal{C}})$ of $N_0^{\vee}$ is an even overlattice of N_0 if and only if*

$$|A| \equiv \begin{cases} 0 \mod 4 & \text{if } \alpha = 0 \\ 1 \mod 4 & \text{if } \alpha = 1 \end{cases}$$

holds for every $(A, \alpha) \in \widetilde{\mathcal{C}}$. □

Let $(X, \mathcal{L})$ be a polarized supersingular $K3$ surface of type $(\sharp)$, and let $NS(X)$ denote the numerical Néron-Severi lattice of X. There exists $G \in \mathcal{U}$ such that $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ factors through $\pi_G : Y_G \to \mathbb{P}^2$. We put

$$Z_{(X,\mathcal{L})} := Z(dG) = \pi_G(\operatorname{Sing} Y_G).$$

There also exists a point $\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$ of $\mathcal{G}$, unique up to the action of $\mathfrak{S}(\mathcal{P})$, that induces a bijection from $\mathcal{P}$ to $Z_{(X,\mathcal{L})}$. We fix such a point $\gamma \in \mathcal{G}$. Let E_i be the (-2)-curve on X such that $\Phi_{|\mathcal{L}|}(E_i)$ is the point $\gamma(P_i) \in Z_{(X,\mathcal{L})}$. Then we obtain an embedding

$$\iota_\gamma \ \ : \ \ N_0 \ \ \hookrightarrow \ \ NS(X)$$

of the lattice N_0 into $NS(X)$ by $e_i \mapsto [E_i]$ and $h \mapsto [\mathcal{L}]$. By the embedding ι_γ, we can regard $NS(X)$ as a submodule of $N_0^{\vee}$. We put

$$\begin{aligned} \widetilde{\mathcal{C}}(X, \mathcal{L}, \gamma) \ &:= \ NS(X)/N_0 \ \subset \ \operatorname{Pow}(\mathcal{P}) \oplus \mathbb{F}_2, \quad \text{and} \\ \mathcal{C}(X, \mathcal{L}, \gamma) \ &:= \ \rho(\widetilde{\mathcal{C}}(X, \mathcal{L}, \gamma)) \ \subset \ \operatorname{Pow}(\mathcal{P}). \end{aligned}$$

Since $NS(X)$ is an even overlattice of N_0, the code $\widetilde{\mathcal{C}}(X, \mathcal{L}, \gamma)$ is uniquely recovered from $\mathcal{C}(X, \mathcal{L}, \gamma)$ by Lemma 2.3, and hence the lattice $NS(X)$ is also uniquely recovered from the code $\mathcal{C}(X, \mathcal{L}, \gamma)$. In particular, the Artin invariant $\sigma(X)$ of X is given by

$$\sigma(X) = 11 - \dim_{\mathbb{F}_2} \mathcal{C}(X, \mathcal{L}, \gamma).$$

Note that the isomorphism class of the code $\mathcal{C}(X, \mathcal{L}, \gamma)$ does not depend on the choice of γ. The following is one of the main results of [11]:

Theorem 2.4. *For an isomorphism class* $[\mathbf{C}]$ *of codes in* $\mathrm{Pow}(\mathcal{P})$*, the following two conditions are equivalent.*

(i) *There exists a polarized supersingular* $K3$ *surface* $(X, \mathcal{L})$ *of type* ($\sharp$) *such that, for a* (*and hence any*) *bijection* γ *from* $\mathcal{P}$ *to* $Z_{(X,\mathcal{L})}$*, the code* $\mathcal{C}(X, \mathcal{L}, \gamma)$ *is in the isomorphism class* $[\mathbf{C}]$.

(ii) *A* (*and hence any*) *code* $\mathbf{C} \in [\mathbf{C}]$ *satisfies the following:*

- $\dim \mathbf{C} \le 10$,
- *the word* $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$ *is contained in* $\mathbf{C}$, *and*
- $|A| \in \{0, 5, 8, 9, 12, 13, 16, 21\}$ *for every word* $A \in \mathbf{C}$.

2.2. **Geometry of $Z_{(X,\mathcal{L})}$ and the code $\mathcal{C}(X, \mathcal{L}, \gamma)$.** Let $(X, \mathcal{L})$ be a polarized supersingular $K3$ surface of type ($\sharp$). We fix a bijection γ from $\mathcal{P}$ to $Z_{(X,\mathcal{L})}$. Let $G \in \mathcal{U}$ be a homogeneous polynomial such that $\Phi_{|\mathcal{L}|}$ factors through Y_G, or equivalently, such that $Z(dG) = Z_{(X,\mathcal{L})}$ holds. For the proofs of the facts stated in this subsection, we refer the reader to [11, §6 and §7].

Definition 2.5. Let $C \subset \mathbb{P}^2$ be a reduced irreducible curve. We say that C *splits in* $(X, \mathcal{L})$ if the proper transform of C by $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ is non-reduced. We say that a reduced (possibly reducible) curve C' *splits in* $(X, \mathcal{L})$ if every irreducible component of C' splits in $(X, \mathcal{L})$.

Since $\Phi_{|\mathcal{L}|}$ is purely inseparable of degree 2, the proper transform of a splitting curve C by $\Phi_{|\mathcal{L}|}$ is written as $2F_C$, where F_C is a reduced divisor of X. We denote by $w(C) \in \mathrm{Pow}(\mathcal{P})$ the image of the numerical equivalence class $[F_C] \in NS(X)$ by

$$NS(X) \quad \longrightarrow \quad NS(X)/N_0 \quad \hookrightarrow \quad N_0^\vee/N_0 \quad \xrightarrow{\rho} \quad \mathrm{Pow}(\mathcal{P}),$$

where $N_0 \hookrightarrow NS(X)$ is obtained from the fixed bijection $\gamma : \mathcal{P} \xrightarrow{\sim} Z_{(X,\mathcal{L})}$. By definition, we have

$$w(C) \in \mathcal{C}(X, \mathcal{L}, \gamma).$$

It is easy to see that

$$w(C) = \{\ P_i \in \mathcal{P} \ \mid \ \text{the multiplicity of } C \text{ at } \gamma(P_i) \text{ is odd}\ \}.$$

If C is a nonsingular curve splitting in $(X, \mathcal{L})$, then

$$w(C) = \gamma^{-1}(C \cap Z_{(X,\mathcal{L})}).$$

If C_1 and C_2 are two splitting curves without common irreducible components, then $w(C_1 \cup C_2) = w(C_1) + w(C_2)$ holds.

Proposition 2.6. *Let* $\mathcal{I}_{Z(dG)} \subset \mathcal{O}_{\mathbb{P}^2}$ *be the ideal sheaf defining the subscheme* $Z(dG)$*. The linear system* $|\mathcal{I}_{Z(dG)}(5)|$ *of quintic curves passing through all the points of* $Z(dG)$ *is of dimension* 2*, and spanned by the curves defined by*

$$\partial G/\partial X = 0, \quad \partial G/\partial Y = 0 \quad \textit{and} \quad \partial G/\partial Z = 0.$$

A general member C *of* $|\mathcal{I}_{Z(dG)}(5)|$ *splits in* $(X, \mathcal{L})$*, and the word* $w(C) \in \mathcal{C}(X, \mathcal{L}, \gamma)$ *is equal to* $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$.

Proposition 2.7. *Let C be a reduced curve splitting in $(X, \mathcal{L})$, and let p be a point of C.* (1) *If p is an ordinary node of C, then $p \in Z_{(X,\mathcal{L})}$.* (2) *If p is an ordinary tacnode of C, then $p \notin Z_{(X,\mathcal{L})}$.*

Proposition 2.8. *Let C be a reduced curve of degree 6 splitting in $(X, \mathcal{L})$, and let $G' = 0$ be a defining equation of C. If C has only ordinary nodes as its singularities, then the homogeneous polynomial G' is a point of $\mathcal{U}$, and the point $[G'] \in \mathfrak{M}$ corresponds to the isomorphism class of $(X, \mathcal{L})$.*

Proposition 2.9. *Let $L \subset \mathbb{P}^2$ be a line. The following conditions are equivalent:* (i) *L splits in $(X, \mathcal{L})$,* (ii) *$|L \cap Z_{(X,\mathcal{L})}| \geq 3$, and* (iii) *$|L \cap Z_{(X,\mathcal{L})}| = 5$.*

Proposition 2.10. *Let $Q \subset \mathbb{P}^2$ be a nonsingular conic curve. The following conditions are equivalent:* (i) *Q splits in $(X, \mathcal{L})$,* (ii) *$|Q \cap Z_{(X,\mathcal{L})}| \geq 6$, and* (iii) *$|Q \cap Z_{(X,\mathcal{L})}| = 8$.*

Corollary 2.11. *The word $w(L) = \gamma^{-1}(L \cap Z_{(X,\mathcal{L})})$ of a splitting line L is of weight 5, and the word $w(Q) = \gamma^{-1}(Q \cap Z_{(X,\mathcal{L})})$ of a splitting nonsingular conic curve Q is of weight 8.*

Definition 2.12. A pencil $\mathcal{E}$ of cubic curves in $\mathbb{P}^2$ is called a *regular pencil* if the following hold:

- the base locus $\mathrm{Bs}(\mathcal{E})$ of $\mathcal{E}$ consists of distinct 9 points, and
- every singular member of $\mathcal{E}$ has only one ordinary node as its singularities.

We say that a regular pencil $\mathcal{E}$ *splits in* $(X, \mathcal{L})$ if every member of $\mathcal{E}$ splits in $(X, \mathcal{L})$.

Proposition 2.13. *Let $\mathcal{E}$ be a regular pencil of cubic curves spanned by E_0 and E_∞. Let $H_0 = 0$ and $H_\infty = 0$ be the defining equations of E_0 and E_∞, respectively. Then $\mathcal{E}$ splits in $(X, \mathcal{L})$ if and only if there exist $c \in k^\times$ and $H \in \mathcal{V}$ such that*

$$G = cH_0H_\infty + H^2 \tag{2.1}$$

holds. If $\mathcal{E}$ splits in $(X, \mathcal{L})$, then $\mathrm{Bs}(\mathcal{E})$ is contained in $Z_{(X,\mathcal{L})}$, and

$$w(E_t) = \gamma^{-1}(\mathrm{Bs}(\mathcal{E}))$$

holds for every member E_t of $\mathcal{E}$. In particular, the word $w(E_t)$ is of weight 9.

Remark 2.14. The condition (2.1) is equivalent to

$$Z(d(H_0H_\infty)) = Z(dG) = Z_{(X,\mathcal{L})}$$

by Proposition 1.3.

Remark 2.15. A regular pencil $\mathcal{E}$ has 12 singular members $E^{(1)}, \dots, E^{(12)}$. We denote by $N^{(i)}$ the ordinary node of $E^{(i)}$. Suppose that $\mathcal{E}$ splits in $(X, \mathcal{L})$. Then $Z_{(X,\mathcal{L})}$ is a disjoint union of $\mathrm{Bs}(\mathcal{E})$ and $\{N^{(1)}, \dots, N^{(12)}\}$.

Let L_1 and L_2 be distinct lines splitting in $(X, \mathcal{L})$. Then the intersection point of L_1 and L_2 is in $Z_{(X,\mathcal{L})}$ by Proposition 2.7, and hence

$$w(L_1 \cup L_2) = w(L_1) + w(L_2)$$

is a word of weight 8.

Let L_1, L_2 and L_3 be lines splitting in $(X, \mathcal{L})$ such that $L_1 \cap L_2 \cap L_3 = \emptyset$. Then the three ordinary nodes of $L_1 \cup L_2 \cup L_3$ are in $Z_{(X,\mathcal{L})}$ by Proposition 2.7, and hence

$$w(L_1 \cup L_2 \cup L_3) = w(L_1) + w(L_2) + w(L_3)$$

is a word of weight 9.

Let Q be a nonsingular conic curve splitting in $(X, \mathcal{L})$, and let L be a line splitting in $(X, \mathcal{L})$. Using Proposition 2.7, we see that L intersects Q transversely if and only if $w(L \cup Q) = w(L) + w(Q)$ is of weight 9. We also see that L is tangent to Q if and only if $w(L) \cap w(Q) = \emptyset$.

Definition 2.16. Let $\mathbf{C} \subset \mathrm{Pow}(\mathcal{P})$ be a code satisfying the conditions in (ii) of Theorem 2.4, and let A be a word of $\mathbf{C}$ with $|A| \in \{5, 8, 9\}$.

(i) We say that A is a *linear word of* $\mathbf{C}$ if $|A| = 5$.

(ii) Suppose $|A| = 8$. If A is *not* a sum of two linear words of $\mathbf{C}$, then we say that A is a *quadratic word of* $\mathbf{C}$.

(iii) Suppose $|A| = 9$. If A is neither a sum of three linear words of $\mathbf{C}$ nor a sum of a linear and a quadratic word of $\mathbf{C}$, then we say that A is a *cubic word of* $\mathbf{C}$.

Proposition 2.17. (1) *The correspondence $L \mapsto w(L)$ yields a bijection from the set of lines splitting in $(X, \mathcal{L})$ to the set of linear words in $\mathcal{C}(X, \mathcal{L}, \gamma)$.*

(2) *The correspondence $Q \mapsto w(Q)$ yields a bijection from the set of nonsingular conic curves splitting in $(X, \mathcal{L})$ to the set of quadratic words in $\mathcal{C}(X, \mathcal{L}, \gamma)$.*

(3) *The correspondence $\mathcal{E} \mapsto \gamma^{-1}(\mathrm{Bs}(\mathcal{E}))$ yields a bijection from the set of regular pencils of cubic curves splitting in $(X, \mathcal{L})$ to the set of cubic words in $\mathcal{C}(X, \mathcal{L}, \gamma)$.*

By Theorem 2.4, the code $\mathcal{C}(X, \mathcal{L}, \gamma)$ is generated by the word $\mathcal{P}$ and by the linear, quadratic and cubic words in $\mathcal{C}(X, \mathcal{L}, \gamma)$. Combining this fact with Proposition 2.17, we obtain the following:

Corollary 2.18. *Let g be an element of the group*

$$\mathrm{Aut}(X, \mathcal{L}) = \{\ h \in PGL(3, k) \ \mid\ h(Z_{(X,\mathcal{L})}) = Z_{(X,\mathcal{L})}\ \}.$$

Then we have $\mathcal{C}(X, \mathcal{L}, \gamma) = \mathcal{C}(X, \mathcal{L}, g \circ \gamma)$. Hence there exists a unique element $\sigma_g \in \mathrm{Aut}(\mathcal{C}(X, \mathcal{L}, \gamma))$ such that $g \circ \gamma = \gamma \circ \sigma_g$ holds. By $g \mapsto \sigma_g$, we can embed $\mathrm{Aut}(X, \mathcal{L})$ into $\mathrm{Aut}(\mathcal{C}(X, \mathcal{L}, \gamma))$.

2.3. **Construction of $\mathfrak{M}_{[\mathbf{C}]}$ from $\mathbf{C}$.** Let $[\mathbf{C}]$ be an isomorphism class of codes satisfying the conditions in (ii) of Theorem 2.4. We denote by

$$\mathfrak{M}_{[\mathbf{C}]} \quad \subset \quad \mathfrak{M}$$

the locus of all isomorphism classes of polarized supersingular $K3$ surfaces $(X, \mathcal{L})$ of type $(\sharp)$ such that $\mathcal{C}(X, \mathcal{L}, \gamma)$ is contained in $[\mathbf{C}]$ for a (and hence any) bijection γ from $\mathcal{P}$ to $Z_{(X,\mathcal{L})}$. We also denote by $\mathcal{G}_{[\mathbf{C}]}$ the pull-back of $\mathfrak{M}_{[\mathbf{C}]}$ by the quotient map

$$\mathcal{G} \quad \longrightarrow \quad \mathfrak{M} = PGL(3, k) \backslash \mathcal{G} / \mathfrak{S}(\mathcal{P}).$$

We will describe the locus $\mathcal{G}_{[\mathbf{C}]}$.

Definition 2.19. For a point γ of $\mathcal{G}$, let $\mathcal{C}[\gamma]$ denote the code in $\mathrm{Pow}(\mathcal{P})$ generated by the following words:

- $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$,
- words A of weight 5 such that the points $\gamma(A)$ are collinear,
- words A of weight 8 such that there exists a nonsingular conic curve containing $\gamma(A)$, and
- words A of weight 9 such that there exists a regular pencil $\mathcal{E}$ of cubic curves spanned by $E_0 = \{H_0 = 0\}$ and $E_\infty = \{H_\infty = 0\}$ such that $\mathrm{Bs}(\mathcal{E}) = \gamma(A)$ and $Z(d(H_0 H_\infty)) = \gamma(\mathcal{P})$ hold.

From the results above, we obtain the following:

Corollary 2.20. *Suppose that $\gamma \in \mathcal{G}$, and let $(X, \mathcal{L})$ be a polarized supersingular $K3$ surface of type $(\sharp)$ such that $\gamma(\mathcal{P}) = Z_{(X,\mathcal{L})}$. Then the code $\mathcal{C}[\gamma]$ coincides with the code $\mathcal{C}(X, \mathcal{L}, \gamma)$.*

By definition, we have

$$\mathcal{C}[\gamma \circ \sigma] = \sigma^{-1}(\mathcal{C}[\gamma]) \qquad \text{for any } \sigma \in \mathfrak{S}(\mathcal{P}).$$

For each code $\mathbf{C} \in [\mathbf{C}]$, we put

$$\mathcal{G}_{\mathbf{C}} := \{ \gamma \in \mathcal{G} \mid \mathcal{C}[\gamma] = \mathbf{C} \}.$$

Then we have

$$\mathcal{G}_{\mathbf{C}}^{\sigma} = \mathcal{G}_{\sigma^{-1}(\mathbf{C})},$$

where $\mathcal{G}_{\mathbf{C}}^{\sigma}$ denotes the image of $\mathcal{G}_{\mathbf{C}}$ by the action of $\sigma \in \mathfrak{S}(\mathcal{P})$. Therefore we obtain

$$\mathcal{G}_{[\mathbf{C}]} = \bigsqcup_{\mathbf{C}' \in [\mathbf{C}]} \mathcal{G}_{\mathbf{C}'} = \bigsqcup_{\sigma} \mathcal{G}_{\mathbf{C}}^{\sigma} \qquad \text{(disjoint union)},$$

where σ runs through the set of representatives for the right cosets in $\mathfrak{S}(\mathcal{P})$ with respect to the subgroup $\mathrm{Aut}(\mathbf{C}) \subset \mathfrak{S}(\mathcal{P})$. Hence we have

$$\mathfrak{M}_{[\mathbf{C}]} = PGL(3, k) \backslash \mathcal{G}_{\mathbf{C}} / \mathrm{Aut}(\mathbf{C}).$$

For $\gamma \in \mathcal{G}_{\mathbf{C}}$, let $[\gamma] \in PGL(3, k) \backslash \mathcal{G}_{\mathbf{C}}$ denote the projective equivalence class of γ. From Corollary 2.18, we obtain the following:

Corollary 2.21. *Let $(X, \mathcal{L})$ be a polarized supersingular $K3$ surface of type $(\sharp)$ corresponding to the image of $[\gamma] \in PGL(3, k) \backslash \mathcal{G}_{\mathbf{C}}$ by the quotient map*

$$PGL(3, k) \backslash \mathcal{G}_{\mathbf{C}} \;\; \to \;\; \mathfrak{M}_{[\mathbf{C}]} = PGL(3, k) \backslash \mathcal{G}_{\mathbf{C}} / \mathrm{Aut}(\mathbf{C}).$$

Via the natural embedding of $\mathrm{Aut}(X, \mathcal{L})$ into $\mathrm{Aut}(\mathcal{C}(X, \mathcal{L}, \gamma)) = \mathrm{Aut}(\mathbf{C})$, the automorphism group $\mathrm{Aut}(X, \mathcal{L})$ is equal to the stabilizer subgroup of the point $[\gamma]$.

3. The isomorphism classes of codes with Artin invariant 1 and 2

We have classified all isomorphism classes of codes satisfying the conditions in (ii) of Theorem 2.4. The list is given in [11, §8]. Using the classification, we have obtained the following [11, Corollary 1.11]:

Theorem 3.1. *There exists exactly one isomorphism class $[\mathbf{C}_0]$ of codes of dimension 10 satisfying the conditions in (ii) of Theorem 2.4. The moduli space $\mathfrak{M}_{[\mathbf{C}_0]}$ consists of a single point corresponding to the Dolgachev-Kondo polynomial*

$$G_{\mathrm{DK}} := XYZ(X^3 + Y^3 + Z^3).$$

We call the point $[G_{\mathrm{DK}}]$ constituting $\mathfrak{M}_1 = \mathfrak{M}_{[\mathbf{C}_0]}$ the *Dolgachev-Kondo point.* We define the *Dolgachev-Kondo code*

$$\mathbf{C}_{\mathrm{DK}} \;\; \subset \;\; \mathrm{Pow}(\mathbb{P}^2(\mathbb{F}_4))$$

to be the code generated by the words $\Lambda(\mathbb{F}_4)$, where Λ are $\mathbb{F}_4$-rational lines in $\mathbb{P}^2$. The codes in the isomorphism class $[\mathbf{C}_0]$ are precisely the codes $\gamma^{-1}(\mathbf{C}_{\mathrm{DK}})$, where γ runs through the set of all bijections from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4) = Z(dG_{\mathrm{DK}})$. The weight enumerator of any code in $[\mathbf{C}_0]$ is

$$1 + 21z^5 + 210z^8 + 280z^9 + 280z^{12} + 210z^{13} + 21z^{16} + z^{21}.$$

$$
\begin{array}{cccccccccccccccccccccc}
[& 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 &] \\
[& 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 &] \\
[& 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 &]
\end{array}
$$

TABLE 3.1. Generators of the code $\mathbf{C}_A$

$$
\begin{array}{cccccccccccccccccccccc}
[& 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 &] \\
[& 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 &] \\
[& 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 &]
\end{array}
$$

TABLE 3.2. Generators of the code $\mathbf{C}_B$

$$
\begin{array}{cccccccccccccccccccccc}
[& 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 &] \\
[& 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 &] \\
[& 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 &] \\
[& 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 &]
\end{array}
$$

TABLE 3.3. Generators of the code $\mathbf{C}_C$

There are no quadratic nor cubic words in $\mathbf{C}_0$.

From the list in [11, §8], we obtain the following:

Proposition 3.2. *There are exactly three isomorphism classes* $[\mathbf{C}_A]$, $[\mathbf{C}_B]$, $[\mathbf{C}_C]$ *of codes of dimension* 9 *satisfying the conditions in* (ii) *of Theorem 2.4.*

As representatives of these isomorphism classes, we can take codes $\mathbf{C}_A$, $\mathbf{C}_B$ and $\mathbf{C}_C$ generated by vectors in Tables 3.1, 3.2 and 3.3. The numbers of linear, quadratic and cubic words in these codes are given in the following table:

	linear	quadratic	cubic
$\mathbf{C}_A$	13	28	0
$\mathbf{C}_B$	9	66	0
$\mathbf{C}_C$	5	120	0

.

The weight enumerators of these codes are as follows:

$$\begin{aligned}
\mathbf{C}_A &: \quad 1 + 13z^5 + 106z^8 + 136z^9 + 136z^{12} + 106z^{13} + 13z^{16} + z^{21},\\
\mathbf{C}_B &: \quad 1 + 9z^5 + 102z^8 + 144z^9 + 144z^{12} + 102z^{13} + 9z^{16} + z^{21},\\
\mathbf{C}_C &: \quad 1 + 5z^5 + 130z^8 + 120z^9 + 120z^{12} + 130z^{13} + 5z^{16} + z^{21}.
\end{aligned}$$

Remark 3.3. The Dolgachev-Kondo code $\mathbf{C}_{\mathrm{DK}}$ is related to the binary Golay code $\mathbf{C}_{24}$ in the following way. Let $M := \{\mu_1, \dots, \mu_{24}\}$ be the set of positions of the Miracle Octad Generator (MOG) as is indicated in [9, Table 6.1]. The definition of $\mathbf{C}_{24}$ as a subcode of $\mathrm{Pow}(M)$ is described in [3, Chapter 11]. We put $N := \{\mu_{22}, \mu_{23}, \mu_{24}\} \subset M$, and consider the 10-dimensional subcode

$$\mathbf{C}_{22} := \{\ w \in \mathbf{C}_{24} \ \mid\ w \supset N \ \text{ or } \ w \cap N = \emptyset\ \}$$

of $\mathbf{C}_{24}$. We then define a map

$$\mathbb{P}^2(\mathbb{F}_4) \quad\longrightarrow\quad M$$

by [9, Table 6.2]. The pull-back of $\mathbf{C}_{22}$ by this map is just the Dolgachev-Kondo code $\mathbf{C}_{\mathrm{DK}}$.

Remark 3.4. The codes $\mathbf{C}_A$, $\mathbf{C}_B$ and $\mathbf{C}_C$ are isomorphic to linear subcodes of $\mathbf{C}_{\mathrm{DK}}$ defined as follows. Let $F = \{Q_1, Q_2, Q_3, Q_4\}$ be a set of four points of $\mathbb{P}^2(\mathbb{F}_4)$, and let $\mathbf{C}_F$ be the 9-dimensional linear subcode of $\mathbf{C}_{\mathrm{DK}}$ defined by

$$\mathbf{C}_F := \{\ w \in \mathbf{C}_{\mathrm{DK}} \ \mid\ |w \cap F| \text{ is even }\}.$$

If no three points of F are collinear, then $\mathbf{C}_F$ is isomorphic to $\mathbf{C}_A$; if exactly one triplet of the points of F are collinear, then $\mathbf{C}_F$ is isomorphic to $\mathbf{C}_B$; while if F is on a line, then $\mathbf{C}_F$ is isomorphic to $\mathbf{C}_C$.

For $T = A, B$ and C, we will write $\mathfrak{M}_T$ instead of $\mathfrak{M}_{[\mathbf{C}_T]}$, and $\mathcal{G}_T$ instead of $\mathcal{G}_{\mathbf{C}_T}$. In the next three sections, we will construct explicitly the space

$$\mathfrak{M}_T = PGL(3,k)\backslash \mathcal{G}_T / \operatorname{Aut}(\mathbf{C}_T)$$

for $T = A, B, C$, and prove Theorems 1.6, 1.7 and 1.8 stated in Introduction. For this purpose, we have to determine the group $\operatorname{Aut}(\mathbf{C}_T)$ and the space $\mathcal{G}_T$. Since $\mathbf{C}_T$ is generated by $\mathcal{P}$ and the set of linear and quadratic words, we obtain the following:

Proposition 3.5. *Let $W_1(\mathbf{C}_T)$ and $W_2(\mathbf{C}_T)$ be the sets of linear and quadratic words in $\mathbf{C}_T$, respectively. An element σ of $\mathfrak{S}(\mathcal{P})$ is contained in* $\operatorname{Aut}(\mathbf{C}_T)$ *if and only if the following hold:*

$$\sigma(W_1(\mathbf{C}_T)) = W_1(\mathbf{C}_T) \quad \text{and} \quad \sigma(W_2(\mathbf{C}_T)) = W_2(\mathbf{C}_T).$$

Proposition 3.6. *Suppose that a map $\gamma : \mathcal{P} \to \mathbb{P}^2$ is given. Then γ is contained in $\mathcal{G}_T = \{\gamma \in \mathcal{G} \mid \mathcal{C}[\gamma] = \mathbf{C}_T\}$ if and only if the following hold:*

(i) *γ is injective,*
(ii) *there exists a homogeneous polynomial G of degree 6 such that $\gamma(\mathcal{P}) = Z(dG)$,*

(iii) *for every linear word l of $\mathbf{C}_T$, there exists a line $L \subset \mathbb{P}^2$ containing $\gamma(l)$, and*
(iv) *for every quadratic word q of $\mathbf{C}_T$, there exists a nonsingular conic curve $Q \subset \mathbb{P}^2$ containing $\gamma(q)$.*

Proof. The "only if " part is obvious from the definition of $\mathcal{G}_T$. Suppose that γ satisfies (i)-(iv). By (i) and (ii), we have $\gamma \in \mathcal{G}$. Since $\mathbf{C}_T$ is generated by the word $\mathcal{P}$ and linear and quadratic words, the properties (iii) and (iv) implies that $\mathbf{C}_T \subseteq \mathcal{C}[\gamma]$. If $\mathbf{C}_T \neq \mathcal{C}[\gamma]$, then, by Theorem 3.1, the code $\mathcal{C}[\gamma] \subset \mathrm{Pow}(\mathcal{P})$ is isomorphic to the Dolgachev-Kondo code $\mathbf{C}_{\mathrm{DK}} \subset \mathrm{Pow}(\mathbb{P}^2(\mathbb{F}_4))$ by some bijection from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4)$. Hence there exists $g \in PGL(3,k)$ such that

$$g(\gamma(\mathcal{P})) = Z(dG_{\mathrm{DK}}) = \mathbb{P}^2(\mathbb{F}_4).$$

However, there are no eight points in $\mathbb{P}^2(\mathbb{F}_4)$ that are contained in a nonsingular conic curve. □

It will turn out that, for $T = A, B$ and C, the following hold.

(1) The space $PGL(3,k)\backslash\mathcal{G}_T$ has exactly two connected components, both of which are isomorphic to $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$. Let $N_T \subset \mathrm{Aut}(\mathbf{C}_T)$ be the subgroup consisting of the elements that do not interchange the two connected components, and let Γ_T be the image of N_T in $\mathrm{Aut}(\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\})$. Then N_T is of index 2 in $\mathrm{Aut}(\mathbf{C}_T)$. The moduli curve $\mathfrak{M}_T$ is the quotient of $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$ by Γ_T.

(2) The action of Γ_T on the punctured affine line $\mathbb{A}^1\setminus\{0,1,\omega,\bar{\omega}\}$ is free. Hence the order of the stabilizer subgroup $\mathrm{Stab}([\gamma]) \subset \mathrm{Aut}(\mathbf{C}_T)$ of a point $[\gamma] \in PGL(3,k)\backslash\mathcal{G}_T$ is constant on $PGL(3,k)\backslash\mathcal{G}_T$. By Corollary 2.21, $\mathrm{Stab}([\gamma])$ is equal to $\mathrm{Aut}(X,\mathcal{L})$, where $(X,\mathcal{L})$ corresponds to the image of $[\gamma]$ in $\mathfrak{M}_T$. Hence we have an exact sequence

$$1 \;\to\; \mathrm{Aut}(X,\mathcal{L}) \;\to\; N_T \;\to\; \Gamma_T \;\to\; 1$$

for any polarized supersingular $K3$ surface $(X,\mathcal{L})$ corresponding to a point of $\mathfrak{M}_T$.

The orders of the groups above are given as follows.

T	$\lvert\mathrm{Aut}(\mathbf{C}_T)\rvert$	$=$	2	$\times$	$\lvert\Gamma_T\rvert$	$\times$	$\lvert\mathrm{Aut}(X,\mathcal{L})\rvert$
A	1152	$=$	2	$\times$	6	$\times$	96
B	432	$=$	2	$\times$	12	$\times$	18
C	23040	$=$	2	$\times$	12	$\times$	960

Remark 3.7. The following algorithm will be used frequently. Suppose that we are given eight points

$$p_i = [\xi_i, \eta_i, \zeta_i] \qquad (i = 1, \dots, 8)$$

on $\mathbb{P}^2$. In order for them to be on a (possibly singular) conic curve, it is necessary and sufficient that the 8×6 matrix

$$M := \begin{bmatrix} \xi_1^2, & \eta_1^2, & \zeta_1^2, & \xi_1\eta_1, & \eta_1\zeta_1, & \zeta_1\xi_1 \\ & & \cdots\cdots & & & \\ & & \cdots\cdots & & & \\ \xi_8^2, & \eta_8^2, & \zeta_8^2, & \xi_8\eta_8, & \eta_8\zeta_8, & \zeta_8\xi_8 \end{bmatrix}$$

is of rank < 6. When the rank of M is < 6, a non-zero solution

$${}^T[A, B, C, D, E, F]$$

of the linear equation $M\mathbf{x} = 0$ gives us a defining equation

$$AX^2 + BY^2 + CZ^2 + DXY + EYZ + FZX = 0 \tag{3.1}$$

of a conic curve containing $p_1, \dots, p_8$.

The following are phenomena peculiar to projective geometry in characteristic 2.

Remark 3.8. The conic curve defined by the equation (3.1) is singular if and only if the following holds:

$$AE^2 + BF^2 + CD^2 + DEF = 0.$$

Definition 3.9. Let $L \subset \mathbb{P}^2$ be a line, and let $Q \subset \mathbb{P}^2$ be a (possibly singular) conic curve. We say that L and Q are *tangent* if they fail to intersect at distinct two points.

Remark 3.10. Let L be a line. Then the conic curves tangent to L form a *linear* system in $|\mathcal{O}_{\mathbb{P}^2}(2)|$. If three distinct lines L_1, L_2 and L_3 are concurrent, then every conic curve that is tangent to L_1 and L_2 is tangent to L_3.

Remark 3.11. Let $A, B, C, D \in \mathbb{P}^2$ be distinct points. Suppose that no three of them are collinear. Let O (resp. P) (resp. Q) be the intersection point of the lines $\overline{AB}$ and $\overline{CD}$ (resp. $\overline{AC}$ and $\overline{BD}$) (resp. $\overline{AD}$ and $\overline{BC}$). Then O, P and Q are collinear.

4. The moduli curve corresponding to the code $\mathbf{C}_A$

In this section, we prove Theorem 1.6.

The linear words of $\mathbf{C}_A$ are listed in Table 4.1. From now on, we sometimes abbreviate, for example, the set $\{P_8, P_9, P_{12}, P_{15}, P_{19}\}$ to $\{8, 9, 12, 15, 19\}$. The linear word m stands out from the rest in that there are two points P_1 and P_2 in m through which no other linear words pass. We call m the *special linear word.* The other linear words are divided into three groups according to the intersection point with m. For $\nu = 12, 13, 18$ and $i = 1, 2, 3, 4$, the non-special linear word $l_{\nu,i}$

m	:	{	1,	2,	12,	13,	18 },
$l_{12,1}$	:	{	10,	11,	12,	16,	20 },
$l_{12,2}$	:	{	8,	9,	12,	15,	19 },
$l_{12,3}$	:	{	5,	6,	12,	14,	17 },
$l_{12,4}$	:	{	3,	4,	7,	12,	21 },
$l_{13,1}$	:	{	13,	14,	15,	16,	21 },
$l_{13,2}$	:	{	7,	8,	10,	13,	17 },
$l_{13,3}$	:	{	4,	6,	11,	13,	19 },
$l_{13,4}$	:	{	3,	5,	9,	13,	20 },
$l_{18,1}$	:	{	17,	18,	19,	20,	21 },
$l_{18,2}$	:	{	7,	9,	11,	14,	18 },
$l_{18,3}$	:	{	4,	5,	10,	15,	18 },
$l_{18,4}$	:	{	3,	6,	8,	16,	18 }.
	:						

Table 4.1. Linear words in $\mathbf{C}_A$

$\beta \setminus \alpha$	1	2	3	4
1	4	3	2	1
2	3	4	1	2
3	2	1	4	3
4	1	2	3	4

TABLE 4.2. Concurrent triples $(\alpha, \beta, \gamma(\alpha, \beta))$

$\alpha\beta\gamma$	114	123	132	141	213	224	231	242
$T_{\alpha\beta\gamma}$	P_{16}	P_{10}	P_{11}	P_{20}	P_{15}	P_8	P_{19}	P_9
$\alpha\beta\gamma$	312	321	334	343	411	422	433	444
$T_{\alpha\beta\gamma}$	P_{14}	P_{17}	P_6	P_5	P_{21}	P_7	P_4	P_3

TABLE 4.3. Points $T_{\alpha\beta\gamma}$

intersects m at the point P_ν. For each of P_1 and P_2, there exists only one linear word m containing it. For each of P_{12}, P_{13} and P_{18}, there exist exactly five linear words containing it. For each of the other 16 points, there exist exactly three linear words containing it. For each $\alpha, \beta = 1, \dots, 4$, there exists a unique $\gamma = \gamma(\alpha, \beta)$ such that the three linear words $l_{12,\alpha}$, $l_{13,\beta}$ and $l_{18,\gamma}$ have a point in common. We call such a triple (α, β, γ) a *concurrent triple*. The list of concurrent triples is given in Table 4.2. For a concurrent triple (α, β, γ), we denote by $T_{\alpha\beta\gamma}$ the intersection point of $l_{12,\alpha}$, $l_{13,\beta}$ and $l_{18,\gamma}$.

The 28 quadratic words in $\mathbf{C}_A$ are divided into two groups. The quadratic words $q'_1, \dots, q'_{12}$ listed in Table 4.4 are disjoint from the special linear word m, and intersect each of the non-special linear words $l_{\nu,i}$ at distinct two points. On the other hand, for each concurrent triple (α, β, γ), there exists a unique quadratic word $q_{\alpha\beta\gamma}$ that is disjoint from the three linear words $l_{12,\alpha}$, $l_{13,\beta}$, $l_{18,\gamma}$, and intersects other ten linear words at distinct two points. The list of these quadratic words $q_{\alpha\beta\gamma}$ is given in Table 4.5.

In order to study $\mathrm{Aut}(\mathbf{C}_A)$, we embed $\mathbf{C}_A$ into the Dolgachev-Kondo code $\mathbf{C}_{\mathrm{DK}} \subset \mathrm{Pow}(\mathbb{P}^2(\mathbb{F}_4))$ by the bijection $\phi : \mathcal{P} \xrightarrow{\sim} \mathbb{P}^2(\mathbb{F}_4)$ given in Table 4.6. The following can be checked easily.

(1) If l is a linear word of $\mathbf{C}_A$, then the points in $\phi(l)$ are collinear. The linear words of $\mathbf{C}_A$ coincide with $\phi^{-1}(\Lambda(\mathbb{F}_4))$, where Λ are $\mathbb{F}_4$-rational lines containing at least one of $\phi(P_{12})$, $\phi(P_{13})$, $\phi(P_{18})$.

(2) The words $q'_1, \dots, q'_{12}$ coincide with the words written as
$$\phi^{-1}(\Lambda_1(\mathbb{F}_4) + \Lambda_2(\mathbb{F}_4)),$$
where Λ_1 and Λ_2 are distinct $\mathbb{F}_4$-rational lines such that both of $\Lambda_1(\mathbb{F}_4)$ and $\Lambda_2(\mathbb{F}_4)$ are disjoint from $\{\phi(P_{12}), \phi(P_{13}), \phi(P_{18})\}$, and such that the intersection point of $\Lambda_1(\mathbb{F}_4)$ and $\Lambda_2(\mathbb{F}_4)$ is either $\phi(P_1)$ or $\phi(P_2)$.

(3) For a concurrent triple (α, β, γ), let Λ_i be the $\mathbb{F}_4$-rational line passing through $\phi(T_{\alpha\beta\gamma})$ and $\phi(P_i)$ for $i = 1, 2$. Then we have $q_{\alpha\beta\gamma} = \phi^{-1}(\Lambda_1(\mathbb{F}_4) + \Lambda_2(\mathbb{F}_4))$.

q'_1	:	{	5,	6,	7,	9,	10,	16,	19,	21 },
q'_2	:	{	5,	6,	7,	8,	11,	15,	20,	21 },
q'_3	:	{	4,	6,	8,	9,	10,	14,	20,	21 },
q'_4	:	{	4,	6,	7,	9,	15,	16,	17,	20 },
q'_5	:	{	4,	5,	8,	9,	11,	16,	17,	21 },
q'_6	:	{	4,	5,	7,	8,	14,	16,	19,	20 },
q'_7	:	{	3,	6,	9,	10,	11,	15,	17,	21 },
q'_8	:	{	3,	6,	7,	10,	14,	15,	19,	20 },
q'_9	:	{	3,	5,	8,	10,	11,	14,	19,	21 },
q'_{10}	:	{	3,	5,	7,	11,	15,	16,	17,	19 },
q'_{11}	:	{	3,	4,	9,	10,	14,	16,	17,	19 },
q'_{12}	:	{	3,	4,	8,	11,	14,	15,	17,	20 }.

TABLE 4.4. Quadratic words q'_ν in $\mathbf{C}_A$

q_{114}	:	{	1,	2,	4,	5,	7,	9,	17,	19 },
q_{123}	:	{	1,	2,	3,	6,	9,	14,	19,	21 },
q_{132}	:	{	1,	2,	3,	5,	8,	15,	17,	21 },
q_{141}	:	{	1,	2,	4,	6,	7,	8,	14,	15 },
q_{213}	:	{	1,	2,	3,	6,	7,	11,	17,	20 },
q_{224}	:	{	1,	2,	4,	5,	11,	14,	20,	21 },
q_{231}	:	{	1,	2,	3,	5,	7,	10,	14,	16 },
q_{242}	:	{	1,	2,	4,	6,	10,	16,	17,	21 },
q_{312}	:	{	1,	2,	3,	4,	8,	10,	19,	20 },
q_{321}	:	{	1,	2,	3,	4,	9,	11,	15,	16 },
q_{334}	:	{	1,	2,	7,	9,	10,	15,	20,	21 },
q_{343}	:	{	1,	2,	7,	8,	11,	16,	19,	21 },
q_{411}	:	{	1,	2,	5,	6,	8,	9,	10,	11 },
q_{422}	:	{	1,	2,	5,	6,	15,	16,	19,	20 },
q_{433}	:	{	1,	2,	8,	9,	14,	16,	17,	20 },
q_{444}	:	{	1,	2,	10,	11,	14,	15,	17,	19 }.

TABLE 4.5. Quadratic words $q_{\alpha\beta\gamma}$ in $\mathbf{C}_A$

Let PG' be the subgroup of $PGL(3, \mathbb{F}_4)$ consisting of $g \in PGL(3, \mathbb{F}_4)$ satisfying

$$\{g(\phi(P_{12})), g(\phi(P_{13})), g(\phi(P_{18}))\} = \{\phi(P_{12}), \phi(P_{13}), \phi(P_{18})\},$$

and let PG be the subgroup $\phi^{-1} \circ PG' \circ \phi$ of $\mathfrak{S}(\mathcal{P})$. The order of PG is 288. Let $F' \in \mathfrak{S}(\mathbb{P}^2(\mathbb{F}_4))$ be the element of order 2 obtained by the conjugation $\omega \mapsto \bar{\omega}$ of $\mathbb{F}_4$ over $\mathbb{F}_2$. We then put

$$F := \phi^{-1} \circ F' \circ \phi = (P_1P_2)(P_3P_5)(P_4P_6)(P_7P_{14})(P_8P_{15})(P_{10}P_{16})(P_{17}P_{21}) \in \mathfrak{S}(\mathcal{P}).$$

We also put

$$T := (P_1P_2).$$

$$
\begin{array}{rclcrcl}
\phi(P_1) &=& [1,\omega,0], && \phi(P_{12}) &=& [0,1,0],\\
\phi(P_2) &=& [1,\bar{\omega},0], && \phi(P_{13}) &=& [1,1,0],\\
\phi(P_3) &=& [1,1,\omega], && \phi(P_{14}) &=& [1,\bar{\omega},\bar{\omega}],\\
\phi(P_4) &=& [1,\bar{\omega},\omega], && \phi(P_{15}) &=& [1,\omega,1],\\
\phi(P_5) &=& [1,1,\bar{\omega}], && \phi(P_{16}) &=& [0,1,\omega],\\
\phi(P_6) &=& [1,\omega,\bar{\omega}], && \phi(P_{17}) &=& [1,0,\bar{\omega}],\\
\phi(P_7) &=& [1,\omega,\omega], && \phi(P_{18}) &=& [1,0,0],\\
\phi(P_8) &=& [1,\bar{\omega},1], && \phi(P_{19}) &=& [1,0,1],\\
\phi(P_9) &=& [1,1,1], && \phi(P_{20}) &=& [0,0,1],\\
\phi(P_{10}) &=& [0,1,\bar{\omega}], && \phi(P_{21}) &=& [1,0,\omega].\\
\phi(P_{11}) &=& [0,1,1], && &&
\end{array}
$$

TABLE 4.6. Bijection ϕ from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4)$

Proposition 4.1. *The group* $\mathrm{Aut}(\mathbf{C}_A)$ *is of order* 1152*, and is generated by* PG*,* F *and* T*.*

Proof. Since the actions of PG' and F' on $\mathbb{P}^2(\mathbb{F}_4)$ leave the set

$$\{[0,1,0],[1,1,0],[1,0,0]\} = \{\phi(P_{12}),\phi(P_{13}),\phi(P_{18})\}$$

invariant, and preserve the line-point incidence configuration, we see that $PG \subset \mathrm{Aut}(\mathbf{C}_A)$ and $F \in \mathrm{Aut}(\mathbf{C}_A)$. It is obvious that $T \in \mathrm{Aut}(\mathbf{C}_A)$. By direct calculation, we see that the subgroup of $\mathfrak{S}(\mathcal{P})$ generated by PG, F and T is of order 1152.

Every automorphism of $\mathbf{C}_A$ leaves each of the sets $\{P_1,P_2\}$ and $\{P_{12},P_{13},P_{18}\}$ invariant. Hence we have a homomorphism

$$\mathrm{Aut}(\mathbf{C}_A) \;\to\; \mathfrak{S}(\{P_1,P_2\})\times\mathfrak{S}(\{P_{12},P_{13},P_{18}\}). \tag{4.1}$$

Since PG acts on $\{P_{12},P_{13},P_{18}\}$ as the full-symmetric group, and since T is contained in $\mathrm{Aut}(\mathbf{C}_A)$, the homomorphism (4.1) is surjective. Let K denote the kernel of (4.1). We have a homomorphism

$$K \;\to\; \mathfrak{S}_4\times\mathfrak{S}_4, \qquad g \;\mapsto\; (\sigma,\sigma'), \tag{4.2}$$

where σ and σ' are given by

$$g(l_{12,\alpha}) = l_{12,\sigma(\alpha)}, \quad g(l_{13,\beta}) = l_{13,\sigma'(\beta)}.$$

We also have a homomorphism

$$\mathfrak{S}_4\times\mathfrak{S}_4 \;\to\; \mathfrak{S}(\mathcal{P}), \qquad (\sigma,\sigma') \;\mapsto\; g_{\sigma,\sigma'}, \tag{4.3}$$

where $g_{\sigma,\sigma'}$ is given by

$$
\begin{array}{rcl}
g_{\sigma,\sigma'}(P_i) &=& P_i \qquad \text{if } P_i \in m,\\
g_{\sigma,\sigma'}(T_{\alpha\beta\gamma}) &=& T_{\sigma(\alpha)\sigma'(\beta)\gamma'}\\
&& \text{where } (\alpha,\beta,\gamma) \text{ and } (\sigma(\alpha),\sigma'(\beta),\gamma') \text{ are concurrent triples.}
\end{array}
$$

$\gamma_\lambda(P_1)$	$=$	$[1,\omega,0]$,	$\gamma_\lambda(P_{12})$	$=$	$[0,1,0]$,
$\gamma_\lambda(P_2)$	$=$	$[1,\bar\omega,0]$,	$\gamma_\lambda(P_{13})$	$=$	$[1,1,0]$,
$\gamma_\lambda(P_3)$	$=$	$[1+\lambda,1+\lambda,1]$,	$\gamma_\lambda(P_{14})$	$=$	$[\lambda,1,1]$,
$\gamma_\lambda(P_4)$	$=$	$[1+\lambda,\lambda,1]$,	$\gamma_\lambda(P_{15})$	$=$	$[1,\lambda,1]$,
$\gamma_\lambda(P_5)$	$=$	$[\lambda,\lambda,1]$,	$\gamma_\lambda(P_{16})$	$=$	$[0,1+\lambda,1]$,
$\gamma_\lambda(P_6)$	$=$	$[\lambda,1+\lambda,1]$,	$\gamma_\lambda(P_{17})$	$=$	$[\lambda,0,1]$,
$\gamma_\lambda(P_7)$	$=$	$[1+\lambda,1,1]$,	$\gamma_\lambda(P_{18})$	$=$	$[1,0,0]$,
$\gamma_\lambda(P_8)$	$=$	$[1,1+\lambda,1]$,	$\gamma_\lambda(P_{19})$	$=$	$[1,0,1]$,
$\gamma_\lambda(P_9)$	$=$	$[1,1,1]$,	$\gamma_\lambda(P_{20})$	$=$	$[0,0,1]$,
$\gamma_\lambda(P_{10})$	$=$	$[0,\lambda,1]$,	$\gamma_\lambda(P_{21})$	$=$	$[1+\lambda,0,1]$.
$\gamma_\lambda(P_{11})$	$=$	$[0,1,1]$,			

TABLE 4.7. Definition of γ_λ for $\mathbf{C}_A$

Since the composite of (4.2) and (4.3) is the identity of K, the homomorphism (4.2) is injective. For each pair (σ,σ') of $\mathfrak{S}_4\times\mathfrak{S}_4$, we check whether $g_{\sigma,\sigma'}$ is in $\mathrm{Aut}(\mathbf{C}_A)$; that is, whether $g_{\sigma,\sigma'}$ satisfies the following (see Proposition 3.5):

$$g_{\sigma,\sigma'}(W_1(\mathbf{C}_A))=W_1(\mathbf{C}_A) \quad\text{and}\quad g_{\sigma,\sigma'}(W_2(\mathbf{C}_A))=W_2(\mathbf{C}_A). \tag{4.4}$$

Among $(4!)^2=576$ pairs, exactly 96 pairs satisfy (4.4). Hence $\mathrm{Aut}(\mathbf{C}_A)$ is of order $|K||\mathfrak{S}_2||\mathfrak{S}_3|=96\cdot 12=1152$, and is generated by PG, F and T. □

For a parameter λ of the affine line $\mathbb{A}^1$, we define a map

$$\gamma_\lambda\;:\;\mathcal{P}\;\to\;\mathbb{P}^2$$

by Table 4.7. Note that γ_ω coincides with ϕ defined above. We denote by $\widetilde{T}$ the subgroup $\{1,T\}$ of $\mathrm{Aut}(\mathbf{C}_A)$.

Proposition 4.2. *The map $\lambda\mapsto\gamma_\lambda$ induces an isomorphism from $\mathbb{A}^1\setminus\{0,1,\omega,\bar\omega\}$ to $PGL(3,k)\backslash\mathcal{G}_A/\widetilde{T}$.*

Proof. First note that γ_λ is injective if and only if

$$\lambda\neq 0\quad\text{and}\quad\lambda\neq 1. \tag{4.5}$$

From now on, we assume (4.5).

We will show the following:

Claim 4.3. Let γ' be an arbitrary element of $\mathcal{G}_A$. Then there exists a unique triple

$$(g,t,\lambda)\;\in\;PGL(3,k)\times\widetilde{T}\times(k\setminus\{0,1,\omega,\bar\omega\})$$

such that

$$g\circ\gamma'\circ t=\gamma_\lambda.$$

Because the points $\gamma'(P_{18})$, $\gamma'(P_{19})$, $\gamma'(P_{20})$ of $\gamma'(l_{18,1})$ are on a line and the points $\gamma'(P_{12})$, $\gamma'(P_{13})$, $\gamma'(P_{18})$ of $\gamma'(m)$ are on another line, there exists a unique element $g \in PGL(3,k)$ such that $\gamma := g \circ \gamma'$ satisfies the following:

$$
\begin{aligned}
&\gamma(P_{18}) = [1,0,0],\\
&\gamma(P_{12}) = [0,1,0],\ \gamma(P_{13}) = [1,1,0],\\
&\gamma(P_{20}) = [0,0,1],\ \gamma(P_{19}) = [1,0,1].
\end{aligned}
\tag{4.6}
$$

Let $L_{12,\alpha}$, $L_{13,\beta}$ and $L_{18,\gamma}$ be the lines containing $\gamma(l_{12,\alpha})$, $\gamma(l_{13,\beta})$ and $\gamma(l_{18,\gamma})$, respectively. We put $x := X/Z$, $y := Y/Z$. Then the defining equations of these lines can be written as follows:

$$
\begin{aligned}
L_{12,\alpha} &: x + a_\alpha = 0,\\
L_{13,\beta} &: x + y + b_\beta = 0,\\
L_{18,\gamma} &: y + c_\gamma = 0.
\end{aligned}
\tag{4.7}
$$

From (4.6), we have

$$
a_1 = 0, \quad a_2 = 1, \quad b_3 = 1, \quad b_4 = 0, \quad c_1 = 0. \tag{4.8}
$$

The condition that (α, β, γ) is a concurrent triple is equivalent to

$$
a_\alpha + b_\beta + c_\gamma = 0.
$$

Solving the linear equations corresponding to the 16 concurrent triples and combining the result with (4.8), we obtain the following solutions:

$$
\begin{aligned}
(a_1, a_2, a_3, a_4) &= (0, 1, \lambda, 1+\lambda),\\
(b_1, b_2, b_3, b_4) &= (1+\lambda, \lambda, 1, 0),\\
(c_1, c_2, c_3, c_4) &= (0, 1, \lambda, 1+\lambda),
\end{aligned}
\tag{4.9}
$$

where λ is a parameter. The coordinates of the points $T_{\alpha\beta\gamma}$ are given by $[a_\alpha, c_\gamma, 1]$. Using Table 4.3, we see that $\gamma(P_i) = \gamma_\lambda(P_i)$ holds for every i except for $i = 1$ and $i = 2$. The line M containing $\gamma(m)$ is defined by $Z = 0$. Hence we can put

$$
\gamma(P_1) = [1, \tau_1, 0], \quad \gamma(P_2) = [1, \tau_2, 0].
$$

By the algorithm in Remark 3.7, we see that a conic curve containing $\gamma(q_{114})$ exists if and only if the following hold:

$$
\begin{aligned}
&\left(1 + \tau_2 + {\tau_2}^2\right)(\lambda+1)^2 \lambda^2 = 0,\\
&(\tau_1 + \tau_2)\left(1 + \tau_2 + {\tau_2}^2\right)(\lambda+1)\lambda = 0,\\
&(\tau_1 + \tau_2)(\tau_1 + \tau_2 + 1)(\lambda+1)\lambda = 0.
\end{aligned}
\tag{4.10}
$$

Here we have used the Buchberger algorithm to calculate the Gröbner basis of the ideal in $k[\lambda, \tau_1, \tau_2]$ generated by 6×6-minors of the 8×6-matrix corresponding to the eight points in $\gamma(q_{114})$. Replacing γ by $\gamma \circ T$ if necessary, we have

$$
\tau_1 = \omega \quad \text{and} \quad \tau_2 = \bar{\omega}
$$

by (4.5), (4.10) and $\tau_1 \neq \tau_2$. Then the conic curve containing $\gamma(q_{114})$ is defined by

$$
X^2 + Y^2 + \lambda Z^2 + XY + (\lambda+1)ZX = 0,
$$

which is nonsingular if and only if $\lambda^2 + \lambda + 1 \neq 0$. (See Remark 3.8.) Thus we have proved the existence and the uniqueness of the triple (g, t, λ) satisfying $g \circ \gamma' \circ t = \gamma_\lambda$. In particular, for each double coset in $PGL(3,k)\backslash\mathcal{G}_A/\widetilde{T}$, there exists a unique $\lambda \in k \setminus \{0, 1, \omega, \bar{\omega}\}$ such that γ_λ is contained in the coset.

$$\begin{array}{rcl}
Q'_1 & : & \lambda X^2+Y^2+\left(\lambda^2+\lambda\right)Z^2+YZ+\lambda^2 ZX=0,\\
Q'_2 & : & (\lambda+1)X^2+Y^2+YZ+(\lambda^2+1)ZX=0,\\
Q'_3 & : & (\lambda+1)X^2+\lambda Y^2+\lambda^2 YZ+(\lambda^2+1)ZX=0,\\
Q'_4 & : & \lambda X^2+(\lambda+1)Y^2+\left(\lambda^2+1\right)YZ+\lambda^2 ZX=0,\\
Q'_5 & : & X^2+\lambda Y^2+\left(\lambda^2+\lambda\right)Z^2+\lambda^2 YZ+ZX=0,\\
Q'_6 & : & X^2+(\lambda+1)Y^2+\left(\lambda^2+1\right)YZ+ZX=0,\\
Q'_7 & : & X^2+(\lambda+1)Y^2+\left(\lambda^2+\lambda\right)Z^2+\left(\lambda^2+1\right)YZ+ZX=0,\\
Q'_8 & : & X^2+\lambda Y^2+\lambda^2 YZ+ZX=0,\\
Q'_9 & : & \lambda X^2+(\lambda+1)Y^2+\left(\lambda^2+\lambda\right)Z^2+\left(\lambda^2+1\right)YZ+\lambda^2 ZX=0,\\
Q'_{10} & : & (\lambda+1)X^2+\lambda Y^2+(\lambda^2+\lambda)Z^2+\lambda^2 YZ+(\lambda^2+1)ZX=0,\\
Q'_{11} & : & (\lambda+1)X^2+Y^2+(\lambda^2+\lambda)Z^2+YZ+(\lambda^2+1)ZX=0,\\
Q'_{12} & : & \lambda X^2+Y^2+YZ+\lambda^2 ZX=0.
\end{array}$$

TABLE 4.8. Defining equations of the conic curves Q'_i

Conversely, let λ be an element of $k\setminus\{0,1,\omega,\bar\omega\}$. We will show that γ_λ is in $\mathcal{G}_A$. The points $\gamma_\lambda(\mathcal{P})$ coincides with $Z(dGA[\lambda])$, where $GA[\lambda]$ is given in Theorem 1.6. Indeed, we can check that

$$\frac{\partial\, GA[\lambda]}{\partial X}(\gamma_\lambda(P_i))=\frac{\partial\, GA[\lambda]}{\partial Y}(\gamma_\lambda(P_i))=\frac{\partial\, GA[\lambda]}{\partial Z}(\gamma_\lambda(P_i))=0$$

holds for $i=1,\dots,21$. For each linear word l of $\mathbf{C}_A$, there exists a line containing $\gamma_\lambda(l)$. The defining equations of them are given by (4.7) and (4.9). (The line M containing $\gamma_\lambda(m)$ is defined by $Z=0$.) For each quadratic word q'_i of $\mathbf{C}_A$ (resp. $q_{\alpha\beta\gamma}$), there exists a nonsingular conic curve Q'_i (resp. $Q_{\alpha\beta\gamma}$) containing $\gamma_\lambda(q'_i)$ (resp. $\gamma_\lambda(q_{\alpha\beta\gamma})$). The defining equations of them are given in Tables 4.8 and 4.9. Hence $\gamma_\lambda\in\mathcal{G}_A$ by Proposition 3.6. □

Remark 4.4. The polynomial $GA[\lambda]$ defines the nodal splitting curve

$$M\cup L_{12,1}\cup L_{18,1}\cup L_{13,3}\cup Q_{242}.$$

See Proposition 2.8.

Remark 4.5. When $\lambda\in\{\omega,\bar\omega\}$, the set $\gamma_\lambda(\mathcal{P})$ coincides with $\mathbb{P}^2(\mathbb{F}_4)$, and the point $[GA[\lambda]]\in\mathfrak{M}$ is the Dolgachev-Kondo point.

Let $k(\lambda)$ be the rational function field with variable λ. For each $\sigma\in\mathrm{Aut}(\mathbf{C}_A)$, we calculate the unique triple

$$(g_\sigma,t_\sigma,\lambda^\sigma)\ \in\ PGL(3,k(\lambda))\times\widetilde{T}\times k(\lambda)$$

such that

$$g_\sigma\circ(\gamma_\lambda\circ\sigma)\circ t_\sigma=\gamma_{\lambda^\sigma}$$

$$
\begin{array}{lcl}
Q_{114} & : & X^2+Y^2+XY+\lambda\, Z^2+(\lambda+1)\, ZX=0,\\
Q_{123} & : & X^2+Y^2+XY+(\lambda+1)\, Z^2+\lambda\, ZX=0,\\
Q_{132} & : & X^2+Y^2+XY+\left(\lambda^2+\lambda\right) Z^2+ZX=0,\\
Q_{141} & : & X^2+Y^2+XY+\left(\lambda^2+\lambda+1\right) Z^2=0,\\
Q_{213} & : & X^2+Y^2+XY+YZ+\lambda\, ZX=0,\\
Q_{224} & : & X^2+Y^2+XY+YZ+(\lambda+1)\, ZX=0,\\
Q_{231} & : & X^2+Y^2+XY+\left(\lambda^2+\lambda\right) Z^2+YZ=0,\\
Q_{242} & : & X^2+Y^2+XY+\left(\lambda^2+\lambda\right) Z^2+YZ+ZX=0,\\
Q_{312} & : & X^2+Y^2+XY+\lambda\, YZ+ZX=0,\\
Q_{321} & : & X^2+Y^2+XY+(\lambda+1)\, Z^2+\lambda\, YZ=0,\\
Q_{334} & : & X^2+Y^2+XY+\lambda\, YZ+(\lambda+1)\, ZX=0,\\
Q_{343} & : & X^2+Y^2+XY+(\lambda+1)\, Z^2+\lambda\, YZ+\lambda\, ZX=0,\\
Q_{411} & : & X^2+Y^2+XY+\lambda\, Z^2+(\lambda+1)\, YZ=0,\\
Q_{422} & : & X^2+Y^2+XY+(\lambda+1)\, YZ+ZX=0,\\
Q_{433} & : & X^2+Y^2+XY+(\lambda+1)\, YZ+\lambda\, ZX=0,\\
Q_{444} & : & X^2+Y^2+XY+\lambda\, Z^2+(\lambda+1)\, YZ+(\lambda+1)\, ZX=0.
\end{array}
$$

TABLE 4.9. Defining equations of the conic curves $Q_{\alpha\beta\gamma}$

holds (see Claim 4.3.) The calculation is done as follows: g_σ is the unique linear automorphism of $\mathbb{P}^2$ characterized by

$$
\begin{array}{ll}
g_\sigma(\gamma_\lambda(\sigma(P_{18})))=[1,0,0], & \\
g_\sigma(\gamma_\lambda(\sigma(P_{12})))=[0,1,0], & g_\sigma(\gamma_\lambda(\sigma(P_{13})))=[1,1,0] \qquad \text{and}\\
g_\sigma(\gamma_\lambda(\sigma(P_{20})))=[0,0,1], & g_\sigma(\gamma_\lambda(\sigma(P_{19})))=[1,0,1];
\end{array}
$$

$t_\sigma \in \widetilde{T}$ is given by

$$
t_\sigma = \begin{cases} \mathrm{id} & \text{if } g_\sigma(\gamma_\lambda(\sigma(P_1)))=[1,\omega,0],\\ T & \text{if } g_\sigma(\gamma_\lambda(\sigma(P_1)))=[1,\bar{\omega},0]; \end{cases}
$$

and λ^σ is the rational function of the parameter λ satisfying

$$
g_\sigma(\gamma_\lambda(\sigma(P_{10})))=[0,\lambda^\sigma,1].
$$

The map $\sigma\mapsto t_\sigma$ is a homomorphism from $\mathrm{Aut}(\mathbf{C}_A)$ to $\widetilde{T}$. We put

$$
N_A := \mathrm{Ker}(\mathrm{Aut}(\mathbf{C}_A)\to\widetilde{T}).
$$

From the proof of Proposition 4.2, we obtain the following:

Corollary 4.6. *The space $PGL(3,k)\backslash\mathcal{G}_A$ has exactly two connected components, each of which is isomorphic to $\mathbb{A}^1\setminus\{0,1,\omega,\bar{\omega}\}$. Set-theoretically, they are given by*

$$
\begin{array}{lcl}
(PGL(3,k)\backslash\mathcal{G}_A)^+ & := & \{\ [\gamma_\alpha]\ \mid\ \alpha\in k\setminus\{0,1,\omega,\bar{\omega}\}\ \}, \quad \textit{and}\\
(PGL(3,k)\backslash\mathcal{G}_A)^- & := & \{\ [\gamma_\alpha\circ T]\ \mid\ \alpha\in k\setminus\{0,1,\omega,\bar{\omega}\}\ \}.
\end{array}
$$

The group N_A acts on $(PGL(3,k)\backslash\mathcal{G}_A)^+$, and the moduli curve $\mathfrak{M}_A$ is equal to the quotient space $(PGL(3,k)\backslash\mathcal{G}_A)^+/N_A$.

Let

$$p_A : \mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\} \cong (PGL(3,k)\backslash\mathcal{G}_A)^+ \to \mathfrak{M}_A = (PGL(3,k)\backslash\mathcal{G}_A)^+/N_A$$

denote the natural projection. For $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$, let $P[\alpha]$ be the point of $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$ given by $\lambda = \alpha$. Then $p_A(P[\alpha]) \in \mathfrak{M}_A$ corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$.

Proposition 4.7. *The set* $p_A^{-1}(p_A(P[\alpha]))$ *is equal to*

$$\{\, P[\alpha],\ P[1/\alpha],\ P[\alpha+1],\ P[1/(\alpha+1)],\ P[\alpha/(\alpha+1)],\ P[(\alpha+1)/\alpha] \,\}, \tag{4.11}$$

and $\mathrm{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ *is equal to the group* (1.1).

Proof. The set $\{\lambda^\sigma \mid \sigma \in N_A\} \subset k(\lambda)$ coincides with the group Γ_A given in Theorem 1.6. The fiber $p_A^{-1}(p_A(P[\alpha]))$ is therefore equal to (4.11). Note that the fiber $p_A^{-1}(p_A(P[\alpha]))$ consists of six distinct points for any $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$; that is, the action of Γ_A on $(PGL(3,k)\backslash\mathcal{G}_A)^+$ is free. Hence, for any $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$ and any $\sigma \in \mathrm{Aut}(\mathbf{C}_A)$, the projective equivalence classes $[\gamma_\alpha]$ and

$$[\gamma_\alpha \circ \sigma] = [\gamma_{\alpha^\sigma} \circ t_\sigma] \ \in\ PGL(3,k)\backslash\mathcal{G}_C$$

coincide if and only if $t_\sigma = \mathrm{id}$ and $\lambda^\sigma = \lambda$ hold. Therefore, using Corollary 2.21, we can obtain $\mathrm{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ from the subgroup

$$\{\ g_\sigma \ \mid\ t_\sigma = \mathrm{id} \quad \text{and} \quad \lambda^\sigma = \lambda\ \} \ \subset\ PGL(3, k(\lambda))$$

by substituting α for λ. □

Corollary 4.8. *We have* $\mathfrak{M}_A = \mathrm{Spec}\, k[J_A, 1/J_A]$*, where* $J_A = (\lambda^2+\lambda+1)/\lambda^2(\lambda+1)^2$*. The morphism* p_A *is an étale Galois covering with Galois group* Γ_A*, which is isomorphic to* $\mathfrak{S}_3$*.*

5. The moduli curve corresponding to the code $\mathbf{C}_B$

In this section, we prove Theorem 1.7.

Let AF be the affine plane over $\mathbb{F}_3$, $P(AF)$ the set of rational points of AF, and $L(AF)$ the set of rational affine lines of AF. Each element of $P(AF)$ is expressed by a pair aa' of elements of $\mathbb{F}_3$, and each element of $L(AF)$ is expressed as a subset $\{aa', bb', cc'\}$ of $P(AF)$ with cardinality 3. We have

$$|P(AF)| = 9 \quad \text{and} \quad |L(AF)| = 12.$$

The incidence relation

$$\{\ (p,\ell) \in P(AF) \times L(AF) \ \mid\ p \in \ell\ \}$$

is called the *Hesse configuration* ([4]). The automorphism group

$$G_{\mathrm{Hesse}} := \{\ \sigma \in \mathfrak{S}(P(AF)) \ \mid\ \sigma(\ell) \in L(AF) \ \text{ for all } \ell \in L(AF)\ \}$$

of this configuration is isomorphic to the group of affine transformations of AF defined over $\mathbb{F}_3$. In particular, the order of G_{Hesse} is 432.

We define injective maps

$$C : P(AF) \to \mathcal{P} \quad \text{and} \quad T : L(AF) \to \mathcal{P}$$

by Table 5.1. Then $\mathcal{P}$ is a disjoint union of $C(P(AF))$ and $T(L(AF))$. A point $P \in \mathcal{P}$ is called a *C-point* or a *T-point* according to whether $P \in C(P(AF))$ or $P \in T(L(AF))$. The code $\mathbf{C}_B$ is described as follows.

aa'	00	01	02	10	11	12	20	21	22
$C(aa')$	P_{17}	P_{13}	P_5	P_{10}	P_8	P_6	P_2	P_3	P_1

aa'	bb'	cc'	$T(\ell)$
00	01	02	P_{21}
00	10	20	P_{20}
00	11	22	P_{19}
00	12	21	P_{18}
01	10	22	P_{16}
01	11	21	P_{15}

aa'	bb'	cc'	$T(\ell)$
01	12	20	P_{14}
02	10	21	P_{11}
02	11	20	P_9
02	12	22	P_7
10	11	12	P_{12}
20	21	22	P_4

TABLE 5.1. C-points $C(aa')$ and T-points $T(\ell)$ for $\ell = \{aa', bb', cc'\}$

The linear words of $\mathbf{C}_B$ are precisely the words

$$l_{aa'} := \{C(aa'), T(\ell_1), T(\ell_2), T(\ell_3), T(\ell_4)\} \qquad (aa' \in P(AF)),$$

where $\ell_1, \dots, \ell_4 \in L(AF)$ are the four affine lines passing through the point $aa' \in P(AF)$.

There are two types of quadratic words.

(I) Let $\ell = \{aa', bb', cc'\}$ be an element of $L(AF)$. There exists a unique pair of distinct affine lines

$$\ell_1 = \{a_1a_1', b_1b_1', c_1c_1'\} \neq \ell, \qquad \ell_2 = \{a_2a_2', b_2b_2', c_2c_2'\} \neq \ell$$

that are parallel to ℓ. Then the word

$$q_\ell := \{C(a_1a_1'), C(b_1b_1'), C(c_1c_1'), C(a_2a_2'), C(b_2b_2'), C(c_2c_2'), T(\ell_1), T(\ell_2)\}$$

is a quadratic word of $\mathbf{C}_B$.

(II) Let ℓ_1 and ℓ_2 be two distinct elements of $L(AF)$ that are not parallel, and let $aa' \in P(AF)$ be the intersection point of ℓ_1 and ℓ_2. Then there exists a pair $\{m, n\}$ of elements of $L(AF)$ with the following properties:

(i) m and n are parallel,
(ii) $aa' \notin m$, $aa' \notin n$, and
(iii) none of the pairs (ℓ_1, m), (ℓ_2, m), (ℓ_1, n), (ℓ_2, n) are parallel.

For such a pair $\{m, n\}$, there exists a unique line $\ell' \in L(AF)$ such that

(a) $aa' \in \ell'$,
(b) is distinct from ℓ_1 and ℓ_2, and
(c) intersects both of m and n.

We denote the intersection points of these affine lines as in Figure 5.1. Then the word

$$q'_{\ell_1,\ell_2} := \{C(M_1), C(M_2), C(N_1), C(N_2), T(M_1N'), T(M_2N'), T(N_1M'), T(N_2M')\}$$

is a quadratic word of $\mathbf{C}_B$, where $MN \in L(AF)$ denotes the affine line containing the points M and N. For each (ℓ_1, ℓ_2), there exist exactly two pairs satisfying (i), (ii) and (iii). However, the word q'_{ℓ_1,ℓ_2} is independent of the choice of the pair.

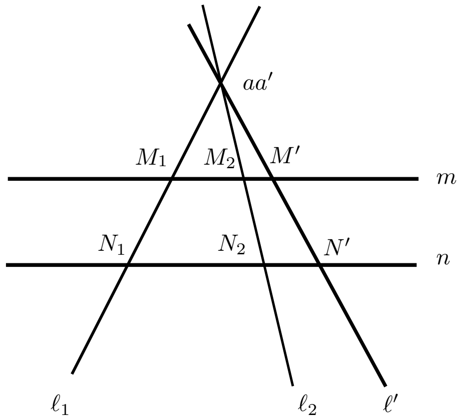

FIGURE 5.1. Intersection points

$$
\begin{array}{llllllll}
l_{00} & : & \{ & 17, & 18, & 19, & 20, & 21 \ \}, \\
l_{01} & : & \{ & 13, & 14, & 15, & 16, & 21 \ \}, \\
l_{02} & : & \{ & 5, & 7, & 9, & 11, & 21 \ \}, \\
l_{10} & : & \{ & 10, & 11, & 12, & 16, & 20 \ \}, \\
l_{11} & : & \{ & 8, & 9, & 12, & 15, & 19 \ \}, \\
l_{12} & : & \{ & 6, & 7, & 12, & 14, & 18 \ \}, \\
l_{20} & : & \{ & 2, & 4, & 9, & 14, & 20 \ \}, \\
l_{21} & : & \{ & 3, & 4, & 11, & 15, & 18 \ \}, \\
l_{22} & : & \{ & 1, & 4, & 7, & 16, & 19 \ \}.
\end{array}
$$

TABLE 5.2. Linear words of $\mathbf{C}_B$

There exist 12 quadratic words of type I, and 54 quadratic words of type II. The quadratic words of $\mathbf{C}_B$ are precisely these 66 words. The linear and quadratic words of $\mathbf{C}_B$ are explicitly presented in Tables 5.2, 5.3 and 5.4.

The following proposition can be checked easily:

Proposition 5.1. *Let $\ell = \{aa', bb', cc'\}$ be an element of $L(AF)$. Then the quadratic word q_ℓ of type* I *is disjoint from the three linear words $l_{aa'}, l_{bb'}, l_{cc'}$ containing $T(\ell) \in \mathcal{P}$.*

We define a homomorphism

$$\Psi : G_{\mathrm{Hesse}} \to \mathfrak{S}(\mathcal{P})$$

by

$$\Psi(g)(C(aa')) := C(g(aa')) \qquad \Psi(g)(T(\ell)) := T(g(\ell)).$$

It is obvious that Ψ is injective.

Proposition 5.2. *The automorphism group* $\mathrm{Aut}(\mathbf{C}_B)$ *of the code* $\mathbf{C}_B$ *coincides with the image of* Ψ*.*

Proof. The above description of the linear and quadratic words in $\mathbf{C}_B$ shows that every element in the image of Ψ preserves the sets of these words. Since $\mathbf{C}_B$ is

$q_{00,01,02}$	:	{	1,	2,	3,	4,	6,	8,	10,	12	},
$q_{00,10,20}$	:	{	1,	3,	5,	6,	7,	8,	13,	15	},
$q_{00,11,22}$	:	{	2,	3,	5,	6,	10,	11,	13,	14	},
$q_{00,12,21}$	:	{	1,	2,	5,	8,	9,	10,	13,	16	},
$q_{01,10,22}$	:	{	2,	3,	5,	6,	8,	9,	17,	18	},
$q_{01,11,21}$	:	{	1,	2,	5,	6,	7,	10,	17,	20	},
$q_{01,12,20}$	:	{	1,	3,	5,	8,	10,	11,	17,	19	},
$q_{02,10,21}$	:	{	1,	2,	6,	8,	13,	14,	17,	19	},
$q_{02,11,20}$	:	{	1,	3,	6,	10,	13,	16,	17,	18	},
$q_{02,12,22}$	:	{	2,	3,	8,	10,	13,	15,	17,	20	},
$q_{10,11,12}$	:	{	1,	2,	3,	4,	5,	13,	17,	21	},
$q_{20,21,22}$	:	{	5,	6,	8,	10,	12,	13,	17,	21	}.

TABLE 5.3. Quadratic words of type I in $\mathbf{C}_B$

generated by the word $\mathcal{P} \in \mathrm{Pow}(\mathcal{P})$ and these words, the image of Ψ is contained in $\mathrm{Aut}(\mathbf{C}_B)$.

Suppose that $\sigma \in \mathrm{Aut}(\mathbf{C}_B)$ is given. A point $P \in \mathcal{P}$ is a C-point if and only if there exists exactly one linear word in $\mathbf{C}_B$ that contains P. Hence σ preserves the set of C-points. Via the bijection $C : P(AF) \cong \mathrm{Im}\, C$, we obtain a unique element $\tilde{\sigma} \in \mathfrak{S}(P(AF))$ such that $\sigma \circ C = C \circ \tilde{\sigma}$ holds. When $P = C(aa')$, the unique linear word in $\mathbf{C}_B$ containing P is just $l_{aa'}$. The Hesse configuration on AF is recovered from $\mathbf{C}_B$ as follows; a set $\{aa', bb', cc'\}$ of cardinality 3 is an element of $L(AF)$ if and only if the three linear words $l_{aa'}$, $l_{bb'}$, $l_{cc'}$ have a point in common. In this case, the common point of $l_{aa'}$, $l_{bb'}$, $l_{cc'}$ is just $T(\{aa', bb', cc'\})$. Therefore we see that $\tilde{\sigma} \in G_{\mathrm{Hesse}}$, and that $\sigma \circ T = T \circ \tilde{\sigma}$ holds. Thus $\sigma = \Psi(\tilde{\sigma})$. $\square$

Let λ be a parameter of the affine line $\mathbb{A}^1$. We define $\gamma_\lambda : \mathcal{P} \to \mathbb{P}^2$ by Table 5.5. We also denote by $\widetilde{T} = \langle T \rangle$ the subgroup of $\mathrm{Aut}(\mathbf{C}_B)$ of order 2 generated by

$$T := (P_2P_5)(P_3P_6)(P_4P_7)(P_{10}P_{13})(P_{11}P_{14})(P_{12}P_{15})(P_{20}P_{21}),$$

which corresponds to the automorphism of the Hesse configuration given by $aa' \mapsto a'a$.

Proposition 5.3. *The map $\lambda \mapsto \gamma_\lambda$ induces an isomorphism from $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ to $PGL(3, k)\backslash \mathcal{G}_B/\widetilde{T}$.*

Proof. First note that γ_λ is injective if and only if

$$\lambda \neq 0, \quad \lambda \neq 1 \quad \text{and} \quad \lambda \neq \bar{\omega}$$

hold.

Suppose that $\lambda \neq 0, 1, \omega$ and $\bar{\omega}$. Then γ_λ is injective, and the image $\gamma_\lambda(\mathcal{P})$ coincides with $Z(dGB[\lambda])$, where $GB[\lambda]$ is given in Theorem 1.7. Moreover, for each linear word $l_{aa'}$ (resp. each quadratic word q_ℓ of type I) (resp. each quadratic word $q'_{\ell,\ell'}$ of type II) of the code $\mathbf{C}_B$, there exists a line $L_{aa'}$ containing $\gamma_\lambda(\ell_{aa'})$ (resp. a conic curve Q_ℓ containing $\gamma_\lambda(q_\ell)$) (resp. a conic curve $Q'_{\ell,\ell'}$ containing $\gamma_\lambda(q'_{\ell,\ell'})$) given in Tables 5.6, 5.7, 5.8. The conic curves in Tables 5.7 and 5.8 are nonsingular because $\lambda \notin \{0, 1, \omega, \bar{\omega}\}$. Hence γ_λ is in $\mathcal{G}_B$ by Proposition 3.6.

$T(\ell)$	$T(\ell')$	$q'_{\ell,\ell'}$
18	19	{ 1, 3, 6, 8, 9, 11, 14, 16 }
18	20	{ 2, 3, 6, 7, 9, 10, 15, 16 }
18	21	{ 3, 4, 5, 6, 9, 12, 13, 16 }
19	20	{ 1, 2, 7, 8, 10, 11, 14, 15 }
19	21	{ 1, 4, 5, 8, 11, 12, 13, 14 }
20	21	{ 2, 4, 5, 7, 10, 12, 13, 15 }
11	12	{ 3, 4, 5, 6, 8, 14, 19, 21 }
11	16	{ 1, 3, 5, 9, 13, 14, 18, 19 }
11	20	{ 2, 3, 5, 7, 14, 15, 17, 19 }
12	16	{ 1, 4, 6, 8, 9, 13, 18, 21 }
12	20	{ 2, 4, 6, 7, 8, 15, 17, 21 }
16	20	{ 1, 2, 7, 9, 13, 15, 17, 18 }
4	9	{ 1, 3, 5, 8, 12, 16, 18, 21 }
4	14	{ 1, 3, 6, 11, 12, 13, 19, 21 }
4	20	{ 1, 3, 7, 10, 12, 15, 17, 21 }
9	14	{ 5, 6, 8, 11, 13, 16, 18, 19 }
9	20	{ 5, 7, 8, 10, 15, 16, 17, 18 }
14	20	{ 6, 7, 10, 11, 13, 15, 17, 19 }
14	15	{ 2, 3, 6, 7, 8, 11, 19, 20 }
14	16	{ 1, 2, 6, 9, 10, 11, 18, 19 }
14	21	{ 2, 4, 5, 6, 11, 12, 17, 19 }
15	16	{ 1, 3, 7, 8, 9, 10, 18, 20 }
15	21	{ 3, 4, 5, 7, 8, 12, 17, 20 }
16	21	{ 1, 4, 5, 9, 10, 12, 17, 18 }
9	12	{ 2, 4, 5, 6, 10, 16, 18, 21 }
9	15	{ 2, 3, 5, 7, 13, 16, 18, 20 }
9	19	{ 1, 2, 5, 11, 14, 16, 17, 18 }
12	15	{ 3, 4, 6, 7, 10, 13, 20, 21 }
12	19	{ 1, 4, 6, 10, 11, 14, 17, 21 }
15	19	{ 1, 3, 7, 11, 13, 14, 17, 20 }
4	11	{ 1, 2, 5, 10, 12, 14, 19, 21 }
4	15	{ 1, 2, 7, 8, 12, 13, 20, 21 }
4	18	{ 1, 2, 6, 9, 12, 16, 17, 21 }
11	15	{ 5, 7, 8, 10, 13, 14, 19, 20 }
11	18	{ 5, 6, 9, 10, 14, 16, 17, 19 }
15	18	{ 6, 7, 8, 9, 13, 16, 17, 20 }
7	9	{ 1, 2, 6, 8, 15, 16, 18, 20 }
7	11	{ 1, 3, 6, 10, 14, 15, 19, 20 }
7	21	{ 1, 4, 6, 12, 13, 15, 17, 20 }
9	11	{ 2, 3, 8, 10, 14, 16, 18, 19 }
9	21	{ 2, 4, 8, 12, 13, 16, 17, 18 }
11	21	{ 3, 4, 10, 12, 13, 14, 17, 19 }
7	12	{ 1, 4, 5, 8, 10, 15, 20, 21 }
7	14	{ 1, 2, 5, 11, 13, 15, 19, 20 }
7	18	{ 1, 3, 5, 9, 15, 16, 17, 20 }
12	14	{ 2, 4, 8, 10, 11, 13, 19, 21 }
12	18	{ 3, 4, 8, 9, 10, 16, 17, 21 }
14	18	{ 2, 3, 9, 11, 13, 16, 17, 19 }
4	7	{ 2, 3, 5, 6, 12, 15, 20, 21 }
4	16	{ 2, 3, 9, 10, 12, 13, 18, 21 }
4	19	{ 2, 3, 8, 11, 12, 14, 17, 21 }
7	16	{ 5, 6, 9, 10, 13, 15, 18, 20 }
7	19	{ 5, 6, 8, 11, 14, 15, 17, 20 }
16	19	{ 8, 9, 10, 11, 13, 14, 17, 18 }

TABLE 5.4. Quadratic words of type II in $\mathbf{C}_B$

Conversely, let γ' be an arbitrary element of $\mathcal{G}_B$. We will show the following:

Claim 5.4. There exists a unique triple

$$(g, t, \lambda) \ \in \ PGL(3, k) \times \widetilde{T} \times (k \setminus \{0, 1, \omega, \bar{\omega}\})$$

such that $g \circ \gamma' \circ t = \gamma_\lambda$ holds.

The points $\gamma'(P_{15})$, $\gamma'(P_{16})$, $\gamma'(P_{21})$ of $\gamma'(\ell_{01})$ are on a line, and the points $\gamma'(P_{12})$, $\gamma'(P_{16})$, $\gamma'(P_{20})$ of $\gamma'(\ell_{10})$ are on another line. Hence there exists a unique element $g \in PGL(3, k)$ such that $\gamma := g \circ \gamma'$ satisfies the following:

$$\begin{aligned} &\gamma(P_{16}) = [1, 0, 0], \\ &\gamma(P_{12}) = [1, 1, 0], \quad \gamma(P_{20}) = [0, 1, 0], \\ &\gamma(P_{15}) = [1, 0, 1], \quad \gamma(P_{21}) = [0, 0, 1]. \end{aligned} \tag{5.1}$$

P_i	$\gamma_\lambda(P_i)$
$P_1 = C(22)$	$[\lambda+1, \bar{\omega}\lambda+\omega, \bar{\omega}\lambda+\omega]$
$P_2 = C(20)$	$[1, \omega\lambda+\omega, \omega]$
$P_3 = C(21)$	$[\lambda+\bar{\omega}, 1, \lambda+1]$
$P_4 = T(20, 21, 22)$	$[1, \omega, \omega]$
$P_5 = C(02)$	$[\lambda, \bar{\omega}\lambda, \bar{\omega}\lambda+\bar{\omega}]$
$P_6 = C(12)$	$[\lambda+\bar{\omega}, \bar{\omega}\lambda+\bar{\omega}, \bar{\omega}\lambda]$
$P_7 = T(02, 12, 22)$	$[1, \bar{\omega}, \bar{\omega}]$
$P_8 = C(11)$	$[\lambda+1, 1, \lambda]$
$P_9 = T(02, 11, 20)$	$[1, \bar{\omega}, \omega]$
$P_{10} = C(10)$	$[1, \omega\lambda+1, 0]$
$P_{11} = T(02, 10, 21)$	$[1, \bar{\omega}, 0]$
$P_{12} = T(10, 11, 12)$	$[1, 1, 0]$
$P_{13} = C(01)$	$[\lambda, 0, \lambda+\bar{\omega}]$
$P_{14} = T(01, 12, 20)$	$[1, 0, \omega]$
$P_{15} = T(01, 11, 21)$	$[1, 0, 1]$
$P_{16} = T(01, 10, 22)$	$[1, 0, 0]$
$P_{17} = C(00)$	$[0, \lambda, 1]$
$P_{18} = T(00, 12, 21)$	$[0, 1, \omega]$
$P_{19} = T(00, 11, 22)$	$[0, 1, 1]$
$P_{20} = T(00, 10, 20)$	$[0, 1, 0]$
$P_{21} = T(00, 01, 02)$	$[0, 0, 1]$

TABLE 5.5. Definition of γ_λ for $\mathbf{C}_B$

aa'	The defining equation of $L_{aa'}$
00	$X = 0$
01	$Y = 0$
02	$X + \omega Y = 0$
10	$Z = 0$
11	$X + Y + Z = 0$
12	$\omega X + \omega Y + Z = 0$
20	$\omega X + Z = 0$
21	$X + \omega Y + Z = 0$
22	$Y + Z = 0$

TABLE 5.6. Defining equations of the lines $L_{aa'}$

For $aa' \in P(AF)$, let $L_{aa'} \subset \mathbb{P}^2$ be the line containing $\gamma(l_{aa'})$, and let

$$\xi_{aa'}X + \eta_{aa'}Y + \zeta_{aa'}Z = 0$$

aa'	bb'	cc'	The defining equation of $Q_{aa',bb',cc'}$
00	01	02	$(\lambda+\bar{\omega})\,X^2+\bar{\omega}Y^2+(\lambda+\omega)\,Z^2+\lambda XY$
00	10	20	$(\omega\lambda+1)\,X^2+(\bar{\omega}\lambda+1)\,Y^2+\omega\lambda Z^2+ZX$
00	11	22	$(\bar{\omega}\lambda+\omega)\,X^2+\bar{\omega}Y^2+\omega\lambda Z^2+(\lambda+1)\,XY+(\lambda+1)\,ZX$
00	12	21	$Y^2+\lambda Z^2+(\omega\lambda+1)\,XY+(\lambda+\bar{\omega})\,ZX$
01	10	22	$(\bar{\omega}\lambda+1)\,X^2+Y^2+\bar{\omega}\lambda Z^2+(\lambda+\bar{\omega})\,YZ$
01	11	21	$(\omega\lambda+1)\,X^2+\lambda Z^2+XY+YZ$
01	12	20	$(\lambda+\bar{\omega})\,X^2+Y^2+\lambda Z^2+(\omega\lambda+\omega)\,XY+(\lambda+1)\,YZ$
02	11	20	$\omega Y^2+\lambda Z^2+(\bar{\omega}\lambda+\omega)\,XY+(\omega\lambda+1)\,YZ+(\lambda+\bar{\omega})\,ZX$
02	12	22	$(\omega\lambda+1)\,X^2+\omega\lambda Z^2+XY+\omega YZ+ZX$
10	11	12	$(\lambda+\bar{\omega})\,X^2+Y^2+\lambda YZ+\lambda ZX$
20	21	22	$(\lambda+\bar{\omega})\,X^2+\bar{\omega}Y^2+\lambda XY+\bar{\omega}\lambda YZ+\lambda ZX$

TABLE 5.7. Defining equations of the conic curves of type I

be the defining equation of $L_{aa'}$. By (5.1), we can put

$$\begin{aligned}
&\xi_{00}=1, && \eta_{00}=0, && \zeta_{00}=0,\\
&\xi_{01}=0, && \eta_{01}=1, && \zeta_{01}=0,\\
&\xi_{02}=1, && && \zeta_{02}=0,\\
&\xi_{10}=0, && \eta_{10}=0, && \zeta_{10}=1,\\
&\xi_{11}=1, && \eta_{11}=1, && \zeta_{11}=1,\\
&\xi_{12}=1, && \eta_{12}=1, && \\
&\xi_{20}=1, && \eta_{20}=0, && \\
&\xi_{21}=1, && && \zeta_{21}=1,\\
&\xi_{22}=0, && \eta_{22}=1. &&
\end{aligned}$$

The three lines $L_{aa'}$, $L_{bb'}$, $L_{cc'}$ are concurrent if $\{aa', bb', cc'\} \in L(AF)$. Hence we obtain a system of equations

$$\det\begin{bmatrix} \xi_{aa'} & \eta_{aa'} & \zeta_{aa'} \\ \xi_{bb'} & \eta_{bb'} & \zeta_{bb'} \\ \xi_{cc'} & \eta_{cc'} & \zeta_{cc'} \end{bmatrix}=0 \qquad \text{for every } \{aa', bb', cc'\} \in L(AF). \tag{5.2}$$

A Gröbner basis of the ideal generated by the left hand side of (5.2) in the polynomial ring $k[\eta_{02}, \eta_{21}, \zeta_{12}, \zeta_{20}, \zeta_{22}]$ is calculated as follows:

$$\langle\ 1+\zeta_{22},\ \ 1+\zeta_{20}+\eta_{21},\ \ 1+\zeta_{12}+\eta_{21},\ \ \eta_{02}+\eta_{21},\ \ 1+\eta_{21}+\eta_{21}^2\ \rangle.$$

Hence there are two solutions of this system of equations,

$$\begin{aligned}
&\eta_{21}=\eta_{02}=\omega, \quad \zeta_{12}=\zeta_{20}=\bar{\omega}, \quad \zeta_{22}=1, \quad \text{or}\\
&\eta_{21}=\eta_{02}=\bar{\omega}, \quad \zeta_{12}=\zeta_{20}=\omega, \quad \zeta_{22}=1,
\end{aligned}$$

$T(\ell)$	$T(\ell')$	The defining equation of $Q'_{\ell,\ell'}$
18	19	$\bar{\omega}\lambda Y^2 + \bar{\omega}Z^2 + \omega\lambda XY + ZX$
18	20	$Y^2 + (\lambda+1)Z^2 + (\omega\lambda+1)XY + (\lambda+1)ZX$
18	21	$(\lambda+1)Y^2 + \lambda Z^2 + (\lambda+1)XY + (\lambda+\bar{\omega})ZX$
19	20	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}Y^2 + (\omega\lambda+1)Z^2 + (\lambda+1)XY + (\lambda+\bar{\omega})ZX$
19	21	$(\bar{\omega}\lambda+\omega)X^2 + (\lambda+\bar{\omega})Y^2 + \omega\lambda Z^2 + (\omega\lambda+1)XY + (\lambda+1)ZX$
20	21	$(\omega\lambda+1)X^2 + Y^2 + \omega\lambda Z^2 + \omega\lambda XY + ZX$
11	12	$\omega X^2 + (\bar{\omega}\lambda+\omega)Y^2 + (\bar{\omega}\lambda+\omega)YZ + ZX$
11	16	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}\lambda Y^2 + \omega\lambda Z^2 + \lambda YZ + (\lambda+1)ZX$
11	20	$\bar{\omega}\lambda X^2 + \omega Y^2 + \omega\lambda Z^2 + \omega(\lambda+1)YZ + \lambda ZX$
12	16	$(\lambda+\bar{\omega})X^2 + \bar{\omega}(\lambda+1)Y^2 + \omega(\lambda+1)YZ + \lambda ZX$
12	20	$(\lambda+1)X^2 + Y^2 + \lambda YZ + (\lambda+1)ZX$
16	20	$(\omega\lambda+1)X^2 + \bar{\omega}Y^2 + \omega\lambda Z^2 + (\bar{\omega}\lambda+\omega)YZ + ZX$
4	9	$\bar{\omega}(\lambda+1)Y^2 + \bar{\omega}(\lambda+1)XY + \omega(\lambda+1)YZ + ZX$
4	14	$(\lambda+\bar{\omega})X^2 + (\omega\lambda+1)Y^2 + (\bar{\omega}\lambda+\omega)XY + (\omega\lambda+1)YZ + \lambda ZX$
4	20	$(\lambda+\bar{\omega})X^2 + \bar{\omega}Y^2 + \lambda XY + \bar{\omega}\lambda YZ + (\lambda+\bar{\omega})ZX$
9	14	$\omega\lambda Y^2 + \lambda Z^2 + \lambda XY + \bar{\omega}\lambda YZ + (\lambda+\bar{\omega})ZX$
9	20	$\omega Y^2 + \lambda Z^2 + (\bar{\omega}\lambda+\omega)XY + (\omega\lambda+1)YZ + \lambda ZX$
14	20	$(\omega\lambda+1)X^2 + \omega Y^2 + \omega\lambda Z^2 + \bar{\omega}(\lambda+1)XY + \omega(\lambda+1)YZ + ZX$
14	15	$\lambda X^2 + (\lambda+\bar{\omega})Z^2 + \omega\lambda XY + (\lambda+\bar{\omega})YZ$
14	16	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}Y^2 + \omega Z^2 + (\lambda+1)XY + YZ$
14	21	$\bar{\omega}X^2 + Y^2 + \lambda Z^2 + \omega XY + (\lambda+1)YZ$
15	16	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}(\lambda+1)Z^2 + \omega XY + (\lambda+1)YZ$
15	21	$(\lambda+1)X^2 + \lambda Z^2 + (\lambda+1)XY + YZ$
16	21	$(\omega\lambda+1)X^2 + Y^2 + \bar{\omega}\lambda Z^2 + \omega\lambda XY + (\lambda+\bar{\omega})YZ$
9	12	$\bar{\omega}Y^2 + (\lambda+\bar{\omega})XY + \omega YZ + (\lambda+1)ZX$
9	15	$\lambda Z^2 + (\bar{\omega}\lambda+\bar{\omega})XY + \omega\lambda YZ + (\lambda+\bar{\omega})ZX$
9	19	$Y^2 + \bar{\omega}\lambda Z^2 + \bar{\omega}XY + (\lambda+\bar{\omega})YZ + \lambda ZX$
12	15	$(\lambda+\bar{\omega})X^2 + \bar{\omega}XY + (\lambda+\bar{\omega})YZ + \lambda ZX$
12	19	$(\omega\lambda+1)X^2 + \omega Y^2 + \bar{\omega}(\lambda+1)XY + \omega\lambda YZ + (\lambda+\bar{\omega})ZX$
15	19	$(\bar{\omega}\lambda+\omega)X^2 + \omega\lambda Z^2 + (\lambda+\bar{\omega})XY + \omega YZ + (\lambda+1)ZX$
4	11	$(\omega\lambda+1)X^2 + Y^2 + \omega\lambda XY + YZ + (\lambda+\bar{\omega})ZX$
4	15	$(\lambda+\bar{\omega})X^2 + (\lambda+\bar{\omega})XY + \bar{\omega}(\lambda+1)YZ + \lambda ZX$
4	18	$\omega Y^2 + \omega XY + \omega\lambda YZ + (\lambda+1)ZX$
11	15	$(\bar{\omega}\lambda+\omega)X^2 + \omega\lambda Z^2 + \omega XY + \omega\lambda YZ + (\lambda+1)ZX$
11	18	$\bar{\omega}Y^2 + \bar{\omega}\lambda Z^2 + (\lambda+\bar{\omega})XY + \bar{\omega}(\lambda+1)YZ + \lambda ZX$
15	18	$\lambda Z^2 + \omega\lambda XY + YZ + (\lambda+\bar{\omega})ZX$
7	9	$(\lambda+1)Z^2 + \bar{\omega}\lambda XY + \omega(\lambda+1)YZ + (\lambda+1)ZX$
7	11	$(\bar{\omega}\lambda+\omega)X^2 + (\omega\lambda+1)Z^2 + \omega XY + (\omega\lambda+1)YZ + (\lambda+\bar{\omega})ZX$
7	21	$(\omega\lambda+1)X^2 + \omega\lambda Z^2 + (\omega\lambda+1)XY + \omega YZ + ZX$
9	11	$Y^2 + \bar{\omega}Z^2 + (\omega\lambda+1)XY + \omega YZ + ZX$
9	21	$\omega Y^2 + \lambda Z^2 + \omega XY + (\omega\lambda+1)YZ + (\lambda+\bar{\omega})ZX$
11	21	$(\bar{\omega}\lambda+\omega)X^2 + \omega Y^2 + \omega\lambda Z^2 + \bar{\omega}\lambda XY + \omega(\lambda+1)YZ + (\lambda+1)ZX$
7	12	$(\lambda+\bar{\omega})X^2 + \bar{\omega}XY + (\lambda+1)YZ + (\lambda+\bar{\omega})ZX$
7	14	$(\omega\lambda+1)X^2 + \omega\lambda Z^2 + (\bar{\omega}\lambda+\omega)XY + \omega\lambda YZ + ZX$
7	18	$\lambda Z^2 + \omega(\lambda+1)XY + YZ + \lambda ZX$
12	14	$(\lambda+\bar{\omega})X^2 + Y^2 + \omega(\lambda+1)XY + YZ + \lambda ZX$
12	18	$\omega Y^2 + (\bar{\omega}\lambda+\omega)XY + \omega\lambda YZ + ZX$
14	18	$Y^2 + \lambda Z^2 + \bar{\omega}XY + (\lambda+1)YZ + (\lambda+\bar{\omega})ZX$
4	7	$(\lambda+1)X^2 + (\lambda+1)XY + (\bar{\omega}\lambda+\omega)YZ + (\lambda+1)ZX$
4	16	$(\lambda+\bar{\omega})X^2 + \bar{\omega}Y^2 + \lambda XY + \omega YZ + \lambda ZX$
4	19	$\omega X^2 + \bar{\omega}Y^2 + XY + \bar{\omega}\lambda YZ + ZX$
7	16	$(\omega\lambda+1)X^2 + \omega\lambda Z^2 + XY + \bar{\omega}\lambda YZ + ZX$
7	19	$\bar{\omega}\lambda X^2 + \omega\lambda Z^2 + \lambda XY + \omega YZ + \lambda ZX$
16	19	$(\bar{\omega}\lambda+\omega)X^2 + \bar{\omega}Y^2 + \omega\lambda Z^2 + (\lambda+1)XY + (\bar{\omega}\lambda+\omega)YZ + (\lambda+1)ZX$

TABLE 5.8. Defining equations of the conic curves of type II

which are conjugate over $\mathbb{F}_2$. If the latter holds, then we replace γ with $g_0 \circ \gamma \circ T$, where

$$g_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

so that we can assume the former always holds. The image of the T-points by γ is therefore equal to the ones given in Table 5.5, and the lines $L_{aa'}$ are given by equations in Table 5.6.

We next determine the coordinates of the image of C-points by γ. The point $\gamma(C(00)) = \gamma(P_{17})$ is on the line $L_{00} = \{X = 0\}$ and is different from $\gamma(P_{20}) = [0, 1, 0]$ and $\gamma(P_{21}) = [0, 0, 1]$. Hence we can put

$$\gamma(P_{17}) = [0, \lambda, 1], \tag{5.3}$$

where λ is a parameter $\neq 0$. Let ℓ, ℓ_1, ℓ_2 be three distinct elements of $L(AF)$ that are parallel to each other. The conic curves Q satisfying the following conditions form a pencil PQ_ℓ:

(i) Q contains $\gamma(T(\ell_1))$ and $\gamma(T(\ell_2))$, and
(ii) Q is tangent to the lines $L_{aa'}$, $L_{bb'}$, $L_{cc'}$, where $\ell = \{aa', bb', cc'\}$. (Recall Definition 3.9 and Remark 3.10.)

Using the coordinates of the points $\gamma(T(\ell))$ and the defining equations of the nine lines $L_{aa'}$ determined so far, we can calculate this pencil explicitly. By Proposition 5.1, the conic curve Q_ℓ containing $\gamma(q_\ell)$ is a nonsingular member of the pencil PQ_ℓ. Starting from (5.3), we can determine the coordinates of $\gamma(C(aa'))$, and see that they coincide with Table 5.5. For example, consider $\ell = \{01, 10, 22\} \in L(AF)$. We have

$$\ell_1 = \{02, 11, 20\}, \quad \ell_2 = \{00, 12, 21\}.$$

The pencil of conic curves passing through the points

$$\gamma(T(\ell_1)) = \gamma(P_9) = [1, \bar{\omega}, \omega], \quad \gamma(T(\ell_2)) = \gamma(P_{18}) = [0, 1, \omega],$$

and tangent to the lines

$$L_{01} = \{Y = 0\}, \quad L_{10} = \{Z = 0\}, \quad L_{22} = \{Y + Z = 0\}$$

is spanned by the two conic curves defined by

$$\omega X^2 + \bar{\omega} Y^2 + Z^2 = 0 \quad \text{and} \quad \omega X^2 + \omega Y^2 + YZ = 0.$$

Because the conic curve Q_ℓ passes through $\gamma(P_{17}) = [0, \lambda, 1]$, it is defined by

$$\lambda(\omega X^2 + \bar{\omega} Y^2 + Z^2) + (\omega\lambda + 1)(\omega X^2 + \omega Y^2 + YZ) = 0.$$

The intersection points of Q_ℓ with the line $L_{12} = \{\omega X + \omega Y + Z = 0\}$ are $\gamma\,(T(\ell_2)) = \gamma(P_{18}) = [0, 1, \omega]$ and $\gamma(C(12)) = \gamma(P_6)$. Hence we obtain

$$\gamma(C(12)) = \gamma(P_6) = [\omega\lambda + 1, \lambda + 1, \lambda].$$

See Table 5.9 for the detail of the calculation. Thus we have proved that γ is equal to γ_λ. Because γ_λ is injective, λ is not among $\{0, 1, \bar{\omega}\}$.

There exists a unique conic curve containing $\gamma_\lambda(q)$ for each quadratic word q of $\mathbf{C}_B$, and the defining equations of those conic curves are given in Tables 5.7 and 5.8. The smoothness of these curves implies that $\lambda \neq \omega$. Thus we have proved Claim 5.4. □

ℓ	F_1, F_2	β_ℓ
$00, 01, 02$	$\omega X^2 + \omega Y^2 + Z^2,\quad \bar\omega X^2 + \omega Y^2 + XY$	$(\lambda^2 + \bar\omega)/\lambda^2$
$00, 10, 20$	$X^2 + \omega Y^2 + Z^2,\quad X^2 + Y^2 + ZX$	$(\omega\lambda^2 + 1)/\lambda^2$
$00, 11, 22$	$\bar\omega X^2 + \omega Y^2 + Z^2,\quad \omega X^2 + \bar\omega Y^2 + XY + ZX$	$(\bar\omega\lambda^2 + \omega)/\lambda^2$
$00, 12, 21$	$\omega Y^2 + Z^2,\quad \omega Y^2 + \omega XY + ZX$	$(\lambda^2 + \bar\omega)/\lambda^2$
$01, 10, 22$	$\omega X^2 + \bar\omega Y^2 + Z^2,\quad \omega X^2 + \omega Y^2 + YZ$	$(\omega\lambda + 1)/\lambda$
$01, 11, 21$	$\omega X^2 + Z^2,\quad X^2 + XY + YZ$	$1/\lambda$
$01, 12, 20$	$\omega X^2 + Y^2 + Z^2,\quad \bar\omega X^2 + Y^2 + \omega XY + YZ$	$(\lambda + 1)/\lambda$
$02, 10, 21$	$\bar\omega X^2 + Y^2 + Z^2,\quad \omega X^2 + \omega Y^2 + \omega YZ + ZX$	$(\bar\omega\lambda + \bar\omega)/\lambda$
$02, 11, 20$	$\bar\omega Y^2 + Z^2,\quad \bar\omega Y^2 + \bar\omega XY + \omega YZ + ZX$	$(\lambda + \bar\omega)/\lambda$
$02, 12, 22$	$X^2 + Z^2,\quad X^2 + XY + \omega YZ + ZX$	$\bar\omega/\lambda$
$10, 11, 12$	$\bar\omega X^2 + Y^2,\quad X^2 + YZ + ZX$	λ
$20, 21, 22$	$X^2 + Y^2,\quad X^2 + XY + \bar\omega YZ + ZX$	$\omega\lambda$

TABLE 5.9. Basis $\{F_1 = 0, F_2 = 0\}$ of the pencil PQ_ℓ and the member $Q_\ell = \{F_1 + \beta_\ell F_2 = 0\}$

Remark 5.5. The polynomial $GB[\lambda]$ is the defining equation of the nodal splitting curve

$$L_{00} \cup L_{01} \cup L_{10} \cup L_{11} \cup Q'_{\ell,\ell'},$$

where $T(\ell) = P_{16}$ and $T(\ell') = P_{19}$. See Proposition 2.8.

Remark 5.6. Consider the projective plane $(\mathbb{P}^2)^\vee$ of lines on $\mathbb{P}^2$. Let $[U, V, W]$ be the homogeneous coordinates of $(\mathbb{P}^2)^\vee$ dual to the homogeneous coordinates $[X, Y, Z]$ of $\mathbb{P}^2$. Let E_λ be the cubic curve in $(\mathbb{P}^2)^\vee$ defined by

$$\bar\omega\, U^2W + UW^2 + \omega\lambda\, UV^2 + (\omega\lambda + 1)\, V^2W + \\ + (\omega\lambda + 1)\, VW^2 + \bar\omega\lambda\, U^2V + (\omega + \lambda)\, UVW = 0.$$

Then the points $\gamma_\lambda(C(P(AF)))$ correspond to the nine flex tangents to E_λ, and the points $\gamma_\lambda(T(L(AF)))$ correspond to the twelve lines containing three flex points of E_λ.

Remark 5.7. When $\lambda = \omega$, the set $\gamma_\lambda(\mathcal{P})$ coincides with $\mathbb{P}^2(\mathbb{F}_4)$, and the point $[GB[\lambda]] \in \mathfrak{M}$ is equal to the Dolgachev-Kondo point.

For each $\sigma \in \mathrm{Aut}(\mathbf{C}_B)$, we calculate the unique triple

$$(g_\sigma, t_\sigma, \lambda^\sigma) \in PGL(3, k(\lambda)) \times \widetilde{T} \times k(\lambda)$$

such that $g_\sigma \circ (\gamma_\lambda \circ \sigma) \circ t_\sigma = \gamma_{\lambda^\sigma}$ holds. The map $\sigma \mapsto t_\sigma$ is a homomorphism from $\mathrm{Aut}(\mathbf{C}_B)$ to $\widetilde{T}$. We put $N_B := \mathrm{Ker}(\mathrm{Aut}(\mathbf{C}_B) \to \widetilde{T})$.

Corollary 5.8. *The space $PGL(3, k)\backslash\mathcal{G}_B$ has exactly two connected components, each of which is isomorphic to $\mathbb{A}^1\backslash\{0, 1, \omega, \bar\omega\}$. One of them is given, set-theoretically, by*

$$(PGL(3, k)\backslash\mathcal{G}_B)^+ := \{\ [\gamma_\alpha]\ \mid\ \alpha \in k \setminus \{0, 1, \omega, \bar\omega\}\ \},$$

$$
\begin{array}{lllllllll}
l_1 & := & \{ & 17, & 18, & 19, & 20, & 21 & \}, \\
l_2 & := & \{ & 13, & 14, & 15, & 16, & 21 & \}, \\
l_3 & := & \{ & 9, & 10, & 11, & 12, & 21 & \}, \\
l_4 & := & \{ & 5, & 6, & 7, & 8, & 21 & \}, \\
l_5 & := & \{ & 1, & 2, & 3, & 4, & 21 & \}.
\end{array}
$$

TABLE 6.1. Linear words of $\mathbf{C}_C$

and the other one is equal to $((PGL(3,k)\backslash\mathcal{G}_B)^+)\cdot T$. *The group* N_B *acts on* $(PGL(3,k)\backslash\mathcal{G}_B)^+$, *and the moduli curve* $\mathfrak{M}_B$ *is equal to the quotient space* $(PGL(3,k)\backslash\mathcal{G}_B)^+/N_B$.

Consider the natural projection

$$p_B : \mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\} \cong (PGL(3,k)\backslash\mathcal{G}_B)^+ \to \mathfrak{M}_B = (PGL(3,k)\backslash\mathcal{G}_B)^+/N_B.$$

For $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$, let $P[\alpha]$ denote the point of $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$ given by $\lambda = \alpha$. Then $p_B(P[\alpha]) \in \mathfrak{M}_B$ corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$. The following is proved in the same way as Proposition 4.7.

Proposition 5.9. *The fiber* $p_B^{-1}(p_B(P[\alpha]))$ *is equal* $\{P[\varphi]\}$, *where* φ *runs through the set* Γ_B *in Theorem 1.7 with* λ *replaced by* α. *The group* $\mathrm{Aut}(X_{GB[\alpha]}, \mathcal{L}_{GB[\alpha]})$ *is equal to the subgroup of* $PGL(3,k)$ *generated by the elements in* (1.2).

Corollary 5.10. *We have* $\mathfrak{M}_B = \mathrm{Spec}\, k[J_B, 1/J_B]$, *where*

$$J_B = (\lambda+\omega)^{12}/\lambda^3(\lambda+1)^3(\lambda+\bar{\omega})^3.$$

The morphism p_B *is an étale Galois covering with Galois group* Γ_B, *which is isomorphic to the alternating group* $\mathfrak{A}_4$.

Indeed the group Γ_B acts on the set $\{0,1,\bar{\omega},\infty\}$ as $\mathfrak{A}_4$.

6. THE MODULI CURVE CORRESPONDING TO THE CODE $\mathbf{C}_C$

In this section, we prove Theorem 1.8.

The linear words of $\mathbf{C}_C$ are listed in Table 6.1. The list of quadratic words in $\mathbf{C}_C$ is omitted. The point P_{21} is special because every linear word contains it. The following can be checked directly by computer:

Proposition 6.1. *Let* $\phi : \mathcal{P} \xrightarrow{\sim} \mathbb{P}^2(\mathbb{F}_4)$ *be the bijection given in Table 6.2.*

(1) *The linear words of* $\mathbf{C}_C$ *are precisely the words* $\phi^{-1}(\Lambda(\mathbb{F}_4))$, *where* Λ *are* $\mathbb{F}_4$*-rational lines passing through*

$$O := [0,0,1] = \phi(P_{21}).$$

(2) *The quadratic words of* $\mathbf{C}_C$ *are precisely the words* $\phi^{-1}(\Lambda(\mathbb{F}_4) + \Lambda'(\mathbb{F}_4))$, *where* Λ *and* Λ' *are distinct* $\mathbb{F}_4$*-rational lines that do not pass through* O.

Note that ϕ embeds $\mathbf{C}_C$ into the Dolgachev-Kondo code $\mathbf{C}_{\mathrm{DK}}$.

Corollary 6.2. *For each quadratic word* q *in* $\mathbf{C}_C$, *there exists a unique linear word* l *in* $\mathbf{C}_C$ *such that* $q \cap l = \emptyset$.

From Remark 3.11, we obtain the following:

$$\begin{array}{rclcrcl}
\phi(P_1) &=& [1,1,0], & \quad & \phi(P_{13}) &=& [1,0,1],\\
\phi(P_2) &=& [1,1,1], & & \phi(P_{14}) &=& [1,0,0],\\
\phi(P_3) &=& [1,1,\omega], & & \phi(P_{15}) &=& [1,0,\bar\omega],\\
\phi(P_4) &=& [1,1,\bar\omega], & & \phi(P_{16}) &=& [1,0,\omega],\\
\phi(P_5) &=& [1,\bar\omega,\omega], & & \phi(P_{17}) &=& [0,1,1],\\
\phi(P_6) &=& [1,\bar\omega,0], & & \phi(P_{18}) &=& [0,1,0],\\
\phi(P_7) &=& [1,\bar\omega,1], & & \phi(P_{19}) &=& [0,1,\bar\omega],\\
\phi(P_8) &=& [1,\bar\omega,\bar\omega], & & \phi(P_{20}) &=& [0,1,\omega],\\
\phi(P_9) &=& [1,\omega,\bar\omega], & & \phi(P_{21}) &=& [0,0,1].\\
\phi(P_{10}) &=& [1,\omega,0], & & & & \\
\phi(P_{11}) &=& [1,\omega,\omega], & & & & \\
\phi(P_{12}) &=& [1,\omega,1], & & & &
\end{array}$$

TABLE 6.2. Bijection ϕ from $\mathcal{P}$ to $\mathbb{P}^2(\mathbb{F}_4)$

.

Corollary 6.3. *Let l and l' be distinct linear words of $\mathbf{C}_C$, and let $A_1, A_2 \in l$ (resp. $B_1, B_2 \in l'$) be distinct points not equal to P_{21}. Then there are exactly two quadratic words q and q' in $\mathbf{C}_C$ containing the points $\{A_1, A_2, B_1, B_2\}$. Moreover, if a linear word $l'' \in \mathbf{C}_C$ is disjoint from q, then l'' is also disjoint from q'.*

For $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_4$, we denote by $\Lambda[\alpha_1\alpha_2\alpha_3]$ the $\mathbb{F}_4$-rational line defined by

$$\alpha_1 X + \alpha_2 Y + \alpha_3 Z = 0,$$

and by $q[\alpha_1\alpha_2\alpha_3, \beta_1\beta_2\beta_3] \in \mathbf{C}_C$ the quadratic word

$$\phi^{-1}(\Lambda[\alpha_1\alpha_2\alpha_3](\mathbb{F}_4) + \Lambda[\beta_1\beta_2\beta_3](\mathbb{F}_4)).$$

We put

$$LG' := \{\ g \in PGL(3, \mathbb{F}_4)\ \mid\ g(O) = O\ \}.$$

The automorphism group $\mathrm{Aut}(\mathbf{C}_C)$ of the code $\mathbf{C}_C$ contains a subgroup

$$LG := \phi^{-1} \circ LG' \circ \phi.$$

The order of LG is 2880. The group $\mathrm{Aut}(\mathbf{C}_C)$ also contains the permutation

$$T := (P_3P_4)(P_5P_9)(P_6P_{10})(P_7P_{12})(P_8P_{11})(P_{15}P_{16})(P_{19}P_{20})$$

of $\mathcal{P}$ that corresponds, via the bijection ϕ, to the action of the conjugation $\omega \mapsto \bar\omega$ over $\mathbb{F}_2$ on $\mathbb{P}^2(\mathbb{F}_4)$. It can be checked easily by computer that the following permutation is also contained in $\mathrm{Aut}(\mathbf{C}_C)$:

$$S := (P_1P_3)(P_2P_4)(P_5P_7)(P_6P_8)(P_9P_{11})(P_{10}P_{12}).$$

The automorphisms T and S of $\mathbf{C}_C$ generate a subgroup isomorphic to the dihedral group of order 8 in $\mathrm{Aut}(\mathbf{C}_C)$. An ordered quartet

$$(R_1, R_2, R_1', R_2')$$

f	$[q, q']$	l_ν
$\{13, 14, 17, 18\}$	$q[101, 011] = \{7, 8, 11, 12, 13, 14, 17, 18\}$	l_5
	$q[111, 001] = \{5, 6, 9, 10, 13, 14, 17, 18\}$	
$\{13, 14, 17, 19\}$	$q[1\bar{\omega}1, 011] = \{2, 3, 10, 11, 13, 14, 17, 19\}$	l_4
	$q[111, 0\bar{\omega}1] = \{1, 4, 9, 12, 13, 14, 17, 19\}$	
$\{13, 14, 17, 20\}$	$q[1\omega 1, 011] = \{2, 4, 6, 8, 13, 14, 17, 20\}$	l_3
	$q[111, 0\omega 1] = \{1, 3, 5, 7, 13, 14, 17, 20\}$	
$\{13, 14, 18, 19\}$	$q[101, 0\bar{\omega}1] = \{2, 4, 5, 7, 13, 14, 18, 19\}$	l_3
	$q[1\bar{\omega}1, 001] = \{1, 3, 6, 8, 13, 14, 18, 19\}$	
$\{13, 14, 18, 20\}$	$q[101, 0\omega 1] = \{2, 3, 9, 12, 13, 14, 18, 20\}$	l_4
	$q[1\omega 1, 001] = \{1, 4, 10, 11, 13, 14, 18, 20\}$	
$\{13, 14, 19, 20\}$	$q[1\bar{\omega}1, 0\omega 1] = \{7, 8, 9, 10, 13, 14, 19, 20\}$	l_5
	$q[1\omega 1, 0\bar{\omega}1] = \{5, 6, 11, 12, 13, 14, 19, 20\}$	

TABLE 6.3. List of the triples $(f, \{q, q'\}, l_\nu)$

of points in $\mathcal{P} \setminus \{P_{21}\}$ is called a *marking quartet* if P_{21}, R_1, R_2 are in a linear word, and P_{21}, R'_1, R'_2 are in another linear word. There are 2880 marking quartets, and the action of LG on the set of marking quartets is simply transitive.

Proposition 6.4. *The group* $\mathrm{Aut}(\mathbf{C}_C)$ *is generated by* LG*,* T *and* S*, and the order of* $\mathrm{Aut}(\mathbf{C}_C)$ *is* 23040*.*

Proof. Let σ be an arbitrary element of $\mathrm{Aut}(\mathbf{C}_C)$. Because $(P_{17}, P_{18}, P_{13}, P_{14})$ and $(\sigma(P_{17}), \sigma(P_{18}), \sigma(P_{13}), \sigma(P_{14}))$ are marking quartets, there exists an element $\tau \in LG$ such that $\tau\sigma(P_i) = P_i$ for $i = 13, 14, 17, 18, 21$. Because $\tau\sigma(l_1) = l_1$ and $\tau\sigma(l_2) = l_2$, we have

$$\{\tau\sigma(P_{19}), \tau\sigma(P_{20})\} = \{P_{19}, P_{20}\} \quad \text{and} \quad \{\tau\sigma(P_{15}), \tau\sigma(P_{16})\} = \{P_{15}, P_{16}\}.$$

If $\tau\sigma(P_{19}) = P_{20}$, then we replace τ by $T\tau$. Therefore, modulo the subgroup generated by LG and T, we can assume that σ has the following properties:

(σ-i) σ fixes each of the seven points $P_{13}, P_{14}, P_{17}, P_{18}, P_{19}, P_{20}, P_{21}$,

(σ-ii) $\{\sigma(P_{15}), \sigma(P_{16})\} = \{P_{15}, P_{16}\}$

Consider, for example, a set of four points $\{P_{13}, P_{14}, P_{17}, P_{18}\}$, each of which is fixed by σ. The two quadratic words containing them are

$$\begin{aligned} q[101, 011] &= \{2, 7, 12, 13, 18\} + \{2, 8, 11, 14, 17\} = \{7, 8, 11, 12, 13, 14, 17, 18\}, \\ q[111, 001] &= \{1, 5, 9, 13, 17\} + \{1, 6, 10, 14, 18\} = \{5, 6, 9, 10, 13, 14, 17, 18\}. \end{aligned}$$

Both of $q[101, 011]$ and $q[111, 001]$ are disjoint from l_5. By Corollary 6.2, we have $\sigma(l_5) = l_5$. Considering other sets of four points fixed by σ, we can show that $\sigma(l_4) = l_4$ and $\sigma(l_3) = l_3$. In Table 6.3, we list the triples $(f, \{q, q'\}, l_\nu)$, where f is a set of four points pointwise fixed by σ, $\{q, q'\}$ is the pair of quadratic words containing f, and l_ν is the linear word disjoint from both of q and q'. Therefore we have the following:

(σ-iii) σ leaves each of the sets

$$\{P_1, P_2, P_3, P_4\}, \ \ \{P_5, P_6, P_7, P_8\}, \ \ \{P_9, P_{10}, P_{11}, P_{12}\}$$

invariant.

Let us consider the quadratic words $q_1 := q[101, 011]$ and $q_2 := q[111, 001]$ again. Since

$$\{\sigma(q_1 \cap l_4), \sigma(q_2 \cap l_4)\} = \{q_1 \cap l_4, q_2 \cap l_4\},$$

the action of σ on $\{P_5, P_6, P_7, P_8\}$ preserves the decomposition

$$\{P_5, P_6, P_7, P_8\} = \{P_5, P_6\} \cup \{P_7, P_8\};$$

that is, $\{\sigma(P_5), \sigma(P_6)\}$ is either $\{P_5, P_6\}$ or $\{P_7, P_8\}$. By the same argument applied to the pairs $\{q, q'\}$ of quadratic words in Table 6.3, we see the following:

(σ-iv) σ preserves the decompositions

$$\begin{aligned} \{P_1, P_2, P_3, P_4\} &= \{P_1, P_4\} \cup \{P_2, P_3\} = \{P_1, P_3\} \cup \{P_2, P_4\}, \\ \{P_5, P_6, P_7, P_8\} &= \{P_5, P_6\} \cup \{P_7, P_8\} = \{P_5, P_7\} \cup \{P_6, P_8\}, \quad \text{and} \\ \{P_9, P_{10}, P_{11}, P_{12}\} &= \{P_9, P_{10}\} \cup \{P_{11}, P_{12}\} = \{P_9, P_{12}\} \cup \{P_{10}, P_{11}\}. \end{aligned}$$

The two quadratic words containing $\{P_{13}, P_{16}, P_{17}, P_{18}\}$ are

$$\begin{aligned} q[\omega 11, 101] &= \{2, 4, 10, 12, 13, 16, 17, 18\} \quad \text{and} \\ q[111, \omega 01] &= \{1, 3, 9, 11, 13, 16, 17, 18\}, \end{aligned}$$

both of which are disjoint from l_4. On the other hand, the two quadratic words containing $\{P_{13}, P_{15}, P_{17}, P_{18}\}$ are

$$\begin{aligned} q[\bar{\omega} 11, 101] &= \{2, 3, 6, 7, 13, 15, 17, 18\} \quad \text{and} \\ q[111, \bar{\omega} 01] &= \{1, 4, 5, 8, 13, 15, 17, 18\}, \end{aligned}$$

both of which are disjoint from l_3. Since σ fixes each of l_4 and l_3, we see that the property (σ-ii) of σ can be strengthened to the following:

(σ-ii)$'$ $\sigma(P_{15}) = P_{15}$, $\sigma(P_{16}) = P_{16}$.

Using computer, we can easily list all $4^3 = 64$ permutations σ satisfying (σ-i), (σ-ii)$'$, (σ-iii) and (σ-iv). We can check that exactly four of them id, S,

$$\begin{aligned} (ST)^2 &= (P_1 P_2)(P_3 P_4)(P_5 P_6)(P_7 P_8)(P_9 P_{10})(P_{11} P_{12}) \quad \text{and} \\ (ST)^2 S &= (P_1 P_4)(P_2 P_3)(P_5 P_8)(P_6 P_7)(P_9 P_{12})(P_{10} P_{11}) \end{aligned}$$

preserve the set of quadratic words in $\mathbf{C}_C$. Hence, by Proposition 3.5, $\mathrm{Aut}(\mathbf{C}_C)$ is generated by LG, S and T. It can be checked by computer that the order of $\mathrm{Aut}(\mathbf{C}_C)$ is 23040. □

Let λ be a parameter of the affine line $\mathbb{A}^1$. We define $\gamma_\lambda : \mathcal{P} \to \mathbb{P}^2$ by Table 6.4. When $\lambda = 0$, the map γ_λ is equal to ϕ. Let $\widetilde{T}$ denote the subgroup of $\mathrm{Aut}(\mathbf{C}_C)$ generated by the involution T.

Proposition 6.5. *The map $\lambda \mapsto \gamma_\lambda$ induces an isomorphism from $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ to $PGL(3, k) \backslash \mathcal{G}_C / \widetilde{T}$.*

Proof. First note that γ_λ is injective for every λ.

Claim 6.6. Let γ' be an arbitrary element of $\mathcal{G}_C$. Then there exists a unique triple

$$(g, t, \lambda) \ \in \ PGL(3, k) \times \widetilde{T} \times (k \setminus \{0, 1, \omega, \bar{\omega}\})$$

such that $g \circ \gamma' \circ t = \gamma_\lambda$.

$$
\begin{array}{rclrcl}
\gamma_\lambda(P_1) & = & [1,1,\lambda], & \gamma_\lambda(P_{13}) & = & [1,0,1],\\
\gamma_\lambda(P_2) & = & [1,1,\lambda+1], & \gamma_\lambda(P_{14}) & = & [1,0,0],\\
\gamma_\lambda(P_3) & = & [1,1,\lambda+\omega], & \gamma_\lambda(P_{15}) & = & [1,0,\bar\omega],\\
\gamma_\lambda(P_4) & = & [1,1,\lambda+\bar\omega], & \gamma_\lambda(P_{16}) & = & [1,0,\omega],\\
\gamma_\lambda(P_5) & = & [1,\bar\omega,\omega\lambda+\omega], & \gamma_\lambda(P_{17}) & = & [0,1,1],\\
\gamma_\lambda(P_6) & = & [1,\bar\omega,\omega\lambda], & \gamma_\lambda(P_{18}) & = & [0,1,0],\\
\gamma_\lambda(P_7) & = & [1,\bar\omega,\omega\lambda+1], & \gamma_\lambda(P_{19}) & = & [0,1,\bar\omega],\\
\gamma_\lambda(P_8) & = & [1,\bar\omega,\omega\lambda+\bar\omega], & \gamma_\lambda(P_{20}) & = & [0,1,\omega],\\
\gamma_\lambda(P_9) & = & [1,\omega,\bar\omega\lambda+\bar\omega], & \gamma_\lambda(P_{21}) & = & [0,0,1].\\
\gamma_\lambda(P_{10}) & = & [1,\omega,\bar\omega\lambda], & & & \\
\gamma_\lambda(P_{11}) & = & [1,\omega,\bar\omega\lambda+\omega], & & & \\
\gamma_\lambda(P_{12}) & = & [1,\omega,\bar\omega\lambda+1], & & &
\end{array}
$$

TABLE 6.4. Definition of γ_λ for $\mathbf{C}_C$

Since $\gamma'(P_{21})$, $\gamma'(P_{13})$, $\gamma'(P_{14})$ are on a line, and $\gamma'(P_{21})$, $\gamma'(P_{17})$, $\gamma'(P_{18})$ are on another line, there exists a unique $g \in PGL(3,k)$ such that $\gamma := g \circ \gamma'$ satisfies

$$
\begin{aligned}
&\gamma(P_{21}) = [0,0,1] = O,\\
&\gamma(P_{17}) = [0,1,1], \quad \gamma(P_{18}) = [0,1,0],\\
&\gamma(P_{13}) = [1,0,1], \quad \gamma(P_{14}) = [1,0,0].
\end{aligned}
$$

The X-coordinate of $\gamma(P_i)$ is not 0 for $i = 1,\dots,16$, because otherwise $\gamma(P_i)$, $\gamma(P_{17})$ and $\gamma(P_{21})$ would be collinear, and hence there would exist a linear word of $\mathbf{C}_C$ containing $\{P_i, P_{17}, P_{21}\}$ by Proposition 2.9. Therefore there exist parameters α_1, α_2, β_1, β_2, t_i, s_{ij} $(i = 3,4,5,\ j = 1,\dots,4)$ such that γ is given by Table 6.5. The lines L_ν containing the points $\gamma(l_\nu)$ are defined by

$$
\begin{aligned}
L_1 = \{X = 0\}, \quad L_2 = \{Y = 0\},&\\
&L_3 = \{Y = t_3 X\}, \quad L_4 = \{Y = t_4 X\}, \quad L_5 = \{Y = t_5 X\}.
\end{aligned}
$$

Claim 6.7. $t_5 = 1$.

Consider the quadratic word

$$q_1 := \{7,8,11,12,13,14,17,18\} = \{2,7,12,13,18\} + \{2,8,11,14,17\},$$

which passes through the four points $P_{13}, P_{14}, P_{17}, P_{18}$, and is disjoint from the linear word l_5. The conic curves containing the points $\gamma(P_{13}) = [1,0,1]$, $\gamma(P_{14}) = [1,0,0]$, $\gamma(P_{17}) = [0,1,1]$ and $\gamma(P_{18}) = [0,1,0]$ form a pencil

$$\sigma Z(X+Y+Z) + XY = 0 \qquad (\sigma \in \mathbb{P}^1).$$

The conic curve $Q_1 \subset \mathbb{P}^2$ containing $\gamma(q_1)$ is a member of this pencil. Since Q_1 is nonsingular, the value of the parameter σ corresponding to Q_1 is not 0 nor ∞. Since Q_1 is tangent to the line $L_5 = \{Y = t_5 X\}$, we have $\sigma(1 + t_5) = 0$. Hence $t_5 = 1$.

From the quadratic words that

$\gamma(P_1)$	$=$	$[1, t_5, s_{5,1}]$,	$\gamma(P_{13})$	$=$	$[1, 0, 1]$,
$\gamma(P_2)$	$=$	$[1, t_5, s_{5,2}]$,	$\gamma(P_{14})$	$=$	$[1, 0, 0]$,
$\gamma(P_3)$	$=$	$[1, t_5, s_{5,3}]$,	$\gamma(P_{15})$	$=$	$[\alpha_2, 0, 1]$,
$\gamma(P_4)$	$=$	$[1, t_5, s_{5,4}]$,	$\gamma(P_{16})$	$=$	$[\beta_2, 0, 1]$,
$\gamma(P_5)$	$=$	$[1, t_4, s_{4,1}]$,	$\gamma(P_{17})$	$=$	$[0, 1, 1]$,
$\gamma(P_6)$	$=$	$[1, t_4, s_{4,2}]$,	$\gamma(P_{18})$	$=$	$[0, 1, 0]$,
$\gamma(P_7)$	$=$	$[1, t_4, s_{4,3}]$,	$\gamma(P_{19})$	$=$	$[0, \alpha_1, 1]$,
$\gamma(P_8)$	$=$	$[1, t_4, s_{4,4}]$,	$\gamma(P_{20})$	$=$	$[0, \beta_1, 1]$,
$\gamma(P_9)$	$=$	$[1, t_3, s_{3,1}]$,	$\gamma(P_{21})$	$=$	$[0, 0, 1]$.
$\gamma(P_{10})$	$=$	$[1, t_3, s_{3,2}]$,			
$\gamma(P_{11})$	$=$	$[1, t_3, s_{3,3}]$,			
$\gamma(P_{12})$	$=$	$[1, t_3, s_{3,4}]$,			

TABLE 6.5. Parametric presentation of γ

- contain exactly one of $\{P_{17}, P_{18}\}$,
- contain exactly one of $\{P_{13}, P_{14}\}$, and
- are disjoint from l_5,

we obtain the following relations:

Claim 6.8. $\alpha_1 = \alpha_2 (=: \alpha)$, $\beta_1 = \beta_2 (=: \beta)$, $\alpha + \beta = \alpha\beta$.

Consider, for example, the quadratic word

$$q_2 := \{6, 8, 9, 11, 14, 16, 17, 19\} = \{2, 6, 9, 16, 19\} + \{2, 8, 11, 14, 17\}.$$

Since the conic curve Q_2 containing $\gamma(q_2)$ passes through the points $\gamma(P_{14}) = [1, 0, 0]$ and $\gamma(P_{17}) = [0, 1, 1]$ and is tangent to $L_5 = \{X = Y\}$, it is a member of the web

$$\sigma_1(Y^2 + Z^2) + \sigma_2 XY + \sigma_3(Y^2 + YZ + ZX) = 0 \qquad ([\sigma_1, \sigma_2, \sigma_3] \in \mathbb{P}^2)$$

of conic curves. Since $\gamma(P_{16}) = [\beta_2, 0, 1] \in Q_2$, we have $\beta_2 = \sigma_1/\sigma_3$. Since $\gamma(P_{19}) = [0, \alpha_1, 1] \in Q_2$ and $\alpha_1 \neq 1$, we have $\alpha_1 = \sigma_1/(\sigma_1 + \sigma_3)$. Therefore we obtain a relation $\alpha_1 + \beta_2 + \alpha_1\beta_2 = 0$.

From the quadratic words that contain exactly three of $P_{17}, P_{18}, P_{13}, P_{14}$, we obtain the following relations:

Claim 6.9. $\alpha_1 + t_3 = 0$, $\quad \beta_1 + t_4 = 0$.
$1 + \alpha_2 t_4 = 0$, $\quad 1 + \beta_2 t_3 = 0$.
$1 + \alpha_2 + \alpha_2 t_3 = 0$, $\quad 1 + \beta_2 + \beta_2 t_4 = 0$.
$\alpha_1 + t_4 + \alpha_1 t_4 = 0$, $\quad \beta_1 + t_3 + \beta_1 t_3 = 0$.

Consider, for example, the quadratic word

$$q_3 := \{2, 4, 10, 12, 13, 16, 17, 18\} = \{4, 7, 10, 16, 17\} + \{2, 7, 12, 13, 18\}.$$

Since the conic curve Q_3 containing $\gamma(q_3)$ passes through the points $\gamma(P_{13}) = [1, 0, 1]$, $\gamma(P_{17}) = [0, 1, 1]$ and $\gamma(P_{18}) = [0, 1, 0]$, it is a member of the web

$$\sigma_1(X^2 + Z^2 + YZ) + \sigma_2(X^2 + ZX) + \sigma_3 XY = 0 \qquad ([\sigma_1, \sigma_2, \sigma_3] \in \mathbb{P}^2)$$

of conic curves. Since $\gamma(P_{16}) = [\beta_2, 0, 1]$ is contained in Q_3, we obtain ${\beta_2}^2(\sigma_1 + \sigma_2) + \beta_2\sigma_2 + \sigma_1 = 0$. Since Q_3 is tangent to the line $L_4 = \{Y = t_4X\}$, we have $t_4\sigma_1 + \sigma_2 = 0$. Combining these two relations and $\beta_2 \neq 1$, we obtain a relation $1 + \beta_2 + \beta_2 t_4 = 0$.

Combining Claims 6.7-6.9, we obtain the following two possibilities for the parameters;

$$\begin{aligned}
&\alpha_1 = \alpha_2 = \omega,\ \ \beta_1 = \beta_2 = \bar{\omega},\ \ t_3 = \omega,\ \ t_4 = \bar{\omega},\ \ t_5 = 1, \quad \text{or} \\
&\alpha_1 = \alpha_2 = \bar{\omega},\ \ \beta_1 = \beta_2 = \omega,\ \ t_3 = \bar{\omega},\ \ t_4 = \omega,\ \ t_5 = 1.
\end{aligned}$$

If the latter holds, then we replace γ by $\gamma \circ T$ so that we assume that the former always holds.

Next we put

$$P_1 = [1, 1, \lambda],$$

where $\lambda = s_{5,1}$ is a parameter. Using quadratic words that

- contain exactly four points among $l_1 \cup l_2$, and
- are not disjoint from l_5,

we obtain the following:

Claim 6.10.

$$\begin{aligned}
&s_{5,1} = \lambda, \quad s_{5,2} = \lambda + 1, \quad s_{5,3} = \lambda + \omega, \quad s_{5,4} = \lambda + \bar{\omega}, \\
&s_{4,1} = \omega\lambda + \omega, \quad s_{4,2} = \omega\lambda, \quad s_{4,3} = \omega\lambda + 1, \quad s_{4,4} = \omega\lambda + \bar{\omega}, \\
&s_{3,1} = \bar{\omega}\lambda + \bar{\omega}, \quad s_{3,2} = \bar{\omega}\lambda, \quad s_{3,3} = \bar{\omega}\lambda + \omega, \quad s_{3,4} = \bar{\omega}\lambda + 1.
\end{aligned}$$

Consider, for example, the quadratic word

$$q_4 := \{1, 2, 11, 12, 14, 16, 17, 20\} = \{1, 8, 12, 16, 20\} + \{2, 8, 11, 14, 17\},$$

which is disjoint from l_4. Because there exists a conic curve Q_4 that contains $\gamma(q_4)$ and is tangent to the line $L_4 = \{Y = \bar{\omega}X\}$, the following matrix $\widetilde{M}$ is of rank < 6.

$$\widetilde{M} := \begin{bmatrix}
1 & 1 & \lambda^2 & 1 & \lambda & \lambda \\
1 & 1 & {s_{5,2}}^2 & 1 & s_{5,2} & s_{5,2} \\
1 & \bar{\omega} & {s_{3,3}}^2 & \omega & \omega\, s_{3,3} & s_{3,3} \\
1 & \bar{\omega} & {s_{3,4}}^2 & \omega & \omega\, s_{3,4} & s_{3,4} \\
1 & 0 & 0 & 0 & 0 & 0 \\
\omega & 0 & 1 & 0 & 0 & \bar{\omega} \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & \omega & 1 & 0 & \bar{\omega} & 0 \\
0 & 0 & 0 & 0 & \bar{\omega} & 1
\end{bmatrix}.$$

Indeed, if the equation

$$a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5YZ + a_6ZX = 0$$

defines a conic curve containing $\gamma(q_4)$ and tangent to L_4, then $\mathbf{a} = {}^t[a_1, a_2, \dots, a_6]$ is a non-zero solution of $\widetilde{M}\mathbf{x} = \mathbf{0}$. (The condition $\bar{\omega}a_5 + a_6 = 0$ is equivalent to the condition that the conic curve is tangent to L_4.) Let $\widetilde{M}[i_1, \dots, i_6]$ denote the submatrix of $\widetilde{M}$ consisting of i_j-th rows of $\widetilde{M}$. Because

$$\det \widetilde{M}[1,2,5,7,8,9] = (s_{5,2} + \lambda)\,(s_{5,2} + \lambda + 1)$$

and $s_{5,2} \neq s_{5,1} = \lambda$, we obtain $s_{5,2} = \lambda + 1$. Because

$$\det \widetilde{M}[1,3,5,7,8,9] = (s_{3,3} + \bar{\omega}\lambda + 1)\,(s_{3,3} + \bar{\omega}\lambda + \omega)\,,$$

we obtain

$$s_{3,3} = \bar{\omega}\lambda + 1 \quad \text{or} \quad \bar{\omega}\lambda + \omega.$$

Continuing the same calculations, we get the relations in Claim 6.10.

Thus we have proved Claim 6.6.

Conversely, suppose that $\lambda \in k \setminus \{0, 1, \omega, \bar{\omega}\}$ is given. Then $\gamma_\lambda(\mathcal{P})$ is equal to $Z(dGC[\lambda])$, where $GC[\lambda]$ is given in Theorem 1.8. Moreover, for every linear word l of $\mathbf{C}_C$, there exists a line containing $\gamma_\lambda(l)$, and for every quadratic word q of $\mathbf{C}_C$, there exists a unique conic curve containing $\gamma_\lambda(q)$. The defining equations of the 120 conic curves are omitted. These conic curves are nonsingular because $\lambda \notin \{0, 1, \omega, \bar{\omega}\}$. Hence γ_λ is in $\mathcal{G}_B$ by Proposition 3.6. □

Remark 6.11. When $\lambda \in \{0, 1, \omega, \bar{\omega}\}$, the set $\gamma_\lambda(\mathcal{P})$ coincides with $\mathbb{P}^2(\mathbb{F}_4)$, and the point $[GC[\lambda]] \in \mathfrak{M}$ is equal to the Dolgachev-Kondo point.

For each $\sigma \in \mathrm{Aut}(\mathbf{C}_C)$, we calculate the unique triple

$$(g_\sigma, t_\sigma, \lambda^\sigma) \;\in\; PGL(3, k(\lambda)) \times \widetilde{T} \times k(\lambda)$$

such that $g_\sigma \circ (\gamma_\lambda \circ \sigma) \circ t_\sigma = \gamma_{\lambda^\sigma}$ holds. The map $\sigma \mapsto t_\sigma$ is a homomorphism from $\mathrm{Aut}(\mathbf{C}_C)$ to $\widetilde{T}$. We put $N_C := \mathrm{Ker}(\mathrm{Aut}(\mathbf{C}_C) \to \widetilde{T})$.

Corollary 6.12. *The space $PGL(3,k)\backslash\mathcal{G}_C$ has exactly two connected components*

$$(PGL(3,k)\backslash\mathcal{G}_C)^+ := \{\ [\gamma_\alpha] \ \mid\ \alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}\ \}$$

and $((PGL(3,k)\backslash\mathcal{G}_C)^+) \cdot T$, each of which is isomorphic to $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$. The group N_C acts on $(PGL(3,k)\backslash\mathcal{G}_C)^+$, and the moduli curve $\mathfrak{M}_C$ is equal to the quotient space $(PGL(3,k)\backslash\mathcal{G}_C)^+/N_C$.

Consider the natural projection

$$p_C\ :\ \mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\} \cong (PGL(3,k)\backslash\mathcal{G}_C)^+ \ \to\ \mathfrak{M}_C = (PGL(3,k)\backslash\mathcal{G}_C)^+/N_C.$$

For $\alpha \in k \setminus \{0,1,\omega,\bar{\omega}\}$, let $P[\alpha]$ denote the point of $\mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$ given by $\lambda = \alpha$. Then $p_C(P[\alpha]) \in \mathfrak{M}_C$ corresponds to the isomorphism class of the polarized supersingular $K3$ surface $(X_{GC[\alpha]}, \mathcal{L}_{GC[\alpha]})$.

Proposition 6.13. *We have*

$$p_C^{-1}(p_C(P[\alpha])) = \{\ P[u\alpha + v] \ \mid\ u \in \mathbb{F}_4^\times, v \in \mathbb{F}_4\ \}.$$

The group $\mathrm{Aut}(X_{GC[\alpha]}, \mathcal{L}_{GC[\alpha]})$ *is equal to the subgroup* (1.3) *of* $PGL(3,k)$.

Corollary 6.14. *We have $\mathfrak{M}_C = \mathrm{Spec}\, k[J_C, 1/J_C]$, where $J_C := (\lambda^4 + \lambda)^3$. The morphism p_C is an étale Galois covering with Galois group Γ_C.*

7. Cremona transformations by quintic curves

7.1. **Preliminaries.** Let Σ_1 and Σ_2 be disjoint sets of reduced points of $\mathbb{P}^2$ with $|\Sigma_1| = n_1$ and $|\Sigma_2| = n_2$, and let $\mathcal{I}_{\Sigma_1} \subset \mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{I}_{\Sigma_2} \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaves defining Σ_1 and Σ_2. We define $\widetilde{\Sigma}$ to be the 0-dimensional subscheme of $\mathbb{P}^2$ defined by the ideal sheaf

$$\mathcal{I}_{\widetilde{\Sigma}} := \mathcal{I}_{\Sigma_1}\mathcal{I}_{\Sigma_2}^2.$$

The length of $\mathcal{O}_{\widetilde{\Sigma}}$ is n_1+3n_2. Let d be a positive integer. The linear system $|\mathcal{I}_{\widetilde{\Sigma}}(d)|$ consists of plane curves of degree d that pass through the points of $\Sigma_1 \cup \Sigma_2$ and are singular at each point of Σ_2.

Proposition 7.1. *Suppose that the linear system $|\mathcal{I}_{\widetilde{\Sigma}}(d)|$ is of dimension ≥ 1 and has no fixed components. If*

$$\dim |\mathcal{I}_{\widetilde{\Sigma}}(d)| > \frac{(d+2)(d+1)}{2} - (n_1+3n_2) - 1, \tag{7.1}$$

then there exists a projective plane curve of degree $d-3$ that passes through all the points of Σ_2.

Corollary 7.2. *Suppose that the linear system $|\mathcal{I}_{\widetilde{\Sigma}}(d)|$ is of dimension ≥ 1 and has no fixed components. If $d \leq 3$ and $n_2 > 0$, then the dimension of the linear system $|\mathcal{I}_{\widetilde{\Sigma}}(d)|$ is equal to $(d+2)(d+1)/2 - (n_1+3n_2) - 1$.*

Proof. We follow the argument in [6, pp.712-714]. From the exact sequence

$$0 \to \mathcal{I}_{\widetilde{\Sigma}}(d) \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_{\widetilde{\Sigma}}(d) \to 0,$$

we obtain

$$h^0(\mathbb{P}^2, \mathcal{I}_{\widetilde{\Sigma}}(d)) = (d+2)(d+1)/2 - (n_1+3n_2) + h^1(\mathbb{P}^2, \mathcal{I}_{\widetilde{\Sigma}}(d)). \tag{7.2}$$

Let $\beta : S \to \mathbb{P}^2$ denote the blowing up of $\mathbb{P}^2$ at the points of $\Sigma_1 \cup \Sigma_2$. We put

$$\Delta_1 := \beta^{-1}(\Sigma_1), \quad \Delta_2 := \beta^{-1}(\Sigma_2),$$

both of which are considered to be reduced divisors. Let $H \subset S$ be the pull-back of a general line on $\mathbb{P}^2$. We put

$$L := \beta^*\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_S(-\Delta_1 - 2\Delta_2) = \mathcal{O}_S(dH - \Delta_1 - 2\Delta_2).$$

Because $K_S = -3H + \Delta_1 + \Delta_2$, we have

$$L^2 = d^2 - n_1 - 4n_2, \qquad LK_S = -3d + n_1 + 2n_2.$$

The complete linear system $|H| = |\beta^*\mathcal{O}_{\mathbb{P}^2}(1)|$ on S is fixed component free. Since $H(K_S - L) = -d - 3 < 0$, we have

$$h^2(S, L) = h^0(S, K_S - L) = 0.$$

By the Riemann-Roch theorem, we obtain

$$h^0(S, L) = (d+2)(d+1)/2 - (n_1+3n_2) + h^1(S, L). \tag{7.3}$$

There exists a canonical isomorphism

$$|\mathcal{I}_{\widetilde{\Sigma}}(d)| \cong |L| \tag{7.4}$$

that maps a member C of $|\mathcal{I}_{\widetilde{\Sigma}}(d)|$ to the member $\beta^*C - \Delta_1 - 2\Delta_2$ of $|L|$. From (7.2), (7.3) and (7.4), we obtain

$$h^1(\mathbb{P}^2, \mathcal{I}_{\widetilde{\Sigma}}(d)) = h^1(S, L). \tag{7.5}$$

Using the assumption (7.1) and the equalities (7.2) and (7.5), we obtain

$$h^1(S, L) > 0. \tag{7.6}$$

Since $|\mathcal{I}_{\widetilde{\Sigma}}(d)|$ is of dimension ≥ 1 and has no fixed components, we obtain by the isomorphism (7.4) global sections s and s' of L such that the subscheme $R = \{s = s' = 0\}$ of S is of dimension 0. Let $\mathcal{I}_R \subset \mathcal{O}_S$ be the ideal sheaf defining R. From the Koszul complex

$$0 \longrightarrow \mathcal{O}_S(K_S - L) \xrightarrow{(s,s')} \mathcal{O}_S(K_S) \oplus \mathcal{O}_S(K_S) \xrightarrow{{}^T(-s',s)} \mathcal{I}_R(K_S + L) \longrightarrow 0$$

and $h^0(S, \mathcal{O}_S(K_S)) = h^1(S, \mathcal{O}_S(K_S)) = 0$, we obtain

$$h^1(S, L) = h^1(S, \mathcal{O}_S(K_S - L)) = h^0(\mathcal{I}_R(K_S + L)).$$

From (7.6), we see that the linear system $|\mathcal{I}_R(K_S+L)|$ is non-empty. Since $K_S+L = \beta^*\mathcal{O}_{\mathbb{P}^2}(d-3)\otimes\mathcal{O}_S(-\Delta_2)$, a member of $|\mathcal{I}_R(K_S+L)|$ is mapped by β to a projective plane curve of degree $d-3$ that passes through the points of Σ_2. □

Definition 7.3. Let F be an effective divisor of $\mathbb{P}^2$. We put

$$\Sigma_1' := (\Sigma_1 \setminus (\Sigma_1 \cap F)) \cup (\Sigma_2 \cap F^0), \qquad \Sigma_2' := \Sigma_2 \setminus (\Sigma_2 \cap F),$$

where F^0 is the locus of all $p \in \operatorname{Supp}(F)$ at which F is reduced and nonsingular. We then define $\widetilde{\Sigma} \setminus F$ to be the 0-dimensional subscheme of $\mathbb{P}^2$ defined by the ideal sheaf

$$\mathcal{I}_{\widetilde{\Sigma}\setminus F} := \mathcal{I}_{\Sigma_1'}\mathcal{I}^2_{\Sigma_2'}.$$

If F is a fixed component of $|\mathcal{I}_{\widetilde{\Sigma}}(d)|$, then $C \mapsto C - F$ gives an isomorphism

$$|\mathcal{I}_{\widetilde{\Sigma}}(d)| \;\cong\; |\mathcal{I}_{\widetilde{\Sigma}\setminus F}(d - \deg F)|$$

of linear systems. By the definition, we have

$$\widetilde{\Sigma} \setminus (F_1 + F_2) = (\widetilde{\Sigma} \setminus F_1) \setminus F_2 \tag{7.7}$$

for any effective (not necessarily distinct) divisors F_1 and F_2 of $\mathbb{P}^2$.

7.2. **A homaloidal system of quintic curves.** Let $\Sigma = \{p_1, \dots, p_6\}$ be a set of distinct six points of $\mathbb{P}^2$ satisfying the following:

(Σ1) no three points of Σ are collinear, and
(Σ2) there are no conic curves containing Σ.

These conditions are equivalent to the following:

(Σ3) for each $p_i \in \Sigma$, there exists a nonsingular conic curve $N_i' \subset \mathbb{P}^2$ that contains $\Sigma \setminus \{p_i\}$ and does not contain p_i.

Proposition 7.4. *The linear system $|\mathcal{I}^2_{\Sigma}(5)|$ of quintic curves that pass through the points of Σ and are singular at each point of Σ is of dimension 2, and has no fixed components.*

Proof. Because each point of Σ imposes three linear conditions on $|\mathcal{O}_{\mathbb{P}^2}(5)|$, we have $\dim |\mathcal{I}_{\widetilde{\Sigma}}(5)| \geq 2$.

Suppose that $|\mathcal{I}^2_{\Sigma}(5)|$ has a fixed component. Let F be the fixed component, and let

$$F = F_1 + \cdots + F_N$$

be the decomposition into the reduced irreducible components of F, where non-reduced components are expressed by repetition of summation. We have

$$\deg F = \deg F_1 + \cdots + \deg F_N \quad > \quad 0.$$

As in the previous subsection, we denote by $\widetilde{\Sigma}$ the 0-dimensional subscheme of $\mathbb{P}^2$ defined by the ideal sheaf $\mathcal{I}_\Sigma^2$. We will consider the linear system $|\mathcal{I}_{\widetilde{\Sigma}\setminus F}(5-\deg F)|$, which has no fixed components and is of dimension equal to $\dim|\mathcal{I}_{\widetilde{\Sigma}}(5)| \geq 2$. For $\nu = 0, \dots, N$, we define reduced 0-dimensional subschemes $\Sigma_1^{(\nu)}$ and $\Sigma_2^{(\nu)}$ of $\mathbb{P}^2$ by

$$\mathcal{I}_{\widetilde{\Sigma}\setminus(F_1+\cdots+F_\nu)} = \mathcal{I}_{\Sigma_1^{(\nu)}}\mathcal{I}^2_{\Sigma_2^{(\nu)}}.$$

Then

$$|\Sigma_1^{(\nu+1)}| = |\Sigma_1^{(\nu)}| - i + j - k, \quad |\Sigma_2^{(\nu+1)}| = |\Sigma_2^{(\nu)}| - j,$$

where

$$i := |\Sigma_1^{(\nu)} \cap F_{\nu+1}|, \quad j := |\Sigma_2^{(\nu)} \cap F_{\nu+1}|, \quad k := |\Sigma_2^{(\nu)} \cap \operatorname{Sing} F_{\nu+1}|.$$

The integers i, j and k are subject to the following conditions:

- $i + j \leq 2$ and $k = 0$ if $\deg F_{\nu+1} = 1$, because of (Σ1),
- $i + j \leq 5$ and $k = 0$ if $\deg F_{\nu+1} = 2$, because of (Σ2),
- $k \leq 1$ if $\deg F_{\nu+1} = 3$, because an irreducible cubic curve has at most one singular point, and
- $k \leq 4$ if $\deg F_{\nu+1} = 4$, because if $k \geq 5$, there would exist a conic curve C with $CF_{\nu+1} \geq 10$.

Since $|\mathcal{I}_\Sigma^2(5)| \cong |\mathcal{I}_{\Sigma_1^{(N)}}\mathcal{I}^2_{\Sigma_2^{(N)}}(5 - \deg F)|$ is of dimension ≥ 2 and fixed component free, we have

$$\begin{aligned}
\deg F = 4 \quad &\Longrightarrow \quad |\Sigma_1^{(N)}| \leq 1 \quad \text{and} \quad |\Sigma_2^{(N)}| = 0,\\
\deg F = 3 \quad &\Longrightarrow \quad |\Sigma_1^{(N)}| \leq 4 \quad \text{and} \quad |\Sigma_2^{(N)}| = 0.
\end{aligned}$$

We put

$$\delta := (6 - \deg F)(7 - \deg F)/2 - (|\Sigma_1^{(N)}| + 3|\Sigma_2^{(N)}|) - 1.$$

From Corollary 7.2, we also have

$$\deg F \geq 2 \quad \text{and} \quad 2 > \delta \quad \Longrightarrow \quad |\Sigma_2^{(N)}| = 0.$$

Using these considerations, we see that the triple $(|\Sigma_1^{(N)}|, |\Sigma_2^{(N)}|, \deg F)$ is one of the following:

$$(0, 6, 1), \quad (1, 5, 1), \quad (2, 4, 1).$$

For these triples, however, we have $|\mathcal{I}_{\widetilde{\Sigma}\setminus F}(5 - \deg F)| = \emptyset$, because otherwise, there would exist an irreducible quartic curve C_4 and a conic curve C_2 such that $C_4C_2 > 8$. Thus we have proved that $|\mathcal{I}_{\widetilde{\Sigma}}(5)|$ is fixed component free.

If $\dim|\mathcal{I}_{\widetilde{\Sigma}}(5)| > 2$, then, by Proposition 7.1, there would exists a conic curve that contains Σ, which contradicts (Σ2). $\square$

Remark 7.5. Recall from (Σ3) that $N_i' \subset \mathbb{P}^2$ is the conic curve such that $N_i' \cap \Sigma = \Sigma \setminus \{p_i\}$. Let Q be a general member of $|\mathcal{I}_\Sigma^2(5)|$. Since $N_i'Q = 10$ for each i, the multiplicity of Q at each point of Σ is 2.

Let $\beta : S \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the points of Σ, and let M_i be the exceptional (reduced) divisor $\beta^{-1}(p_i)$. We put

$$L := \beta^* \mathcal{O}_{\mathbb{P}^2}(5) \otimes \mathcal{O}_S(-2M_1 - \cdots - 2M_6).$$

Let N_i be the strict transform of N'_i by β. We have $L^2 = 1$, $N_i L = 0$ and $N_i^2 = -1$.

Proposition 7.6. *The complete linear system $|L|$ on S has no base points, and the morphism $\Phi_{|L|} : S \to \mathbb{P}^2$ defined by $|L|$ is the contraction of the curves $N_1, \dots, N_6$. Let p'_i be the image of N_i by $\Phi_{|L|}$. Then $\Sigma' = \{p'_1, \dots, p'_6\}$ satisfies the condition* ($\Sigma 3$).

Proof. By Proposition 7.4, the complete linear system $|L|$ on S is of dimension 2 and has no fixed components. Suppose that $|L|$ has a base point $p \in S$. Let $\tilde{\beta} : \widetilde{S} \to S$ be the blowing up of S at p, and let M' be the exceptional divisor $\tilde{\beta}^{-1}(p)$ of $\tilde{\beta}$. Since $L^2 = 1$, the complete linear system $|\widetilde{L}|$ of the line bundle $\widetilde{L} := \tilde{\beta}^* L \otimes \mathcal{O}_{\widetilde{S}}(-M')$ is of dimension 2 and has no fixed components. We have $K_{\widetilde{S}} \widetilde{L} = -2$, and hence $h^2(\widetilde{S}, \widetilde{L}) = h^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}} - \widetilde{L})) = 0$ follows. By Riemann-Roch theorem, we have $h^1(\widetilde{S}, \widetilde{L}) = h^1(\widetilde{S}, K_{\widetilde{S}} - \widetilde{L}) = 1$. Using the argument of Koszul complex as in the proof of Proposition 7.1, we see that $h^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{L})) > 0$. Hence there exists a conic curve in $\mathbb{P}^2$ that passes through the points of Σ, which contradicts ($\Sigma 2$). Thus $|L|$ has no base points.

Since $L^2 = 1$, the morphism $\Phi_{|L|}$ is of degree 1. Because $N_i L = 0$, the curves N_i are contracted by $\Phi_{|L|}$. Let C be a reduced irreducible curve on S that is contracted by $\Phi_{|L|}$. Because $M_i L = 2$, we have $C \neq M_i$ and hence $C' := \beta(C) \subset \mathbb{P}^2$ is a reduced irreducible curve. We will show that C' is equal to one of the conic curves N'_i. Let d be the degree of C'. We have

$$\beta^*(C') = C + m_1 M_1 + \cdots + m_6 M_6,$$

where m_j is the multiplicity of C' at p_j. The condition $CL = 0$ implies

$$5d = 2(m_1 + \cdots + m_6).$$

If C' is not equal to N'_i for any i, then

$$C' N'_i = 2d \geq (m_1 + \cdots + m_6) - m_i = 5d/2 - m_i$$

holds for each i. Hence $2m_i \geq d$ for $i = 1, \dots, 6$. Therefore $5d = 2 \sum m_j \geq 6d$, which is absurd. Thus we have proved that $\Phi_{|L|}$ is the contraction of the (-1)-curves $N_1, \dots, N_6$.

Since $M_i L = 2$, the image of M_i by $\Phi_{|L|}$ is a nonsingular conic curve. Because $M_i N_j = 0$ if and only if $i = j$, the conic curve $\Phi_{|L|}(M_i)$ satisfies

$$\Phi_{|L|}(M_i) \cap \Sigma' = \Sigma' \setminus \{p'_i\}.$$

Hence Σ' satisfies ($\Sigma 3$). □

Corollary 7.7. *The rational map*

$$\mathrm{CT}_\Sigma : \mathbb{P}^2 \cdots \to \mathbb{P}^2$$

defined by the linear system $|\mathcal{I}_\Sigma^2(5)|$ is birational, and the inverse map is given by $\mathrm{CT}_{\Sigma'}$.

We will write

$$\beta' \,:\, S \to \mathbb{P}^2$$

instead of $\Phi_{|L|}$. Let H and H' be the pull-backs of a general line of $\mathbb{P}^2$ by β and β', respectively. We put

$$M'_i := \beta'(M_i),$$

which is a nonsingular conic curve containing $\Sigma' \setminus \{p'_i\}$ and not passing through p'_i. We also put

$$U := S \setminus \left(\bigcup_{i=1}^{6} N_i \cup \bigcup_{i=1}^{6} M_i \right).$$

The morphisms β and β' induce isomorphisms

$$\mathbb{P}^2 \setminus \cup N'_i \;\cong\; U \;\cong\; \mathbb{P}^2 \setminus \cup M'_i. \tag{7.8}$$

The Picard group $\operatorname{Pic} S$ of S is a free $\mathbb{Z}$-module of rank 7, and is generated by the linear equivalence classes $[H], [M_1], \dots, [M_6]$, or by the linear equivalence classes $[H'], [N_1], \dots, [N_6]$. They are related by

$$[H'] = 5[H] - 2\sum_{j=1}^{6}[M_j], \quad [N_i] = 2[H] - \sum_{j=1}^{6}[M_j] + [M_i] \quad (i = 1, \dots, 6).$$

In particular, we have

$$3[H] - \sum_{j=1}^{6}[M_j] = 3[H'] - \sum_{j=1}^{6}[N_j] \tag{7.9}$$

in $\operatorname{Pic} S$.

7.3. **Cremona transformations of supersingular $K3$ surfaces.** Let G be a homogeneous polynomial in $\mathcal{U}$, and let $\Sigma = \{p_1, \dots, p_6\}$ be a subset of $Z(dG)$ with $|\Sigma| = 6$. We assume that Σ satisfies the condition $(\Sigma 1)$ and

$(\Sigma 2)'$ for each $p_i \in \Sigma$, the nonsingular conic curve N'_i containing $\Sigma \setminus \{p_i\}$ satisfies $N'_i \cap Z(dG) = \Sigma \setminus \{p_i\}$.

Then the subset

$$Z' := \mathrm{CT}_\Sigma(Z(dG) \setminus \Sigma) \,\cup\, \Sigma'$$

of $\mathbb{P}^2$ is well-defined and consists of 21 points.

Proposition 7.8. *There exists $G' \in \mathcal{U}$ such that $Z' = Z(dG')$.*

For the proof of Proposition 7.8, we first show the following:

Lemma 7.9. *There exists $G_1 \in \mathcal{U}$ that satisfies $Z(dG) = Z(dG_1)$ and $G_1(p_i) = 0$ for each $p_i \in \Sigma$.*

Proof. By $(\Sigma 1)$ and $(\Sigma 2)'$, the points of Σ impose independent linear conditions on the linear system $|\mathcal{O}_{\mathbb{P}^2}(3)|$. (See [6, p. 715].) Hence there exists a homogeneous cubic polynomial H such that $(G + H^2)(p_i) = 0$ holds for each $p_i \in \Sigma$. Then $G_1 := G + H^2$ satisfies $Z(dG) = Z(dG_1)$. $\square$

We replace G by G_1 in Lemma 7.9. Then the sextic curve D defined by $G = 0$ is reduced and has an ordinary node at each point of Σ. Hence

$$\widetilde{D} := \beta^* D - 2\sum_{j=1}^{6} M_j$$

is a reduced effective divisor of S, and it does not contain any of M_j.

Proof of Proposition 7.8. Let D' be the image of $\widetilde{D}$ by β'. Since $\widetilde{D}H' = 6$, D' is a reduced curve of degree 6. Let $G' = 0$ be the defining equation of D'. We will show that $Z' = Z(dG')$. It is enough to show that $Z(dG')$ is of dimension 0 and that $Z' \subseteq Z(dG')$.

Since $\widetilde{D}N_j = 2$ for each N_j, we have

$$\widetilde{D} := \beta'^* D' - 2\sum_{j=1}^{6} N_j.$$

Because $\widetilde{D}$ is effective, we have $\operatorname{Sing} D' \supset \Sigma'$. Hence $\Sigma' \subseteq Z(dG')$. We put

$$\sqrt{L(H)} := \mathcal{O}_S(3H - \sum_{j=1}^{6} M_j) = \mathcal{O}_S(3H' - \sum_{j=1}^{6} N_j).$$

Let $\widetilde{G}$ be a global section of

$$\sqrt{L(H)}^{\otimes 2} = L(H) = \mathcal{O}_S(6H - 2\sum M_j) = \mathcal{O}_S(6H' - 2\sum N_j)$$

such that $\widetilde{G} = 0$ defines $\widetilde{D}$. Let m and n be global sections of $\mathcal{O}_S(\sum M_j)$ and $\mathcal{O}_S(\sum N_j)$ such that $\sum M_j = \{m = 0\}$ and $\sum N_j = \{n = 0\}$. We can choose them so that

$$\beta^* G = \widetilde{G} \cdot m^2 \quad \text{and} \quad \beta'^* G' = \widetilde{G} \cdot n^2 \tag{7.10}$$

hold. We define isomorphisms

$$\beta^* \mathcal{O}_{\mathbb{P}^2}(3)|U \;\cong\; \sqrt{L(H)}|U \;\cong\; \beta'^* \mathcal{O}_{\mathbb{P}^2}(3)|U \tag{7.11}$$

of line bundles on U by multiplications by m and n. We can define $d\widetilde{G}$, $d(\beta^* G)$ and $d(\beta'^* G')$ as global sections of the vector bundles $\Omega^1_S \otimes \sqrt{L(H)}^{\otimes 2}$, $\Omega^1_S \otimes \beta^* \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2}$ and $\Omega^1_S \otimes \beta'^* \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2}$, respectively. By (7.10) and (7.11), we get

$$\begin{aligned} & \beta^{-1}(Z(dG) \setminus \Sigma) = \beta^{-1}(Z(dG) \cap (\mathbb{P}^2 \setminus \cup N'_j)) = \beta^{-1}(Z(dG)) \cap U \\ = \;& Z(d(\beta^* G|U)) = Z(d(\widetilde{G}|U)) = Z(d(\beta'^* G'|U)) \\ = \;& \beta'^{-1}(Z(dG')) \cap U = \beta'^{-1}(Z(dG') \cap (\mathbb{P}^2 \setminus \cup M'_j)). \end{aligned}$$

Hence we get $Z(dG') \cap (\mathbb{P}^2 \setminus \cup M'_j) = \mathrm{CT}_\Sigma(Z(dG) \setminus \Sigma)$. In particular, we have $\mathrm{CT}_\Sigma(Z(dG) \setminus \Sigma) \subset Z(dG')$.

If $\dim Z(dG') > 0$, then one of the conic curves M'_j is contained in $Z(dG')$. Suppose that $M'_k \subset Z(dG')$. Then $M_k \subset Z(d\widetilde{G})$ holds. We choose affine coordinates (x, y) of $\mathbb{P}^2$ such that $p_k = \beta(M_k)$ is the origin. Let

$$g(x, y) = \sum_{i+j \le 6} a_{ij} x^i y^j$$

be the inhomogeneous polynomial corresponding to G. Since $p_k = (0, 0) \in \Sigma$ is contained in $\operatorname{Sing} D$, we have

$$a_{ij} = 0 \quad \text{for} \quad i + j \le 1.$$

Let the blowing up β be given by

$$(u, v) \mapsto (x, y) = (uv, v)$$

around a point of M_k. Then $\widetilde{G}$ is written in terms of the coordinates (u,v) as

$$\tilde{g}(u,v)=\beta^* g/v^2=\sum_{2\le i+j\le 6} a_{ij}u^i v^{i+j-2}.$$

Since $d\widetilde{G}$ is zero along the curve $M_k=\{v=0\}$ by the assumption, we have

$$\frac{\partial \tilde{g}}{\partial u}(u,0)=a_{11}=0.$$

This contradicts the fact that p_k is a reduced point of $Z(dG)$. □

Proposition 7.10. *Let G and G' be as above. Then the Cremona transformation CT_Σ of $\mathbb{P}^2$ lifts to an isomorphism*

$$\widetilde{\mathrm{CT}}_\Sigma \;:\; X_G \;\xrightarrow{\sim}\; X_{G'}$$

of supersingular $K3$ surfaces.

Proof. Let $\widetilde{Y}$ be the subvariety of the total space of the line bundle $\sqrt{L(H)}$ defined by $W^2=\widetilde{G}$, where $\widetilde{G}$ is the global section of $\sqrt{L(H)}^{\otimes 2}=L(H)$ introduced in the proof of Proposition 7.8, and W is the fiber coordinate of $\sqrt{L(H)}$. From (7.10) and (7.11), we obtain isomorphisms

$$X_G|(\mathbb{P}^2\setminus\cup N'_j)\;\cong\;\widetilde{Y}|U\;\cong\;X_{G'}|(\mathbb{P}^2\setminus\cup M'_j)$$

that are compatible with the isomorphisms (7.8). Since $K3$ surfaces are minimal, the isomorphism between Zariski open subsets of X_G and $X_{G'}$ extends to an isomorphism between X_G and $X_{G'}$. □

Remark 7.11. We describe the action of $\widetilde{\mathrm{CT}}_\Sigma$ on the numerical Néron-Severi lattices of the supersingular $K3$ surfaces. We number the points of $Z(dG)$ and $Z(dG')$ in such a way that

$$\Sigma=\{p_1,\dots,p_6\},\qquad Z(dG)=\Sigma\cup\{p_7,\dots,p_{21}\},$$
$$\Sigma'=\{p'_1,\dots,p'_6\},\qquad Z(dG')=\Sigma'\cup\{p'_7,\dots,p'_{21}\},$$

where $p'_i=\mathrm{CT}_\Sigma(p_i)$ for $i=7,\dots,21$. Let $E_i\subset X_G$ be the (-2)-curve that is contracted to $p_i\in\mathbb{P}^2$, and $E'_i\subset X_{G'}$ the (-2)-curve that is contracted to $p'_i\in\mathbb{P}^2$. Then $NS(X_G)\otimes_{\mathbb{Z}}\mathbb{Q}$ is generated by $[E_1],\dots,[E_{21}],[\mathcal{L}_G]$, and $NS(X_{G'})\otimes_{\mathbb{Z}}\mathbb{Q}$ is generated by $[E'_1],\dots,[E'_{21}],[\mathcal{L}_{G'}]$. Since $\mathrm{CT}_\Sigma(p_i)=p'_i$ for $i>6$, we have

$$\widetilde{\mathrm{CT}}_\Sigma^*([E'_i])=[E_i]\quad\text{for}\quad i>6.$$

The exceptional curve N_i on S contracted to p'_i by $\beta' : S\to\mathbb{P}^2$ is mapped by $\beta : S\to\mathbb{P}^2$ to the nonsingular conic curve N'_i such that $N'_i\cap Z(dG)=\Sigma\setminus\{p_i\}$. Hence

$$\widetilde{\mathrm{CT}}_\Sigma^*([E'_i])=2[\mathcal{L}_G]-\sum_{j=1}^{6}[E_j]+[E_i]\quad\text{for}\quad i=1,\dots,6.$$

The pull-back of a general line of $\mathbb{P}^2$ by $\mathrm{CT}_\Sigma:\mathbb{P}^2\cdots\to\mathbb{P}^2$ is a quintic curve Q such that $Q\cap Z(dG)=\Sigma$ and that the multiplicity of Q at each point of Σ is 2 (Remark 7.5). Thus

$$\widetilde{\mathrm{CT}}_\Sigma^*([\mathcal{L}_{G'}])=5[\mathcal{L}_G]-2\sum_{j=1}^{6}[E_j].$$

These formula describe the homomorphism $\widetilde{\mathrm{CT}}_{\Sigma}^{*} \otimes_{\mathbb{Z}} \mathbb{Q}$ from $NS(X_{G'}) \otimes_{\mathbb{Z}} \mathbb{Q}$ to $NS(X_G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 7.12. Suppose that the point $[G'] \in \mathfrak{M}$ corresponding to $G' \in \mathcal{U}$ in Proposition 7.8 coincides with the point $[G] \in \mathfrak{M}$. Then the Cremona transformation $\widetilde{\mathrm{CT}}_{\Sigma}$ defines a right coset in $\mathrm{Aut}(X_G)$ with respect to the subgroup $\mathrm{Aut}(X_G, \mathcal{L}_G) \subset \mathrm{Aut}(X_G)$. Indeed, the assumption $[G] = [G']$ implies the existence of a linear isomorphism $g : \mathbb{P}^2 \xrightarrow{\sim} \mathbb{P}^2$ such that $g(Z(dG')) = Z(dG)$. Let $\hat{g} \in GL(3, k)$ be a lift of $g \in PGL(3, k)$. Then there exists $c \in k^{\times}$ and $H \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ such that $\hat{g}^* G = c\, G' + H^2$. Let X_G and $X_{G'}$ be defined by $W^2 = G(X, Y, Z)$ and $W'^2 = G'(X, Y, Z)$, respectively. We have a lift $\tilde{g} : X_{G'} \xrightarrow{\sim} X_G$ of g given by

$$\tilde{g}^* W = \sqrt{c}\, W' + H.$$

The composite $\tilde{g} \circ \widetilde{\mathrm{CT}}_{\Sigma}$ is an automorphism of X_G. Since the linear isomorphism g is unique up to the group

$$\{ \ h \in PGL(3, k) \ \mid \ h(Z(dG)) = Z(dG) \ \} = \mathrm{Aut}(X_G, \mathcal{L}_G),$$

the automorphism $\tilde{g} \circ \widetilde{\mathrm{CT}}_{\Sigma} \in \mathrm{Aut}(X_G)$ is also unique up to $\mathrm{Aut}(X_G, \mathcal{L}_G)$.

8. The isomorphism correspondences by Cremona transformations

8.1. **The action of Cremona transformations on the moduli space.** Let $\mathbf{C}$ be a code satisfying the conditions in (ii) of Theorem 2.4. For $\gamma \in \mathcal{G}_{\mathbf{C}}$, we denote by $G_\gamma \in \mathcal{U}$ a homogeneous polynomial such that $\gamma(\mathcal{P}) = Z(dG_\gamma)$. Let $c \in \mathrm{Pow}(\mathcal{P})$ be a word of weight 6. Recall from Definition 1.11 that $\gamma(c)$ is a center of Cremona transformation for G_γ if no three points of $\gamma(c)$ are collinear and there are no nonsingular conic curves $C \subset \mathbb{P}^2$ such that $|C \cap \gamma(c)| \geq 5$ and $|C \cap \gamma(\mathcal{P})| \geq 6$. By Propositions 2.9, 2.10 and 2.17, we see that the following conditions on a word $c \in \mathrm{Pow}(\mathcal{P})$ of weight 6 is equivalent to each other:

- the word c satisfies the following:
  - (i) $|c \cap l| \leq 2$ for any linear word l of $\mathbf{C}$, and
  - (ii) $|c \cap q| \leq 4$ for any quadratic word q of $\mathbf{C}$,
- there exists $\gamma \in \mathcal{G}_{\mathbf{C}}$ such that $\gamma(c)$ is a center of Cremona transformation for G_γ, and
- for arbitrary $\gamma \in \mathcal{G}_{\mathbf{C}}$, $\gamma(c)$ is a center of Cremona transformation for G_γ.

Definition 8.1. A word $c \in \mathrm{Pow}(\mathcal{P})$ of weight 6 is called a *center of Cremona transformation with respect to* $\mathbf{C}$ if the above conditions are satisfied.

Let c be a center of Cremona transformation with respect to $\mathbf{C}$. For $\gamma \in \mathcal{G}_{\mathbf{C}}$, we put

$$\Sigma := \gamma(c),$$

and consider the Cremona transformation CT_Σ. We put

$$Z'_{\gamma,c} := \{ \ \mathrm{CT}_\Sigma(\gamma(P)) \ \mid \ P \in \mathcal{P} \setminus c \ \} \cup \{p'_1, \ldots, p'_6\},$$

where p'_i is the image of the strict transform $N_i \subset S$ of the conic curve N'_i that contains $\gamma(c) \setminus \{p_i\}$. By Proposition 7.8, there exists a polynomial $G'_{\gamma,c} \in \mathcal{U}$ such that $Z'_{\gamma,c} = Z(dG'_{\gamma,c})$. Even though the polynomial $G'_{\gamma,c}$ is not uniquely determined, the corresponding point $[G'_{\gamma,c}] \in \mathfrak{M}$ is uniquely determined by c and γ. The map

$\gamma \mapsto [G'_{\gamma,c}]$ gives a morphism from $\mathcal{G}_{\mathbf{C}}$ to $\mathfrak{M}$. It is obvious that this morphism descends to the morphism

$$\mathrm{ct}_c : PGL(3,k)\backslash\mathcal{G}_{\mathbf{C}} \to \mathfrak{M}.$$

8.2. **The case where Artin invariant is 2.** Let T be A, B or C, and let $c \in \mathrm{Pow}(\mathcal{P})$ be a center of Cremona transformation with respect to $\mathbf{C}_T$. The image by ct_c of the connected component

$$(PGL(3,k)\backslash\mathcal{G}_T)^+ = \{\ [\gamma_\lambda]\ \mid\ \lambda \in \mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}\ \}$$

of $PGL(3,k)\backslash\mathcal{G}_T$ is a connected component of $\mathfrak{M}_2 = \mathfrak{M}_A \sqcup \mathfrak{M}_B \sqcup \mathfrak{M}_C$, and hence there exists $T' \in \{A,B,C\}$ such that ct_c yields a morphism

$$\mathrm{ct}^+_{T,c} : (PGL(3,k)\backslash\mathcal{G}_T)^+ \to \mathfrak{M}_{T'}.$$

Using $\mathrm{ct}^+_{T,c}$ and the quotient morphism

$$p_T : (PGL(3,k)\backslash\mathcal{G}_T)^+ \to \mathfrak{M}_T$$

by $N_T = \mathrm{Ker}(\mathrm{Aut}(\mathbf{C}_T) \to \widetilde{T})$, we obtain an irreducible isomorphism correspondence

$$D_{T,T'}[c] := \{\ (p_T([\gamma]), \mathrm{ct}^+_{T,c}([\gamma]))\ \mid\ [\gamma] \in (PGL(3,k)\backslash\mathcal{G}_T)^+\ \}\quad\subset\quad \mathfrak{M}_T \times \mathfrak{M}_{T'}.$$

For $\sigma \in N_T$, we have

$$\mathrm{ct}_c([\gamma \circ \sigma]) = \mathrm{ct}_{\sigma(c)}([\gamma]).$$

Hence the type T' and the correspondence $D_{T,T'}[c]$ depends only on the orbit of c under the action of N_T. We present in Table 8.1 the decomposition of the set of centers of Cremona transformation with respect to $\mathbf{C}_T$ into the orbits under the action of N_T. For each orbit, the type T' and the defining equation of the isomorphism correspondence $D_{T,T'}[c]$ are also given.

We will explain the algorithm for obtaining the defining equation of $D_{T,T'}[c]$. For example, consider the case where $T = A$ and $c = \{P_1, P_4, P_6, P_9, P_{12}, P_{20}\}$. The six points $\Sigma = \gamma_\lambda(c) = \{p_1, \dots, p_6\}$ are as follows:

$$\begin{aligned} p_1 &:= \gamma_\lambda(P_1) = [1,\omega,0], & p_4 &:= \gamma_\lambda(P_9) = [1,1,1],\\ p_2 &:= \gamma_\lambda(P_4) = [1+\lambda,\lambda,1], & p_5 &:= \gamma_\lambda(P_{12}) = [0,1,0],\\ p_3 &:= \gamma_\lambda(P_6) = [\lambda,1+\lambda,1], & p_6 &:= \gamma_\lambda(P_{20}) = [0,0,1]. \end{aligned}$$

Solving linear equations, we see that the 3-dimensional linear space $H^0(\mathbb{P}^2, \mathcal{I}^2_\Sigma(5))$ is generated by the homogeneous quintic polynomials in Table 8.2. The Cremona transformation $\mathrm{CT}_\Sigma : \mathbb{P}^2 \cdots\to \mathbb{P}^2$ is given by

$$[X,Y,Z] \mapsto [F_1,F_2,F_3].$$

The points $\gamma_\lambda(P_i)$ $(P_i \notin c)$ is mapped by CT_Σ to the points in Table 8.3. The conic curve $N'_1 \subset \mathbb{P}^2$ containing $\Sigma \setminus \{p_1\}$ is defined by

$$E_1 := X^2 + (\lambda^2+\lambda)YZ + (\lambda^2+\lambda+1)ZX = 0.$$

Let V_1 be the vector space of cubic homogeneous polynomials C such that $E_1 C$ is a member of $H^0(\mathbb{P}^2, \mathcal{I}^2_\Sigma(5))$. Then we have $\dim V_1 = 2$, and the image of the linear map $V_1 \to H^0(\mathbb{P}^2, \mathcal{I}^2_\Sigma(5))$ given by $C \mapsto E_1 C$ is spanned by F_1 and F_3. Hence the image $\beta'(N_1)$ of the strict transform $N_1 \subset S$ of N'_1 is $[0,1,0]$. In the same way, we calculate $\beta'(N_i)$ as in Table 8.3. The set LW of collinear 5-tuples of the points in $Z' = \{q_1, \dots, q_{21}\}$ and the set QW of 8-tuples of the points in Z' that are on a nonsingular conic curve are given in Table 8.4, where $\{1,3,5,11,17\}$ means

T	c	$\lvert N_T \cdot c\rvert$	T'	$D_{T,T'}[c]$
A	$\{1,2,8,10,15,16\}$	12	A	$\Delta = 0$
A	$\{1,7,8,15,18,21\}$	144	A	$D1 = 0$
A	$\{2,4,8,11,14,16\}$	576	B	$D2 = 0$
A	$\{1,4,6,9,12,20\}$	72	A	$D3 = 0$
A	$\{2,8,10,12,14,21\}$	72	A	$D3 = 0$
A	$\{5,8,9,10,14,16\}$	48	A	$\Delta = 0$
A	$\{4,9,12,16,17,18\}$	24	A	$\Delta = 0$
A	$\{1,2,9,10,16,19\}$	36	A	$\Delta = 0$
A	$\{7,12,13,14,19,20\}$	36	A	$\Delta = 0$
A	$\{2,6,9,10,13,21\}$	48	C	$D4 = 0$
A	$\{2,5,11,13,17,21\}$	576	A	$D1 = 0$
B	$\{2,7,8,9,10,17\}$	216	B	$\Delta = 0$
B	$\{1,2,11,12,13,18\}$	72	B	$D5 = 0$
B	$\{4,5,6,10,13,19\}$	54	B	$\Delta = 0$
B	$\{4,7,12,15,20,21\}$	6	B	$\Delta = 0$
B	$\{1,2,6,10,14,16\}$	54	B	$\Delta = 0$
B	$\{3,5,14,16,19,20\}$	108	B	$\Delta = 0$
B	$\{1,3,8,12,13,17\}$	108	B	$\Delta = 0$
B	$\{1,5,6,16,20,21\}$	216	A	$D6 = 0$
B	$\{2,6,9,13,16,18\}$	36	B	$\Delta = 0$
B	$\{3,7,8,10,19,21\}$	216	B	$D5 = 0$
B	$\{2,5,9,16,18,19\}$	108	B	$\Delta = 0$
B	$\{1,3,5,15,19,21\}$	72	B	$D5 = 0$
B	$\{2,6,7,16,20,21\}$	108	B	$\Delta = 0$
C	$\{3,5,9,13,17,21\}$	960	A	$D7 = 0$
C	$\{3,5,10,14,17,21\}$	64	C	$D8 = 0$
C	$\{1,5,8,10,14,18\}$	960	C	$\Delta = 0$
C	$\{1,2,5,8,18,19\}$	240	C	$\Delta = 0$

$$
\begin{aligned}
\Delta &:= J_T + J_{T'},\\
D1 &:= J_{T'}^6 J_T^2 + J_{T'}^4 J_T^4 + J_{T'}^2 J_T^6 + J_{T'}^4 J_T^3 + J_{T'}^3 J_T^4 + J_{T'}^4 J_T^2 + J_{T'}^3 J_T^3 + J_{T'}^2 J_T^4 +\\
&\quad + J_{T'}^4 J_T + J_{T'} J_T^4 + J_{T'}^3 J_T + J_{T'}^2 J_T^2 + J_{T'} J_T^3 + J_{T'}^3 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_T^3,\\
D2 &:= J_T^4 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_{T'} J_T + J_{T'},\\
D3 &:= J_{T'}^3 J_T^2 + J_{T'}^2 J_T^3 + J_{T'}^2 J_T^2 + J_{T'} J_T + 1,\\
D4 &:= J_{T'}^2 J_T^4 + J_{T'} J_T^2 + J_{T'} J_T + J_{T'} + J_T,\\
D5 &:= J_{T'}^4 J_T + J_{T'}^3 J_T^2 + J_{T'}^2 J_T^3 + J_{T'} J_T^4 + J_{T'}^3 J_T + J_{T'}^2 J_T^2 + J_{T'} J_T^3 + 1,\\
D6 &:= J_{T'}^4 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_{T'} J_T + J_T,\\
D7 &:= J_{T'}^4 J_T^2 + J_{T'}^2 J_T + J_{T'} J_T + J_{T'} + J_T,\\
D8 &:= J_{T'}^4 J_T^4 + J_{T'}^3 J_T + J_{T'}^2 J_T^2 + J_{T'} J_T^3 + J_{T'}^3 + J_{T'}^2 J_T + J_{T'} J_T^2 + J_T^3.
\end{aligned}
$$

TABLE 8.1. Isomorphism correspondences by Cremona transformations

$$
\begin{aligned}
F_1 \quad := \quad & X^4Z + X^3Z^2\lambda^2\omega + X^3Z^2\lambda\,\omega + X^2Y^2Z\omega + X^2YZ^2\lambda^2\omega + X^2YZ^2\lambda\,\omega + \\
& + X^2YZ^2\omega + X^2Z^3\lambda^4\omega + X^2Z^3\lambda^4 + X^2Z^3\lambda^2 + X^2Z^3\lambda\,\omega + X^2Z^3 + \\
& + XY^2Z^2\lambda^2\omega + XY^2Z^2\lambda\,\omega + XY^2Z^2\omega + XYZ^3\omega + Y^3Z^2\lambda^2\omega + \\
& + Y^3Z^2\lambda\,\omega + Y^2Z^3\lambda^4\omega + Y^2Z^3\lambda^4 + Y^2Z^3\lambda^2 + Y^2Z^3\lambda\,\omega,
\end{aligned}
$$

$$
\begin{aligned}
F_2 \quad := \quad & XY^3Z\lambda^2\omega + X^3Z^2\lambda^2 + X^2YZ^2\lambda + X^2Z^3\lambda + XY^2Z^2\lambda^2 + XYZ^3 + \\
& + X^2Y^2Z + X^2Z^3\lambda^4 + Y^3Z^2\lambda^4 + Y^2Z^3\lambda^4 + Y^2Z^3\lambda + X^2YZ^2\lambda^4 + \\
& + X^3Z^2\lambda + XY^2Z^2\lambda + X^4Y + XY^2Z^2 + Y^3Z^2\lambda + \ Y^2Z^3\lambda^3\omega + \\
& + X^2Z^3\lambda^5\omega + X^2Z^3\lambda^3\omega + Y^2Z^3\lambda^5\omega + Y^2Z^3\lambda^2\omega + Y^2Z^3\lambda\,\omega + Y^2Z^3\lambda^6\omega + \\
& + X^2Y^3\omega + X^3YZ\lambda^2\omega + X^2Z^3\lambda^4\omega + X^2YZ^2\lambda\,\omega + Y^3Z^2\lambda\,\omega + XY^2Z^2\omega + \\
& + X^2Z^3\lambda^6\omega + XY^3Z\omega + Y^3Z^2\lambda^4\omega + XY^3Z\lambda\,\omega + Y^2Z^3\lambda^4\omega + X^3YZ\lambda\,\omega + \\
& + X^2Y^2Z\lambda\,\omega + X^2Y^2Z\omega + X^2Y^2Z\lambda^2\omega + X^2YZ^2\lambda^4\omega,
\end{aligned}
$$

$$
\begin{aligned}
F_3 \quad := \quad & XY^3Z\lambda^2\omega + X^2YZ^2\lambda^2\omega + X^3Y^2\omega + X^3Z^2 + X^2YZ^2\lambda + X^2Z^3\lambda + \\
& + XYZ^3 + X^2YZ^2 + X^2Y^2Z + X^2Z^3\lambda^4 + Y^3Z^2\lambda^2 + Y^2Z^3\lambda^4 + Y^2Z^3\lambda + \\
& + XY^2Z^2\lambda^4 + X^3Z^2\lambda^4 + X^3Z^2\lambda + XY^2Z^2\lambda + X^2YZ^2\lambda^2 + XY^2Z^2 + \\
& + Y^3Z^2\lambda + Y^2Z^3\lambda^3\omega + X^5 + X^2Z^3\lambda^5\omega + X^2Z^3\lambda^3\omega + Y^2Z^3\lambda^5\omega + \\
& + X^2Z^3\lambda^2\omega + Y^2Z^3\lambda\,\omega + X^3Z^2\lambda^4\omega + Y^2Z^3\lambda^6\omega + X^3YZ\omega + X^3YZ\lambda^2\omega + \\
& + X^2YZ^2\lambda\,\omega + Y^3Z^2\lambda\,\omega + Y^3Z^2\lambda^2\omega + XY^2Z^2\omega + XYZ^3\omega + X^2Z^3\lambda^6\omega + \\
& + XY^3Z\lambda\,\omega + XY^2Z^2\lambda^4\omega + X^3YZ\lambda\,\omega + X^3Z^2\lambda^2\omega + X^2Y^2Z\lambda\,\omega + \\
& + XY^2Z^2\lambda^2\omega + X^2Y^2Z\lambda^2\omega.
\end{aligned}
$$

TABLE 8.2. Generators of $H^0(\mathbb{P}^2, \mathcal{I}_\Sigma^2(5))$

$\{q_1, q_3, q_5, q_{11}, q_{17}\}$, for example. Since $|LW| = 13$ and $|QW| = 28$, we see that the type T' of the target moduli curve is A. Let σ be the following permutation:

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\
13 & 16 & 2 & 6 & 21 & 4 & 18 & 17 & 3 & 7 & 12 & 1 & 8 & 9 & 11 & 15 & 14 & 20 & 5 & 19 & 10
\end{pmatrix}.
$$

Then the map

$$\gamma' : \mathcal{P} \to \mathbb{P}^2$$

defined by $\gamma'(P_i) = q_{\sigma(i)}$ yields bijections from the set of linear words in $\mathbf{C}_A$ (see Table 4.1) to LW and from the set of quadratic words in $\mathbf{C}_A$ (see Tables 4.4 and 4.5) to QW. Hence the map γ' is an element of $\mathcal{G}_A$. We make the linear change of homogeneous coordinates of $\mathbb{P}^2$ so that

$$
\begin{aligned}
&\gamma'(P_{18}) = q_{20} = [1,0,0], \\
&\gamma'(P_{12}) = q_1 = [0,1,0], \quad \gamma'(P_{13}) = q_8 = [1,1,0], \\
&\gamma'(P_{20}) = q_{19} = [0,0,1], \quad \gamma'(P_{19}) = q_5 = [1,0,1],
\end{aligned}
$$

$$
\begin{array}{rclcl}
q_1 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_2)) & = & [\,0,1,\omega\,],\\
q_2 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_3)) & = & [\,\omega,\lambda^2+\bar{\omega}\lambda+1,\lambda(\lambda+\bar{\omega})\,],\\
q_3 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_5)) & = & [\,\omega,\lambda^2+\bar{\omega}\lambda+\bar{\omega},(\lambda+1)(\lambda+\omega)\,],\\
q_4 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_7)) & = & [\,\omega,\lambda^2+\omega\lambda+\omega,(\lambda+\bar{\omega})^2\,],\\
q_5 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_8)) & = & [\,\omega,(\lambda+1)^2,\lambda^2+\omega\lambda+1\,],\\
q_6 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{10})) & = & [\,\omega,(\lambda+\omega)^2,(\lambda+1)(\lambda+\bar{\omega})\,],\\
q_7 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{11})) & = & [\,\omega,(\lambda+\omega)(\lambda+\bar{\omega}),\lambda^2+\lambda+\omega\,],\\
q_8 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{13})) & = & [\,0,1,1\,],\\
q_9 & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{14})) & = & [\,\omega,\lambda^2+\omega\lambda+1,(\lambda+\omega)^2\,],\\
q_{10} & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{15})) & = & [\,\omega,\lambda^2,\lambda^2+\omega\lambda+\omega\,],\\
q_{11} & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{16})) & = & [\,\omega,(\lambda+\bar{\omega})^2,\lambda(\lambda+\omega)\,],\\
q_{12} & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{17})) & = & [\,\omega,\lambda(\lambda+\omega),\lambda^2\,],\\
q_{13} & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{18})) & = & [\,0,0,1\,],\\
q_{14} & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{19})) & = & [\,\omega,\lambda^2+\lambda+\bar{\omega},\lambda(\lambda+1)\,],\\
q_{15} & := & \mathrm{CT}_\Sigma(\gamma_\lambda(P_{21})) & = & [\,\omega,(\lambda+1)(\lambda+\bar{\omega}),(\lambda+1)^2\,].
\end{array}
$$

$$
\begin{array}{rclcl}
q_{16} & := & \beta'(N_1) & = & [\,0,1,0\,],\\
q_{17} & := & \beta'(N_2) & = & [\,\omega,(\lambda+\bar{\omega})\lambda,\lambda^2+\bar{\omega}\lambda+\bar{\omega}\,],\\
q_{18} & := & \beta'(N_3) & = & [\,\omega,(\lambda+1)(\lambda+\omega),\lambda^2+\bar{\omega}\lambda+1\,],\\
q_{19} & := & \beta'(N_4) & = & [\,\omega,\lambda^2+\lambda+\omega,\lambda^2+\lambda+\bar{\omega}\,],\\
q_{20} & := & \beta'(N_5) & = & [\,0,1,\bar{\omega}\,],\\
q_{21} & := & \beta'(N_6) & = & [\,\omega,\lambda(\lambda+1),(\lambda+\omega)(\lambda+\bar{\omega})\,].
\end{array}
$$

TABLE 8.3. Points q_i

hold (see (4.6)); that is, we multiply the matrix

$$
\begin{bmatrix}
\lambda^2+\lambda & 1 & \bar{\omega}\\
\lambda^2+\lambda+1 & \bar{\omega} & 1\\
\omega\lambda+1 & 0 & 0
\end{bmatrix}
$$

from the left to the vectors $\gamma'(P_i) = q_{\sigma(i)}$. Then we have

$$\gamma'(P_1) = q_{13} = [1,\omega,0].$$

Therefore the projective equivalence class $[\gamma'] \in PGL(3,k)\backslash\mathcal{G}_A$ of γ' is contained in the connected component $(PGL(3,k)\backslash\mathcal{G}_A)^+$, because otherwise we would have $\gamma'(P_1) = [1,\bar{\omega},0]$. Since

$$\gamma'(P_{10}) = q_7 = [0,1,\lambda+\bar{\omega}],$$

the point $[\gamma']$ corresponds to $1/(\lambda+\bar{\omega})$ under the isomorphism $(PGL(3,k)\backslash\mathcal{G}_A)^+ \cong \mathbb{A}^1 \setminus \{0,1,\omega,\bar{\omega}\}$. Substituting $1/(\lambda+\bar{\omega})$ for λ in

$$J_A = \frac{(\lambda^2+\lambda+1)^3}{\lambda^2(\lambda+1)^2},$$

$$
\begin{aligned}
LW = \quad & \{\{1,3,5,11,17\},\{4,5,6,8,12\},\{1,2,6,10,18\},\{2,3,8,19,21\},\\
& \{1,4,9,14,21\},\{8,9,10,11,15\},\{6,7,11,20,21\},\{1,7,12,15,19\},\\
& \{3,9,12,18,20\},\{1,8,13,16,20\},\{7,8,14,17,18\},\{5,10,14,19,20\},\\
& \{2,4,15,17,20\}\}.
\end{aligned}
$$

$$
\begin{aligned}
QW = \quad & \{\{2,3,4,5,9,10,13,16\},\{2,3,4,7,10,11,12,14\},\{2,3,5,6,7,9,14,15\},\\
& \{2,4,5,7,9,11,18,19\},\{3,4,5,7,10,15,18,21\},\{3,4,6,7,9,10,17,19\},\\
& \{2,5,6,7,13,16,17,19\},\{2,3,6,11,12,13,15,16\},\{3,4,6,11,14,15,18,19\},\\
& \{3,5,6,13,14,16,18,21\},\{2,5,7,9,10,12,17,21\},\{4,6,7,10,13,14,15,16\},\\
& \{3,4,7,12,13,16,17,21\},\{5,7,9,11,12,13,14,16\},\{2,6,9,11,12,14,17,19\},\\
& \{4,6,9,11,13,16,17,18\},\{2,7,9,13,15,16,18,21\},\{5,6,9,15,17,18,19,21\},\\
& \{3,7,10,11,13,16,18,19\},\{6,9,10,12,13,16,19,21\},\{3,6,10,12,14,15,17,21\},\\
& \{2,5,11,12,14,15,18,21\},\{4,10,11,12,17,18,19,21\},\{2,10,11,13,14,16,17,21\},\\
& \{4,5,11,13,15,16,19,21\},\{2,4,12,13,14,16,18,19\},\{5,10,12,13,15,16,17,18\},\\
& \{3,9,13,14,15,16,17,19\}\}.
\end{aligned}
$$

TABLE 8.4. Sets LW and QW

we see that the J_A-invariant of $[\gamma']$ is equal to

$$J'_A = \frac{\lambda^3(\lambda+1)^3}{(\lambda^2+\lambda+1)^2}.$$

Eliminating λ from J_A and J'_A, we obtain the defining equation

$$1 + J_A J'_A + J_A^2 J_A'^2 + J_A^3 J_A'^2 + J_A^2 J_A'^3 = 0$$

of the isomorphism correspondence given by the Cremona transformation with the center $c = \{P_1, P_4, P_6, P_9, P_{12}, P_{20}\}$.

Putting

$$
\begin{aligned}
& D_{A,A,1} := \{D3=0\}, \quad D_{A,A,2} := \{D1=0\},\\
& D_{B,B,1} := \{D5=0\}, \quad D_{C,C,1} := \{D8=0\},\\
& D_{A,B,1} := \{D2=0\} = {}^tD_{B,A,1} = {}^t\{D6=0\},\\
& D_{A,C,1} := \{D4=0\} = {}^tD_{C,A,1} = {}^t\{D7=0\},
\end{aligned}
$$

we obtain Theorem 1.13. The composite $D_1 * D_2$ of correspondences

$$
\begin{aligned}
D_1 &= \{f_1(J_T, J_{T'}) = 0\} \subset \mathfrak{M}_T \times \mathfrak{M}_{T'} \quad \text{and}\\
D_2 &= \{f_2(J_{T'}, J_{T''}) = 0\} \subset \mathfrak{M}_{T'} \times \mathfrak{M}_{T''}
\end{aligned}
$$

is obtained by eliminating the variable $J_{T'}$ from $f_1(J_T, J_{T'}) = f_2(J_{T'}, J_{T''}) = 0$. Starting from the eight isomorphism correspondences above and making composites, we obtain irreducible isomorphism correspondences listed in Table 1.1, which have the relations in Table 8.5. This table also shows that the isomorphism correspondences Δ_A, Δ_B, Δ_C and the ones in Table 1.1 are closed under compositions of correspondences.

$D_{A,A,1} * D_{A,A,1}$	$=$	$\Delta_A + D_{A,A,2}$,	$D_{B,A,2} * D_{A,C,2}$	$=$	$D_{B,C,1} + D_{B,C,2}$,
$D_{A,A,1} * D_{A,A,2}$	$=$	$D_{A,A,1} + D_{A,A,2}$,	$D_{B,C,1} * D_{C,C,1}$	$=$	$D_{B,C,2}$,
$D_{A,A,2} * D_{A,A,1}$	$=$	$D_{A,A,1} + D_{A,A,2}$,	$D_{B,C,2} * D_{C,C,1}$	$=$	$D_{B,C,1} + D_{B,C,2}$,
$D_{A,A,2} * D_{A,A,2}$	$=$	$\Delta_A + D_{A,A,1} + D_{A,A,2}$,	$D_{B,C,1} * D_{C,B,1}$	$=$	Δ_B,
$D_{A,A,1} * D_{A,B,1}$	$=$	$D_{A,B,2}$,	$D_{B,C,1} * D_{C,B,2}$	$=$	$D_{B,B,1}$,
$D_{A,A,1} * D_{A,B,2}$	$=$	$D_{A,B,1} + D_{A,B,2}$,	$D_{B,C,2} * D_{C,B,1}$	$=$	$D_{B,B,1}$,
$D_{A,A,2} * D_{A,B,1}$	$=$	$D_{A,B,1} + D_{A,B,2}$,	$D_{B,C,2} * D_{C,B,2}$	$=$	$\Delta_B + D_{B,B,1}$,
$D_{A,A,2} * D_{A,B,2}$	$=$	$D_{A,B,1} + D_{A,B,2}$,	$D_{B,C,1} * D_{C,A,1}$	$=$	$D_{B,A,1}$,
$D_{A,A,1} * D_{A,C,1}$	$=$	$D_{A,C,2}$,	$D_{B,C,1} * D_{C,A,2}$	$=$	$D_{B,A,2}$,
$D_{A,A,1} * D_{A,C,2}$	$=$	$D_{A,C,1} + D_{A,C,2}$,	$D_{B,C,2} * D_{C,A,1}$	$=$	$D_{B,A,1} + D_{B,A,2}$,
$D_{A,A,2} * D_{A,C,1}$	$=$	$D_{A,C,1} + D_{A,C,2}$,	$D_{B,C,2} * D_{C,A,2}$	$=$	$D_{B,A,1} + D_{B,A,2}$,
$D_{A,A,2} * D_{A,C,2}$	$=$	$D_{A,C,1} + D_{A,C,2}$,	$D_{C,B,1} * D_{B,B,1}$	$=$	$D_{C,B,2}$,
$D_{B,B,1} * D_{B,B,1}$	$=$	$\Delta_B + D_{B,B,1}$,	$D_{C,B,2} * D_{B,B,1}$	$=$	$D_{C,B,1} + D_{C,B,2}$,
$D_{B,B,1} * D_{B,A,1}$	$=$	$D_{B,A,1} + D_{B,A,2}$,	$D_{C,B,1} * D_{B,A,1}$	$=$	$D_{C,A,1}$,
$D_{B,B,1} * D_{B,A,2}$	$=$	$D_{B,A,1} + D_{B,A,2}$,	$D_{C,B,1} * D_{B,A,2}$	$=$	$D_{C,A,2}$,
$D_{B,B,1} * D_{B,C,1}$	$=$	$D_{B,C,2}$,	$D_{C,B,2} * D_{B,A,1}$	$=$	$D_{C,A,1} + D_{C,A,2}$,
$D_{B,B,1} * D_{B,C,2}$	$=$	$D_{B,C,1} + D_{B,C,2}$,	$D_{C,B,2} * D_{B,A,2}$	$=$	$D_{C,A,1} + D_{C,A,2}$,
$D_{C,C,1} * D_{C,C,1}$	$=$	$\Delta_C + D_{C,C,1}$,	$D_{C,B,1} * D_{B,C,1}$	$=$	Δ_C,
$D_{C,C,1} * D_{C,B,1}$	$=$	$D_{C,B,2}$,	$D_{C,B,1} * D_{B,C,2}$	$=$	$D_{C,C,1}$,
$D_{C,C,1} * D_{C,B,2}$	$=$	$D_{C,B,1} + D_{C,B,2}$,	$D_{C,B,2} * D_{B,C,1}$	$=$	$D_{C,C,1}$,
$D_{C,C,1} * D_{C,A,1}$	$=$	$D_{C,A,1} + D_{C,A,2}$,	$D_{C,B,2} * D_{B,C,2}$	$=$	$\Delta_C + D_{C,C,1}$,
$D_{C,C,1} * D_{C,A,2}$	$=$	$D_{C,A,1} + D_{C,A,2}$,	$D_{C,A,1} * D_{A,A,1}$	$=$	$D_{C,A,2}$,
$D_{A,B,1} * D_{B,B,1}$	$=$	$D_{A,B,1} + D_{A,B,2}$,	$D_{C,A,1} * D_{A,A,2}$	$=$	$D_{C,A,1} + D_{C,A,2}$,
$D_{A,B,2} * D_{B,B,1}$	$=$	$D_{A,B,1} + D_{A,B,2}$,	$D_{C,A,2} * D_{A,A,1}$	$=$	$D_{C,A,1} + D_{C,A,2}$,
$D_{A,B,1} * D_{B,A,1}$	$=$	$\Delta_A + D_{A,A,2}$,	$D_{C,A,2} * D_{A,A,2}$	$=$	$D_{C,A,1} + D_{C,A,2}$,
$D_{A,B,1} * D_{B,A,2}$	$=$	$D_{A,A,1} + D_{A,A,2}$,	$D_{C,A,1} * D_{A,B,1}$	$=$	$D_{C,B,1} + D_{C,B,2}$,
$D_{A,B,2} * D_{B,A,1}$	$=$	$D_{A,A,1} + D_{A,A,2}$,	$D_{C,A,1} * D_{A,B,2}$	$=$	$D_{C,B,2}$,
$D_{A,B,2} * D_{B,A,2}$	$=$	$\Delta_A + D_{A,A,1} + D_{A,A,2}$,	$D_{C,A,2} * D_{A,B,1}$	$=$	$D_{C,B,2}$,
$D_{A,B,1} * D_{B,C,1}$	$=$	$D_{A,C,1}$,	$D_{C,A,2} * D_{A,B,2}$	$=$	$D_{C,B,1} + D_{C,B,2}$,
$D_{A,B,1} * D_{B,C,2}$	$=$	$D_{A,C,1} + D_{A,C,2}$,	$D_{C,A,1} * D_{A,C,1}$	$=$	$\Delta_C + D_{C,C,1}$,
$D_{A,B,2} * D_{B,C,1}$	$=$	$D_{A,C,2}$,	$D_{C,A,1} * D_{A,C,2}$	$=$	$D_{C,C,1}$,
$D_{A,B,2} * D_{B,C,2}$	$=$	$D_{A,C,1} + D_{A,C,2}$,	$D_{C,A,2} * D_{A,C,1}$	$=$	$D_{C,C,1}$,
$D_{B,A,1} * D_{A,A,1}$	$=$	$D_{B,A,2}$,	$D_{C,A,2} * D_{A,C,2}$	$=$	$\Delta_C + D_{C,C,1}$,
$D_{B,A,1} * D_{A,A,2}$	$=$	$D_{B,A,1} + D_{B,A,2}$,	$D_{A,C,1} * D_{C,C,1}$	$=$	$D_{A,C,1} + D_{A,C,2}$,
$D_{B,A,2} * D_{A,A,1}$	$=$	$D_{B,A,1} + D_{B,A,2}$,	$D_{A,C,2} * D_{C,C,1}$	$=$	$D_{A,C,1} + D_{A,C,2}$,
$D_{B,A,2} * D_{A,A,2}$	$=$	$D_{B,A,1} + D_{B,A,2}$,	$D_{A,C,1} * D_{C,B,1}$	$=$	$D_{A,B,1}$,
$D_{B,A,1} * D_{A,B,1}$	$=$	$\Delta_B + D_{B,B,1}$,	$D_{A,C,1} * D_{C,B,2}$	$=$	$D_{A,B,1} + D_{A,B,2}$,
$D_{B,A,1} * D_{A,B,2}$	$=$	$D_{B,B,1}$,	$D_{A,C,2} * D_{C,B,1}$	$=$	$D_{A,B,2}$,
$D_{B,A,2} * D_{A,B,1}$	$=$	$D_{B,B,1}$,	$D_{A,C,2} * D_{C,B,2}$	$=$	$D_{A,B,1} + D_{A,B,2}$,
$D_{B,A,2} * D_{A,B,2}$	$=$	$\Delta_B + D_{B,B,1}$,	$D_{A,C,1} * D_{C,A,1}$	$=$	$\Delta_A + D_{A,A,2}$,
$D_{B,A,1} * D_{A,C,1}$	$=$	$D_{B,C,1} + D_{B,C,2}$,	$D_{A,C,1} * D_{C,A,2}$	$=$	$D_{A,A,1} + D_{A,A,2}$,
$D_{B,A,1} * D_{A,C,2}$	$=$	$D_{B,C,2}$,	$D_{A,C,2} * D_{C,A,1}$	$=$	$D_{A,A,1} + D_{A,A,2}$,
$D_{B,A,2} * D_{A,C,1}$	$=$	$D_{B,C,2}$,	$D_{A,C,2} * D_{C,A,2}$	$=$	$\Delta_A + D_{A,A,1} + D_{A,A,2}$.

TABLE 8.5. Relations between non-trivial isomorphism correspondences

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