# Topology of curves on a surface and lattice-theoretic invariants of coverings of the surface

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**Abstract.**

Let $S$ be a smooth simply-connected complex projective surface, and let $A$ be a finite abelian group. We define invariants $T_A$, $F_A$ and $\sigma_A$ for curves $B$ on $S$ by means of étale Galois coverings of the complement of $B$ with the Galois group $A$, and show that they are useful in finding examples of Zariski pairs of curves on $S$. We also investigate the relation between these invariants and the fundamental group of the complement of $B$.

## §1. Introduction

We work over the complex number field $\mathbb{C}$. Let $S$ be a smooth projective surface. Throughout this paper, *we assume that $S$ is simply-connected*. By a *curve* on $S$, we mean a reduced (possibly reducible) curve on $S$.

Let $B$ and $B'$ be curves on $S$.

**Definition 1.1.** We say that a homeomorphism $f : B \xrightarrow{\sim} B'$ *preserves the classes of irreducible components* if we have $[B_i] = [f(B_i)]$ in $H^2(S, \mathbb{Z})$ for any irreducible component $B_i$ of $B$.

Note that, since $S$ is simply-connected, the equality $[B_i] = [f(B_i)]$ in $H^2(S, \mathbb{Z})$ is equivalent to the equality $[B_i] = [f(B_i)]$ in the Picard group $\mathrm{Pic}(S)$ of $S$.

Following [5, Definition 2], we make the following:

**Definition 1.2.** We say that $B$ and $B'$ have *the same embedding topology* and write $B \sim_{\mathrm{top}} B'$ if there exists a homeomorphism between $(S, B)$ and $(S, B')$ such that the induced homeomorphism $B \xrightarrow{\sim} B'$ preserves the classes of irreducible components.

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**Definition 1.3.** A *map of equi-configuration* is a homeomorphism $(\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$, where $\mathcal{T} \subset S$ is a tubular neighborhood of $B$ and $\mathcal{T}' \subset S$ is a tubular neighborhood of $B'$, such that the induced homeomorphism $B \xrightarrow{\sim} B'$ preserves the classes of irreducible components.

**Definition 1.4.** We say that $B$ and $B'$ are *of the same configuration type* and write $B \sim_{\mathrm{cfg}} B'$ if there exist a tubular neighborhood $\mathcal{T} \subset S$ of $B$, a tubular neighborhood $\mathcal{T}' \subset S$ of $B'$, and a map of equi-configuration $(\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$.

It is obvious that $B \sim_{\mathrm{top}} B'$ implies $B \sim_{\mathrm{cfg}} B'$.

**Definition 1.5.** A pair $[B, B']$ of curves on $S$ is said to be a *Zariski pair* if $B \sim_{\mathrm{cfg}} B'$ but $B \not\sim_{\mathrm{top}} B'$.

By a *plane curve*, we mean a curve on $\mathbb{P}^2$. Since the work of Artal-Bartolo [2], Zariski pairs of plane curves have been studied by many authors. See the survey paper [5]. The most classical example of Zariski pairs is the following (Zariski [28], see also Oka [13] and Shimada [15]):

**Example 1.6.** There exist irreducible plane curves $B$ and $B'$ of degree 6 with six ordinary cusps as their only singularities such that $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, while $\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

As in this example, the fundamental group $\pi_1(\mathbb{P}^2 \setminus B)$ has been a main tool in finding the examples of Zariski pairs of plane curves.

In this paper, we fix a finite abelian group $A$ and define three invariants $T_A(S, B, \gamma)$, $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$ of curves $B$ on $S$ by means of étale Galois coverings $W_\gamma \to S \setminus B$ with the Galois group $A$, where $\gamma$ is a homomorphism $H^2(B, \mathbb{Z}) \to A$ describing the Galois covering. The invariants $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$ are defined in terms of the algebraic cycles on a smooth projective completion $X$ of $W_\gamma$, while the invariant $T_A(S, B, \gamma)$ involves the transcendental cycles of $X$. Using these invariants, we can distinguish topological types of curves on $S$ in the same configuration type, and find many Zariski pairs.

The idea of the invariant $T_A(S, B, \gamma)$ comes from Shioda's observation [20, Lemma 3.1] that the transcendental lattice of a smooth projective surface is a birational invariant.

These invariants have been defined and studied for the double coverings of the projective plane branching along plane curves of degree 6 with only simple singularities ([1], [19], [16], [18]). In particular, the invariant $F_A(S, B, \gamma)$ was intensively studied in [18] in terms of $Z$-splitting curves.

The plan of this paper is as follows. In §2, we describe all étale Galois coverings of $S \setminus B$ with the Galois group $A$, and define the invariants $F_A(S, B, \gamma)$ and $T_A(S, B, \gamma)$ in Definition 2.3. In §3, we investigate $T_A(S, B, \gamma)$, and show that, under certain conditions, $T_A(S, B, \gamma)$ is an invariant of the embedding topology of curves (Theorem 3.1). In §4, we define a new invariant $\sigma_A(S, B, \gamma)$, and show that it is an invariant of the configuration types of curves (Theorem 4.3). The invariants $F_A(S, B, \gamma)$ and $T_A(S, B, \gamma)$ are related via $\sigma_A(S, B, \gamma)$ (Proposition 4.7). We then present a method of finding examples of Zariski pairs by means of these invariants (Corollary 4.9). In §5, a relation between $T_A(S, B, \gamma)$, $F_A(S, B, \gamma)$, $\sigma_A(S, B, \gamma)$ and $\pi_1(S \setminus B)$ is presented. We then give several sufficient conditions for $\pi_1(S \setminus B)$ to be non-abelian (Corollaries 5.11 and 5.12). This result generalizes the theory of dihedral coverings, which has been studied by several authors. (See, for example, Artal et al. [3], [4], [6], Tokunaga [22], [23], [24], Degtyarev [8], [9], Degtyarev-Oka [10]). We conclude this paper by a remark on the computation of these invariants in §6.

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**Conventions.**

- A *lattice* is a free $\mathbb{Z}$-module $L$ of finite rank with a non-degenerate symmetric bilinear form $L \times L \to \mathbb{Z}$. For a subset $R$ of a lattice $L$, we denote by $\langle R \rangle$ the submodule generated by $R$.
- Every (co)homology group is the singular (co)homology group with coefficients in $\mathbb{Z}$, unless otherwise stated.
- Let $A$ be a finite abelian group. For a prime number $p$, we denote by $A_p$ the $p$-part of $A$, and by $\mathrm{leng}_p(A)$ the minimal number of generators of $A_p$.
- For a smooth projective surface $Y$, we denote by $\mathrm{NS}(Y) \subset H^2(Y)/(\text{the torsion part})$ the *Néron-Severi lattice* of $Y$.

## §2. Definition of the invariants $F_A(S, B, \gamma)$ and $T_A(S, B, \gamma)$

We fix a finite abelian group $A$ once and for all.

Let $S$ be a smooth simply-connected projective surface, and let $B$ be a curve on $S$ with the irreducible components $B_1, \dots, B_m$. We classify all étale Galois coverings of $S \setminus B$ with the Galois group $A$; that is, we describe all surjective homomorphisms $\pi_1(S \setminus B) \twoheadrightarrow A$. We have

$$H^2(B) = \bigoplus_{i=1}^{m} \mathbb{Z}[B_i].$$

Since $S$ is smooth and projective, we have $H_1(S \setminus B) \cong H^3(S, B)$. Since $S$ is simply-connected, we have $H^3(S) = 0$ and obtain an exact sequence

$$H^2(S) \xrightarrow{\ r\ } H^2(B) \longrightarrow H_1(S \setminus B) \longrightarrow 0,$$

where $r$ is the restriction homomorphism. Hence all étale Galois coverings of $S \setminus B$ with the Galois group $A$ are in one-to-one correspondence with the set

$$\mathcal{C}_A(S, B) := \left\{ \gamma \;\middle|\; \begin{array}{l} \gamma \text{ is a surjective homomorphism} \\ H^2(B) \twoheadrightarrow A \text{ such that } \operatorname{Im} r \subset \operatorname{Ker} \gamma \end{array} \right\}.$$

For an element $\gamma$ of $\mathcal{C}_A(S, B)$, we denote by

$$\varphi_\gamma : W_\gamma \to S \setminus B$$

the étale Galois covering corresponding to $\gamma$.

Since $S$ is simply-connected, $H^2(S)$ is torsion-free and we have a canonical isomorphism

$$H^2(S) \xrightarrow{\sim} \operatorname{Hom}(H^2(S), \mathbb{Z}) \tag{2.1}$$

by the cup-product. The restriction homomorphism

$$r_i : H^2(S) \to H^2(B_i) = \mathbb{Z}[B_i] \cong \mathbb{Z}$$

is given by $[B_i] \in H^2(S)$ under (2.1). If $\tau : (\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$ is a map of equi-configuration, then $[B_i] = [\tau(B_i)]$ holds in $H^2(S)$ and hence we have the following commutative diagram:

$$\begin{array}{ccc} H^2(S) & \xrightarrow{\ r\ } & H^2(B') \\ \| & & \downarrow \tau^* \\ H^2(S) & \xrightarrow{\ r\ } & H^2(B). \end{array}$$

Therefore $\tau$ induces a bijection

$$\tau_* : \mathcal{C}_A(S, B) \xrightarrow{\sim} \mathcal{C}_A(S, B').$$

Let

$$h : (S, B) \xrightarrow{\sim} (S, B')$$

be a homeomorphism. Restricting $h$ to a tubular neighborhood $\mathcal{T}$ of $B$, we obtain a map of equi-configuration $h|_{\mathcal{T}}$, and hence we have a bijection

$$h^* = (h|_{\mathcal{T}}^{-1})_* : \mathcal{C}_A(S, B') \;\xrightarrow{\sim}\; \mathcal{C}_A(S, B).$$

For $\gamma \in \mathcal{C}_A(S, B')$, the étale Galois covering

$$\varphi_{h^*\gamma} : W_{h^*\gamma} \to S \setminus B$$

corresponding to $h^*\gamma \in \mathcal{C}_A(S, B)$ is obtained as the pull-back of the étale Galois covering $\varphi_\gamma : W_\gamma \to S \setminus B'$ by the homeomorphism of the complement $h : S \setminus B \cong S \setminus B'$. In particular, we see that $W_{h^*\gamma}$ is homeomorphic to $W_\gamma$.

**Definition 2.1.** A *smooth projective completion* of $\varphi_\gamma : W_\gamma \to S \setminus B$ is a morphism

$$\phi : X \to S$$

from a smooth projective surface $X$ such that $X$ contains $W_\gamma$ as a Zariski open dense subset, and that $\phi$ extends $\varphi_\gamma : W_\gamma \to S \setminus B$.

**Definition 2.2.** A smooth projective completion $\phi : X \to S$ of $\varphi_\gamma : W_\gamma \to S \setminus B$ is said to be *$A$-equivariant* if the action of $A$ on $W_\gamma$ is extended to the action on $X$.

We choose a smooth projective completion $\phi : X \to S$ of $\varphi_\gamma$ (not necessarily $A$-equivariant), and put

$$\mathcal{E}(X) := \left\{ \; E \subset X \;\middle|\; \begin{array}{l} E \text{ is a reduced irreducible curve on} \\ X \text{ such that } \phi(E) \text{ is a point on } S \end{array} \right\}.$$

We consider

$$H^2(X)' := H^2(X)/(\text{the torsion part})$$

as a lattice under the cup-product. In this lattice, we have two submodules

$$\begin{array}{rcl} \phi^*\mathrm{NS}(S) & = & \langle\, [\phi^* C] \;|\; C \text{ is a curve on } S \,\rangle, \quad \text{and} \\ \langle \mathcal{E}(X) \rangle & = & \langle\, [E] \;|\; E \in \mathcal{E}(X) \,\rangle, \end{array}$$

which are perpendicular to each other by the cup-product. Note that $\phi^*\mathrm{NS}(S)$ is a hyperbolic lattice by the Hodge index theorem, and that the intersection pairing on $\langle \mathcal{E}(X) \rangle$ is negative-definite by Mumford's result [11]. In particular, the cup-product is non-degenerate on

$$\Sigma(X) := \phi^*\mathrm{NS}(S) \oplus \langle \mathcal{E}(X) \rangle \;\subset\; H^2(X)',$$

that is, $\Sigma(X)$ is a sublattice of $H^2(X)'$. We denote by

$$\Lambda(X) := (\Sigma(X) \otimes \mathbb{Q}) \cap H^2(X)'$$

the primitive closure of $\Sigma(X)$ in $H^2(X)'$.

**Definition 2.3.** We put

$$F_A(S, B, \gamma) := \Lambda(X)/\Sigma(X),$$

which is a finite abelian group, and denote by

$$T_A(S, B, \gamma) := \Sigma(X)^\perp = \Lambda(X)^\perp \ \subset \ H^2(X)'$$

the orthogonal complement of $\Sigma(X)$, which is a primitive sublattice of $H^2(X)'$.

**Proposition 2.4.** *Neither the isomorphism class of the finite abelian group $F_A(S, B, \gamma)$ nor the isomorphism class of the lattice $T_A(S, B, \gamma)$ does depend on the choice of the smooth projective completion $\phi : X \to S$ of $\varphi_\gamma : W_\gamma \to S \setminus B$.*

*Proof.* Suppose that $\phi' : X' \to S$ is another smooth projective completion of $\varphi_\gamma : W_\gamma \to S \setminus B$. Then there is a commutative diagram

$$\begin{array}{ccccc} & & X'' & & \\ & \swarrow & & \searrow & \\ X & & & & X' \\ & \searrow & & \swarrow & \\ & & S & & , \end{array}$$

where $X''$ is a smooth projective surface, and $X'' \to X$ and $X'' \to X'$ are birational morphisms that are isomorphisms over $S \setminus B$. Since a birational morphism between smooth surfaces are composite of blowing-ups at points, we obtain orthogonal direct-sum decompositions

$$\begin{aligned} &(2.2) \qquad & \Sigma(X'') &= \Sigma(X) \ \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle \quad \text{and} \\ &(2.3) \qquad & H^2(X'')' &= H^2(X)' \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle, \end{aligned}$$

where $e_1, \ldots, e_N$ are classes with $e_i^2 = -1$. Hence we obtain

$$\Lambda(X)/\Sigma(X) \ \cong \ \Lambda(X'')/\Sigma(X'') \quad \text{and} \quad \Sigma(X)^\perp \ \cong \ \Sigma(X'')^\perp.$$

The same isomorphisms hold between $X'$ and $X''$. Q.E.D.

We investigate the action of $A$ on these invariants.

**Proposition 2.5.** *There always exists an $A$-equivariant smooth projective completion.*

For the proof, we need the following:

**Lemma 2.6.** *There exist a vector bundle $\eta_\gamma : V_\gamma \to S$ on $S$ and a closed subvariety $\overline{W}_\gamma \subset V_\gamma$ finite over $S$ such that $A$ acts on $V_\gamma$ over $S$, that $\overline{W}_\gamma$ is stable under this action, and that there exists an $A$-equivariant isomorphism $W_\gamma \cong \overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$.*

*Proof.* First we prove the case where $A$ is cyclic of order $d$. We choose a generator $g$ of $A$ and fix an isomorphism $A \cong \mathbb{Z}/d\mathbb{Z}$ by $g \mapsto 1$. We also embed $A$ into $\mathbb{C}^\times$ by $g \mapsto \exp\left(2\pi\sqrt{-1}/d\right)$. Let $a_i$ be an integer such that

$$\gamma([B_i]) \equiv a_i \bmod d$$

in $A = \mathbb{Z}/d\mathbb{Z}$. Recall that the restriction map $r_i : H^2(S) \to H^2(B_i) \cong \mathbb{Z}$ is given by $[B_i] \in H^2(S)$ under (2.1). The condition $\operatorname{Im} r \subset \operatorname{Ker} \gamma$ for $\gamma$ implies that there exists a line bundle $\eta_\gamma : V_\gamma \to S$ on $S$ such that

$$a_1[B_1] + \cdots + a_m[B_m] = d\,[V_\gamma]$$

holds in $\operatorname{Pic}(S) \subset H^2(S)$. We have a section $s$ of $V_\gamma^{\otimes d}$ such that $s = 0$ defines the divisor

$$a_1 B_1 + \cdots + a_m B_m.$$

We denote by $S_\gamma \subset V_\gamma^{\otimes d}$ the image of the section $s : S \to V_\gamma$. We have a morphism

$$\delta : V_\gamma \to V_\gamma^{\otimes d}$$

given by $\xi \mapsto \xi^d$, where $\xi$ is a fiber coordinate of $V_\gamma$. Let $\overline{W}_\gamma$ be the pull-back of $S_\gamma$ by $\delta$. Then $W_\gamma$ is isomorphic to $\overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$. The natural action of $\mathbb{C}^\times$ on $V_\gamma$ and the embedding $A \hookrightarrow \mathbb{C}^\times$ induces an $A$-action on $V_\gamma$ over $S$, under which $\overline{W}_\gamma$ is stable and the isomorphism $W_\gamma \xrightarrow{\sim} \overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$ is $A$-equivariant.

In the general case, we decompose $A$ into a direct sum of cyclic groups $A \cong A_1 \times \cdots \times A_l$, and let

$$\gamma(j) : H^2(B) \to A_j$$

be the composite of $\gamma$ with the projection $A \to A_j$. We put

$$V_\gamma := V_{\gamma(1)} \oplus \cdots \oplus V_{\gamma(l)},$$

on which $A$ acts over $S$, and define the closed subvariety $\overline{W}_\gamma \subset V_\gamma$ by

$$\overline{W}_\gamma = \{\ (\xi_1, \ldots, \xi_l) \in V_\gamma \ \mid\ \xi_j \in \overline{W}_{\gamma(j)} \subset V_{\gamma(j)} \ \text{ for } j = 1, \ldots, l\ \},$$

which is stable under the action of $A$. Then $W_\gamma$ is $A$-equivariantly isomorphic to $\overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$. Q.E.D.

*Proof of Proposition 2.5.* By means of the celebrated theorem of Villamayor [25, Corollary 7.6.3], we can make an equivariant embedded desingularization of $\overline{W}_\gamma \subset V_\gamma$. Q.E.D.

Combining Propositions 2.4 and 2.5, we obtain the following:

**Corollary 2.7.** *The Galois group $A$ acts on the finite abelian group $F_A(S, B, \gamma)$ and on the lattice $T_A(S, B, \gamma)$.*

## §3. The invariant $T_A(S, B, \gamma)$

The invariant $T_A(S, B, \gamma)$ is a topological invariant. Recall that a homeomorphism

$$h : (S, B) \xrightarrow{\sim} (S, B')$$

induces a bijection $h^* : \mathcal{C}_A(S, B') \xrightarrow{\sim} \mathcal{C}_A(S, B)$.

**Theorem 3.1.** *Suppose that the classes $[B_i]$ of the irreducible components of $B$ span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$. If $h : (S, B) \xrightarrow{\sim} (S, B')$ is a homeomorphism, then the lattices $T_A(S, B, h^*\gamma)$ and $T_A(S, B', \gamma)$ are isomorphic.*

*Proof.* Remark that the classes $[h(B_i)]$ of the irreducible components of $B'$ also span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$.

Since $W_{h^*\gamma}$ is homeomorphic to $W_\gamma$, it is enough to show that the lattice $T_A(S, B, \gamma)$ is determined by the homeomorphism type of the open surface $W_\gamma$. We consider the intersection pairing

$$\iota_W : H_2(W_\gamma) \times H_2(W_\gamma) \to \mathbb{Z},$$

which may be degenerate since $W_\gamma$ is not compact. We put

$$\mathrm{Ker}(\iota_W) := \{\ x \in H_2(W_\gamma) \ \mid\ \iota_W(x, y) = 0 \ \text{ for all }\ y \in H_2(W_\gamma)\ \}.$$

Then $\iota_W$ induces a non-degenerate symmetric bilinear form

$$\bar{\iota}_W \ :\ H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W) \ \times\ H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W) \ \to\ \mathbb{Z}$$

on the free $\mathbb{Z}$-module $H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W)$. Since the lattice

$$(H_2(W_\gamma)/\,\mathrm{Ker}(\iota_W),\ \ \bar{\iota}_W)$$

is determined by the homeomorphism type of $W_\gamma$, the proof is completed by Proposition 3.2 below, which was proved in a slightly different situation in [16] and [19]. Q.E.D.

**Proposition 3.2.** *Suppose that the classes* $[B_i]$ *span* $\mathrm{NS}(S)\otimes\mathbb{Q}$ *over* $\mathbb{Q}$. *Then the lattice* $H_2(W_\gamma)/\operatorname{Ker}(\iota_W)$ *is isomorphic to* $T_A(S, B, \gamma)$.

*Proof.* We put $D := X \setminus W_\gamma$, and let $D_1, \dots, D_M$ be the reduced irreducible components of $D$. Since the classes $[B_i]$ span $\mathrm{NS}(S)\otimes\mathbb{Q}$, the classes $[D_1], \dots, [D_M]$ span $\Sigma(X)\otimes\mathbb{Q}$ over $\mathbb{Q}$. We put

$$\widetilde{T} := \{\ x \in H_2(X)\ \mid\ (x, [D_i])_X = 0 \quad \text{for all } i = 1, \dots, M\ \},$$

where $(\ ,\ )_X$ is the intersection pairing on $X$. Then we have an isomorphism

$$T_A(S, B, \gamma) \cong \widetilde{T}/(\text{the torsion part})$$

of lattices. The image of the homomorphism

$$j_* : H_2(W_\gamma) \to H_2(X)$$

induced by $j : W_\gamma \hookrightarrow X$ is contained in $\widetilde{T}$. Note that, by definition, the homomorphism $j_*$ preserves the intersection pairings. On the other hand, from the Poincaré-Lefschetz duality isomorphisms

$$H_2(W_\gamma) \cong H^2(X, D) \quad \text{and} \quad H_2(X) \cong H^2(X)$$

and the cohomology exact sequence

$$H^2(X, D) \ \to\ H^2(X)\ \to\ H^2(D) = \bigoplus \mathbb{Z}[D_i],$$

we see that every homology class in $\widetilde{T}$ is represented by a topological 2-cycle on $W_\gamma$. Thus the inclusion $j$ induces a surjective homomorphism

$$\bar{j}_* : H_2(W_\gamma) \twoheadrightarrow T_A(S, B, \gamma),$$

which preserves the intersection pairings. Since the symmetric bilinear form on $T_A(S, B, \gamma)$ is non-degenerate, we can easily prove that $\operatorname{Ker}\bar{j}_*$ is equal to $\operatorname{Ker}(\iota_W)$. Q.E.D.

**Definition 3.3.** A plane curve $B \subset \mathbb{P}^2$ of degree 6 is called a *simple sextic* if $B$ has only simple singularities. A simple sextic $B$ is called a *maximizing sextic* if the total Milnor number $\mu(B)$ attains the possible maximum 19.

**Example 3.4.** Suppose that $B \subset \mathbb{P}^2$ is a maximizing sextic. We consider the double covering of $\mathbb{P}^2$ corresponding to

$$\gamma : H^2(B) \to A = \mathbb{Z}/2\mathbb{Z}$$

such that $\gamma([B_i]) \neq 0$ for any $B_i$. Then we have a $K3$ surface with the Picard number being the possible maximum 20 (i.e. a *singular* $K3$ surface in the sense of Shioda) as a smooth projective completion $X$ of $\varphi_\gamma : W_\gamma \to \mathbb{P}^2 \setminus B$, and the invariant $T_A(\mathbb{P}^2, B, \gamma)$ is the *transcendental lattice* of $X$, which is a positive-definite even lattice of rank 2. Using the result of transcendental lattices of conjugate $K3$ surfaces with the maximal Picard number [14, 17], we have obtained many arithmetic Zariski pairs of degree 6 in [19].

In [1], we have exhibited a pair $[B_+, B_-]$ of maximizing sextics with the singularities of type $A_9 + A_{10}$ that are defied over $\mathbb{Q}(\sqrt{5})$ and are conjugate under the action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. The invariants $T_A$ for them are calculated as follows:

$$T_A(\mathbb{P}^2, B_+, \gamma) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T_A(\mathbb{P}^2, B_-, \gamma) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

## §4. The invariants $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$

We investigate the algebraicity of the invariant $F_A(S, B, \gamma)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $\gamma \in \mathcal{C}_A(S, B)$, we denote by $\gamma^\sigma \in \mathcal{C}_A(S^\sigma, B^\sigma)$ the element corresponding to the étale Galois covering of $S^\sigma \setminus B^\sigma$ obtained as the pull-back of the morphism $\varphi_\gamma : W_\gamma \to S \setminus B$ over $\operatorname{Spec} \mathbb{C}$ by $\sigma^* : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}$; that is,

$$\gamma^\sigma : H^2(B^\sigma) \to A$$

is given by $\gamma^\sigma([B_i^\sigma]) = \gamma([B_i])$, where $B_i^\sigma$ is the conjugate of $B_i$ by $\sigma$. The following is obvious from the definition:

**Proposition 4.1.** *For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the finite abelian groups $F_A(S, B, \gamma)$ and $F_A(S^\sigma, B^\sigma, \gamma^\sigma)$ are isomorphic.*

Next we define a new invariant $\sigma_A(S, B, \gamma)$, which is an invariant of the configuration type of $B$. The invariants $T_A(S, B, \gamma)$ and $F_A(S, B, \gamma)$ are related via this invariant.

We recall the definition of the *discriminant group* of a lattice. Let $L$ be a lattice. Then we can canonically embed $L$ into its dual lattice

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}).$$

The *discriminant group* $\mathrm{disc}(L)$ of $L$ is defined by

$$\mathrm{disc}(L) := L^\vee / L.$$

**Proposition 4.2.** *The isomorphism class of the discriminant group* $\mathrm{disc}(\Sigma(X))$ *does not depend on the choice of the smooth projective completion* $\phi : X \to S$ *of* $\varphi_\gamma : W_\gamma \to S \setminus B$.

*Proof.* The discriminant group of a lattice $\langle e \rangle$ of rank 1 with $e^2 = -1$ is trivial. Proposition 4.2 then follows from (2.2) by the same argument as in the proof of Proposition 2.4. Q.E.D.

Thus the following is well-defined:

$$\sigma_A(S, B, \gamma) := \mathrm{disc}(\Sigma(X)).$$

We will show that $\sigma_A(S, B, \gamma)$ is an invariant of the configuration type.

**Theorem 4.3.** *Suppose that* $\tau : (\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$ *is a map of equi-configuration. Then* $\sigma_A(S, B, \gamma)$ *is isomorphic to* $\sigma_A(S, B', \tau_*\gamma)$.

For the proof, we recall the definition of *equisingularity* of plane curve singularities. See [26, Proposition 4.3.9] for details.

Let $P \in \mathrm{Sing}\, B$ be a singular point of $B$, and let $P' \in \mathrm{Sing}\, B'$ be a singular point of $B'$. Let $B^{(1)}, \dots, B^{(k)}$ be the local branches of $B$ at $P$, and let $B'^{(1)}, \dots, B'^{(k')}$ be the local branches of $B'$ at $P'$.

**Definition 4.4.** We say that the two germs $(B, P)$ and $(B', P')$ of the plane curve singularity are *equisingular* if $k = k'$ holds and there exists a bijection from $\{B^{(1)}, \dots, B^{(k)}\}$ to $\{B'^{(1)}, \dots, B'^{(k')}\}$, given by $B^{(\kappa)} \mapsto B'^{(\kappa)}$ after permutations of indices, such that $B^{(\kappa)}$ and $B'^{(\kappa)}$ have the same Puiseux characteristic for $\kappa = 1, \dots, k$ and that the equalities of intersection numbers $B^{(i)} \cdot B^{(j)} = B'^{(i)} \cdot B'^{(j)}$ hold for all $i \neq j$.

*Proof of Theorem 4.3.* By the equivalence of (i) and (iv) in [26, Theorem 5.5.9], we see that $(B, P)$ and $(B', \tau(P))$ are equisingular for any singular point $P$ of $B$. Let $\mu : (\widetilde{S}, \widetilde{B}) \to (S, B)$ be the minimal good embedded resolution of $B$, and let $\mu' : (\widetilde{S}', \widetilde{B}') \to (S, B')$ be the minimal good embedded resolution of $B'$. (See [26, §3.4] for the definition of minimal good embedded resolutions.) Note that $\mu$ induces an analytic isomorphism $\widetilde{S} \setminus \widetilde{B} \cong S \setminus B$, and hence induces a bijection

$$\mu_* : \mathcal{C}_A(\widetilde{S}, \widetilde{B}) \;\xrightarrow{\sim}\; \mathcal{C}_A(S, B)$$

via the isomorphism $\mu_* : \pi_1(\widetilde{S} \setminus \widetilde{B}) \xrightarrow{\sim} \pi_1(S \setminus B)$. By Theorem 8.1.7 or Proposition 8.3.1 of [26], we have a map of equi-configuration

$$\tilde{\tau} : (\widetilde{T}, \widetilde{B}) \to (\widetilde{T}', \widetilde{B}'),$$

which induces a commutative diagram of bijections

$$\begin{array}{ccc} \mathcal{C}_A(\widetilde{S}, \widetilde{B}) & \xrightarrow{\mu_*} & \mathcal{C}_A(S, B) \\ {\scriptstyle \tilde{\tau}_*} \downarrow & & \downarrow {\scriptstyle \tau_*} \\ \mathcal{C}_A(\widetilde{S}', \widetilde{B}') & \xrightarrow{\mu'_*} & \mathcal{C}_A(S, B'). \end{array}$$

A smooth projective completion $\widetilde{X} \to \widetilde{S}$ of an étale Galois covering $W_{\tilde{\gamma}} \to \widetilde{S} \setminus \widetilde{B}$ corresponding to $\tilde{\gamma} \in \mathcal{C}_A(\widetilde{S}, \widetilde{B})$ is a smooth projective completion of $W_{\mu_* \tilde{\gamma}} \to S \setminus B$. Therefore, by Proposition 4.2, it is enough to prove

$$\sigma_A(\widetilde{S}, \widetilde{B}, \tilde{\gamma}) \cong \sigma_A(\widetilde{S}', \widetilde{B}', \tilde{\tau}_* \tilde{\gamma})$$

for any $\tilde{\gamma} \in \mathcal{C}_A(\widetilde{S}, \widetilde{B})$; that is,*we can assume that $B$ and $B'$ are normal crossing divisors on $S$.*

Suppose that $B$ and $B'$ are normal crossing divisors. Recall the *finite* covering

$$\bar{\varphi}_\gamma : \overline{W}_\gamma \to S$$

constructed in Lemma 2.6. Let $\nu : Y_\gamma \to \overline{W}_\gamma$ be the normalization of $\overline{W}_\gamma$, and consider the finite covering

$$\bar{\varphi}_\gamma \circ \nu : Y_\gamma \to S.$$

Then $\operatorname{Sing} Y_\gamma$ is located over $\operatorname{Sing} B$. If $P \in \operatorname{Sing} B$ is an intersection point of $B_i$ and $B_j$, then the number and the analytic isomorphism classes of singular points of $Y_\gamma$ over $P$ are determined by $\gamma([B_i]) \in A$ and $\gamma([B_j]) \in A$. We construct the finite covering

$$\bar{\varphi}_{\tau_* \gamma} \circ \nu' : Y_{\tau_* \gamma} \to S$$

of $S$ by a normal surface $Y_{\tau_* \gamma}$ branching along $B'$ in the same way. Then there exists a bijection

$$\operatorname{Sing} Y_\gamma \cong \operatorname{Sing} Y_{\tau_* \gamma}$$

that covers the bijection $\operatorname{Sing} B \cong \operatorname{Sing} B'$ by $\tau$ and preserves the analytic isomorphism classes of the surface singularities. Hence there exist desingularizations

$$X_\gamma \to Y_\gamma, \quad \text{and} \quad X_{\tau_* \gamma} \to Y_{\tau_* \gamma}$$

such that the sets $\mathcal{E}(X_\gamma)$ and $\mathcal{E}(X_{\tau_* \gamma})$ of exceptional curves have the same configuration. Therefore we have

$$\Sigma(X_\gamma) \cong \Sigma(X_{\tau_* \gamma}).$$

By Proposition 4.2, we complete the proof. Q.E.D.

The isomorphism class of the discriminant group $\mathrm{disc}(\Lambda(X))$ of the primitive closure $\Lambda(X)$ of $\Sigma(X)$ is also independent of the choice of the smooth projective completion $X$. More precisely, we have the following:

**Proposition 4.5.** *The discriminant group* $\mathrm{disc}(\Lambda(X))$ *is isomorphic to* $\mathrm{disc}(T_A(S,B,\gamma))$ *for any smooth projective completion* $X$.

The proof follows from Lemma 4.6 below and the fact that the lattice $H^2(X)'$ is unimodular.

**Lemma 4.6.** *Let* $L$ *and* $L'$ *be primitive sublattices of a unimodular lattice* $M$ *such that* $L \perp L'$ *and that* $[M : L \oplus L'] < \infty$. *Then* $\mathrm{disc}(L)$ *and* $\mathrm{disc}(L')$ *are isomorphic.*

Lemma 4.6 is [12, Proposition 1.6.1] without the assumption that lattices be even. See also the proof of [17, Proposition 2.1.1].

The three invariants $T_A(S,B,\gamma)$, $F_A(S,B,\gamma)$ and $\sigma_A(S,B,\gamma)$ are related as follows:

**Proposition 4.7.** *For any* $\gamma \in \mathcal{C}_A(S,B)$, *we have*

$$|\,\mathrm{disc}(T_A(S,B,\gamma))| \cdot |F_A(S,B,\gamma)|^2 = |\sigma_A(S,B,\gamma)|.$$

*Moreover, for any prime integer* $p$, *we have*

$$\begin{aligned}\mathrm{leng}_p(\mathrm{disc}(T_A(S,B,\gamma))) &\le \mathrm{leng}_p(\sigma_A(S,B,\gamma)) \\ &\le \mathrm{leng}_p(\mathrm{disc}(T_A(S,B,\gamma))) + 2\,\mathrm{leng}_p(F_A(S,B,\gamma)).\end{aligned}$$

This proposition follows from the following elementary lemma [12] and Proposition 4.5.

**Lemma 4.8.** *Let* $L$ *be a lattice, and let* $M$ *be a sublattice of* $L$ *with finite index. Then we have*

$$M \subset L \subset L^\vee \subset M^\vee.$$

*Since* $M^\vee/L^\vee \cong L/M$, *we have* $|\,\mathrm{disc}(M)| = |\,\mathrm{disc}(L)| \cdot [L : M]^2$, *and*

$$\mathrm{leng}_p(\mathrm{disc}(L)) \le \mathrm{leng}_p(\mathrm{disc}(M)) \le \mathrm{leng}_p(\mathrm{disc}(L)) + 2\,\mathrm{leng}_p(L/M).$$

As a corollary of Theorems 3.1, 4.3 and Proposition 4.7, we obtain the following generalization of [18, Theorem 8.5] and the idea of Xie and Yang in [27]. This corollary shows that the algebraic invariant $F_A(S,B,\gamma)$ can be used to distinguish the topological types of $B$.

**Corollary 4.9.** *Suppose that the classes* $[B_i]$ *span* $\mathrm{NS}(S) \otimes \mathbb{Q}$ *over* $\mathbb{Q}$. *Let* $\tau : (\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$ *be a map of equi-configuration. If we have* $|F_A(S,B,\gamma)| \ne |F_A(S,B',\tau_*\gamma)|$, *then* $\tau$ *cannot be extended to a homeomorphism* $(S,B) \xrightarrow{\sim} (S,B')$.

**Example 4.10.** Let $B$ and $B'$ be the plane curves of degree 6 in Example 1.6. Consider the finite abelian group $A = \mathbb{Z}/2\mathbb{Z}$. Then each of $\mathcal{C}_A(\mathbb{P}^2, B)$ and $\mathcal{C}_A(\mathbb{P}^2, B')$ consists of a single element $\gamma$. We have $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}$ while $F_A(\mathbb{P}^2, B', \gamma) = 0$.

The six-cuspidal sextic $B$ is defined by the torus-type equation

$$f^3 + g^2 = 0,$$

where $\deg f = 2$ and $\deg g = 3$, and $f$ and $g$ are chosen generally. The conic $Q$ defined by $f = 0$ passes through $\operatorname{Sing} B$, and hence $B$ is called a *conical six-cuspidal sextic*. The proper transform of $Q$ by $\phi : X \to \mathbb{P}^2$ splits into the union two irreducible components $\tilde{Q}^+$ and $\tilde{Q}^-$. The class $[\tilde{Q}^+]$ is contained in the primitive closure $\Lambda(X)$, and $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by $[\tilde{Q}^+]$.

On the other hand, there exist no conics passing through the 6 cusps $\operatorname{Sing} B'$. The existence of such a non-conical six-cuspidal sextic $B'$ was stated by Del Pezzo without proof, and was proved by B. Segre (see [21, page 407]). Zariski also proved the existence in [28]. The explicit defining equation of a non-conical six-cuspidal sextics was given by Oka [13].

Many Zariski pairs of simple sextics have been discovered in [18] and by Xie and Yang in [27] using the idea of Corollary 4.9.

We also have the following corollary, which plays an important role in the next section:

**Corollary 4.11.** *Let $p$ be a prime integer. If we have*

$$\operatorname{leng}_p(\operatorname{disc}(T_A(S, B, \gamma))) < \operatorname{leng}_p(\sigma_A(S, B, \gamma)),$$

*then we have $F_A(S, B, \gamma)_p \neq 0$. In particular, if*

$$\operatorname{rank}(T_A(S, B, \gamma)) < \operatorname{leng}_p(\sigma_A(S, B, \gamma))$$

*holds, then $F_A(S, B, \gamma)_p \neq 0$.*

The second assertion follows from the observation that, for a lattice $L$, we have $\operatorname{leng}_p(\operatorname{disc}(L)) \leq \operatorname{rank}(L)$.

## §5. The fundamental group $\pi_1(S \setminus B)$

In this section, we give a result on a relation between $\pi_1(S \setminus B)$ and the invariants $T_A(S, B, \gamma)$, $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$.

**Definition 5.1.** Let $M$ be an abelian group, and let $G$ be a group. Suppose that there exists an exact sequence

$$(5.1) \qquad 0 \longrightarrow M \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Then we have an action $\sigma_\Gamma : G \to \mathrm{Aut}(M)$ of $G$ on $M$ defined by

$$\bar{\gamma}(a) := \gamma a \gamma^{-1},$$

where $\bar{\gamma} \in G$ is the image of $\gamma \in \Gamma$, and $M$ is regarded as a normal subgroup of $\Gamma$. We call $\sigma_\Gamma$ the *action associated with* (5.1).

In this section, we put

$$U := S \setminus B,$$

and fix a base point $b \in U$. Let $\varphi : W \to U$ be a finite étale Galois covering with the Galois group $G$, which is not necessarily abelian. Then the group $G$ acts on $W$ and hence on $H_1(W)$ in a natural way. Let

$$N := \mathrm{Ker}(\rho : \pi_1(U, b) \twoheadrightarrow G),$$

be the kernel of the surjective homomorphism $\rho : \pi_1(U, b) \twoheadrightarrow G$ associated with $\varphi$. Then $N$ is (non-canonically) isomorphic to the fundamental group of $W$, and $H_1(W)$ is *canonically* identified with $N/[N, N]$.

The following is well-known, for example, in the study of Alexander polynomials [7].

**Proposition 5.2.** *The action of $G$ on $H_1(W)$ is associated with the exact sequence*

$$0 \to H_1(W) \to \pi_1(U, b)/[N, N] \to G \to 1. \tag{5.2}$$

**Corollary 5.3.** *Suppose that there exists a finite étale Galois covering $W \to U$ with the Galois group $G$ acting on $H_1(W)$ non-trivially. Then $\pi_1(U, b)$ is non-abelian.*

**Corollary 5.4.** *Let $\Gamma$ be a group that fits in an exact sequence*

$$0 \longrightarrow M \longrightarrow \Gamma \xrightarrow{g} G \longrightarrow 1$$

*with $M$ being abelian. Suppose that there is a surjective homomorphism $\gamma : \pi_1(U, b) \twoheadrightarrow \Gamma$. Let $W \to U$ be the finite étale Galois covering associated with the composite $g \circ \gamma : \pi_1(U, b) \twoheadrightarrow G$. Then there exists a surjective homomorphism of $G$-modules $H_1(W) \twoheadrightarrow M$, where $M$ is considered as a $G$-module by $\sigma_\Gamma$.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccccccccccc} 1 & \to & \pi_1(W) & \to & \pi_1(U) & \to & G & \to & 1 & \text{(exact)} \\ & & & & \downarrow\!\!\!\downarrow & & \| & & & \\ 0 & \to & M & \to & \Gamma & \to & G & \to & 1 & \text{(exact).} \end{array}$$

Hence we have a surjective homomorphism $\pi_1(W) \twoheadrightarrow M$, which factors through the homomorphism of $G$-modules $H_1(W) \twoheadrightarrow M$. Q.E.D.

We now return to the finite abelian Galois covering

$$\varphi_\gamma : W_\gamma \to U = S \setminus B$$

with the Galois group $G = A$ associated with an element $\gamma \in \mathcal{C}_A(S, B)$. Let $\phi : X \to S$ be a smooth projective completion. We put

$$D := X \setminus W_\gamma = \phi^{-1}(B),$$

and let $D_1, \dots, D_M$ be the reduced irreducible components of $D$. We consider the submodule

$$\Theta(X) := \langle [D_1], \dots, [D_M] \rangle \subset H^2(X)'$$

of $H^2(X)'$ generated by $[D_1], \dots, [D_M]$, and its primitive closure

$$\Xi(X) := (\Theta(X) \otimes \mathbb{Q}) \cap H^2(X)'.$$

We put

$$F'_A(S, B, \gamma) := \Xi(X)/\Theta(X).$$

We can prove the following by the same argument as Proposition 2.4:

**Proposition 5.5.** *The isomorphism class of the finite abelian group $F'_A(S, B, \gamma)$ is independent of the choice of the smooth projective completion $\phi : X \to S$.*

Therefore, by choosing an $A$-equivariant smooth completion, we see that $A$ acts on $F'_A(S, B, \gamma)$.

**Proposition 5.6.** *There exists a natural $A$-equivariant embedding*

$$F'_A(S, B, \gamma)^\vee \hookrightarrow H_1(W_\gamma),$$

*where* $F'_A(S, B, \gamma)^\vee := \mathrm{Hom}(F'_A(S, B, \gamma), \mathbb{Q}/\mathbb{Z})$.

*Proof.* We have a canonical isomorphism $H_1(W_\gamma) \cong H^3(X, D)$. Hence the cokernel of the restriction homomorphism

$$r_X : H^2(X) \to H^2(D) = \bigoplus \mathbb{Z}[D_i]$$

is contained in $H_1(W_\gamma)$. Note that $r_X$ factors through

$$s : H^2(X)' \to H^2(D)$$

and that $H^2(X)'$ is a unimodular lattice by the cup-product. Hence $H^2(X)'$ is self-dual. The submodule $\Theta(X)$ is the image of the dual homomorphism

$$s^\vee : H^2(D)^\vee \to H^2(X)'$$

of $s$. Thus we have a decomposition

$$H^2(D)^\vee \twoheadrightarrow \Theta(X) \hookrightarrow \Xi(X) \hookrightarrow H^2(X)'$$

of $s^\vee$, where $H^2(D)^\vee = \mathrm{Hom}(H^2(D), \mathbb{Z})$. The dual homomorphism $H^2(X)' \to \Xi(X)^\vee$ of the primitive embedding $\Xi(X) \hookrightarrow H^2(X)'$ is surjective. The dual homomorphism $\Xi(X)^\vee \to \Theta(X)^\vee$ of $\Theta(X) \hookrightarrow \Xi(X)$ is injective and its cokernel is canonically isomorphic to

$$F'_A(S, B, \gamma)^\vee = \mathrm{Hom}(\Xi(X)/\Theta(X), \mathbb{Q}/\mathbb{Z}).$$

The dual homomorphism $\Theta(X)^\vee \to H^2(D)$ of the surjective homomorphism $H^2(D)^\vee \twoheadrightarrow \Theta(X)$ is injective. Thus $\mathrm{Coker}(s) = \mathrm{Coker}(r_X)$ contains $F'_A(S, B, \gamma)^\vee$ in a natural way, and hence so does $H_1(W_\gamma)$. Q.E.D.

We investigate the relation between $F'_A(S, B, \gamma)$ and $F_A(S, B, \gamma)$.

**Definition 5.7.** For a reduced irreducible curve $F$ on $S$, the *strict transform of* $F$ is the total transform of $F$ by $\phi : X \to S$ minus the components that are contracted to points by $\phi$.

Remark that the class of the strict transform of any reduced irreducible curve on $S$ is contained in $\Sigma(X)$.

Suppose that $A$ is a cyclic group of prime order $l$. Then, for any reduced irreducible curve $F$ on $S$, the strict transform of $F$ is either reduced, or of the form $lC$ with $C$ being reduced and irreducible. The later occurs if and only if $F$ is an irreducible component $B_i$ of $B$ such that $\gamma([B_i]) \neq 0$ in $A \cong \mathbb{Z}/l\mathbb{Z}$.

**Assumption 5.8.** We consider the following assumptions:

(a) the finite abelian group $A$ is cyclic of prime order $l$,
(b) the classes $[B_1], \dots, [B_m]$ span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$, and
(c) $\gamma([B_i]) \neq 0$ for $i = 1, \dots, m$.

**Proposition 5.9.** *Suppose that Assumption 5.8 holds. Then, for any prime $p \neq l$, there exists a surjective homomorphism of $A$-modules from $F'_A(S, B, \gamma)_p$ to $F_A(S, B, \gamma)_p$.*

*Proof.* The assumption (b) implies that $\Theta(X) \otimes \mathbb{Q} = \Sigma(X) \otimes \mathbb{Q}$. Hence we have $\Xi(X) = \Lambda(X)$. Moreover $\Theta(X) \cap \Sigma(X)$ is of finite index in $\Lambda(X)$. We put

$$\widetilde{F}_A := \Lambda(X)/(\Theta(X) \cap \Sigma(X)).$$

The assumptions (a) and (c) imply that

$$\Theta(X)/(\Theta(X)\cap\Sigma(X)) = \operatorname{Ker}(\widetilde{F}_A \twoheadrightarrow F'_A(S,B,\gamma))$$

is an elementary $l$-group. Indeed, if $D_i \in \mathcal{E}(X)$, then $[D_i] \in \Sigma(X)$, while if $D_i \notin \mathcal{E}(X)$, then $D_i$ is the reduced part of the strict transform of an irreducible component $B_j$ of $B$, and hence $l[D_i] \in \Sigma(X)$. In particular, the natural projection $\widetilde{F}_A \twoheadrightarrow F'_A(S,B,\gamma)$ induces $(\widetilde{F}_A)_p \cong F'_A(S,B,\gamma)_p$ for $p \neq l$. Therefore the natural projection

$$\widetilde{F}_A \twoheadrightarrow F_A(S,B,\gamma)$$

induces a surjective homomorphism from $F'_A(S,B,\gamma)_p$ to $F_A(S,B,\gamma)_p$ for any $p \neq l$. Q.E.D.

On the other hand, we have the following:

**Proposition 5.10.** *Suppose that Assumption 5.8 holds. If the order of a non-zero element $f \in F_A(S,B,\gamma)$ is not equal to $l$, then $A$ acts on $f$ non-trivially.*

*Proof.* We choose an $A$-equivariant smooth projective completion $\phi : X \to S$. Suppose that $R$ is a divisor on $X$ such that

$$f = [R] \bmod \Sigma(X).$$

Let $H$ be an ample divisor on $S$. Since $[\phi^* H] \in \Sigma(X)$, we can replace $R$ by $R + n(\phi^* H)$ with sufficiently large $n$ if necessary, and assume that $R$ is effective. We write

$$R = R_1 + \cdots + R_N,$$

where $R_1, \dots, R_N$ are reduced and irreducible. Since $\langle \mathcal{E}(X) \rangle \subset \Sigma(X)$, we can assume that each $R_i$ is not in $\mathcal{E}(X)$ and hence is mapped by $\phi$ to a curve on $S$. Let $\overline{R}_i$ be the reduced irreducible curve on $S$ that is the image of $R_i$. Let $d_i$ be the degree of $R_i \to \overline{R}_i$, which is either 1 or $l$. The divisor $\sum_{g\in A} g(R_i)$ on $X$ is equal to $d_i$ times the strict transform of $\overline{R}_i$, and hence its class is contained in $\Sigma(X)$. Therefore we have $\sum_{g\in A} g(f) = 0$. Since the order of $f \neq 0$ is not equal to $|A| = l$, we have $g(f) \neq f$. Q.E.D.

Combining all the results, we obtain the following:

**Corollary 5.11.** *Suppose that Assumption 5.8 holds. If we have $F_A(S,B,\gamma)_p \neq 0$ for some $p \neq l$, then $\pi_1(S \setminus B)$ acts on $H_1(W_\gamma)$ non-trivially and hence is non-abelian.*

By Corollary 4.11, we obtain the following:

**Corollary 5.12.** *Suppose that Assumption 5.8 holds. If we have*

$$\mathrm{leng}_p(\sigma_A(S,B,\gamma)) > \mathrm{leng}_p(\mathrm{disc}(T_A(S,B,\gamma)))$$

*for some $p \neq l$, then $\pi_1(S \setminus B)$ is non-abelian. In particular, if*

$$\mathrm{leng}_p(\sigma_A(S,B,\gamma)) > \mathrm{rank}(T_A(S,B,\gamma))$$

*for some $p \neq l$, then $\pi_1(S \setminus B)$ is non-abelian.*

We apply these corollaries to the double covering of $\mathbb{P}^2$ branching along a curve with only simple singularities. Let $B \subset \mathbb{P}^2$ be a plane curve of even degree $d$. Consider the double covering $\varphi_\gamma : W_\gamma \to \mathbb{P}^2 \setminus B$ corresponding to $\gamma : H^2(B) \to \mathbb{Z}/2\mathbb{Z}$ with $\gamma([B_i]) \neq 0$ for any irreducible component $B_i$ of $B$. Suppose that $B$ has only simple singularities, and let $\mu_B$ be the total Milnor number of $\mathrm{Sing}\, B$. Then the normal surface $Y_\gamma$ constructed in the proof of Theorem 4.3 has only rational double points of the total Milnor number equal to $\mu_B$. We choose the minimal resolution $X$ of $Y_\gamma$ as the smooth projective completion of $\varphi_\gamma : W_\gamma \to \mathbb{P}^2 \setminus B$. Then we have

$$\begin{aligned}&\mathrm{rank}(\Sigma(X)) = 1 + \mu_B \quad \text{and}\\ &b_2(X) = \mathrm{rank}(\Sigma(X)) + \mathrm{rank}(T_A(\mathbb{P}^2, B, \gamma)) = d^2 - 3d + 4.\end{aligned}$$

Therefore we obtain the following corollary, which has been proved in [23].

**Corollary 5.13.** *If $\mu_B + \mathrm{leng}_p(\sigma_A(\mathbb{P}^2, B, \gamma)) > d^2 - 3d + 3$ for some odd prime $p$, then $\pi_1(\mathbb{P}^2 \setminus B)$ is non-abelian.*

See [23] also for various applications of this corollary.

Note that $\mathrm{leng}_p(\sigma_A(\mathbb{P}^2, B, \gamma))$ is easily calculated from the *ADE*-type of $\mathrm{Sing}\, B$. Note also that $\mu_B$ and $\mathrm{leng}_p(\sigma_A(\mathbb{P}^2, B, \gamma))$ are both invariants of the configuration type of $B$.

Another corollary is about the relation between the existence of $Z$-splitting curves and $\pi_1(\mathbb{P}^2 \setminus B)$.

**Definition 5.14.** Let $B \subset \mathbb{P}^2$ be as above. A reduced irreducible curve $\Gamma \subset \mathbb{P}^2$ is said to be *$Z$-splitting* if the strict transform $\widetilde{\Gamma} \subset X$ of $\Gamma$ splits into two irreducible components $\widetilde{\Gamma}^+$, $\widetilde{\Gamma}^-$ and their classes $[\widetilde{\Gamma}^+]$ and $[\widetilde{\Gamma}^-]$ are distinct elements of $\Lambda(X)$. The *class order* of a $Z$-splitting curve $\Gamma$ is the order of $[\widetilde{\Gamma}^+]$ in the finite abelian group $F_A(\mathbb{P}^2, B, \gamma)$.

**Corollary 5.15.** *If $B$ has a $Z$-splitting curve of class order not equal to a power of* 2*, then* $\pi_1(\mathbb{P}^2 \setminus B)$ *is non-abelian.*

**Example 5.16.** In [18], we have completely classified all $Z$-splitting curves of degree $\leq 3$ for simple sextics by means of period mapping for complex $K3$ surfaces.

For example, we have found a maximizing sextic $B = C + Q$ of type $A_3 + A_5 + A_{11}$ (a union of a conic $C$ and a quartic $Q$ with $A_5$) with a $Z$-splitting line of class order 12. By Corollary 5.15, we see that $\pi_1(\mathbb{P}^2 \setminus B)$ is non-abelian.

## §6. Computation of the invariants

We close this paper with a remark on the computation of the invariants $T_A$, $F_A$ and $\sigma_A$. Suppose that we know the structure of $\mathrm{NS}(S)$. The lattice $\Sigma(X)$ and hence its discriminant group $\sigma_A(S, B, \gamma)$ can be calculated from the configuration type of $B$. In [1], we have developed a general method of Zariski-van Kampen type to calculate the lattice $T_A(S, B, \gamma)$. Hence the order of the finite abelian group $F_A(S, B, \gamma)$ can be also calculated. We also obtain some information about the structure of $F_A(S, B, \gamma)$ from the discriminant groups of $T_A(S, B, \gamma)$ and of $\Sigma(X)$ by using Lemma 4.8.

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