Moduli of supersingular K3 surfaces in characteristic 2

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We work over an algebraically closed field k of characteristic 2.

§1. Construction of the Moduli Space

Let X be a supersingular $K3$ surface.

Let $\mathcal{L}$ be a line bundle on X with $\mathcal{L}^2 = 2$. We say that $\mathcal{L}$ is a *polarization of type* ($\sharp$) if the following conditions are satisfied:

- **the complete linear system $|\mathcal{L}|$ has no fixed components, and**
- **the set of curves contracted by the morphism**

$$\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$$

defined by $|\mathcal{L}|$ consists of 21 disjoint (-2)-curves.

If $(X, \mathcal{L})$ is a polarized supersingular $K3$ surface of type ($\sharp$), then $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$ is purely inseparable.

Every supersingular $K3$ surface has a polarization of type ($\sharp$).

We will construct the moduli space $\mathcal{M}$ of polarized supersingular $K3$ surfaces of type ($\sharp$).

Let $G = G(X_0, X_1, X_2)$ be a non-zero homogeneous polynomial of degree 6.

We can define

$$dG \in \Gamma(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(6)),$$

because we are in characteristic 2 and we have $\mathcal{O}_{\mathbb{P}^2}(6) \cong \mathcal{O}_{\mathbb{P}^2}(3)^{\otimes 2}$.

We put

$$Z(dG) := \{dG = 0\} = \left\{ \frac{\partial G}{\partial X_0} = \frac{\partial G}{\partial X_1} = \frac{\partial G}{\partial X_2} = 0 \right\} \subset \mathbb{P}^2.$$

If $\dim Z(dG) = 0$, then

$$\text{length}\, \mathcal{O}_{Z(dG)} = c_2(\Omega^1_{\mathbb{P}^2}(6)) = 21.$$

We put

$$\mathcal{U} := \{ G \mid Z(dG) \text{ is reduced of dimension } 0 \} \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)).$$

For $G \in \mathcal{U}$, we put

$$Y_G := \{W^2 = G(X_0, X_1, X_2)\} \xrightarrow{\pi_G} \mathbb{P}^2,$$

and let

$$\rho_G : X_G \to Y_G$$

be the minimal resolution of Y_G.

We have

$$\text{Sing}(Y_G) = \pi_G^{-1}(Z(dG)) = \{ 21 \text{ ordinary nodes} \}.$$

We then put

$$\mathcal{L}_G := (\pi_G \circ \rho_G)^* \mathcal{O}_{\mathbb{P}^2}(1).$$

$(X, \mathcal{L})$ is a polarized supersingular $K3$ surface of type $(\sharp)$

$$\Updownarrow$$

there exists $G \in \mathcal{U}$ such that $(X, \mathcal{L}) \cong (X_G, \mathcal{L}_G)$

We put

$$\mathcal{V} := H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)).$$

Because we have $d(G + H^2) = dG$ for $H \in \mathcal{V}$, the additive group $\mathcal{V}$ acts on the space $\mathcal{U}$ by

$$(G, H) \in \mathcal{U} \times \mathcal{V} \quad \mapsto \quad G + H^2 \in \mathcal{U}.$$

Let G and G' be homogeneous polynomials in $\mathcal{U}$. Then the following conditions are equivalent:

(i) Y_G and $Y_{G'}$ are isomorphic over $\mathbb{P}^2$,

(ii) $Z(dG) = Z(dG')$, and

(iii) there exist $c \in k^\times$ and $H \in \mathcal{V}$ such that $G' = c\,G + H^2$.

Therefore the moduli space $\mathcal{M}$ of polarized supersingular $K3$ surfaces of type ($\sharp$) is constructed by

$$\mathcal{M} = PGL(3, k)\backslash\mathbb{P}_*(\mathcal{U}/\mathcal{V}).$$

We put

$$\mathcal{P} := \{P_1, \dots, P_{21}\},$$

on which the full symmetric group S_{21} acts from left.

We denote by $\mathcal{G}$ the space of all injective maps

$$\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$$

such that there exists $G \in \mathcal{U}$ satisfying $\gamma(\mathcal{P}) = Z(dG)$.

Then we can construct $\mathcal{M}$ by

$$\mathcal{M} = PGL(3, k)\backslash\mathcal{G}/S_{21}.$$

Example by Dolgachev-Kondo:

$$\begin{aligned} G_{\mathrm{DK}} &:= X_0X_1X_2(X_0^3 + X_1^3 + X_2^3), \\ Z(dG_{\mathrm{DK}}) &= \mathbb{P}^2(\mathbb{F}_4). \end{aligned}$$

The Artin invariant of the supersingular $K3$ surface $X_{G_{\mathrm{DK}}}$ is 1.

$[G_{\mathrm{DK}}] \in \mathcal{M}$: the Dolgachev-Kondo point.

§2. Stratification by Isomorphism Classes of Codes

Let G be a polynomial in $\mathcal{U}$.

$$NS(X_G) : \text{the Néron-Severi lattice of } X_G,$$

$$\operatorname{disc} NS(X_G) = -2^{2\sigma(X_G)},$$

$$(\sigma(X_G) \text{ is the Artin invariant of } X_G).$$

Let $\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$ be an injective map such that

$$\gamma(\mathcal{P}) = Z(dG) = \pi_G(\operatorname{Sing} Y_G),$$

that is, γ is a numbering of the singular points of Y_G.

$$E_i \subset X_G : \text{the } (-2)\text{-curve that is contracted to } \gamma(P_i).$$

Then $NS(X_G)$ contains a sublattice

$$S_0 \;=\; \langle\ [E_1], \dots, [E_{21}], [\mathcal{L}_G]\ \rangle = \begin{bmatrix} -2 & & & \\ & -2 & & \\ & & \ddots & \\ & & & -2 & \\ & & & & 2 \end{bmatrix}.$$

$$\begin{aligned} S_0^{\vee} \quad &= \operatorname{Hom}(S_0, \mathbb{Z}) = \langle\ [E_1]/2, \dots, [E_{21}]/2, [\mathcal{L}_G]/2\ \rangle \\ &\supset\ NS(X_G). \end{aligned}$$

We put

$$\begin{aligned} \widetilde{\mathcal{C}}_G &:= NS(X_G)/S_0 \quad \subset \quad {S_0}^{\vee}/S_0 \; = \; \mathbb{F}_2^{\oplus 21} \oplus \mathbb{F}_2, \\ \mathcal{C}_G &:= \mathrm{pr}(\widetilde{\mathcal{C}}_G) \quad \subset \quad \mathbb{F}_2^{\oplus 21} \; \cong \; 2^{\mathcal{P}} (\text{the power set of } \mathcal{P}). \end{aligned}$$

Here the identification $\mathbb{F}_2^{\oplus 21} \cong 2^{\mathcal{P}}$ is given by

$$v \mapsto \{ \, P_i \in \mathcal{P} \;\; | \;\; \text{the } i\text{-th coordinate of } v \text{ is } 1 \, \}.$$

We have

$$\dim \widetilde{\mathcal{C}}_G = \dim \mathcal{C}_G = 11 - \sigma(X_G).$$

We say that a reduced irreducible curve $C \subset \mathbb{P}^2$ *splits in* X_G if the proper transform of C in X_G is non-reduced, that is, of the form $2F_C$, where $F_C \subset X_G$ is a reduced curve in X_G.

We say that a reduced curve $C \subset \mathbb{P}^2$ *splits in* X_G if every irreducible component of C splits in X_G.

$C \subset \mathbb{P}^2$: a curve of degree d splitting in X_G,

$m_i(C)$: the multiplicity of C at $\gamma(P_i) \in Z(dG)$.

$$
\begin{aligned}
[F_C] &= \frac{1}{2}(d \cdot [\mathcal{L}_G] - \sum_{i=1}^{21} m_i(C)[E_i]) \quad \in \quad NS(X_G), \\
\tilde{w}(C) &:= [F_C] \bmod S_0 \quad \in \quad \widetilde{\mathcal{C}}_G = NS(X_G)/S_0, \\
w(C) &:= \mathrm{pr}(\tilde{w}(C)) \\
&= \{ P_i \in \mathcal{P} \mid m_i(C) \text{ is odd} \} \quad \in \quad \mathcal{C}_G.
\end{aligned}
$$

A general member Q of the linear system

$$|\mathcal{I}_{Z(dG)}(5)| = \left\langle \frac{\partial G}{\partial X_0}, \frac{\partial G}{\partial X_1}, \frac{\partial G}{\partial X_2} \right\rangle$$

splits in X_G.

In particular,

$$w(Q) = \mathcal{P} = (1, 1, \ldots, 1) \in \mathcal{C}_G.$$

What kind of codes can appear as $\mathcal{C}_G$ for some $G \in \mathcal{U}$?

$NS(X_G)$ has the following properties;

- type II (that is, $v^2 \in \mathbb{Z}$ for any $v \in NS(X_G)^\vee$),
- there are no $u \in NS(X_G)$ such that $u \cdot [\mathcal{L}_G] = 1$ and $u^2 = 0$ (that is, $|\mathcal{L}_G|$ is fixed component free), and
- if $u \in NS(X_G)$ satisfies $u \cdot [\mathcal{L}_G] = 0$ and $u^2 = -2$, then $u = [E_i]$ or $-[E_i]$ for some i (that is, $\operatorname{Sing} Y_G$ consists of 21 ordinary nodes).

$\mathcal{C}_G$ has the following properties;

- $\mathcal{P} = (1, 1, \ldots, 1) \in \mathcal{C}_G$, and
- $|w| \in \{0, 5, 8, 9, 12, 13, 16, 21\}$ for any $w \in \mathcal{C}_G$.

The isomorphism classes $[\mathcal{C}]$ of codes $\mathcal{C} \subset \mathbb{F}_2^{\oplus 21} = 2^{\mathcal{P}}$ satisfying these conditions are classified:

$\sigma = 11 - \dim \mathcal{C}$,

$r(\sigma) =$ the number of the isomorphism classes.

σ	1	2	3	4	5	6	7	8	9	10	total
$r(\sigma)$	1	3	13	41	58	43	21	8	3	1	193

.

the isomorphism class of $(X_G, \mathcal{L}_G) \in \mathcal{M}_{[\mathcal{C}]}$

$$\iff \quad \mathcal{C}_G \in [\mathcal{C}]$$

$$\mathcal{M} = PGL(3,k)\backslash \mathbb{P}_*(\mathcal{U}/\mathcal{V}) = \bigsqcup_{\text{the isom. classes}} \mathcal{M}_{[\mathcal{C}]}.$$

Each $\mathcal{M}_{[\mathcal{C}]}$ is non-empty.

$$\dim \mathcal{M}_{[\mathcal{C}]} = \sigma - 1 = 10 - \dim \mathcal{C}.$$

Case of $\sigma = 1$.
There exists only one isomorphism class $[\mathcal{C}_{\mathrm{DK}}]$ with dimension 10.

$$\mathcal{P} \cong \mathbb{P}^2(\mathbb{F}_4),$$

$$\mathcal{C}_{\mathrm{DK}} := \langle\, L(\mathbb{F}_4) \mid L : \mathbb{F}_4\text{-rational lines}\rangle \quad \subset \quad 2^{\mathcal{P}}.$$

The weight enumerator of $\mathcal{C}_{\mathrm{DK}}$ is

$1 + 21z^5 + 210z^8 + 280z^9 + 280z^{12} + 210z^{13} + 21z^{16} + z^{21}$.

The 0-dimensional stratum $\mathcal{M}_{\mathrm{DK}}$ consists of a single point $[(X_{\mathrm{DK}}, \mathcal{L}_{\mathrm{DK}})]$, where X_{DK} is the resolution of

$$W^2 = X_0X_1X_2(X_0^3 + X_1^3 + X_2^3).$$

§3. Geometry of Splitting Curves and Codes

$G \in \mathcal{U}$.
We fix a bijection

$$\gamma : \mathcal{P} \xrightarrow{\sim} Z(dG) = \pi_G(\operatorname{Sing} Y_G).$$

Let $L \subset \mathbb{P}^2$ be a line.
L splits in $(X_G, \mathcal{L}_G)$,
$\iff |L \cap Z(dG)| \geq 3$,
$\iff |L \cap Z(dG)| = 5$.

Let $Q \subset \mathbb{P}^2$ be a non-singular conic curve.
Q splits in $(X_G, \mathcal{L}_G)$,
$\iff |Q \cap Z(dG)| \geq 6$, and
$\iff |Q \cap Z(dG)| = 8$.

The word $w(L) = \gamma^{-1}(L \cap Z(dG))$ of a splitting line L is of weight 5.
The word $w(Q) = \gamma^{-1}(Q \cap Z(dG))$ of a splitting non-singular conic curve Q is of weight 8.

A pencil $\mathcal{E}$ of cubic curves in $\mathbb{P}^2$ is called a *regular pencil* if the following hold:

- the base locus $\mathrm{Bs}(\mathcal{E})$ consists of distinct 9 points, and
- every singular member has only one ordinary node.

We say that a regular pencil $\mathcal{E}$ *splits in* $(X_G, \mathcal{L}_G)$ if every member of $\mathcal{E}$ splits in $(X_G, \mathcal{L}_G)$.

Let $\mathcal{E}$ be a regular pencil of cubic curves spanned by E_0 and E_∞. Let $H_0 = 0$ and $H_\infty = 0$ be the defining equations of E_0 and E_∞, respectively. Then $\mathcal{E}$ splits in $(X_G, \mathcal{L}_G)$ if and only if

$$Z(dG) = Z(d(H_0 H_\infty)),$$

or equivalently

$$Y_G \text{ and } Y_{H_0 H_\infty} \text{ are isomorphisc over } \mathbb{P}^2,$$

or equivalently

$$\exists c \in k^\times, \ \ \exists H \in \mathcal{V}, \ \ H_0 H_\infty = cG + H^2.$$

If $\mathcal{E}$ splits in $(X_G, \mathcal{L}_G)$, then $\mathrm{Bs}(\mathcal{E})$ is contained in $Z(dG)$, and

$$w(E_t) = \gamma^{-1}(\mathrm{Bs}(\mathcal{E}))$$

holds for every member E_t of $\mathcal{E}$. In particular, the word $w(E_t)$ is of weight 9.

Let A be a word of $\mathcal{C}_G$.

(i) We say that A is a *linear word* if $|A| = 5$.
(ii) Suppose $|A| = 8$. If A is *not* a sum of two linear words, then we say that A is a *quadratic word*.
(iii) Suppose $|A| = 9$. If A is neither a sum of three linear words nor a sum of a linear and a quadratic words, then we say that A is a *cubic word*.

By $C \mapsto w(C)$, we obtain the following bijections:

$$\{ \text{ lines splitting in } (X_G, \mathcal{L}_G) \} \cong \{ \text{ linear words in } \mathcal{C}_G \},$$

$$\{ \text{ non-singular conic curves splitting in } (X_G, \mathcal{L}_G) \} \cong \{ \text{ quadratic words in } \mathcal{C}_G \}.$$

By $\mathcal{E} \mapsto w(E_t) = \gamma^{-1}(\mathrm{Bs}(\mathcal{E}))$, we obtain the bijection

$$\{ \text{ regular pencils of cubic curves splitting in } (X_G, \mathcal{L}_G) \} \cong \{ \text{ cubic words in } \mathcal{C}_G \}.$$

§4. The Case of Artin Invariant 2

We start from a code $\mathcal{C} \subset 2^{\mathcal{P}}$ such that

- $\mathcal{P} = (1, 1, \ldots, 1) \in \mathcal{C}$, and
- $|w| \in \{0, 5, 8, 9, 12, 13, 16, 21\}$ for any $w \in \mathcal{C}$,

and construct the stratum $\mathcal{M}_{[\mathcal{C}]}$.

For simplicity, we assume that $\mathcal{C}$ is generated by $\mathcal{P}$ and words of weight 5 and 8.

We denote by $\mathcal{G}_{\mathcal{C}}$ the space of all injective maps

$$\gamma : \mathcal{P} \hookrightarrow \mathbb{P}^2$$

with the following properties:

(i) $\gamma(\mathcal{P}) = Z(dG)$ for some $G \in \mathcal{U}$ (that is, $\gamma \in \mathcal{G}$),

(ii) for a subset $A \subset \mathcal{P}$ of weight 5, $\gamma(A)$ is collinear if and only if $A \in \mathcal{C}$,

(iii) for a subset $A \subset \mathcal{P}$ of weight 8, $\gamma(A)$ is on a non-singular conic curve if and only if $A \in \mathcal{C}$ and A is not a sum of words of weight 5 in $\mathcal{C}$.

$$\mathcal{M} = PGL(3,k)\backslash\mathcal{G}/S_{21} \quad \supset$$
$$\mathcal{M}_{[\mathcal{C}]} = PGL(3,k)\backslash\mathcal{G}_{\mathcal{C}}/\operatorname{Aut}(\mathcal{C}).$$

Suppose that the isomorphism class of $(X_G, \mathcal{L}_G)$ is a point of $\mathcal{M}_{[\mathcal{C}]}$.
Let $\gamma \in \mathcal{G}_{\mathcal{C}}$ be the injective map such that $\gamma(\mathcal{P}) = Z(dG)$.

Then

$\operatorname{Aut}(X_G, \mathcal{L}_G) = \{\, g \in PGL(3,k) \mid g(Z(dG)) = Z(dG) \,\}$
is the stabilizer subgroup

$$\operatorname{Stab}(\langle\gamma\rangle) \quad \subset \quad \operatorname{Aut}(\mathcal{C})$$

of the projective equivalence class $\langle\gamma\rangle \in PGL(3,k)\backslash\mathcal{G}_{\mathcal{C}}$.

We carry out this construction of $\mathcal{M}_{[\mathcal{C}]}$ for the three isomorphism classes $[\mathcal{C}_A]$, $[\mathcal{C}_B]$, $[\mathcal{C}_C]$ of codes with dimension 9, that is, the Artin invariant 2.

Generators of the code $\mathcal{C}_A$

[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]

[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1]

[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1]

[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 0 0 1 0]

[0 0 0 0 0 0 0 1 1 0 0 1 0 0 1 0 0 0 1 0 0]

[0 0 0 0 0 0 1 0 1 0 1 0 0 1 0 0 0 1 0 0 0]

[0 0 0 0 1 1 0 0 0 0 0 1 0 1 0 0 0 1 1 1 1]

[0 0 0 1 0 1 0 0 0 0 1 0 0 1 1 1 0 0 1 0 1]

[0 0 1 0 0 1 0 0 1 0 0 1 0 0 1 1 0 1 1 0 0]

Generators of the code $\mathcal{C}_B$

[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]

[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1]

[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1]

[0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 0 0 1 0]

[0 0 0 0 0 0 0 1 1 0 0 1 0 0 1 0 0 0 1 0 0]

[0 0 0 0 0 1 1 0 0 0 0 1 0 1 0 0 0 1 0 0 0]

[0 0 0 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 1]

[0 0 1 1 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0]

[0 1 0 1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 0 1 0]

Generators of the code $\mathcal{C}_C$

[1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1]

[0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1]

[0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 1]

[0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 1]

[0 0 0 0 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0]

[0 0 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0]

[0 0 0 0 1 0 0 1 0 1 1 0 0 1 1 0 0 1 1 0 1]

[0 0 1 1 0 0 0 0 0 0 1 1 0 1 0 1 0 1 1 0 0]

[0 1 0 1 0 0 0 0 0 1 0 1 0 1 1 0 0 0 1 1 0]

The weight enumerators of these codes are as follows:

$$\mathcal{C}_A \ : \ 1 + z^{21} + 13(z^5 + z^{16}) + 106(z^8 + z^{13}) + 136(z^9 + z^{12}),$$

$$\mathcal{C}_B \ : \ 1 + z^{21} + 9(z^5 + z^{16}) + 102(z^8 + z^{13}) + 144(z^9 + z^{12}),$$

$$\mathcal{C}_C \ : \ 1 + z^{21} + 5(z^5 + z^{16}) + 130(z^8 + z^{13}) + 120(z^9 + z^{12}).$$

The numbers of linear, quadratic and cubic words in these codes, and the order of the automorphism group are given in the following table:

	linear	quadratic	cubic	$\vert\mathrm{Aut}(\mathcal{C})\vert$
$\mathcal{C}_A$	13	28	0	1152
$\mathcal{C}_B$	9	66	0	432
$\mathcal{C}_C$	5	120	0	23040

.

These codes are generated by $\mathcal{P}$ and linear and quadratic words.

For $T = A, B$ and C, the following hold.
(ω is the third root of unity, and $\bar{\omega} = \omega + 1$.)

The space $PGL(3, k)\backslash\mathcal{G}_T$ has exactly two connected components, both of which are isomorphic to

$$\operatorname{Spec} k[\lambda, 1/(\lambda^4 + \lambda)] = \mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}.$$

Let $N_T \subset \operatorname{Aut}(\mathcal{C}_T)$ be the subgroup of index 2 that preserves the connected components, and let Γ_T be the image of N_T in

$$\operatorname{Aut}(\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}).$$

The moduli curve

$$\mathcal{M}_T = (\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\})/\Gamma_T$$

is isomorphic to a punctured affine line

$$\operatorname{Spec} k[J_T, 1/J_T] = \mathbb{A}^1 \setminus \{0\}.$$

The punctured origin $J_T = 0$ corresponds to the Dolgachev-Kondo point.

The action of Γ_T on $\mathbb{A}^1 \setminus \{0, 1, \omega, \bar{\omega}\}$ is free. Hence the order of $\operatorname{Stab}(\langle\gamma\rangle) \subset \operatorname{Aut}(\mathcal{C}_T)$ is constant on $PGL(3, k)\backslash\mathcal{G}_T$. We have an exact sequence

$$1 \;\to\; \operatorname{Aut}(X, \mathcal{L}) \;\to\; N_T \;\to\; \Gamma_T \;\to\; 1.$$

The case $\boldsymbol{A}$:

$$\Gamma_A = \left\{\lambda, \lambda+1, \frac{1}{\lambda}, \frac{1}{\lambda+1}, \frac{\lambda}{\lambda+1}, \frac{\lambda+1}{\lambda}\right\} \cong S_3,$$

$$J_A = \frac{(\lambda^2+\lambda+1)^3}{\lambda^2\,(\lambda+1)^2},$$

$$GA[\lambda] := X_0X_1X_2\,(X_0+X_1+X_2)\cdot$$
$$\left({X_0}^2+{X_1}^2+\left(\lambda^2+\lambda\right){X_2}^2+X_0X_1+X_1X_2+X_2X_0\right).$$

The family

$$W^2 = GA[\lambda]$$

is the universal family of polarized supersingular $K3$ surfaces over the λ-line.

For $\alpha \in k \setminus \{0, 1, \omega, \bar{\omega}\}$, $\mathrm{Aut}(X_{GA[\alpha]}, \mathcal{L}_{GA[\alpha]})$ is equal to the group

$$\left\{ \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix} a \\ b \end{matrix} \\ \hline 0 & 0 & 1 \end{array}\right] \in PGL(3,k) \;\middle|\; \begin{array}{l} A \in GL(2, \mathbb{F}_2), \\ a, b \in \{0, 1, \alpha, \alpha+1\} \end{array} \right\}.$$

Γ_B is isomorphic to the alternating group A_4.
$J_B = (\lambda+\omega)^{12}/\big(\lambda^3(\lambda+1)^3(\lambda+\bar{\omega})^3\big)$.

$$GB[\lambda] = X_0X_1X_2\,(X_0+X_1+X_2)\cdot \big((\bar{\omega}\lambda+\omega)\,{X_0}^2+\bar{\omega}\,{X_1}^2+\omega\lambda\,{X_2}^2+ (\lambda+1)\,X_0X_1+(\bar{\omega}\lambda+\omega)X_1X_2+(\lambda+1)\,X_2X_0\big).$$

Γ_C is the group of affine transformations of an affine line over $\mathbb{F}_4$.
$J_C = (\lambda^4+\lambda)^3$.

$$GC[\lambda] = X_0X_1X_2\left(X_0^3+X_1^3+X_2^3\right)+(\lambda^4+\lambda)X_0^3X_1^3.$$

The orders of the groups above are given as follows.

T	$\lvert \mathrm{Aut}(\mathcal{C}_T)\rvert$	$=$	2	$\times$	$\lvert\Gamma_T\rvert$	$\times$	$\lvert \mathrm{Aut}(X,\mathcal{L})\rvert$
A	1152	$=$	2	$\times$	6	$\times$	96
B	432	$=$	2	$\times$	12	$\times$	18
C	23040	$=$	2	$\times$	12	$\times$	960

§5. Cremona transformations

Let $\Sigma = \{p_1, \ldots, p_6\} \subset Z(dG)$ be a subset with $|\Sigma| = 6$ satisfying the following:

- **no three points of Σ are collinear, and**
- **for each i, the non-singular conic curve Q_i containing $\Sigma \setminus \{p_i\}$ satisfies $Q_i \cap Z(dG) = \Sigma \setminus \{p_i\}$.**

Let $\beta : S \to \mathbb{P}^2$ be the blowing up at the points in Σ, and

let $\beta' : S \to \mathbb{P}^2$ be the blowing down of the strict transforms Q'_i of the conic curves Q_i.

The birational map

$$c := \beta' \circ \beta^{-1} \;:\; \mathbb{P}^2 \cdots \longrightarrow \mathbb{P}^2$$

is called the *Cremona transformation with the center* Σ.

There exists $G' \in \mathcal{U}$ such that

$$c(Z(dG) \setminus \Sigma) \cup \{\beta'(Q'_i) \mid i = 1, \ldots, 6\} = Z(dG').$$

Obviously, X_G and $X_{G'}$ are isomorphic.

But $(X_G, \mathcal{L}_G)$ and $(X_{G'}, \mathcal{L}_{G'})$ may fail to be isomorphic.

A curve $D \subset \mathcal{M}_T \times \mathcal{M}_{T'}$ is called an *isomorphism correspondence* if, for any pair

$$([X, \mathcal{L}], [X', \mathcal{L}']) \quad \in \quad D,$$

the $K3$ surfaces X and X' are isomorphic as non-polarized surfaces.

Using Cremona transformations, we obtain an example of non-trivial isomorphism correspondences.

Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be polarized supersingular $K3$ surfaces of type ($\sharp$) with Artin invariant 2, and let J_T and $J_{T'}$ be their J-invariants.

If $T = T' = A$ and

$$1 + J_A\, J'_A + {J_A}^2 {J'_A}^2 + {J_A}^2 {J'_A}^3 + {J_A}^3 {J'_A}^2 = 0,$$

then X and X' are isomorphic.

If $T = A$ and $T' = B$ and

$$J_B + J_A\, J_B + J_A\, {J_B}^2 + {J_A}^2 J_B + {J_A}^4 = 0,$$

then X and X' are isomorphic.

The isomorphism correspondence

$$1 + J_A J'_A + {J_A}^2 {J'_A}^2 + {J_A}^2 {J'_A}^3 + {J_A}^3 {J'_A}^2 = 0$$

intersects with the diagonal $\Delta_A \subset \mathcal{M}_A \times \mathcal{M}_A$ at two points $(J_A, J'_A) = (\omega, \omega)$ and $(\bar{\omega}, \bar{\omega})$.

At these points, the automorphism group $\mathrm{Aut}(X)$ of the supersingular $K3$ surface *jumps*.

Do all isomorphism correspondences come from Cremona transformations?