PROJECTIVE MODELS OF THE SUPERSINGULAR $K3$ SURFACE WITH ARTIN INVARIANT 1 IN CHARACTERISTIC 5

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Abstract. Let X be a supersingular $K3$ surface in characteristic 5 with Artin invariant 1. Then X has a polarization that realizes X as the Fermat sextic double plane. We present a list of polarizations of X with degree 2 whose intersection number with this Fermat sextic polarization is less than or equal to 5, and give the defining equations of the corresponding projective models. We also present a method to describe birational morphisms between these projective models explicitly. As a by-product, a non-projective automorphism of the Fermat sextic double plane is obtained.

1. Introduction

Let Y be a supersingular $K3$ surface (in the sense of Shioda) in characteristic $p > 0$. Artin [3] showed that the discriminant of the Néron-Severi lattice $\mathrm{NS}(Y)$ is written as $-p^{2\sigma}$, where σ is a positive integer ≤ 10. This integer σ in called the *Artin invariant* of Y. It is proved in [14, 15, 17] that, for each prime p, a supersingular $K3$ surface with Artin invariant 1 in characteristic p exists and is unique up to isomorphisms. Recently, many detailed study of supersingular $K3$ surfaces with Artin invariant 1 in small characteristics have appeared (see [6, 7, 9, 11, 12, 21]).

The purpose of this paper is to investigate projective models of degree 2 of the supersingular $K3$ surface X with Artin invariant 1 in characteristic 5. It is well-known that the Fermat sextic double plane in characteristic 5 is isomorphic to X. Starting from this projective model, we obtain many other projective models of degree 2, and describe birational morphisms between them.

Our method is computational, and can be easily adapted to any $K3$ surface. In particular, combining the algorithms developed in this paper with the Borcherds-Kondo method [4, 10], we have succeeded in obtaining a set of generators of the automorphism group of the supersingular $K3$ surface in characteristic 3 with Artin invariant 1 in [12].

We fix terminologies and explain our motivation. Let Y be a $K3$ surface defined over an algebraically closed field of arbitrary characteristic. Let $(\ ,\ )_{\mathrm{NS}}$ denote the intersection form of the Néron-Severi lattice $\mathrm{NS}(Y)$ of Y. For $v \in \mathrm{NS}(Y)$, we denote

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by $\mathcal{L}_v \to Y$ the corresponding line bundle. Let d be an even positive integer. We say that a vector $h \in \mathrm{NS}(Y)$ is a *polarization of degree* d if $(h, h)_{\mathrm{NS}}$ is equal to d and the complete linear system $|\mathcal{L}_h|$ is non-empty and has no fixed-components. Let h be a polarization of degree d. Then $|\mathcal{L}_h|$ is base-point free by Corollary 3.2 of [18], and hence defines a morphism Φ_h from Y to a projective space of dimension $1 + d/2$. We denote by

$$Y \xrightarrow{\phi_h} Y_h \xrightarrow{\psi_h} \mathbb{P}^{1+d/2}$$

the Stein factorization of Φ_h. By [1, 2], the normal surface Y_h has only rational double points as its singularities, and ϕ_h is a contraction of an ADE-configuration of smooth rational curves. We say that $\psi_h : Y_h \to \mathbb{P}^{1+d/2}$ is the *projective model* of Y corresponding to h. We put

$$\mathcal{P}_d(Y) := \{\, h \in \mathrm{NS}(Y) \mid h \text{ is a polarization of degree } d \,\}.$$

The automorphism group $\mathrm{Aut}(Y)$ of Y acts on $\mathcal{P}_d(Y)$. For $h, h' \in \mathcal{P}_d(Y)$, we say that h and h' are *projectively equivalent* and write $h \sim h'$ if there exist an isomorphism $Y_h \xrightarrow{\sim} Y_{h'}$ and a linear automorphism $\mathbb{P}^{1+d/2} \xrightarrow{\sim} \mathbb{P}^{1+d/2}$ that make the following diagram commutative:

$$\begin{array}{ccc} Y_h & \xrightarrow{\psi_h} & \mathbb{P}^{1+d/2} \\ \downarrow\wr & & \downarrow\wr \\ Y_{h'} & \xrightarrow{\psi_{h'}} & \mathbb{P}^{1+d/2}. \end{array} \tag{1.1}$$

It is obvious that the equivalence classes of $\sim$ in $\mathcal{P}_d(Y)$ are just the $\mathrm{Aut}(Y)$-orbits. For $h \in \mathcal{P}_d(Y)$, the stabilizer subgroup $\mathrm{Aut}(Y, h)$ of h in $\mathrm{Aut}(Y)$ is the projective automorphism group of the projective model ψ_h. It is usually easy to determine $\mathrm{Aut}(Y, h)$. Hence it is important to study the equivalence classes of $\sim$ for the study of $\mathrm{Aut}(Y)$. Moreover, to obtain an element of $\mathrm{Aut}(Y)$ *not* contained in $\mathrm{Aut}(Y, h)$, we need to write the isomorphism $Y_h \xrightarrow{\sim} Y_{h'}$ in (1.1) explicitly.

We concentrate upon the supersingular $K3$ surface X with Artin invariant 1 in characteristic 5. It is known that X has a projective model $\psi_F : X_F \to \mathbb{P}^2$ of degree 2, where X_F is defined by

$$X_F := \{w^2 = x^6 + y^6 + z^6\} \subset \mathbb{P}(3, 1, 1, 1) \tag{1.2}$$

in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, and the double covering ψ_F is given by $[w : x : y : z] \mapsto [x : y : z]$, which is branching along the Fermat sextic curve

$$B_F : x^6 + y^6 + z^6 = 0.$$

We denote by $h_F \in \mathrm{NS}(X)$ a polarization of the projective model $\psi_F : X_F \to \mathbb{P}^2$, and by

$$\Phi_F \quad : \quad X \xrightarrow{\phi_F} X_F \xrightarrow{\psi_F} \mathbb{P}^2$$

the Stein factorization of the morphism given by $|\mathcal{L}_{h_F}|$. Note that $\phi_F : X \to X_F$ is an isomorphism. The group $\mathrm{Aut}(X, h_F)$ is an extension of the projective automorphism group $\mathrm{PGU}_3(\mathbb{F}_{25})$ of $B_F \subset \mathbb{P}^2$ by $\mathrm{Gal}(X_F/\mathbb{P}^2) \cong \mathbb{Z}/2\mathbb{Z}$. In particular, the order of $\mathrm{Aut}(X, h_F)$ is $756,000$. Using this projective model ψ_F, we obtain a set of generators of $\mathrm{NS}(X)$ (see Section 2). It turns out that $\mathrm{NS}(X)$ is generated by the numerical equivalence classes of curves on X_F defined over $\mathbb{F}_{25}$. In particular, every projective model of X is projectively equivalent to a projective model defined over $\mathbb{F}_{25}$ (see [19]). Moreover, the Frobenius action of $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ on X_F induces an action of $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ on $\mathrm{NS}(X)$, which we denote by $v \mapsto \bar{v}$. It is easy to see that $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ acts on the set of $\mathrm{Aut}(X)$-orbits in $\mathcal{P}_d(X)$.

We will study the polarizations of degree 2 on X. Consider neighborhoods

$$\mathcal{B}_r := \{\, v \in \mathrm{NS}(X) \mid (v, h_F)_{\mathrm{NS}} \le r \,\}$$

of h_F in $\mathrm{NS}(X)$. Since $\overline{h_F} = h_F$, $\mathrm{Aut}(X, h_F)$ and $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ act on $\mathcal{P}_2(X) \cap \mathcal{B}_r$. By computer-aided calculation, we have obtained the following:

Theorem 1.1. *The set $\mathcal{P}_2(X) \cap \mathcal{B}_5$ consists of* $146,945,851$ *vectors, and they are decomposed into the equivalence classes $\mathcal{E}_0, \dots, \mathcal{E}_{64}$ under the relation $\sim$. The details of these equivalence classes are described in Section 7.*

We explain the items of the table in Section 7. For $h \in \mathcal{P}_2(X)$, let B_h denote the branch curve of the double covering $\psi_h : X_h \to \mathbb{P}^2$.

- $\mathcal{E}_i = \overline{\mathcal{E}}_j$ means that $\mathcal{E}_i$ is equal to the image of $\mathcal{E}_j$ under the action of $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ defined above. In particular, $\mathcal{E}_i = \overline{\mathcal{E}}_i$ means that $\mathcal{E}_i$ is self-conjugate, while $\mathcal{E}_i = \overline{\mathcal{E}}_{i+1}$ means that $\mathcal{E}_i$ is *not* self-conjugate, that the items RT, |aut| and N explained below are the same for $\mathcal{E}_i$ and $\mathcal{E}_{i+1}$, and that the defining equation of B_h for $\mathcal{E}_{i+1}$ is obtained from that for $\mathcal{E}_i$ by changing the sign of $\sqrt{2}$.
- RT denotes the ADE-type of the singular points of B_h.
- |aut| denotes the order of the projective automorphism group of the plane curve $B_h \subset \mathbb{P}^2$. Hence the order of $\mathrm{Aut}(X, h)$ is equal to $2\,|\mathrm{aut}|$.
- N is the total number of the vectors in $\mathcal{E}_i \subset \mathcal{P}_2(X) \cap \mathcal{B}_5$.
- h is a sample element of $\mathcal{E}_i$ written in a row vector with respect to the basis of $\mathrm{NS}(X)$ given in Section 2.
- An affine defining equation of B_h with coefficients in $\mathbb{F}_{25}$ is given in the framed box.

Each of the 65 projective models in Theorem 1.1 exhibits interesting properties that are peculiar to characteristic 5. One of these properties is the existence of *splitting lines*. A (-2)-*curve* on X is a smooth rational curve on X. Let h be a polarization of degree 2 on X. We say that a (-2)-curve C on X is h-*exceptional* if C is mapped to a point by $\Phi_h : X \to \mathbb{P}^2$, while C is said to be an h-*line* if Φ_h maps C to a line on $\mathbb{P}^2$ isomorphically. A line l on $\mathbb{P}^2$ is said to be *h-splitting* if l is the image of an h-line by Φ_h. In other words, a line $l \subset \mathbb{P}^2$ is h-splitting if and only if either l is an irreducible component of B_h, or $l \not\subset B_h$ and the intersection multiplicity at each point of $l \cap B_h$ is even. We observe the following:

Proposition 1.2. *For each $h \in \mathcal{P}_2(X) \cap \mathcal{B}_5$, the lattice $\mathrm{NS}(X)$ is generated by the classes of h-exceptional curves and h-lines.*

In fact, we establish a method to write the birational morphism $\phi_h : X \to X_h$ explicitly as a list of rational functions on $X \cong X_F$ for any $h \in \mathcal{P}_2(X)$. Applying this method to a polarization $h \in \mathcal{E}_0$ with $(h_F, h)_{\mathrm{NS}} = 4$, we obtain the following:

Theorem 1.3. *Let (w, x, y) be the affine coordinates of $\mathbb{P}(3, 1, 1, 1)$ with $z = 1$ in (1.2). Then the rational map $g : X_F \to \mathbb{P}(3, 1, 1, 1)$ given by*

$$(w, x, y) \mapsto [\omega(w, x, y) : \xi_0(w, x, y) : \xi_1(w, x, y) : \xi_2(w, x, y)],$$

where $\omega, \xi_0, \xi_1, \xi_2$ are the polynomials presented in Table 1.1, induces an automorphism of X_F with order 2 such that $(h_F, g^* h_F)_{\mathrm{NS}} = 4$. In particular, this automorphism g is not contained in $\mathrm{Aut}(X, h_F)$.

The study of singularities of sextic double plane models of complex $K3$ surfaces using lattice theory and computer-aided calculation was initiated by Urabe [26] and Yang [27]. The idea of h-splitting lines was used in [24] for the classification of Zariski pairs of simple sextic curves. On the other hand, in [16, 22, 23], sextic double plane models of supersingular $K3$ surfaces were studied by lattice theory. A shortcoming of the method in these works is that it gives only combinatorial data of the singularities of the projective models, and does not yield their defining equations explicitly.

The new devices in this article are the following: (i) Using the ample class $h_F \in \mathrm{NS}(X)$, we can determine whether a given vector $v \in \mathrm{NS}(X)$ is a polarization or not. (ii) The fact that the classes of h_F-lines span $\mathrm{NS}(X)$ enables us to calculate the equation of X_h explicitly and algorithmically. (iii) To deal with the large number of polarizations, we decompose them into $\mathrm{Aut}(X, h_F)$-orbits and calculate the projective model only for a representative polarization of each orbit.

This paper is organized as follows. In Section 2, we give a set of h_F-lines whose classes form a basis of $\mathrm{NS}(X)$. In Section 3, we present algorithms that can be applied to lattices in general. In Section 4, we apply them to $\mathrm{NS}(X)$ and describe algorithms to calculate geometric data of X. In Section 5, we explain how to calculate the morphisms $\phi_h : X \to X_h$ and $\psi_h : X_h \to \mathbb{P}^2$ for a given polarization $h \in \mathcal{P}_2(X)$. In Sections 6 and 8, the computation we carried out to prove Theorems 1.1 and 1.3 are explained. Section 7 is for the list of projective models.

In this table, $a+b\sqrt{2} \in \mathbb{F}_{25}$ with $0 \le a, b < 5$ is denoted by $\bar{ab}$.

$\omega := wf_\omega + h_\omega$, where

$f_\omega := \bar{10}x^{12} + \bar{23}x^{11}y + \bar{21}x^{10}y^2 + \bar{02}x^9y^3 + \bar{11}x^8y^4 + \bar{33}x^7y^5 + \bar{22}x^6y^6 + \bar{40}x^5y^7 + \bar{14}x^4y^8 + \bar{13}x^2y^{10} + \bar{12}xy^{11} + \bar{24}y^{12} + \bar{22}x^{11} + \bar{11}x^{10}y + \bar{44}x^9y^2 + \bar{42}x^8y^3 + \bar{44}x^7y^4 + \bar{31}x^6y^5 + \bar{40}x^5y^6 + \bar{24}x^4y^7 + \bar{41}x^3y^8 + \bar{24}x^2y^9 + \bar{24}xy^{10} + \bar{33}y^{11} + \bar{03}x^{10} + \bar{10}x^9y + \bar{43}x^8y^2 + \bar{43}x^7y^3 + \bar{34}x^6y^4 + \bar{20}x^5y^5 + \bar{21}x^3y^7 + \bar{30}x^2y^8 + \bar{10}xy^9 + \bar{24}y^{10} + \bar{34}x^9 + \bar{23}x^8y + \bar{02}x^7y^2 + \bar{10}x^6y^3 + \bar{14}x^5y^4 + \bar{31}x^4y^5 + \bar{23}x^3y^6 + \bar{03}x^2y^7 + \bar{23}xy^8 + \bar{22}y^9 + \bar{20}x^8 + \bar{23}x^7y + \bar{32}x^6y^2 + \bar{44}x^5y^3 + \bar{42}x^3y^5 + \bar{12}x^2y^6 + \bar{22}xy^7 + \bar{42}y^8 + \bar{33}x^7 + \bar{12}x^6y + \bar{20}x^5y^2 + \bar{01}x^4y^3 + \bar{44}x^3y^4 + \bar{13}x^2y^5 + \bar{31}xy^6 + \bar{02}y^7 + \bar{30}x^6 + \bar{31}x^5y + \bar{33}x^4y^2 + \bar{23}x^3y^3 + \bar{31}x^2y^4 + \bar{41}xy^5 + \bar{31}y^6 + \bar{40}x^5 + \bar{32}x^4y + \bar{24}x^3y^2 + \bar{12}x^2y^3 + \bar{44}xy^4 + \bar{13}y^5 + \bar{14}x^3y + \bar{34}x^2y^2 + \bar{30}y^4 + \bar{31}x^3 + \bar{30}x^2y + \bar{41}xy^2 + \bar{43}y^3 + \bar{42}x^2 + \bar{40}xy + \bar{23}y^2 + \bar{03}x + \bar{02}y$,

and

$h_\omega := \bar{10}x^{15} + \bar{23}x^{14}y + \bar{21}x^{13}y^2 + \bar{02}x^{12}y^3 + \bar{11}x^{11}y^4 + \bar{33}x^{10}y^5 + \bar{02}x^9y^6 + \bar{04}x^8y^7 + \bar{22}x^7y^8 + \bar{11}x^6y^9 + \bar{24}x^5y^{10} + \bar{20}x^4y^{11} + \bar{12}x^3y^{12} + \bar{43}x^2y^{13} + \bar{13}xy^{14} + \bar{34}y^{15} + \bar{22}x^{14} + \bar{11}x^{13}y + \bar{44}x^{12}y^2 + \bar{42}x^{11}y^3 + \bar{44}x^{10}y^4 + \bar{31}x^9y^5 + \bar{01}x^8y^6 + \bar{02}x^7y^7 + \bar{22}x^6y^8 + \bar{20}x^5y^9 + \bar{03}x^4y^{10} + \bar{03}x^3y^{11} + \bar{10}x^2y^{12} + \bar{21}xy^{13} + \bar{41}y^{14} + \bar{03}x^{13} + \bar{10}x^{12}y + \bar{43}x^{11}y^2 + \bar{43}x^{10}y^3 + \bar{34}x^9y^4 + \bar{20}x^8y^5 + \bar{04}x^7y^6 + \bar{40}x^6y^7 + \bar{22}x^5y^8 + \bar{20}x^4y^9 + \bar{24}x^3y^{10} + \bar{02}x^2y^{11} + \bar{41}xy^{12} + \bar{31}y^{13} + \bar{34}x^{12} + \bar{23}x^{11}y + \bar{02}x^{10}y^2 + \bar{10}x^9y^3 + \bar{14}x^8y^4 + \bar{31}x^7y^5 + \bar{10}x^6y^6 + \bar{01}x^5y^7 + \bar{44}x^4y^8 + \bar{20}x^2y^{10} + \bar{43}xy^{11} + \bar{41}y^{12} + \bar{20}x^{11} + \bar{23}x^{10}y + \bar{32}x^9y^2 + \bar{44}x^8y^3 + \bar{01}x^6y^5 + \bar{42}x^5y^6 + \bar{22}x^4y^7 + \bar{33}x^3y^8 + \bar{43}x^2y^9 + \bar{02}xy^{10} + \bar{21}y^{11} + \bar{33}x^{10} + \bar{12}x^9y + \bar{20}x^8y^2 + \bar{01}x^7y^3 + \bar{22}x^6y^4 + \bar{40}x^5y^5 + \bar{41}x^4y^6 + \bar{23}x^3y^7 + \bar{30}x^2y^8 + \bar{20}xy^9 + \bar{04}y^{10} + \bar{10}x^9 + \bar{40}x^8y + \bar{41}x^7y^2 + \bar{24}x^6y^3 + \bar{42}x^5y^4 + \bar{33}x^4y^5 + \bar{42}x^3y^6 + \bar{02}x^2y^7 + \bar{22}xy^8 + \bar{13}y^9 + \bar{01}x^8 + \bar{10}x^7y + \bar{14}x^6y^2 + \bar{23}x^5y^3 + \bar{43}x^4y^4 + \bar{43}x^3y^5 + \bar{01}x^2y^6 + \bar{20}xy^7 + \bar{44}y^8 + \bar{04}x^7 + \bar{11}x^6y + \bar{03}x^5y^2 + \bar{12}x^4y^3 + \bar{44}x^3y^4 + \bar{30}x^2y^5 + \bar{22}xy^6 + \bar{20}y^7 + \bar{03}x^6 + \bar{22}x^4y^2 + \bar{41}x^3y^3 + \bar{22}x^2y^4 + \bar{14}xy^5 + \bar{12}y^6 + \bar{32}x^5 + \bar{11}x^4y + \bar{30}x^3y^2 + \bar{02}x^2y^3 + \bar{22}xy^4 + \bar{21}y^5 + \bar{04}x^4 + \bar{22}x^3y + \bar{10}x^2y^2 + \bar{04}xy^3 + \bar{13}y^4 + \bar{13}x^3 + \bar{32}x^2y + \bar{31}xy^2 + \bar{32}y^3 + \bar{03}x^2 + \bar{42}xy + \bar{44}y^2 + \bar{43}x + \bar{43}y$.

$\xi_0 := wf_0 + h_0$, where

$f_0 := \bar{40}x^2 + \bar{14}xy + \bar{41}y^2 + \bar{11}x + \bar{13}y + \bar{31}$, and

$h_0 := \bar{40}x^5 + \bar{14}x^4y + \bar{41}x^3y^2 + \bar{12}xy^4 + \bar{30}y^5 + \bar{11}x^4 + \bar{13}x^3y + \bar{34}xy^3 + \bar{03}y^4 + \bar{31}x^3 + \bar{04}xy^2 + \bar{22}y^3 + \bar{12}y^2 + \bar{34}x + \bar{34}y + \bar{43}$.

$\xi_1 := wf_1 + h_1$, where

$f_1 := \bar{10}xy + \bar{44}y^2 + \bar{20}y + \bar{21}$, and

$h_1 := \bar{10}x^4y + \bar{44}x^3y^2 + \bar{12}x^2y^3 + \bar{12}xy^4 + \bar{12}y^5 + \bar{20}x^3y + \bar{42}x^2y^2 + \bar{32}y^4 + \bar{21}x^3 + \bar{03}x^2y + \bar{02}xy^2 + \bar{33}y^3 + \bar{24}x^2 + \bar{43}xy + \bar{44}y^2 + \bar{21}x + \bar{43}y + \bar{01}$.

$\xi_2 := wf_2 + h_2$, where

$f_2 := \bar{42}y^2 + \bar{10}x + \bar{40}y + \bar{01}$, and

$h_2 := \bar{42}x^3y^2 + \bar{04}x^2y^3 + \bar{14}xy^4 + \bar{14}y^5 + \bar{10}x^4 + \bar{40}x^3y + \bar{43}x^2y^2 + \bar{04}y^4 + \bar{01}x^3 + \bar{34}x^2y + \bar{10}xy^2 + \bar{04}y^3 + \bar{03}x^2 + \bar{41}xy + \bar{32}y^2 + \bar{33}x + \bar{02}y + \bar{02}$.

TABLE 1.1. Non-projective automorphism g of X_F

Notation. (1) A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $(\ ,\ )_L : L \times L \to \mathbb{Z}$.

(2) The numerical equivalence class of a divisor D on X is denoted by $[D] \in \mathrm{NS}(X)$. The intersection number of divisors D and D' is written as $(D, D')_{\mathrm{NS}}$.

$\ell_1 := \ell^+([0:1:1+\sqrt{2}])$	$\ell_2 := \ell^-([0:1:1+\sqrt{2}])$
$\ell_3 := \ell^+([0:1:1+4\sqrt{2}])$	$\ell_4 := \ell^+([0:1:2])$
$\ell_5 := \ell^+([0:1:3])$	$\ell_6 := \ell^+([0:1:4+\sqrt{2}])$
$\ell_7 := \ell^+([1:0:1+\sqrt{2}])$	$\ell_8 := \ell^+([1:0:1+4\sqrt{2}])$
$\ell_9 := \ell^+([1:0:2])$	$\ell_{10} := \ell^+([1:0:4+\sqrt{2}])$
$\ell_{11} := \ell^+([1:\sqrt{2}:1])$	$\ell_{12} := \ell^-([1:\sqrt{2}:2+2\sqrt{2}])$
$\ell_{13} := \ell^-([1:\sqrt{2}:2+3\sqrt{2}])$	$\ell_{14} := \ell^+([1:\sqrt{2}:3+2\sqrt{2}])$
$\ell_{15} := \ell^-([1:\sqrt{2}:3+3\sqrt{2}])$	$\ell_{16} := \ell^+([1:2\sqrt{2}:2\sqrt{2}])$
$\ell_{17} := \ell^+([1:2\sqrt{2}:3\sqrt{2}])$	$\ell_{18} := \ell^-([1:2\sqrt{2}:2+\sqrt{2}])$
$\ell_{19} := \ell^+([1:2\sqrt{2}:2+4\sqrt{2}])$	$\ell_{20} := \ell^+([1:2\sqrt{2}:3+\sqrt{2}])$
$\ell_{21} := \ell^+([1:1+\sqrt{2}:0])$	$\ell_{22} := \ell^+([1:1+3\sqrt{2}:1])$

TABLE 2.1. Basis of $\mathrm{NS}(X)$

2. THE NÉRON-SEVERI LATTICE OF X

Recall that $B_F \subset \mathbb{P}^2$ is the Fermat curve of degree 6 in characteristic 5, which is the branch curve of the projective model $\psi_F : X_F \to \mathbb{P}^2$ corresponding to the polarization $h_F \in \mathrm{NS}(X)$ of degree 2. We denote by $B_F(\mathbb{F}_{25})$ the set of $\mathbb{F}_{25}$-rational points of B_F. It is known that $|B_F(\mathbb{F}_{25})| = 126$.

Let l be a line on $\mathbb{P}^2$ tangent to B_F. Since B_F is the *Hermitian curve* over $\mathbb{F}_{25}$, either one of the following holds (see [20] or Chapter 23 of [8]):

(1) l is tangent to B_F at a point $[a:b:c] \notin B_F(\mathbb{F}_{25})$ with intersection multiplicity 5, and intersects B_F at the point $[a^{25}:b^{25}:c^{25}]$ transversely.
(2) l is tangent to B_F at $P \in B_F(\mathbb{F}_{25})$ with intersection multiplicity 6.

In the case (2), the inverse image of l by the double covering $\Phi_F : X \to \mathbb{P}^2$ decomposes into two h_F-lines $\ell^+(P)$ and $\ell^-(P)$ such that

$$(\ell^+(P), \ell^-(P))_{\mathrm{NS}} = 3.$$

All h_F-lines on X are obtained as $\ell^{\pm}(P)$ with $P \in B_F(\mathbb{F}_{25})$. In particular, the number of h_F-lines on X is 252. We put

$$P_0 := [0:1:1+\sqrt{2}] \in B_F(\mathbb{F}_{25}) \quad \text{and} \quad \ell^+(P_0) := \{x^3 - w = 0, y + (1-\sqrt{2})z = 0\}.$$

For $P \in B_F(\mathbb{F}_{25}) \setminus \{P_0\}$, we choose the sign of $\ell^{\pm}(P)$ in such a way that

$$(\ell^+(P), \ell^+(P_0))_{\mathrm{NS}} = 1 \qquad (\text{and hence } (\ell^-(P), \ell^+(P_0))_{\mathrm{NS}} = 0).$$

From among these h_F-lines, we choose the 22 curves $\ell_1, \dots, \ell_{22}$ in Table 2.1. Then their intersection matrix M_{NS} is calculated as in Table 2.2. Since $\det M_{\mathrm{NS}} = -25$, the classes of $\ell_1, \dots, \ell_{22}$ form a $\mathbb{Z}$-basis of $\mathrm{NS}(X)$. We fix this basis throughout the paper. Each element of $\mathrm{NS}(X)$ is written as a *row* vector with respect to this basis.

$$
\left[\begin{array}{cccccccccccccccccccccc}
-2 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -2
\end{array}\right]
$$

Table 2.2. Matrix M_{NS}

In particular, the orthogonal group $\mathrm{O}(\mathrm{NS}(X))$ of the lattice $\mathrm{NS}(X)$ acts on $\mathrm{NS}(X)$ from the right. Since $h_F = [\ell^+(P)] + [\ell^-(P)]$ for any $P \in B_F(\mathbb{F}_{25})$, we have

$$h_F = [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]. \tag{2.1}$$

We calculate the vector representations of the classes of all h_F-lines.

Example 2.1. The class of the h_F-line $\ell^-([1 : 4 + 4\sqrt{2} : 0])$ is

$$[-4, -6, 3, 1, 1, 2, 1, -1, 2, 1, 1, 4, 1, 0, -3, 0, 2, -1, 3, -1, -2, -3].$$

From the action of $\mathrm{PGU}_3(\mathbb{F}_{25})$ on the set $B_F(\mathbb{F}_{25})$, we can calculate the action of $\mathrm{Aut}(X, h_F)$ on the set of h_F-lines. Using this permutation representation, we can write explicitly the linear representation

$$\mathrm{Aut}(X, h_F) \to \{\, T \in \mathrm{GL}_{22}(\mathbb{Z}) \mid T M_{\mathrm{NS}}\, {}^tT = M_{\mathrm{NS}} \,\} \cong \mathrm{O}(\mathrm{NS}(X)). \tag{2.2}$$

This representation is faithful (see Proposition 3 in Section 8 of [17]).

$$
\begin{bmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&3&-1&-1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&3&-1&0&-1&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
4&5&-2&-1&-1&-1&-1&-1&0&0&0&-1&-1&1&1&-1&0&0&-1&1&0&0\\
-1&-1&1&1&1&0&0&1&0&0&0&0&1&-1&0&1&-1&0&0&-1&0&0\\
0&-1&1&0&0&1&1&0&0&0&0&1&0&0&-1&0&0&-1&1&-1&0&0\\
2&2&-1&1&0&0&0&0&-1&-1&0&-1&0&0&1&1&-1&-1&0&0&0&0\\
2&2&0&-1&-1&-1&0&0&0&1&0&0&-1&1&0&-1&1&0&-1&0&0&0\\
2&3&-1&0&0&0&-1&0&0&-1&-1&-1&0&0&1&1&-1&0&0&0&0&0\\
1&1&0&-1&0&0&0&-1&1&1&-1&0&-1&1&0&-1&1&0&0&0&0&0\\
3&4&-1&0&-1&-1&-1&0&-1&0&0&-1&-1&0&1&0&0&0&-1&1&0&0\\
3&3&-1&0&-1&0&0&-1&-1&0&0&-1&-1&1&0&-1&1&0&-1&1&0&0\\
-1&-2&1&1&1&1&1&1&0&-1&0&0&1&-1&-1&1&-1&0&1&-1&0&0\\
-2&-3&2&1&0&2&1&-1&2&1&1&4&1&0&-3&0&1&-2&3&-2&-2&-3\\
3&4&-2&0&0&-1&-1&0&-1&-1&-1&-3&0&0&2&0&-1&1&-2&1&1&2
\end{bmatrix}
$$

Table 2.3. Matrix Γ_{NS}

Remark 2.2. The representation (2.2) is encoded as follows. We number the h_F-lines as $\ell_1, \dots, \ell_{22}, \ell_{23}, \dots, \ell_{252}$ once and for all. Then each $\gamma \in \mathrm{Aut}(X, h_F)$ is labelled by a list of 22 integers $[n_\gamma(1), \dots, n_\gamma(22)]$ in such a way that the image ℓ_i^γ of ℓ_i by γ is equal to $\ell_{n_\gamma(i)}$ for $i = 1, \dots, 22$. Then the action of γ on $\mathrm{NS}(X)$ is given by $v \mapsto vT_\gamma$, where T_γ is the 22×22 matrix whose ith row vector is $[\ell_{n_\gamma(i)}]$.

The Galois group $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$ also acts on the set of h_F-lines by the Frobenius action on X_F. The matrix Γ_{NS} in Table 2.3 represents this Frobenius conjugate action $v \mapsto \bar{v} = v\Gamma_{\mathrm{NS}}$ on $\mathrm{NS}(X)$.

3. Algorithms for lattices

3.1. **An algorithm for a positive quadratic triple.** By a *quadratic triple* of n-variables, we mean a triple $[Q, L, c]$, where Q is an $n \times n$ symmetric matrix with entries in $\mathbb{Q}$, L is a column vector of length n with entries in $\mathbb{Q}$, and c is a rational number. An element of $\mathbb{R}^n$ is written as a row vector $\boldsymbol{x} = [x_1, \dots, x_n]$.

The *inhomogeneous quadratic function* $q_{QT} : \mathbb{Q}^n \to \mathbb{Q}$ associated with a quadratic triple $QT = [Q, L, c]$ is defined by

$$q_{QT}(\boldsymbol{x}) := \boldsymbol{x}\, Q\, {}^t\boldsymbol{x} + 2\, \boldsymbol{x}\, L + c.$$

We say that $QT = [Q, L, c]$ and q_{QT} are *positive* or *negative* according to whether the symmetric matrix Q is positive-definite or negative-definite.

Let $QT = [Q, L, c]$ be a positive quadratic triple of n-variables. In this section, we describe an algorithm to calculate the finite set

$$E(QT) := \{\, \boldsymbol{x} \in \mathbb{Z}^n \;\mid\; q_{QT}(\boldsymbol{x}) \le 0 \,\}.$$

Suppose that $QT = [Q, L, c]$ is written as follows:

$$Q = \left[\begin{array}{c|c} Q' & \boldsymbol{p}' \\ \hline {}^t\boldsymbol{p}' & r' \end{array}\right] = \left[\begin{array}{c|c} r'' & {}^t\boldsymbol{p}'' \\ \hline \boldsymbol{p}'' & Q'' \end{array}\right], \quad L = \left[\begin{array}{c} L' \\ \hline m' \end{array}\right] = \left[\begin{array}{c} m'' \\ \hline L'' \end{array}\right],$$

where Q' and Q'' are square matrices of size $n-1$, $\boldsymbol{p}'$, $\boldsymbol{p}''$, L' and L'' are column vectors of length $n-1$, and r', r'', m' and m'' are rational numbers. Note that, since Q is positive-definite, we have $r' > 0$ and $r'' > 0$. We define a positive quadratic triple $\mathrm{pr}(QT)$ of $(n-1)$-variables by

$$\mathrm{pr}(QT) := \left[\; Q' - \frac{1}{r'}(\boldsymbol{p}'\, {}^t\boldsymbol{p}'),\; L' - \frac{m'}{r'}\boldsymbol{p}',\; c - \frac{m'^2}{r'} \;\right].$$

Then, for each $t \in \mathbb{R}$, the compact subset $\{\boldsymbol{x} \in \mathbb{R}^n \mid q_{QT}(\boldsymbol{x}) \le t\}$ of $\mathbb{R}^n$ is mapped by the projection $[x_1, \dots, x_n] \mapsto [x_1, \dots, x_{n-1}]$ to the compact subset

$$\{\, \boldsymbol{y} \in \mathbb{R}^{n-1} \;\mid\; q_{\mathrm{pr}(QT)}(\boldsymbol{y}) \le t \,\}$$

of $\mathbb{R}^{n-1}$. For $a \in \mathbb{Q}$, we define a positive quadratic triple $\iota^*(a, QT)$ of $(n-1)$-variables by

$$\iota^*(a, QT) := [\; Q'',\; a\, \boldsymbol{p}'' + L'',\; a^2\, r'' + 2\, a\, m'' + c \;],$$

and, for $\boldsymbol{a} = [a_1, \dots, a_m] \in \mathbb{Q}^m$ with $m < n$, we define a positive quadratic triple $\iota^*(\boldsymbol{a}, QT)$ of $(n-m)$-variables by

$$QT^0 := QT, \quad QT^{\nu+1} := \iota^*(a_{\nu+1}, QT^\nu) \;\; (\nu = 0, \dots, m-1), \quad \iota^*(\boldsymbol{a}, QT) := QT^m.$$

Then the positive inhomogeneous quadratic function $q_{\iota^*(\boldsymbol{a}, QT)} : \mathbb{Q}^{n-m} \to \mathbb{Q}$ is equal to the composite $q_{QT} \circ \iota_{\boldsymbol{a}}$, where $\iota_{\boldsymbol{a}}$ is the inclusion $\mathbb{Q}^{n-m} \hookrightarrow \mathbb{Q}^n$ given by

$$[y_1, \dots, y_{n-m}] \mapsto [a_1, \dots, a_m, y_1, \dots, y_{n-m}].$$

Suppose that $\boldsymbol{a} = [a_1, \dots, a_{n-1}] \in E(\mathrm{pr}(QT))$ is given. Then the positive quadratic triple $\iota^*(\boldsymbol{a}, QT)$ is of *one* variable, and the fiber of the projection $E(QT) \to E(\mathrm{pr}(QT))$ over $\boldsymbol{a}$ is equal to

$$\{\, [a_1, \dots, a_{n-1}, b] \mid b \in E(\iota^*(\boldsymbol{a}, QT)) \,\}.$$

Since $E(\iota^*(\boldsymbol{a}, QT))$ is easily calculated, we can obtain $E(QT)$ if we know $E(\mathrm{pr}(QT))$. Using this idea iteratively, we carry out the following computation.

Starting from the given positive quadratic triple $QT^0_n := QT$ of n-variables, we compute positive quadratic triples QT^0_μ of μ-variables by

$$QT^0_\mu := \mathrm{pr}(QT^0_{\mu+1}) \qquad (\mu = n-1, \dots, 1).$$

We prepare an empty set $E := \{\ \}$. We then write a program $\mathcal{Q}(\nu, \boldsymbol{a})$ that takes an integer $\nu \le n+1$ and a vector $\boldsymbol{a} = [a_1, \dots, a_{\nu-1}] \in \mathbb{Z}^{\nu-1}$ as input, and carries out the task below. Note that, when $\mathcal{Q}(\nu, \boldsymbol{a})$ starts with $\nu > 1$, $\boldsymbol{a}$ is an element of $E(QT^0_{\nu-1})$, and for $\mu > \nu - 1$, $QT^{\nu-1}_\mu$ is the positive quadratic triple $\iota^*(\boldsymbol{a}, QT^0_\mu)$ of $(\mu - \nu + 1)$-variables. In particular, $QT^{\nu-1}_\nu$ is of one variable.

The task of $\mathcal{Q}(\nu, \boldsymbol{a})$:

(1) If $\nu = n+1$, then $\mathcal{Q}(\nu, \boldsymbol{a})$ appends $\boldsymbol{a}$ to the set E.
(2) If $\nu \le n$, then the program $\mathcal{Q}(\nu, \boldsymbol{a})$
  (2-i) calculates the set $E(QT^{\nu-1}_\nu) = \{b_1, \dots, b_N\}$, and
  (2-ii) for each $b_i \in E(QT^{\nu-1}_\nu)$,
    (2-ii-a) computes $QT^\nu_\mu := \iota^*(b_i, QT^{\nu-1}_\mu)$ for $\mu = \nu+1, \dots, n$, and
    (2-ii-b) proceeds to execute $\mathcal{Q}(\nu+1, [a_1, \dots, a_{\nu-1}, b_i])$.

We execute $\mathcal{Q}(1, [\,])$. Since each $E(QT^{\nu-1}_\nu)$ is finite, this program certainly terminates. When the whole computation halts, the set E is equal to $E(QT)$.

3.2. **An application to hyperbolic lattices I.** Changing the sign, we can apply the algorithm above to *negative* inhomogeneous quadratic functions.

Suppose that N is a hyperbolic lattice of rank n, that is, the signature of $(\ ,\ )_N$ is $(1, n-1)$. Let $\{[v_i, a_i] \mid i = 1, \dots, k\}$ be a finite set of pairs of $v_i \in N$ and $a_i \in \mathbb{Z}$ such that $(v_i, v_i)_N > 0$ for at least one i, and let d be an integer. We can calculate the set

$$\{\, x \in N \mid (x, v_i)_N = a_i \ \text{ for } \ i = 1, \dots, k, \ \text{ and } \ (x, x)_N = d \,\} \tag{3.1}$$

by the following method. We put

$$M := \{\, x \in N \mid (x, v_i)_N = a_i \ \text{ for } \ i = 1, \dots, k \,\}.$$

It is easy to determine whether M is empty or not. Suppose that $M \ne \emptyset$. By choosing a point $c \in M$ as an origin, we can regard M as a free $\mathbb{Z}$-module of finite rank. By the assumption on v_i, the restriction of $(\ ,\ )_N$ to $M \subset N$ defines a negative inhomogeneous quadratic function on M. Therefore we can calculate the set (3.1) by the algorithm in Section 3.1.

3.3. **An application to hyperbolic lattices II.** Let N be as in the previous subsection. Suppose that we are given vectors $h, v \in N$ satisfying

$$(h,h)_N > 0, \quad (v,v)_N > 0, \quad (h,v)_N > 0. \tag{3.2}$$

We describe an algorithm that calculates, for a given integer d, the set

$$S := \{\, r \in N \mid (r,h)_N > 0,\ (r,v)_N < 0,\ (r,r)_N = d \,\}. \tag{3.3}$$

Consider the orthogonal direct-sum decomposition $N \otimes \mathbb{R} = \langle h \rangle \oplus \langle h \rangle^{\perp}$. We denote the second projection by $\mathrm{pr}_2 : N \otimes \mathbb{R} \to \langle h \rangle^{\perp}$, and put

$$W := \mathrm{pr}_2(N),$$

which is a free $\mathbb{Z}$-module of rank $n-1$ such that $W \otimes \mathbb{R} = \langle h \rangle^{\perp}$. Note that $W \subset N \otimes \mathbb{Q}$. We denote by

$$(\ ,\ )_W : W \times W \to \mathbb{Q}$$

the restriction of $(\ ,\ )_N$ to W. Suppose that $x \in N \otimes \mathbb{R}$ satisfies $(h,x)_N \neq 0$ and $(x,x)_N > 0$. Then the composite

$$\langle x \rangle^{\perp} \hookrightarrow N \otimes \mathbb{R} \xrightarrow{\mathrm{pr}_2} \langle h \rangle^{\perp} \tag{3.4}$$

is an isomorphism of $\mathbb{R}$-vector spaces. Let $\varphi_x : \langle h \rangle^{\perp} \xrightarrow{\sim} \langle x \rangle^{\perp}$ denote the inverse of the isomorphism (3.4), that is,

$$\varphi_x(y) = y - \frac{(y,x)_N}{(h,x)_N} h \quad \text{for} \quad y \in \langle h \rangle^{\perp}.$$

We then define $f_x : \langle h \rangle^{\perp} \to \mathbb{R}$ by

$$f_x(y) := (\varphi_x(y), \varphi_x(y))_N = (y,y)_W + \frac{(y,x)_N^2}{(h,x)_N^2}\,(h,h)_N \quad \text{for} \quad y \in \langle h \rangle^{\perp} = W \otimes \mathbb{R}.$$

Since $(x,x)_N > 0$, the real quadratic form $(\ ,\ )_N$ restricted to $\langle x \rangle^{\perp}$ is negative-definite, and hence so is f_x. By the condition (3.2), we see that f_{h+tv} is negative-definite on $W \otimes \mathbb{R}$ for any $t \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. (Here we understand that $f_{h+\infty v} = f_v$.)

For simplicity, we put

$$c_h := (h,h)_N, \quad c_v := (h,v)_N, \quad v_W := \mathrm{pr}_2(v) \in W.$$

Let x' be a vector in $\langle h \rangle^{\perp} = W \otimes \mathbb{R}$. Since $v - v_W \in \langle h \rangle$, we have

$$f_{h+tv}(x') = (x',x')_W + \frac{t^2 (x', v_W)_W^2}{(c_h + t c_v)^2} c_h. \tag{3.5}$$

By (3.2), we have $c_h/c_v > 0$, and hence, for a fixed $x' \in \langle h \rangle^{\perp}$, $f_{h+tv}(x')$ is a non-decreasing function with respect to $t \in \mathbb{R}_{\geq 0}$ bounded from above by

$$f_{h+\infty v}(x') = (x',x')_W + \frac{(x', v_W)_W^2}{c_v^2} c_h.$$

Note that $f_{h+\infty v}$ restricted to $W \subset W \otimes \mathbb{R}$ is $\mathbb{Q}$-valued, and hence $f_{h+\infty v}$ is a negative inhomogeneous quadratic function on $W \otimes \mathbb{Q}$. Applying the algorithm in Section 3.1 to $f_{h+\infty v}$, we can calculate the finite set

$$S_W := \{\, r' \in W \mid f_{h+\infty v}(r') \geq d \,\},$$

where d is the integer given as input.

Suppose that r is an element of the set S in (3.3). We put

$$t_r := -\frac{(r,h)_N}{(r,v)_N} \quad \in \quad \mathbb{R}_{>0}.$$

Then we have $r \in \langle h + t_r v \rangle^{\perp}$. We put $r' := \mathrm{pr}_2(r) \in W$. Since $\varphi_{h+t_r v}(r') = r$, we have

$$d = (r,r)_N = f_{h+t_r v}(r') \leq f_{h+\infty v}(r').$$

Therefore $r' \in S_W$ holds. Let $\rho \in \mathbb{Q}$ be the rational number such that $r = \rho h + r'$. Since $(r,r)_N = d$, $(r',h)_N = 0$ and $(r,h)_N > 0$, we have

$$\rho = \frac{(r,h)_N}{c_h} = \sqrt{\frac{d-(r',r')_W}{c_h}}. \tag{3.6}$$

The right-hand side of (3.6) can be calculated if we know $r' \in W$.

Therefore we obtain S from S_W by the following method. First we set $S = \{\ \}$. For each $r' \in S_W$, we put

$$\rho' := \sqrt{\frac{d-(r',r')_W}{c_h}} \quad \text{and} \quad r := \rho' h + r' \in N \otimes \mathbb{R}.$$

We then determine whether r is contained in N or not. (If $\rho' \notin \mathbb{Q}$, then we obviously have $r \notin N$.) If $r \in N$, $(r,h)_N > 0$ and $(r,v)_N < 0$, we append r to S. When this calculation is done for all $r' \in S_W$, the set S is equal to the set (3.3).

4. Geometric applications

We apply the algorithms in the previous section to the hyperbolic lattice $\mathrm{NS}(X)$.

4.1. **Polarizations.** If $v \in \mathrm{NS}(X)$ is a polarization, then we necessarily have $(v,v)_{\mathrm{NS}} > 0$ and $(v,h_F)_{\mathrm{NS}} > 0$. It is well-known that the nef cone of X is bounded by the hyperplanes perpendicular to classes of (-2)-curves (see Section 3 of [17], for example). If v with $(v,v)_{\mathrm{NS}} > 0$ is nef, then Proposition 0.1 of [13] gives a criterion for v to be a polarization. Thus we obtain the following:

Proposition 4.1. *Suppose that a vector $v \in \mathrm{NS}(X)$ satisfies $(v,v)_{\mathrm{NS}} > 0$ and $(v,h_F)_{\mathrm{NS}} > 0$. Consider the sets*

$$\begin{aligned} S_1 &:= \{\, r \in \mathrm{NS}(X) \mid (r,r)_{\mathrm{NS}} = -2,\ (r,h_F)_{\mathrm{NS}} > 0,\ (r,v)_{\mathrm{NS}} < 0 \,\} \quad \textit{and} \\ S_2 &:= \{\, e \in \mathrm{NS}(X) \mid (e,e)_{\mathrm{NS}} = 0,\ (e,v)_{\mathrm{NS}} = 1 \,\}. \end{aligned}$$

Then v is nef if and only if $S_1 = \emptyset$. If v is nef, then v is a polarization if and only if $S_2 = \emptyset$.

The sets S_1 and S_2 can be calculated by the algorithms in Sections 3.3 and 3.2, respectively. Hence we Proposition 4.1 enables us to determine whether a given vector $v \in \mathrm{NS}(X)$ is a polarization or not.

4.2. **h-Exceptional curves.** Let $h \in \mathrm{NS}(X)$ be a polarization of arbitrary degree. A (-2)-curve C on X is called *h-exceptional* if Φ_h contracts C. The set $\mathrm{Exc}(h) \subset \mathrm{NS}(X)$ of the classes of h-exceptional curves is calculated by the following algorithm. We calculate the finite set

$$R := \{\, r \in \mathrm{NS}(X) \mid (r, r)_{\mathrm{NS}} = -2, \;\; (r, h)_{\mathrm{NS}} = 0 \,\}$$

by the algorithm in Section 3.2, and classify the elements of R by the degree with respect to the ample class h_F as follows:

$$R[m] := \{\, r \in R \mid (r, h_F)_{\mathrm{NS}} = m \,\} \quad \text{and} \quad R^+ := \bigcup_{m>0} R[m].$$

We say that $r \in R^+$ is *indecomposable* if there are no vectors $r_1, \dots, r_k \in R^+$ with $k > 1$ such that $r = r_1 + \cdots + r_k$. Since each $R[m]$ is finite, we can determine whether a given vector $r \in R^+$ is indecomposable or not. It is obvious that $r \in R^+$ is contained in $\mathrm{Exc}(h)$ if and only if r is an indecomposable element of R^+.

4.3. **h-Lines.** Let $h \in \mathrm{NS}(X)$ be a polarization of arbitrary degree. A (-2)-curve C on X is called an *h-line* if Φ_h maps C to a line isomorphically. The set $\mathrm{Lin}(h) \subset \mathrm{NS}(X)$ of the classes of h-lines is calculated by the following algorithm. We calculate the finite sets

$$\begin{aligned} L &:= \{\, r \in \mathrm{NS}(X) \mid (r, r)_{\mathrm{NS}} = -2, \;\; (r, h)_{\mathrm{NS}} = 1 \,\}, \\ L[m] &:= \{\, r \in L \mid (r, h_F)_{\mathrm{NS}} = m \,\}, \quad L^+ := \bigcup_{m>0} L[m]. \end{aligned}$$

It is obvious that $\mathrm{Lin}(h) \subset L^+$. If $r \in L^+$, then we see that r is the class of an effective divisor D, that exactly one irreducible component D_0 of D is an h-line, and that $D - D_0$ is a finite sum of h-exceptional curves. Hence $r \in L^+$ is contained in $\mathrm{Lin}(h)$ if and only if there are no $r' \in L[m']$ with $m' < (r, h_F)_{\mathrm{NS}}$ and $r_1, \dots, r_k \in \mathrm{Exc}(h)$ with $k \ge 1$ such that $r = r' + r_1 + \cdots + r_k$. Since each of $L[m']$ and $\mathrm{Exc}(h)$ are finite, we can determine the subset $\mathrm{Lin}(h) \subset L^+$.

5. Explicit defining equations

We identify X with X_F by the isomorphism $\phi_F : X \xrightarrow{\sim} X_F$, so that, for a polarization $h \in \mathcal{P}_2(X)$, we consider $\Phi_h : X \to \mathbb{P}^2$ and $\phi_h : X \to X_h$ as morphisms from X_F. In this section, we describe a method to write the morphisms Φ_h and ϕ_h as lists of rational functions on X_F over $\mathbb{F}_{25}$.

5.1. **The global sections of a line bundle.** Let $H_\infty \subset X_F$ denote the hyperplane section defined by $z = 0$ in (1.2). We use the affine coordinates (w, x, y) of $\mathbb{P}(3, 1, 1, 1)$ with $z = 1$, and put

$$F := w^2 - x^6 - y^6 - 1 \quad \in \quad \mathbb{F}_{25}[w, x, y].$$

For any $g \in \mathbb{F}_{25}[w, x, y]$, there exists a unique polynomial $\bar{g}^F$ of the form $wf + h$ with $f, h \in \mathbb{F}_{25}[x, y]$ such that

$$g \equiv \bar{g}^F \bmod (F) \quad \text{in} \quad \mathbb{F}_{25}[w, x, y].$$

We call $\bar{g}^F$ the *normal form* of g. Let m be an integer. By identifying the line bundle $\mathcal{L}_{mh_F} \to X$ with the invertible sheaf $\mathcal{O}_{X_F}(mH_\infty)$, the vector space $\Gamma(X, \mathcal{L}_{mh_F})$ of the global sections of $\mathcal{L}_{mh_F}$ defined over $\mathbb{F}_{25}$ is naturally identified with the vector subspace

$$V_m := \{\, wf + h \mid f, h \in \mathbb{F}_{25}[x, y],\ \ \deg f \le m - 3,\ \ \deg h \le m \,\}$$

of $\mathbb{F}_{25}[w, x, y]$. Recall that all h_F-lines are defined over $\mathbb{F}_{25}$, and that no h_F-lines are contained in H_∞. We have indexed the h_F-lines as $\ell_1, \dots, \ell_{252}$ in Remark 2.2. For $j = 1, \dots, 252$, we denote by

$$I_j \subset \mathbb{F}_{25}[w, x, y]$$

the inhomogeneous ideal defining ℓ_j in $\mathbb{P}(3, 1, 1, 1)$, and put

$$I_j^{(\nu)} := I_j^\nu + (F) \subset \mathbb{F}_{25}[w, x, y] \quad \text{for} \ \ \nu \in \mathbb{Z}_{>0}.$$

We describe an algorithm that takes a vector $v \in \mathrm{NS}(X)$ as input, and calculates the vector space $\Gamma(X, \mathcal{L}_v)$ of the global sections of the corresponding line bundle $\mathcal{L}_v \to X$ defined over $\mathbb{F}_{25}$. Using the $\mathbb{Z}$-basis $[\ell_1], \dots, [\ell_{22}]$ of $\mathrm{NS}(X)$, v is uniquely written as

$$v = \sum_{i \in J^+} a_i[\ell_i] - \sum_{j \in J^-} b_j[\ell_j],$$

where J^+ and J^- are disjoint subsets of $\{1, \dots, 22\}$, and a_i, b_j are positive integers. Let i' be the index of the h_F-line $\ell_{i'}$ that is the image of ℓ_i by the deck-transformation of X_F over $\mathbb{P}^2$. Since $[\ell_i] + [\ell_{i'}] = h_F$ for any i, we have

$$v = d'(v)h_F - \sum_{i \in J^+} a_i[\ell_{i'}] - \sum_{j \in J^-} b_j[\ell_j], \quad \text{where } d'(v) := \sum_{i \in J^+} a_i.$$

Thus we have an expression

$$v = d(v)h_F - \sum_{j \in J} c_j[\ell_j], \tag{5.1}$$

where $d(v)$ is a non-negative integer, J is a subset of $\{1, \dots, 252\}$, and c_j are positive integers. (Since there are linear relations among $[\ell_j]$, this expression is not unique.)

Then the vector space $\Gamma(X, \mathcal{L}_v)$ is identified with the space of global sections of $\mathcal{O}_{X_F}(d(v)H_\infty)$ that vanish along ℓ_j with order c_j for each $j \in J$, that is,

$$\Gamma(X, \mathcal{L}_v) \;\cong\; V_{d(v)} \cap \bigcap_{j \in J} I_j^{(c_j)}, \tag{5.2}$$

where the intersections are taken in $\mathbb{F}_{25}[w, x, y]$. From now on, we regard $\Gamma(X, \mathcal{L}_v)$ as a subspace of $V_{d(v)}$ by (5.2). The vector space $V_{d(v)}$ has a basis

$$m_\alpha := wM \;\text{ or }\; N \qquad (\alpha = 1, \dots, 2 + d(v)^2),$$

where M and N are the monomials of x and y with $\deg M \le d(v) - 3$ and $\deg N \le d(v)$. We calculate the Gröbner basis G_j of the ideal $I_j^{(c_j)} \subset \mathbb{F}_{25}[w, x, y]$ for each $j \in J$. (In the actual calculation, we used the graded reverse lexicographic order $\mathtt{grevlex}(w, x, y)$. See p. 56 of [5].) We then calculate the remainders $\overline{m_\alpha}^{G_j}$ of the monomials m_α by these Gröbner bases G_j. An element $\sum_\alpha u_\alpha m_\alpha$ of $V_{d(v)}$ with $u_\alpha \in \mathbb{F}_{25}$ is contained in $\Gamma(X, \mathcal{L}_v)$ if and only if

$$\sum_\alpha u_\alpha \overline{m_\alpha}^{G_j} = 0 \quad \text{for each } j \in J.$$

These equalities constitute a system of linear equations with unknowns u_α. Solving these equations, we obtain a basis of $\Gamma(X, \mathcal{L}_v)$ as a list of polynomials in $V_{d(v)}$.

Let k be a positive integer. Then we can write the vector $kv \in \mathrm{NS}(X)$ as

$$kv := kd(v)h_F - \sum_{j \in J} kc_j[\ell_j]$$

using the same $d(v)$ and J that appeared in (5.1). Under this choice, the natural homomorphism

$$\Gamma(X, \mathcal{L}_v)^{\otimes k} \to \Gamma(X, \mathcal{L}_{kv})$$

is given by restricting the linear homomorphism

$$g_1 \otimes \cdots \otimes g_k \;\mapsto\; \overline{g_1 \cdots g_k}^F$$

from $V_{d(v)}^{\otimes k}$ to $V_{kd(v)}$.

5.2. **The morphisms Φ_h and ϕ_h.** We describe an algorithm that takes a vector $h \in \mathcal{P}_2(X)$ as input, and calculates the morphisms Φ_h, ϕ_h and a defining equation

$$w^2 = s_h(x, y, z)$$

of X_h in $\mathbb{P}(3, 1, 1, 1)$. We have

$$\dim \Gamma(X, \mathcal{L}_h) = 3, \quad \dim \Gamma(X, \mathcal{L}_{3h}) = 11, \quad \dim \Gamma(X, \mathcal{L}_{6h}) = 38.$$

We find an expression $h = d(h)h_F - \sum_{j \in J} c_j[\ell_j]$ of h in the form (5.1). By the method described above, we obtain three polynomials

$$\xi_i(w, x, y) \in V_{d(h)} \qquad (i = 0, 1, 2)$$

that form a basis of $\Gamma(X, \mathcal{L}_h)$. The rational map $(w, x, y) \mapsto [\xi_0 : \xi_1 : \xi_2]$ gives the morphism $\Phi_h : X_F \to \mathbb{P}^2$.

Next we calculate eleven polynomials that form a basis of $\Gamma(X, \mathcal{L}_{3h}) \subset V_{3d(h)}$ using the expression $3h = 3d(h)h_F - \sum_{j\in J} 3c_j[\ell_j]$. We compute the normal forms

$$\overline{\xi_i \xi_{i'} \xi_{i''}}^F \qquad (i, i', i'' \in \{0, 1, 2\})$$

of the ten polynomials $\xi_i \xi_{i'} \xi_{i''}$. These normal forms are contained in $\Gamma(X, \mathcal{L}_{3h})$. Then we find a polynomial $\omega \in V_{3d(h)}$ that is contained in $\Gamma(X, \mathcal{L}_{3h})$, but is *not* contained in the 10-dimensional subspace spanned by $\overline{\xi_i \xi_{i'} \xi_{i''}}^F$. The rational map

$$(w, x, y) \mapsto [\omega : \xi_0 : \xi_1 : \xi_2] \in \mathbb{P}(3, 1, 1, 1)$$

gives the morphism $\phi_h : X_F \to X_h$.

We then compute the 39 normal forms

$$\overline{\omega^2}^F, \qquad \overline{\omega \xi_i \xi_{i'} \xi_{i''}}^F, \qquad \overline{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_6}}^F \quad (i, i', i'', i_1, \ldots, i_6 \in \{0, 1, 2\}),$$

which are contained in $\Gamma(X, \mathcal{L}_{6h}) \subset V_{6d(h)}$. Since $\dim \Gamma(X, \mathcal{L}_{6h}) = 38$, there exists a non-trivial linear relation over $\mathbb{F}_{25}$ among these 39 polynomials. Using homogeneous polynomials $b(x, y, z)$ of degree 3 and $c(x, y, z)$ of degree 6 with coefficients in $\mathbb{F}_{25}$, we write this linear relation as

$$\overline{a\,\omega^2 + b(\xi_0, \xi_1, \xi_2)\,\omega + c(\xi_0, \xi_1, \xi_2)}^F = 0, \tag{5.3}$$

where $a \in \mathbb{F}_{25}$. Since ω is not invariant under the deck-transformation of X_F over $\mathbb{P}^2$, we may assume that $a = 1$. We replace ω by

$$\omega - 2\,\overline{b(\xi_0, \xi_1, \xi_2)}^F \in V_{3d(h)}.$$

Then the linear relation (5.3) is written as

$$\overline{\omega^2}^F = \overline{s_h(\xi_0, \xi_1, \xi_2)}^F, \quad \text{where} \quad s_h(x, y, z) := -\,b(x, y, z)^2 - c(x, y, z).$$

The projective model $\psi_h : X_h \to \mathbb{P}^2$ is defined by $w^2 = s_h(x, y, z)$.

Remark 5.1. The computational difficulty of this method grows rapidly as $d(h)$ increases.

5.3. **The projective equivalence.** Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}_{25}$. For $T \in \mathrm{GL}_3(\overline{\mathbb{F}})$, we denote by $[T] \in \mathrm{PGL}_3(\overline{\mathbb{F}})$ the image of T by the natural homomorphism $\mathrm{GL}_3(\overline{\mathbb{F}}) \to \mathrm{PGL}_3(\overline{\mathbb{F}})$, and by $P \mapsto P^{[T]}$ the linear transformation of $\mathbb{P}^2$ given by $[a : b : c] \mapsto [a : b : c]\,T$. Let $\mathbb{H}_6$ denote the set of homogeneous polynomials of degree 6 in variables x, y, z with coefficients in $\mathbb{F}_{25}$. For $f \in \mathbb{H}_6 \otimes \overline{\mathbb{F}}$, we put

$$f^T(x, y, z) := f(x', y', z'), \quad \text{where} \quad (x', y', z') = (x, y, z)\,T^{-1}.$$

If $f = 0$ defines a curve $C \subset \mathbb{P}^2$, then $f^T = 0$ defines the image $C^{[T]}$ of the curve C by the projective linear transformation $P \mapsto P^{[T]}$.

Let h and h' be elements of $\mathcal{P}_2(X)$. By definition, we have the following:

$$h \sim h' \iff \text{there exist } T \in \mathrm{GL}_3(\overline{\mathbb{F}}) \text{ and } c \in \overline{\mathbb{F}}^{\times} \text{ such that } s_{h'} = c\, s_h^T. \tag{5.4}$$

The polynomials $\omega, \xi_0, \xi_1, \xi_2$ giving $\phi_h : X_F \to X_h$ that are obtained in the previous subsection are unique up to the following transformations:

$$\begin{aligned} \omega &\mapsto \lambda\omega, \quad \text{where} \quad \lambda \in \mathbb{F}_{25}^{\times}, \\ (\xi_0, \xi_1, \xi_2) &\mapsto (\xi_0, \xi_1, \xi_2)T, \quad \text{where} \quad T \in \mathrm{GL}_3(\mathbb{F}_{25}). \end{aligned}$$

Under this transformation, the sextic polynomial $s_h \in \mathbb{H}_6$ is changed to $\lambda^2 s_h^T$. Therefore we can define the following relation $\sim_{\mathbb{F}}$ on $\mathcal{P}_2(X)$:

$$h \sim_{\mathbb{F}} h' \iff \text{there exist } T \in \mathrm{GL}_3(\mathbb{F}_{25}) \text{ and } \lambda \in \mathbb{F}_{25}^{\times} \text{ such that } s_{h'} = \lambda^2 s_h^T. \tag{5.5}$$

We investigate the relation between $\sim$ and $\sim_{\mathbb{F}}$.

Lemma 5.2. *Suppose that there exist $T \in \mathrm{GL}_3(\mathbb{F}_{25})$ and $c \in \overline{\mathbb{F}}^{\times}$ that satisfy $s_{h'} = c\, s_h^T$. Then $h \sim_{\mathbb{F}} h'$ holds.*

Proof. Let K denote the quotient field of the integral domain $\mathbb{F}_{25}[w, x, y]/(F)$. Then we have $\overline{\mathbb{F}} \cap K = \mathbb{F}_{25}$. By the assumption $s_{h'} = c\, s_h^T$, we see that $c \in \mathbb{F}_{25}^{\times}$ and that there exist non-zero elements ω and ω' of K such that $\omega'^2 = c\,\omega^2$. Hence c is a non-zero square in $\mathbb{F}_{25}$. □

Let $B_1 = \{f_1 = 0\}$ and $B_2 = \{f_2 = 0\}$ be reduced plane curves defined by $f_1 \in \mathbb{H}_6$ and $f_2 \in \mathbb{H}_6$, respectively. We consider the set

$$\mathrm{isom}(B_1, B_2) := \{\, \tau \in \mathrm{PGL}_3(\overline{\mathbb{F}}) \mid B_1^{\tau} = B_2 \,\}$$

of projective isomorphisms from B_1 to B_2 defined over $\overline{\mathbb{F}}$. By definitions and Lemma 5.2, we have

$$h \sim h' \iff \mathrm{isom}(B_h, B_{h'}) \neq \emptyset, \tag{5.6}$$

$$h \sim_{\mathbb{F}} h' \iff \mathrm{isom}(B_h, B_{h'}) \cap \mathrm{PGL}_3(\mathbb{F}_{25}) \neq \emptyset. \tag{5.7}$$

Definition 5.3. Let $Q = [Q_0, Q_1, Q_2, Q_3]$ and $Q' = [Q'_0, Q'_1, Q'_2, Q'_3]$ be two ordered 4-tuples of points of $\mathbb{P}^2$ such that no three points of Q are colinear and no three points of Q' are colinear. Then there exists a unique projective transformation $\tau_{QQ'} \in \mathrm{PGL}_3(\overline{\mathbb{F}})$ such that

$$Q^{\tau_{QQ'}} := [Q_0^{\tau_{QQ'}}, Q_1^{\tau_{QQ'}}, Q_2^{\tau_{QQ'}}, Q_3^{\tau_{QQ'}}]$$

is equal to Q'. Let $T_{QQ'} \in \mathrm{GL}_3(\overline{\mathbb{F}})$ denote a matrix such that $[T_{QQ'}] = \tau_{QQ'}$.

Let B be a reduced plane curve defined over $\overline{\mathbb{F}}$. We define $\mathcal{Q}(B)$ to be the set

$$\left\{ \; [Q_0, Q_1, Q_2, Q_3] \;\middle|\; \begin{array}{l} Q_i \in \mathrm{Sing}(B) \text{ for } i = 0, \dots, 3, \text{ and no three of} \\ Q_0, \dots, Q_3 \text{ are colinear} \end{array} \right\}. \tag{5.8}$$

Let R be an element of $\mathcal{Q}(B_1)$. Then the map $\tau \mapsto R^\tau$ induces a bijection

$$\mathrm{isom}(B_1, B_2) \;\cong\; \{\, Q' \in \mathcal{Q}(B_2) \;\mid\; f_2 = c\, f_1^{\tau_{RQ'}} \text{ for some } c \in \overline{\mathbb{F}}^\times \,\}. \tag{5.9}$$

If all points of Q and Q' are $\mathbb{F}_{25}$-rational, then we have $\tau_{QQ'} \in \mathrm{PGL}_3(\mathbb{F}_{25})$. Hence we obtain the following:

Lemma 5.4. *Suppose that every singular point of B_h and $B_{h'}$ is $\mathbb{F}_{25}$-rational, and that $\mathcal{Q}(B_h)$ and $\mathcal{Q}(B_{h'})$ are non-empty. Then* $\mathrm{isom}(B_h, B_{h'})$ *is contained in* $\mathrm{PGL}_3(\mathbb{F}_{25})$. □

The bijection (5.9) also provides us with a practical method to calculate the group $\mathrm{aut}(B) = \mathrm{isom}(B, B)$ for a plane curve B defined over $\mathbb{F}_{25}$ satisfying $\mathrm{Sing}(B) \subset \mathbb{P}^2(\mathbb{F}_{25})$ and $\mathcal{Q}(B) \neq \emptyset$.

6. Proof of Theorem 1.1

6.1. **Step 1.** First note that $\mathcal{P}_2(X) \cap \mathcal{B}_3 = \{h_F\}$.

6.2. **Step 2.** We calculate the sets

$$\mathcal{V}_\delta := \{\, v \in \mathrm{NS}(X) \;\mid\; (v, v)_{\mathrm{NS}} = 2,\;\; (v, h_F)_{\mathrm{NS}} = \delta \,\},$$

for $\delta = 4$ and 5 by the algorithm in Section 3.2. The cardinalities of these sets are $|\mathcal{V}_4| = 1{,}020{,}600$ and $|\mathcal{V}_5| = 208{,}059{,}000$. We put

$$\mathcal{V} := \{h_F\} \cup \mathcal{V}_4 \cup \mathcal{V}_5.$$

Our goal is to calculate the subset $\mathcal{P}_2(X) \cap \mathcal{B}_5 = \mathcal{P}_2(X) \cap \mathcal{V}$ of $\mathcal{V}$, and decompose it into the equivalence classes of the relation $\sim$ of the projective equivalence. Note that $\mathrm{Aut}(X, h_F)$ acts on $\mathcal{V}_4$, $\mathcal{V}_5$ and $\mathcal{P}_2(X)$, and that, if h and h' are in the same $\mathrm{Aut}(X, h_F)$-orbit, then we have $h \sim_{\mathbb{F}} h'$, because every element of $\mathrm{Aut}(X, h_F)$ is defined over $\mathbb{F}_{25}$.

6.3. **Step 3.** We have embedded $\mathrm{Aut}(X, h_F)$ in $\mathrm{O}(\mathrm{NS}(X))$ by (2.2). Let $\boldsymbol{x} = [x_1, \dots, x_{22}]$ and $\boldsymbol{y} = [y_1, \dots, y_{22}]$ be vectors in $\mathrm{NS}(X)$. We put

$$\boldsymbol{x} <_{\mathrm{lex}} \boldsymbol{y} \quad\Longleftrightarrow\quad \text{there exists } k \text{ such that } x_k < y_k \text{ and } x_j = y_j \text{ for } j < k,$$

and define a total order $<$ on $\mathrm{NS}(X)$ by

$$\boldsymbol{x} < \boldsymbol{y} \quad\Longleftrightarrow\quad \sum_{i=1}^{22} |x_i| < \sum_{i=1}^{22} |y_i| \quad \text{or} \quad \left(\sum_{i=1}^{22} |x_i| = \sum_{i=1}^{22} |y_i| \quad \text{and} \quad \boldsymbol{x} <_{\mathrm{lex}} \boldsymbol{y} \right).$$

We then denote by $\mathcal{R}$ the set of vectors $v \in \mathrm{NS}(X)$ that are minimal in the $\mathrm{Aut}(X, h_F)$-orbit containing v:

$$\mathcal{R} := \{\, v \in \mathrm{NS}(X) \;\mid\; v \le vT \;\text{ for all }\; T \in \mathrm{Aut}(X, h_F) \,\}.$$

We define the representative vector v_o of each $\mathrm{Aut}(X, h_F)$-orbit $o \subset \mathrm{NS}(X)$ by

$$o \cap \mathcal{R} = \{v_o\}.$$

We calculate the list $\mathcal{R} \cap \mathcal{V}_4$, $\mathcal{R} \cap \mathcal{V}_5$, and the order of the stabilizer subgroup $\mathrm{Stab}(v) \subset \mathrm{Aut}(X, h_F)$ for each $v \in \mathcal{R} \cap \mathcal{V}$. We obtain

$$|\mathcal{R} \cap \mathcal{V}_4| = |\mathcal{V}_4 / \mathrm{Aut}(X, h_F)| = 8 \quad \text{and} \quad |\mathcal{R} \cap \mathcal{V}_5| = |\mathcal{V}_5 / \mathrm{Aut}(X, h_F)| = 312.$$

Remark 6.1. We choose this total order $<$ on $\mathrm{NS}(X)$ so that we can express each $v \in \mathcal{R} \cap \mathcal{V}$ in the form (5.1) with $d(v)$ small. See Remark 5.1.

6.4. **Step 4.** For each $v \in \mathcal{R} \cap \mathcal{V}$, we calculate the $\mathrm{Gal}(\mathbb{F}_{25}/\mathbb{F}_5)$-conjugate $\bar{v} = v\Gamma_{\mathrm{NS}}$ of v, where Γ_{NS} is the matrix given in Table 2.3, and find the representative vector $v^{\Gamma} \in \mathcal{R} \cap \mathcal{V}$ of the $\mathrm{Aut}(X, h_F)$-orbit containing $\bar{v}$.

6.5. **Step 5.** For each $v \in \mathcal{R} \cap \mathcal{V}$, we calculate the sets S_1 and S_2 in Proposition 4.1, and determine whether v is a polarization or not. We obtain

$$|\mathcal{P}_2(X) \cap \mathcal{R} \cap \mathcal{V}_4| = 7 \quad \text{and} \quad |\mathcal{P}_2(X) \cap \mathcal{R} \cap \mathcal{V}_5| = 224.$$

6.6. **Step 6.** For simplicity, we put

$$\mathcal{H} := \mathcal{P}_2(X) \cap \mathcal{R} \cap \mathcal{V}.$$

By means of the algorithms in Sections 4.2 and 4.3, we calculate, for each $h \in \mathcal{H}$, the set $\mathrm{Exc}(h)$ of the classes of h-exceptional curves, and the set $\mathrm{Lin}(h)$ of the classes of h-lines. From $\mathrm{Exc}(h)$, we determine the ADE-type $\mathrm{RT}(h)$ of $\mathrm{Sing}(B_h)$. We then confirm that the union of $\mathrm{Exc}(h)$ and $\mathrm{Lin}(h)$ spans $\mathrm{NS}(X)$ for any $h \in \mathcal{H}$. Thus Proposition 1.2 is proved.

6.7. **Step 7.** For each $h \in \mathcal{H}$, we carry out the computation in Section 5, and calculate polynomials $\omega, \xi_0, \xi_1, \xi_2 \in \mathbb{F}_{25}[w, x, y]$ that give the morphism $\phi_h : X_F \to X_h$, and $s_h(x, y, z) \in \mathbb{H}_6$ such that $w^2 = s_h(x, y, z)$ defines X_h. Then we compute the coordinates of the singular points of $B_h = \{s_h = 0\}$.

Remark 6.2. By this computation, we observe the following fact. For any $h \in \mathcal{H}$ with $\mathrm{RT}(h) \neq 0$, every singular point of B_h is $\mathbb{F}_{25}$-rational, and the set $\mathcal{Q}(B_h)$ defined by (5.8) is non-empty. By Lemma 5.4, it follows that $\mathrm{isom}(B_h, B_{h'})$ is contained in $\mathrm{PGL}_3(\mathbb{F}_{25})$ for any $h, h' \in \mathcal{H}$ with $\mathrm{RT}(h) \neq 0$ and $\mathrm{RT}(h') \neq 0$.

Remark 6.3. It turns out that each (-2)-curve contracted by ϕ_h is either an h_F-line or an irreducible component of the pull-back by ψ_F of a plane conic totally tangent to B_F (see [25]). We can calculate the coordinates of the singular points of B_h using this fact.

6.8. **Step 8.** We decompose $\mathcal{H}$ into the equivalence classes under the relation $\sim_{\mathbb{F}}$ defined by (5.5), and confirm that the relations $\sim$ and $\sim_{\mathbb{F}}$ are the same on $\mathcal{H}$.

6.8.1. *The case where B_h is non-singular.* In $\mathcal{H}$, there are exactly three polarizations h such that $\mathrm{RT}(h) = 0$: h_F and

$$\begin{aligned} h'_F &= [1,0,0,1,0,1,0,0,0,0,1,0,1,0,-1,0,0,0,0,0,0,0] \in \mathcal{V}_4, \quad \text{and} \\ h''_F &= [0,-1,0,2,1,0,0,0,0,0,1,0,1,0,1,1,0,-1,0,0,0,0] \in \mathcal{V}_5. \end{aligned}$$

Applying the following result, which is a corollary of n. 3 of [20], to $s_{h'_F}$ and $s_{h''_F}$, we see that $h'_F \sim_{\mathbb{F}} h_F$ and $h''_F \sim_{\mathbb{F}} h_F$.

Corollary 6.4. *For $h \in \mathcal{P}_2(X)$, we have $h \sim_{\mathbb{F}} h_F$ if and only if there exist a 3×3 non-degenerate matrix (a_{ij}) over $\mathbb{F}_{25}$ with $a_{ij} = a_{ji}^5$ and $\lambda \in \mathbb{F}_{25}^\times$ such that $s_h = s_h(x_0, x_1, x_2)$ is of the form $\lambda^2 \sum_{i,j=0}^2 a_{ij} x_i x_j^5$.*

6.8.2. *The case where B_h is singular.* We introduce a total order $\prec$ on the set $\mathbb{H}_6$. (Any total order will do.) We fix four reference points

$$P_0 := [1:0:0], \ \ P_1 := [0:1:0], \ \ P_2 := [0:0:1], \ \ P_3 := [1:1:1],$$

and put $P := [P_0, P_1, P_2, P_3]$. For $h \in \mathcal{H}$ with $\mathrm{RT}(h) \neq 0$, we put

$$\begin{aligned} \mathcal{T}(h) &:= \{\, \tau \in \mathrm{PGL}_3(\overline{\mathbb{F}}) \mid \mathrm{Sing}(B_h^\tau) \ni P_i \ \text{ for } \ i = 0,1,2,3 \,\} = \{\tau_{QP} \,|\, Q \in \mathcal{Q}(B_h)\}, \\ S(h) &:= \{\, \lambda^2 s_h^T \mid \lambda \in \mathbb{F}_{25}^\times, \ T \in \mathrm{GL}_3(\mathbb{F}_{25}) \,\}, \\ S^P(h) &:= \{\, s'_h \in S(h) \mid \text{the curve } s'_h = 0 \text{ is singular at } P_0, \dots, P_3 \,\}. \end{aligned}$$

By Remark 6.2, we have $\mathcal{T}(h) \subset \mathrm{PGL}_3(\mathbb{F}_{25})$ and $\mathcal{T}(h) \neq \emptyset$, and hence

$$S^P(h) = \{\, \lambda^2 s_h^T \mid \lambda \in \mathbb{F}_{25}^\times, \ T \in \mathrm{GL}_3(\mathbb{F}_{25}), \ \ [T] \in \mathcal{T}(h) \,\} \neq \emptyset$$

holds. Since $\mathcal{Q}(B_h)$ is easily calculated, so is $S^P(h)$. We put

$$s_h^{\min} := \text{the minimal element of } S^P(h) \text{ with respect to the fixed total order } \prec.$$

By definition, we have $h \sim_{\mathbb{F}} h'$ if and only if $S(h) = S(h')$. Hence we have

$$h \sim_{\mathbb{F}} h' \iff s_h^{\min} = s_{h'}^{\min}.$$

By this method, we decompose $\mathcal{H}$ into the equivalence classes of the relation $\sim_{\mathbb{F}}$.

Remark 6.2 combined with (5.6), (5.7) imply that $\sim$ and $\sim_{\mathbb{F}}$ define the same relation on $\mathcal{H}$. Thus the equivalence classes $\mathcal{E}_0, \dots, \mathcal{E}_{64}$ of $\sim$ are obtained.

For $h \in \mathcal{H}$, we denote by $[h] \subset \mathcal{H}$ the equivalence class of $\sim$ containing h, by $s_{[h]}$ the polynomial $s_h^{\min}$ obtained above, and by $B_{[h]}$ the plane curve $\{s_{[h]} = 0\}$.

6.9. **Step 9.** For each equivalence class $[h] \subset \mathcal{H}$, we calculate the group $\mathrm{aut}(B_{[h]}) = \mathrm{isom}(B_{[h]}, B_{[h]})$ and the set $\mathrm{isom}(B_{[h]}, \overline{B_{[h]}})$ by the method given in Section 5.3, where $\overline{B_{[h]}}$ is the plane curve defined by the polynomial $\overline{s_{[h]}} \in \mathbb{H}_6$ obtained from $s_{[h]}$ by $\sqrt{2} \mapsto -\sqrt{2}$.

6.10. **Step 10.** We search for $(T, \lambda) \in \mathrm{GL}_3(\mathbb{F}_{25}) \times \mathbb{F}_{25}^{\times}$ such that $\lambda^2 s_{[h]}^T$ has coefficients in $\mathbb{F}_5$. If such (T, λ) exists, then we necessarily have $h \sim h^{\Gamma}$.

Proposition 6.5. *For $f \in \mathbb{H}_6$, the following conditions are equivalent.* (i) There exist $T \in \mathrm{GL}_3(\mathbb{F}_{25})$ and $\lambda \in \mathbb{F}_{25}^{\times}$ such that $\lambda^2 f^T$ has coefficients in $\mathbb{F}_5$. (ii) There exist $M \in \mathrm{GL}_3(\mathbb{F}_{25})$ and $c \in \mathbb{F}_{25}^{\times}$ such that $f^M = c\,\bar{f}$, $M\overline{M} = \mathrm{Id}_3$ and $c^3 = 1$.

Since we have already calculated the set $\mathrm{isom}(B_{[h]}, \overline{B_{[h]}})$ for every $[h] \subset \mathcal{H}$, we can make the list of $(M, c) \in \mathrm{GL}_3(\mathbb{F}_{25}) \times \mathbb{F}_{25}^{\times}$ such that $s_{[h]}^M = c\,\overline{s_{[h]}}$. Therefore we can determine whether the condition (ii) is satisfied or not for $f = s_{[h]}$. The proof below shows how to find (T, λ) in the condition (i) from (M, c) in the condition (ii).

Proof of Proposition 6.5. Suppose that (i) holds. Since $\bar{\lambda}^2 \bar{f}^{\overline{T}} = \lambda^2 f^T$, we have $(\lambda^{-1}\bar{\lambda})^2 \bar{f} = f^{T\overline{T}^{-1}}$. Then $M := T\overline{T}^{-1}$ and $c := (\lambda^{-1}\bar{\lambda})^2 = \lambda^8$ satisfy the equalities in (ii). Conversely, suppose that (ii) holds. Then there exists $T \in \mathrm{GL}_3(\mathbb{F}_{25})$ such that $M = T\overline{T}^{-1}$. Indeed, let $m : \mathbb{F}_{25}^3 \to \mathbb{F}_{25}^3$ be defined by $m(\boldsymbol{x}) := \boldsymbol{x} + \bar{\boldsymbol{x}}M$, where vectors of $\mathbb{F}_{25}^3$ are written as row vectors. Then there exist $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ such that $m(\boldsymbol{x}_1), m(\boldsymbol{x}_2), m(\boldsymbol{x}_3)$ are linearly independent. Let C denote the 3×3 matrix whose row vectors are $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$. We put

$$S := C + \overline{C}M,$$

which is non-degenerate. Then we have $\overline{S} = SM^{-1}$. Therefore, putting $T := \overline{S}^{-1}$, we have $M = T\overline{T}^{-1}$. Since $f^M = c\,\bar{f}$, we have $f^T = c\,\overline{f^T}$. Since $c^3 = 1$, there exists $\lambda \in \mathbb{F}_{25}^{\times}$ such that $c = \lambda^8 = (\lambda^{-1}\bar{\lambda})^2$. Then we have $\lambda^2 f^T = \bar{\lambda}^2 \overline{f^T}$, and hence $\lambda^2 f^T$ has coefficients in $\mathbb{F}_5$. $\square$

Remark 6.6. Except for the equivalence class $\mathcal{E}_7 = \overline{\mathcal{E}}_7$, we have found a defining equation $s_{\mathbb{F},[h]}$ of B_h with coefficients in $\mathbb{F}_5$ for each $\mathcal{E}_n$ with $\mathcal{E}_n = \overline{\mathcal{E}}_n$.

7. The list of projective models $\mathcal{E}_0, \ldots, \mathcal{E}_{64}$

$\mathcal{E}_0 = \overline{\mathcal{E}}_0$: RT = 0: |aut| = 378000: N = 13051: h = [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] :

$$\boxed{x^6 + y^6 + 1}$$

$\mathcal{E}_1 = \overline{\mathcal{E}}_1$: RT = $6A_1$: |aut| = 12: N = 5607000: h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1] :

$$\boxed{x^6 + 3\,x^5y + x^4y^2 + 2\,x^3y^3 + y^6 + 3\,x^4 + 3\,x^2y^2 + xy^3 + 3\,xy + 2\,y^2 + 4}$$

$\mathcal{E}_2 = \overline{\mathcal{E}}_2$: RT = $7A_1$: |aut| = 6: N = 6678000: h = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] :

$$\boxed{x^6 + 2\,x^4y^2 + x^2y^4 + x^2y^3 + 2\,y^5 + x^4 + 2\,y^4 + 2\,x^2y + 2\,y^3 + 3\,y^2 + 3\,y + 2}$$

$\mathcal{E}_3 = \overline{\mathcal{E}}_3$: RT = $3A_1 + 2A_2$: |aut| = 6: N = 2268000: h = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0] :

$$\boxed{x^6 + 3\,x^3y^3 + y^6 + 3\,x^3y + 2\,y^2 + 2}$$

$\mathcal{E}_4 = \overline{\mathcal{E}}_4$: RT = $8A_1$: |aut| = 8: N = 2457000: h = [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0] :

$$\boxed{x^6 + 3\,x^4y^2 + x^2y^4 + 4\,x^2y^3 + 4\,y^5 + x^4 + 2\,x^2y^2 + 3\,y^4 + 2\,x^2y + 4\,x^2 + y^2 + 4\,y}$$

$\mathcal{E}_5 = \overline{\mathcal{E}}_5$: RT $= 8A_1$: $|\mathrm{aut}| = 4$: N $= 2268000$: h $= [0,0,0,0,0,1,0,0,1,0,0,0,1,0,0,1,0,0,1,0,0,0]$:

$x^4y^2 + x^2y^4 + 2\,x^4 + 4\,x^2y^2 + y^4 + x^2 + 4\,y^2 + 4$

$\mathcal{E}_6 = \overline{\mathcal{E}}_6$: RT $= 6A_1 + A_2$: $|\mathrm{aut}| = 6$: N $= 1512000$: h $= [0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,1,1,0,1,0,0]$:

$x^6 + 4\,x^4y^2 + 2\,x^2y^4 + 2\,x^2y + y^3 + 4$

$\mathcal{E}_7 = \overline{\mathcal{E}}_7$: RT $= 6A_1 + A_2$: $|\mathrm{aut}| = 2$: N $= 4914000$: h $= [0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,1,0,1]$:

$\sqrt{2}x^3y^3 + \left(1+3\sqrt{2}\right)x^2y^4 + x^4 + \left(2+2\sqrt{2}\right)x^3y + \left(1+4\sqrt{2}\right)x^2y^2 + xy^3 + \left(2+2\sqrt{2}\right)y^4 + \sqrt{2}x^2 + \left(1+3\sqrt{2}\right)xy$

$\mathcal{E}_8 = \overline{\mathcal{E}}_8$: RT $= 6A_1 + A_2$: $|\mathrm{aut}| = 1$: N $= 9828000$: h $= [0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,1,1,0,0,0,1]$:

$x^6 + 2\,x^5y + x^4y^2 + 3\,x^5 + 2\,xy^4 + x^3y + 3\,x^2y^2 + 4\,xy^2 + y^3 + 3\,y^2 + 3\,x + 3\,y$

$\mathcal{E}_9 = \overline{\mathcal{E}}_{10}$: RT $= 4A_1 + 2A_2$: $|\mathrm{aut}| = 2$: N $= 4158000$: h $= [0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,1,1,0,0,0,0,0,1]$:

$x^5y + \left(2+\sqrt{2}\right)x^4y^2 + \left(1+4\sqrt{2}\right)x^3y^3 + \left(3+\sqrt{2}\right)x^2y^4 + \left(2+4\sqrt{2}\right)xy^5 + \left(2+\sqrt{2}\right)y^6 + \left(2+3\sqrt{2}\right)x^4 + \left(1+4\sqrt{2}\right)x^3y + \left(3+\sqrt{2}\right)y^4 + \left(1+4\sqrt{2}\right)x^2 + \left(3+\sqrt{2}\right)xy + 3\,y^2 + 2 + 3\sqrt{2}$

$\mathcal{E}_{11} = \overline{\mathcal{E}}_{11}$: RT $= 9A_1$: $|\mathrm{aut}| = 54$: N $= 84000$: h $= [0,0,0,0,0,0,0,0,0,1,1,0,0,0,1,1,0,1,1,-1,0,0,0]$:

$x^6 + 4\,x^3y^3 + 4\,y^6 + x^4 + 4\,xy^3 + 3\,x^2 + 4$

$\mathcal{E}_{12} = \overline{\mathcal{E}}_{12}$: RT $= 9A_1$: $|\mathrm{aut}| = 9$: N $= 1596000$: h $= [0,0,0,0,0,0,0,0,0,1,1,0,1,0,0,1,0,0,0,0,0,0,0]$:

$4\,x^4y^2 + 3\,x^2y^4 + 4\,y^6 + x^5 + 3\,x^3y^2 + 2\,xy^4 + x^4 + 2\,x^2y^2 + 4\,xy^3 + 2\,xy^2 + 4\,y^3 + 4\,x^2 + 2\,xy + 1$

$\mathcal{E}_{13} = \overline{\mathcal{E}}_{14}$: RT $= 9A_1$: $|\mathrm{aut}| = 6$: N $= 882000$: h $= [0,0,0,0,1,0,0,0,0,1,0,0,1,0,0,0,0,0,0,0,0,1,1]$:

$\sqrt{2}x^5y + 2\,x^4y^2 + \left(3+2\sqrt{2}\right)x^3y^3 + \left(4+2\sqrt{2}\right)x^2y^4 + \left(4+4\sqrt{2}\right)xy^5 + \sqrt{2}y^6 + \left(1+\sqrt{2}\right)x^4 + \left(4+3\sqrt{2}\right)x^3y + \left(1+4\sqrt{2}\right)x^2y^2 + \left(1+4\sqrt{2}\right)y^4 + \left(3+3\sqrt{2}\right)x^2 + \left(1+\sqrt{2}\right)xy + \left(3+4\sqrt{2}\right)y^2 + 1 + \sqrt{2}$

$\mathcal{E}_{15} = \overline{\mathcal{E}}_{16}$: RT $= 9A_1$: $|\mathrm{aut}| = 3$: N $= 2268000$: h $= [0,-1,0,0,0,0,1,1,0,0,1,0,0,1,0,0,1,0,0,1,0,0]$:

$\left(2+2\sqrt{2}\right)x^2y^4 + x^4y + \left(4+4\sqrt{2}\right)x^3y^2 + \left(1+\sqrt{2}\right)x^2y^3 + \left(2+4\sqrt{2}\right)xy^4 + \left(1+\sqrt{2}\right)x^4 + \left(1+2\sqrt{2}\right)x^3y + \left(2+3\sqrt{2}\right)xy^3 + \left(2+4\sqrt{2}\right)x^2y + \left(2+\sqrt{2}\right)xy^2 + \left(2+\sqrt{2}\right)xy + 2\,y^2$

$\mathcal{E}_{17} = \overline{\mathcal{E}}_{17}$: RT $= 9A_1$: $|\mathrm{aut}| = 2$: N $= 3402000$: h $= [1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,0,1]$:

$x^5y + 2\,x^4y^2 + 4\,x^3y^3 + 2\,x^2y^4 + 4\,xy^5 + 3\,y^6 + 2\,x^2y^2 + 2\,x^2 + xy$

$\mathcal{E}_{18} = \overline{\mathcal{E}}_{19}$: RT $= 7A_1 + A_2$: $|\mathrm{aut}| = 2$: N $= 3024000$: h $= [0,0,0,0,0,0,0,0,0,1,0,0,0,1,1,1,1,0,0,0,0,0,0]$:

$\sqrt{2}x^4y^2 + \left(1+2\sqrt{2}\right)x^3y^3 + \left(3+4\sqrt{2}\right)x^2y^4 + 3\sqrt{2}xy^5 + \left(2+2\sqrt{2}\right)x^4 + \sqrt{2}x^3y + 4\,x^2y^2 + 3\sqrt{2}xy^3 + \left(2+2\sqrt{2}\right)y^4 + \left(1+\sqrt{2}\right)x^2 + 4\sqrt{2}xy + \left(1+\sqrt{2}\right)y^2 + 2 + 2\sqrt{2}$

$\mathcal{E}_{20} = \overline{\mathcal{E}}_{21}$: RT $= 7A_1 + A_2$: $|\mathrm{aut}| = 1$: N $= 5292000$: h $= [0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,1,0,1]$:

$2\sqrt{2}x^3y^3 + \left(3+\sqrt{2}\right)x^2y^4 + x^4y + \left(4+2\sqrt{2}\right)x^3y^2 + \left(3+4\sqrt{2}\right)x^2y^3 + \left(4+4\sqrt{2}\right)xy^4 + x^4 + 3\sqrt{2}x^2y^2 + 3\sqrt{2}xy^3 + 4\,y^4 + \sqrt{2}x^3 + 2\sqrt{2}x^2y + \sqrt{2}xy^2 + \left(2+2\sqrt{2}\right)y^3 + 3\,x^2 + \left(3+2\sqrt{2}\right)xy + \left(2+3\sqrt{2}\right)y^2$

$\mathcal{E}_{22} = \overline{\mathcal{E}}_{23}$: RT $= 7A_1 + A_2$: $|\mathrm{aut}| = 1$: N $= 5292000$: h $= [0,0,0,0,0,0,0,0,0,1,1,0,1,0,0,0,0,1,0,0,0,1]$:

$x^3y^3 + \left(1+3\sqrt{2}\right)x^2y^4 + x^4y + \left(3+2\sqrt{2}\right)x^3y^2 + 3\sqrt{2}x^2y^3 + \left(2+4\sqrt{2}\right)xy^4 + \sqrt{2}x^4 + \left(2+4\sqrt{2}\right)x^3y + 4\,xy^3 + \left(1+3\sqrt{2}\right)y^4 + \left(2+\sqrt{2}\right)x^3 + \left(3+3\sqrt{2}\right)x^2y + \sqrt{2}y^3 + \left(4+2\sqrt{2}\right)x^2 + 4\sqrt{2}xy + \left(1+4\sqrt{2}\right)y^2$

$\mathcal{E}_{24} = \overline{\mathcal{E}}_{24}$: RT $= 5A_1 + 2A_2$: $|\mathrm{aut}| = 8$: N $= 378000$: h $= [0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,0,1,1,0,1,0,0]$:

$x^3y^3 + x^4 + x^2y^2 + y^4 + xy$

$\mathcal{E}_{25} = \overline{\mathcal{E}}_{26}$: RT $= 5A_1 + 2A_2$: $|\mathrm{aut}| = 2$: N $= 2268000$: h $= [0,0,0,0,0,0,0,0,0,1,0,0,1,0,1,0,0,1,0,0,0,1]$:

$x^2y^4 + x^4y + \left(1+\sqrt{2}\right)x^3y^2 + \left(3+4\sqrt{2}\right)x^2y^3 + \left(3+2\sqrt{2}\right)xy^4 + \left(1+\sqrt{2}\right)x^3y + \left(1+2\sqrt{2}\right)x^2y^2 + \left(3+\sqrt{2}\right)xy^3 + \left(1+4\sqrt{2}\right)x^2y + \left(1+2\sqrt{2}\right)xy^2 + 3\,x^2 + 4\sqrt{2}xy + \left(1+4\sqrt{2}\right)y^2$

$\mathcal{E}_{27} = \overline{\mathcal{E}}_{27}$: RT $= 5A_1 + 2A_2$: $|\mathrm{aut}| = 1$: N $= 3780000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1]$:

$$x^6 + 3\,x^4y^2 + x^2y^4 + x^3y^2 + 3\,x^2y^3 + xy^4 + 2\,x^3y + 3\,xy^3 + 4\,x^3 + 3\,x^2y + 4\,xy^2 + 4\,y^2$$

$\mathcal{E}_{28} = \overline{\mathcal{E}}_{29}$: RT $= 5A_1 + 2A_2$: $|\mathrm{aut}| = 1$: N $= 4536000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1]$:

$$x^4y^2 + \left(2 + 2\sqrt{2}\right)x^3y^3 + \left(3 + 2\sqrt{2}\right)x^2y^4 + \left(1 + \sqrt{2}\right)x^4y + 2\sqrt{2}x^3y^2 + \left(2 + \sqrt{2}\right)xy^4 + \left(2 + 3\sqrt{2}\right)x^4 + 4\,x^2y^2 + \left(1 + 3\sqrt{2}\right)y^4 + \left(3 + 4\sqrt{2}\right)x^3 + 4\sqrt{2}xy^2 + \left(1 + \sqrt{2}\right)y^3 + \left(4 + 2\sqrt{2}\right)x^2 + \left(3 + 3\sqrt{2}\right)xy + \left(1 + 2\sqrt{2}\right)y^2$$

$\mathcal{E}_{30} = \overline{\mathcal{E}}_{31}$: RT $= 3A_1 + 3A_2$: $|\mathrm{aut}| = 3$: N $= 1260000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1]$:

$$x^4y^2 + \left(1 + \sqrt{2}\right)x^3y^3 + \left(2 + 3\sqrt{2}\right)x^2y^4 + x^4y + 4\,x^3y^2 + \left(3 + 3\sqrt{2}\right)x^2y^3 + 4\sqrt{2}xy^4 + 4\,x^4 + \left(2 + 3\sqrt{2}\right)x^3y + x^2y^2 + \left(4 + 2\sqrt{2}\right)y^4 + \left(3 + 2\sqrt{2}\right)x^3 + \left(4 + 3\sqrt{2}\right)x^2y + \left(4 + 4\sqrt{2}\right)xy^2 + \left(2 + 4\sqrt{2}\right)y^3 + x^2 + \sqrt{2}xy + 3\,y^2$$

$\mathcal{E}_{32} = \overline{\mathcal{E}}_{32}$: RT $= 10A_1$: $|\mathrm{aut}| = 20$: N $= 226800$: h $= [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1]$:

$$x^6 + 2\,x^4y + y^5 + 4\,x^2y^2 + y^3 + 4\,x^2 + 4\,y$$

$\mathcal{E}_{33} = \overline{\mathcal{E}}_{33}$: RT $= 10A_1$: $|\mathrm{aut}| = 4$: N $= 756000$: h $= [0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0]$:

$$x^6 + x^4y^2 + 3\,x^3y^3 + 3\,x^2y^4 + 2\,y^6 + x^2y^2 + 4\,xy + 4$$

$\mathcal{E}_{34} = \overline{\mathcal{E}}_{35}$: RT $= 10A_1$: $|\mathrm{aut}| = 2$: N $= 1890000$: h $= [0, -1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0]$:

$$x^5y + x^4y^2 + 3\,x^3y^3 + \left(4 + \sqrt{2}\right)x^2y^4 + \left(1 + \sqrt{2}\right)xy^5 + 4\sqrt{2}y^6 + 2\,x^4 + 4\,x^3y + \left(4 + 4\sqrt{2}\right)xy^3 + \left(2 + 2\sqrt{2}\right)y^4 + x^2 + \left(1 + 4\sqrt{2}\right)y^2 + 2$$

$\mathcal{E}_{36} = \overline{\mathcal{E}}_{36}$: RT $= 8A_1 + A_2$: $|\mathrm{aut}| = 1$: N $= 3780000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1]$:

$$x^5y + 4\,x^2y^4 + x^5 + 3\,x^4y + 2\,x^2y^3 + 3\,x^4 + 2\,y^4 + 2\,xy^2 + 2\,y^3 + 2\,x^2 + 3\,xy + 4\,y$$

$\mathcal{E}_{37} = \overline{\mathcal{E}}_{37}$: RT $= 8A_1 + A_2$: $|\mathrm{aut}| = 1$: N $= 3024000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1]$:

$$x^4y^2 + 4\,x^3y^3 + 4\,x^2y^4 + 3\,xy^4 + y^5 + 4\,xy^3 + 4\,x^3 + 4\,x^2y + 4\,x^2 + xy + 3\,y^2 + 3\,x + 3\,y$$

$\mathcal{E}_{38} = \overline{\mathcal{E}}_{39}$: RT $= 8A_1 + A_2$: $|\mathrm{aut}| = 1$: N $= 3024000$: h $= [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1]$:

$$\left(1 + 4\sqrt{2}\right)x^2y^4 + x^4y + \left(1 + \sqrt{2}\right)x^3y^2 + 3\,x^2y^3 + \left(2 + \sqrt{2}\right)xy^4 + x^4 + \left(2 + 2\sqrt{2}\right)x^3y + 3\,x^2y^2 + \sqrt{2}y^4 + 4\sqrt{2}x^3 + \left(2 + 3\sqrt{2}\right)x^2y + y^3 + 3\,x^2 + \left(2 + 4\sqrt{2}\right)xy + 3\,y^2$$

$\mathcal{E}_{40} = \overline{\mathcal{E}}_{41}$: RT $= 6A_1 + 2A_2$: $|\mathrm{aut}| = 6$: N $= 378000$: h $= [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0]$:

$$\sqrt{2}x^6 + \left(1 + \sqrt{2}\right)x^5y + \left(1 + 4\sqrt{2}\right)x^3y^3 + \sqrt{2}x^2y^4 + 2\sqrt{2}xy^5 + \left(3 + \sqrt{2}\right)y^6 + \left(4 + 3\sqrt{2}\right)x^4 + 3\,x^3y + \left(2 + \sqrt{2}\right)x^2y^2 + \left(4 + 4\sqrt{2}\right)xy^3 + \left(3 + 3\sqrt{2}\right)y^4 + \left(2 + \sqrt{2}\right)x^2 + \left(1 + 4\sqrt{2}\right)xy + \sqrt{2}y^2 + 4$$

$\mathcal{E}_{42} = \overline{\mathcal{E}}_{43}$: RT $= 6A_1 + 2A_2$: $|\mathrm{aut}| = 2$: N $= 1512000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]$:

$$x^4y^2 + \left(2 + 3\sqrt{2}\right)x^3y^3 + \left(3 + 3\sqrt{2}\right)x^2y^4 + x^4y + \left(3 + 3\sqrt{2}\right)x^2y^3 + 3\sqrt{2}xy^4 + 2\sqrt{2}x^4 + \left(3 + 4\sqrt{2}\right)x^2y^2 + 2\sqrt{2}xy^3 + \left(4 + \sqrt{2}\right)y^4 + \left(3 + 3\sqrt{2}\right)x^3 + \left(4 + 3\sqrt{2}\right)y^3 + \left(4 + 2\sqrt{2}\right)x^2 + 4\sqrt{2}xy + \left(3 + 2\sqrt{2}\right)y^2$$

$\mathcal{E}_{44} = \overline{\mathcal{E}}_{45}$: RT $= 6A_1 + 2A_2$: $|\mathrm{aut}| = 1$: N $= 2268000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1]$:

$$2\,x^3y^3 + 3\sqrt{2}x^2y^4 + \left(4 + 2\sqrt{2}\right)x^3y^2 + \left(3 + \sqrt{2}\right)x^2y^3 + \left(1 + 2\sqrt{2}\right)xy^4 + x^4 + \left(3 + \sqrt{2}\right)x^3y + 3\,x^2y^2 + 3\,y^4 + \left(1 + 4\sqrt{2}\right)x^3 + \left(1 + 3\sqrt{2}\right)x^2y + 4\,xy^2 + \left(2 + 4\sqrt{2}\right)y^3 + \left(3 + \sqrt{2}\right)x^2 + \left(1 + \sqrt{2}\right)xy + \left(3 + 3\sqrt{2}\right)y^2$$

$\mathcal{E}_{46} = \overline{\mathcal{E}}_{46}$: RT $= 4A_1 + 3A_2$: $|\mathrm{aut}| = 3$: N $= 756000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1]$:

$$x^6 + 3\,x^3y^3 + 4\,x^4y + xy^4 + 3\,x^2y^2 + 4\,x^3 + 3\,xy + 4$$

$\mathcal{E}_{47} = \overline{\mathcal{E}}_{47}$: RT $= 4A_1 + 3A_2$: $|\mathrm{aut}| = 2$: N $= 1134000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0]$:

$$x^6 + 3\,x^4y^2 + 4\,x^2y^4 + 2\,y^6 + 4\,x^2y^3 + 2\,x^4 + 3\,x^2y^2 + 4\,x^2y + y^3 + 3\,x^2$$

$\mathcal{E}_{48} = \overline{\mathcal{E}}_{48}$: RT $= 4A_1 + 3A_2$: $|\mathrm{aut}| = 1$: N $= 2268000$: h $= [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1]$:

$$2\,x^4y^2 + x^5 + 2\,x^2y^3 + 4\,xy^4 + 2\,x^3y + 3\,x^2y^2 + 2\,xy^3 + 2\,xy^2 + 3\,x^2 + 2\,xy + 2\,y^2$$

$\mathcal{E}_{49} = \overline{\mathcal{E}}_{49}$: RT $= 11A_1$: $|\mathrm{aut}| = 4$: N $= 378000$: h $= [0,0,0,0,0,0,0,1,0,0,0,1,1,0,0,1,0,1,0,0,0,0]$:

$$\boxed{x^6 + x^4y^2 + 4\,x^2y^4 + 3\,x^5 + 3\,xy^4 + x^2y^2 + 2\,y^4 + x^3 + 4\,y^2 + 2\,x + 2}$$

$\mathcal{E}_{50} = \overline{\mathcal{E}}_{51}$: RT $= 9A_1 + A_2$: $|\mathrm{aut}| = 1$: N $= 1512000$: h $= [0,0,0,0,0,0,0,0,0,1,0,1,1,0,0,0,0,1,0,0,0,1]$:

$$\left(4+\sqrt{2}\right)x^3y^3 + \left(4+2\sqrt{2}\right)x^2y^4 + x^4y + 4\,xy^4 + \sqrt{2}x^4 + \left(3+3\sqrt{2}\right)x^2y^2 + 4\,xy^3 + \left(4+2\sqrt{2}\right)y^4 + \left(2+3\sqrt{2}\right)x^3 + \left(4+4\sqrt{2}\right)x^2y + \left(4+3\sqrt{2}\right)y^3 + \left(1+2\sqrt{2}\right)x^2 + 3\sqrt{2}xy + \left(2+3\sqrt{2}\right)y^2$$

$\mathcal{E}_{52} = \overline{\mathcal{E}}_{52}$: RT $= 7A_1 + 2A_2$: $|\mathrm{aut}| = 2$: N $= 378000$: h $= [0,0,0,0,0,0,1,1,1,0,0,1,1,0,0,0,0,0,0,0,0,0]$:

$$\boxed{x^6 + x^5y + 2\,x^4y^2 + x^2y^4 + 3\,y^6 + x^4 + x^2y^2 + xy^3 + 4\,xy + y^2 + 3}$$

$\mathcal{E}_{53} = \overline{\mathcal{E}}_{54}$: RT $= 7A_1 + 2A_2$: $|\mathrm{aut}| = 1$: N $= 1512000$: h $= [0,0,0,0,0,1,0,0,0,1,0,1,0,0,1,0,0,0,0,0,0,1]$:

$$\left(2+2\sqrt{2}\right)x^2y^4 + x^4y + \left(4+\sqrt{2}\right)x^3y^2 + \left(2+2\sqrt{2}\right)x^2y^3 + 4\,xy^4 + 3\,x^4 + \left(3+2\sqrt{2}\right)x^3y + \sqrt{2}x^2y^2 + \left(3+4\sqrt{2}\right)xy^3 + \left(2+2\sqrt{2}\right)y^4 + \left(3+4\sqrt{2}\right)x^2y + \left(2+3\sqrt{2}\right)xy^2 + \left(2+3\sqrt{2}\right)y^3 + \sqrt{2}xy + 4\,y^2$$

$\mathcal{E}_{55} = \overline{\mathcal{E}}_{56}$: RT $= 7A_1 + 2A_2$: $|\mathrm{aut}| = 1$: N $= 1512000$: h $= [0,0,0,0,0,0,0,0,0,1,0,0,0,1,1,0,1,0,0,1,0,0]$:

$$\left(3+2\sqrt{2}\right)x^2y^4 + x^4y + x^3y^2 + \left(2+\sqrt{2}\right)x^2y^3 + \left(2+4\sqrt{2}\right)xy^4 + \sqrt{2}x^4 + \left(3+4\sqrt{2}\right)x^3y + \left(2+4\sqrt{2}\right)xy^3 + 4\sqrt{2}y^4 + \sqrt{2}x^3 + \left(1+4\sqrt{2}\right)x^2y + \left(4+\sqrt{2}\right)y^3 + 4\,x^2 + 4\sqrt{2}xy + 2\,y^2$$

$\mathcal{E}_{57} = \overline{\mathcal{E}}_{58}$: RT $= 5A_1 + 3A_2$: $|\mathrm{aut}| = 2$: N $= 756000$: h $= [0,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,0,1,0,1,0,1]$:

$$\sqrt{2}x^4y^2 + \left(4+2\sqrt{2}\right)x^3y^3 + 4\sqrt{2}x^2y^4 + \left(3+\sqrt{2}\right)xy^5 + 4\sqrt{2}y^6 + \left(1+4\sqrt{2}\right)x^4 + \left(3+\sqrt{2}\right)x^3y + 2\sqrt{2}x^2y^2 + \left(1+\sqrt{2}\right)y^4 + \left(3+2\sqrt{2}\right)x^2 + \left(2+4\sqrt{2}\right)xy + \left(3+\sqrt{2}\right)y^2 + 1 + 4\sqrt{2}$$

$\mathcal{E}_{59} = \overline{\mathcal{E}}_{60}$: RT $= 8A_1 + 2A_2$: $|\mathrm{aut}| = 2$: N $= 378000$: h $= [0,0,0,0,1,0,0,1,0,0,0,0,1,0,0,1,0,0,1,0,0,0]$:

$$x^5y + \left(1+\sqrt{2}\right)x^4y^2 + 2\,x^2y^4 + \left(2+\sqrt{2}\right)xy^5 + \left(4+3\sqrt{2}\right)y^6 + 3\,x^4 + \left(4+4\sqrt{2}\right)x^3y + \left(1+3\sqrt{2}\right)x^2y^2 + \left(3+3\sqrt{2}\right)xy^3 + 4\sqrt{2}y^4 + 4\,x^2 + \sqrt{2}xy + \left(3+3\sqrt{2}\right)y^2 + 3$$

$\mathcal{E}_{61} = \overline{\mathcal{E}}_{62}$: RT $= 8A_1 + 2A_2$: $|\mathrm{aut}| = 1$: N $= 756000$: h $= [0,0,0,0,0,0,0,1,0,1,0,1,0,0,0,0,0,1,0,0,0,1]$:

$$x^3y^3 + \left(3+4\sqrt{2}\right)x^2y^4 + 2\sqrt{2}x^3y^2 + 2\,x^2y^3 + 2\,xy^4 + x^4 + \left(4+2\sqrt{2}\right)x^3y + \left(1+\sqrt{2}\right)x^2y^2 + \left(3+2\sqrt{2}\right)y^4 + \left(3+\sqrt{2}\right)x^3 + \left(3+4\sqrt{2}\right)x^2y + 4\sqrt{2}y^3 + \left(2+4\sqrt{2}\right)x^2 + \left(3+2\sqrt{2}\right)xy + \left(4+4\sqrt{2}\right)y^2$$

$\mathcal{E}_{63} = \overline{\mathcal{E}}_{63}$: RT $= 6A_1 + 3A_2$: $|\mathrm{aut}| = 3$: N $= 252000$: h $= [0,0,0,1,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,0,0,0]$:

$$\boxed{x^4y^2 + x^4y + x^3y^2 + 2\,y^5 + 2\,x^4 + 4\,x^2y^2 + 3\,xy^3 + 4\,y^4 + 4\,x^3 + 2\,xy^2 + y^3 + 2\,x^2 + y^2}$$

$\mathcal{E}_{64} = \overline{\mathcal{E}}_{64}$: RT $= 6A_1 + 3A_2$: $|\mathrm{aut}| = 3$: N $= 252000$: h $= [0,-1,0,0,1,0,1,0,1,1,0,0,0,0,0,0,1,0,0,0,0,1]$:

$$\boxed{x^4y^2 + x^3y^3 + x^2y^4 + 3\,x^3y^2 + x^2y^3 + 3\,x^3y + x^2y^2 + 2\,x^3 + 2\,x^2y + 3\,y^3 + 2\,x^2 + 3\,xy + 4\,y + 4}$$

8. Proof of Theorem 1.3

The polynomials in Table 1.1 that give a non-projective involution g of X_F are calculated by the following method. Recall that h'_F in Step 8 of Section 6 is the representative vector of the $\mathrm{Aut}(X_F, h_F)$-orbit $\mathcal{V}_4 \cap \mathcal{E}_0$. We have already calculated a birational morphism

$$\phi_{h'_F} = (\omega : \xi_0 : \xi_1 : \xi_2) \ : \ X_F \to X_{h'_F},$$

and the defining equation $s_{h'_F}$ of $B_{h'_F}$. We have observed that $s_{h'_F}$ is written as

$$s_{h'_F}(x,y,z) = \lambda^2 \, \boldsymbol{x} \, H \, {}^t\overline{\boldsymbol{x}},$$

where $\lambda \in \mathbb{F}_{25}^{\times}$, $\boldsymbol{x} = (x, y, z)$, $\overline{\boldsymbol{x}} = (x^5, y^5, z^5)$ and H satisfies $H = {}^t\overline{H}$. We search for $M \in \mathrm{GL}_3(\mathbb{F}_{25})$ such that $H = M\,{}^t\overline{M}$ (see n. 3 of [20]), and put

$$\omega' := \lambda^{-1}\omega, \quad (\xi'_0, \xi'_1, \xi'_2) := (\xi_0, \xi_1, \xi_2)\, M.$$

Then the polynomials $\omega', \xi'_0, \xi'_1, \xi'_2$ satisfy

$$\omega'^2 = \xi_0'^6 + \xi_1'^6 + \xi_2'^6.$$

Hence the rational map from X_F to $\mathbb{P}(3, 1, 1, 1)$ given by $(\omega' : \xi'_0 : \xi'_1 : \xi'_2)$ defines an automorphism γ of X_F. We choose h_F-lines $\ell_{i_1}, \dots, \ell_{i_{22}}$ such that $[\ell_{i_1}], \dots, [\ell_{i_{22}}]$ span $\mathrm{NS}(X) \otimes \mathbb{Q}$, and that none of $i_1, \dots, i_{22}$ is contained in the set J of indices in the expression (5.1) for h'_F that was used in the calculation of $\phi_{h'_F}$. Then we can calculate the images $\ell_{i_\nu}^{\gamma}$ of ℓ_{i_ν} by γ using the parametric representations of ℓ_{i_ν} and the polynomials $(\omega' : \xi'_0 : \xi'_1 : \xi'_2)$. Computing the intersection numbers of $\ell_{i_\nu}^{\gamma}$ with $\ell_1, \dots, \ell_{22}$, we calculate the action of γ on $\mathrm{NS}(X)$. Let $v \mapsto v\Gamma$ denote the matrix representation of this action. We then search for $\tau \in \mathrm{Aut}(X, h_F)$ such that its action on X_F is given by

$$w \mapsto \sigma w, \quad (x, y, z) \mapsto (x, y, z)\, T_\tau,$$

where $\sigma \in \mathbb{F}_{25}^{\times}$, $T_\tau \in \mathrm{GU}_3(\mathbb{F}_{25})$, and its action on $\mathrm{NS}(X)$ is given by $v \mapsto vN_\tau$, where N_τ is a matrix satisfying $(\Gamma N_\tau)^2 = \mathrm{Id}_{22}$. We define $(\omega'', \xi''_0, \xi''_1, \xi''_2)$ by

$$\omega'' := \sigma\omega', \quad (\xi''_0, \xi''_1, \xi''_2) := (\xi'_0, \xi'_1, \xi'_2)\, T_\tau,$$

and replace the original polynomials $(\omega, \xi_0, \xi_1, \xi_2)$ by $(\omega'', \xi''_0, \xi''_1, \xi''_2)$. Then the automorphism $X_F \to X_F$ given by $(\omega : \xi_0 : \xi_1 : \xi_2)$ is of order 2, because its action $v \mapsto v\Gamma N_\tau$ on $\mathrm{NS}(X)$ is of order 2.

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