Singularity of discriminant varieties in characteristic 2 and 3

Ichiro Shimada

(Hokkaido University, Sapporo, JAPAN)

We work over an algebraically closed field k.

§1. An Example

Let $E \subset \mathbb{P}^2$ be a smooth cubic plane curve.
We fix a flex point $O \in E$, and consider the elliptic curve (E, O).

Let $(\mathbb{P}^2)^\vee$ be the dual projective plane, and let $E^\vee \subset (\mathbb{P}^2)^\vee$ be the dual curve of E.
We denote by

$$\phi : E \longrightarrow E^\vee$$

the morphism that maps a point $P \in E$ to the tangent line $T_P(E) \in E^\vee$ to E at P.

Suppose that $\mathrm{char}(k) \neq 2$.

Then $E^\vee$ is of degree 6, and ϕ is birational.
The singular points $\mathrm{Sing}(E^\vee)$ of $E^\vee$ are in one-to-one correspondence with the flex points of E via ϕ.
On the other hand, the flex points of E are in one-to-one correspondence with the 3-torsion subgroup $E[3]$ of (E, O).

We have

$$E[3] \cong \begin{cases} \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} & \text{if } \mathrm{char}(k) \neq 3, \\ \mathbb{Z}/3\mathbb{Z} & \text{if } \mathrm{char}(k) = 3 \text{ and } E \text{ is not supersingular}, \\ 0 & \text{if } \mathrm{char}(k) = 3 \text{ and } E \text{ is supersingular}. \end{cases}$$

Then we have

$$\mathrm{Sing}(E^\vee) \text{ consists of } \begin{cases} 9 \text{ points of type } A_2 & \text{if } \mathrm{char}(k) \neq 3, \\ 3 \text{ points of type } E_6 & \text{if } \mathrm{char}(k) = 3 \text{ and } E \text{ is not s-singular}, \\ 1 \text{ point of type } T_3 & \text{if } \mathrm{char}(k) = 3 \text{ and } E \text{ is s-singular}. \end{cases}$$

type	defining equation	normalization
A_2	$x^2 + y^3 = 0$	$t \mapsto (t^3, t^2)$
E_6	$x^4 + y^3 + x^2y^2 = 0$ or $x^4 + y^3 = 0$	$t \mapsto (t^4, t^3 + t^5)$ or $t \mapsto (t^4, t^3)$
T_3	$x^{10} + y^3 + x^6y^2 = 0$	$t \mapsto (t^{10}, t^3 + t^{11})$

Remark. When $\mathrm{char}(k) \neq 3$, then the two types of the E_6-singular point are isomorphic.

Suppose that char$(k) = 2$.

Then $E^\vee$ is a smooth cubic curve, and $\phi : E \to E^\vee$ is a purely inseparable finite morphism of degree 2.

If E is defined by

$$x^3 + y^3 + z^3 + \; a\, xyz = 0,$$

then $E^\vee$ is defined by

$$\xi^3 + \eta^3 + \zeta^3 + \; a^2\, \xi\eta\zeta = 0,$$

where $[\xi : \eta : \zeta]$ are the homogeneous coordinates dual to $[x : y : z]$ (C. T. C. Wall).

§2. Introduction

The aim of this talk is to investigate the singularity of the *discriminant variety* of a smooth projective variety $X \subset \mathbb{P}^m$ in arbitrary characteristics.

It turns out that the nature of the singularity differs according to the following cases:

- $\operatorname{char}(k) > 3$ **or** $\operatorname{char}(k) = 0$ **(the classical case),**
- $\operatorname{char}(k) = 3$**,**
- $\operatorname{char}(k) = 2$ **and** $\dim X$ **is even,**
- $\operatorname{char}(k) = 2$ **and** $\dim X$ **is odd (I could not analyze the singularity in this case).**

§3. Definition of the discriminant variety

We need some preparation.

Let V be a variety, and let E and F be vector bundles on V with rank e and f, respectively. For a bundle homomorphism $\sigma : E \to F$, we define the *degeneracy subscheme* of σ to be the closed subscheme of V defined locally on V by all r-minors of the $f \times e$-matrix expressing σ, where $r := \min(e, f)$.

Let V and W be smooth varieties, and let $\phi : V \to W$ be a morphism.

The *critical subscheme* of ϕ is the degeneracy subscheme of the homomorphism $d\phi : T(V) \to \phi^* T(W)$.

Suppose that $\dim V \leq \dim W$. We say that ϕ is a *closed immersion formally at* $P \in V$ if $d_P\phi : T_P(V) \to T_{\phi(P)}(W)$ is injective, or equivalently, the induced homomorphism $(\mathcal{O}_{W,\phi(P)})^\wedge \to (\mathcal{O}_{V,P})^\wedge$ is surjective.

When $\dim V \leq \dim W$, a point $P \in V$ is in the support of the critical subscheme of ϕ if and only if ϕ is not a closed immersion formally at P.

Let $X \subset \mathbb{P}^m$ be a smooth projective variety with $\dim X = n > 0$. We put

$$\mathcal{L} := \mathcal{O}_X(1).$$

We assume that X is not contained in any hyperplane of $\mathbb{P}^m$. Then the dual projective space

$$\mathbb{P} := (\mathbb{P}^m)^\vee$$

is regarded as a linear system $|M|$ of divisors on X, where M is a linear subspace of $H^0(X, \mathcal{L})$.

Let $\mathcal{D} \subset X \times \mathbb{P}$ be the universal family of the hyperplane sections of X, which is smooth of dimension $n + m - 1$. The support of $\mathcal{D}$ is equal to

$$\{ (p, H) \in X \times \mathbb{P} \mid p \in H \cap X \}.$$

Let $\mathcal{C} \subset \mathcal{D}$ be the critical subscheme of the second projection $\mathcal{D} \to \mathbb{P}$. It turns out that $\mathcal{C}$ is smooth of dimension $m - 1$. The support of $\mathcal{C}$ is equal to

$$\{ (p, H) \in \mathcal{D} \mid H \cap X \text{ is singular at } p \}.$$

Let $\mathcal{E} \subset \mathcal{C}$ be the critical subscheme of the second projection $\pi_2 : \mathcal{C} \to \mathbb{P}$. The support of $\mathcal{E}$ is equal to

$$\{ (p, H) \in \mathcal{C} \mid \text{the Hessian of } H \cap X \text{ at } p \text{ is degenerate} \}.$$

The image of $\pi_2 : \mathcal{C} \to \mathbb{P}$ is called the *discriminant variety* of $X \subset \mathbb{P}^m$.

We will study the singularity of the discriminant variety by investigating the morphism $\pi_2 : \mathcal{C} \to \mathbb{P}$ at a point of the critical subscheme $\mathcal{E}$

Let $P = (p, H) \in X \times \mathbb{P}$ be a point of $\mathcal{E}$, so that $H \cap X$ has a degenerate singularity at p.
Let $\Lambda \subset \mathbb{P}$ be a general plane passing through the point $\pi_2(P) = H \in \mathbb{P}$.
We denote by $C_\Lambda \subset \mathcal{C}$ the pull-back of Λ by π_2, and by $\pi_\Lambda : C_\Lambda \to \Lambda$ the restriction of π_2 to C_Λ.

- **What type of singular point does the plane curve $\Lambda \cap \pi_2(\mathcal{C})$ have at H?**
- **Does there exist any normal form for the morphism $\pi_\Lambda : C_\Lambda \to \Lambda$ at P?**

§4. The scheme $\mathcal{E}$

For $P = (p, H) \in \mathcal{C}$, we have the *Hessian*

$$H_P : T_p(X) \times T_p(X) \to k$$

of the hypersurface singularity $p \in H \cap X \subset X$. If $H \cap X$ is defined locally by $f = 0$ in X, then H_P is expressed by the symmetric matrix

$$M_P := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right).$$

Over $\mathcal{C}$, we can define the *universal Hessian*

$$\mathcal{H} : \pi_1^* T(X) \otimes \pi_1^* T(X) \to \widetilde{\mathcal{L}} := \pi_1^* \mathcal{L} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}}(1),$$

where $\pi_1 : \mathcal{C} \to X$ and $\pi_2 : \mathcal{C} \to \mathbb{P}$ are the projections.

The critical subscheme $\mathcal{E}$ of $\pi_2 : \mathcal{C} \to \mathbb{P}$ coincides with the degeneracy subscheme of the homomorphism $\pi_1^* T(X) \to \pi_1^* T(X)^\vee \otimes \widetilde{\mathcal{L}}$ induced from $\mathcal{H}$.

From this proposition, we see that $\mathcal{E} \subset \mathcal{C}$ is either empty or of codimension ≤ 1. In positive characteristics, we sometimes have $\mathcal{E} = \mathcal{C}$.

Example.

Suppose that $\mathrm{char}(k) = 2$. Then the Hessian H_P is not only symmetric but also anti-symmetric, because we have

$$M_P = {}^tM_P = -{}^tM_P \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x_i^2}(p) = 0.$$

On the other hand, the rank of an anti-symmetric bilinear form is always even. Hence we obtain the following:

If $\mathrm{char}(k) = 2$ and $\dim X$ is odd, then $\mathcal{C} = \mathcal{E}$.

Example.

Let $X \subset \mathbb{P}^{n+1}$ be the Fermat hypersurface of degree $q+1$, where q is a power of the characteristic of the base field k. Then, at every point (p, H) of $\mathcal{C}$, the singularity of $H \cap X$ at p is always degenerate. In particular, we have $\mathcal{C} = \mathcal{E}$.

The discriminant variety of a hypersurface is the dual hypersurface. The dual hypersurface $X^\vee$ of the Fermat hypersurface X of degree $q + 1$ is isomorphic to the Fermat hypersurface of degree $q + 1$, and the natural morphism $X \to X^\vee$ is purely inseparable of degree q^n.

§5. The quotient morphism by an integrable tangent subbundle

In order to describe the situation in characteristic 2 and 3, we need the notion of the *quotient morphism by an integrable tangent subbundle.*

In this section, we assume that k is of characteristic $p > 0$. Let V be a smooth variety.

A subbundle $\mathcal{N}$ of $T(V)$ is called *integrable* if $\mathcal{N}$ is closed under the p-th power operation and the bracket product of Lie.

The following is due to Seshadri:

Let $\mathcal{N}$ be an integrable subbundle of $T(V)$. Then there exists a unique morphism $q : V \to V^{\mathcal{N}}$ with the following properties;

(i) q induces a homeomorphism on the underlying topological spaces,

(ii) q is a radical covering of height 1, and

(iii) the kernel of $dq : T(V) \to q^* T(V^{\mathcal{N}})$ is equal to $\mathcal{N}$.

Moreover, the variety $V^{\mathcal{N}}$ is smooth, and the morphism q is finite of degree p^r, where $r = \operatorname{rank} \mathcal{N}$.

For an integrable subbundle $\mathcal{N}$ of $T(V)$, the morphism $q : V \to V^{\mathcal{N}}$ is called the *quotient morphism* by $\mathcal{N}$.

The construction of $q : V \to V^{\mathcal{N}}$.

Let V be covered by affine schemes $U_i := \operatorname{Spec} A_i$. We put

$$A_i^{\mathcal{N}} := \{\ f \in A_i \ \mid\ Df = 0 \ \text{ for all } \ D \in \Gamma(U_i, \mathcal{N}\,)\ \}.$$

Then the natural morphisms $\operatorname{Spec} A_i \to \operatorname{Spec} A_i^{\mathcal{N}}$ patch together to form $q : V \to V^{\mathcal{N}}$.

Let $\phi : V \to W$ be a morphism from a smooth variety V to a smooth variety W. Suppose that the kernel $\mathcal{K}$ of $d\phi : T(V) \to \phi^* T(W)$ is a subbundle of $T(V)$, which is always the case if we restrict ϕ to a Zariski open dense subset of V. Then $\mathcal{K}$ is integrable, and ϕ factors through the quotient morphism by $\mathcal{K}$.

The case where $\mathrm{char}(k) = 2$ and $\dim X$ is odd.

Suppose that $\mathrm{char}(k) = 2$ and $\dim X$ is odd, so that $\mathcal{C} = \mathcal{E}$ holds. Let $\mathcal{K}$ be the kernel of the homomorphism $\pi_1^* T(X) \to \pi_1^* T(X)^\vee \otimes \widetilde{\mathcal{L}}$ induced from the universal Hessian $\mathcal{H}$, which is of rank ≥ 1 at the generic point of every irreducible component of $\mathcal{C}$. Then the subsheaf

$$\mathcal{K} \quad\subset\quad \pi_1^* T(X) \quad\subset\quad \pi_1^* T(X) \oplus \pi_2^* T(\mathbb{P}) = T(X \times \mathbb{P})|\mathcal{C}$$

is in fact contained in $T(\mathcal{C}) \subset T(X \times \mathbb{P})|\mathcal{C}$.

Let $U \subset \mathcal{C}$ be a Zariski open dense subset of $\mathcal{C}$ over which $\mathcal{K}$ is a subbundle of $T(\mathcal{C})$. Then the restriction of π_2 to U factors through the quotient morphism by $\mathcal{K}$. In particular, the projection $\mathcal{C} \to \mathbb{P}$ is inseparable onto its image.

§6. The case where char$(k) \neq 2$

Suppose that the characteristic of k is *not* 2.

Let (p, H) be a point of $\mathcal{E}$, so that the divisor $H \cap X$ has a degenerate singularity at p.

We say that the singularity of $H \cap X$ at p is *of type* A_2 if there exists a formal parameter system $(x_1, \dots, x_n)$ of X at p such that $H \cap X$ is given as the zero of the function of the form

$$x_1^2 + \cdots + x_{n-1}^2 + x_n^3 + \text{(higher degree terms)}.$$

We then put

$$\mathcal{E}^{A_2} := \left\{ \ (p, H) \in \mathcal{E} \ \middle| \ \begin{array}{l} \text{the singularity of } H \cap X \text{ at} \\ p \text{ is of type } A_2 \end{array} \right\}.$$

We also put

$$\mathcal{E}^{\mathrm{sm}} := \left\{ \ (p, H) \in \mathcal{E} \ \middle| \ \begin{array}{l} \mathcal{E} \text{ is smooth of dimension} \\ m-2 \text{ at } (p, H) \end{array} \right\}.$$

We see that $\mathcal{E}$ is irreducible and the loci $\mathcal{E}^{A_2}$ and $\mathcal{E}^{\mathrm{sm}}$ are dense in $\mathcal{E}$ if the linear system $|M|$ is sufficiently ample; e.g., if the evaluation homomorphism

$$v_p^{[3]} \ : \ M \ \longrightarrow \ \mathcal{L}_p / m_p^4 \mathcal{L}_p$$

is surjective at every point p of X, where $m_p \subset \mathcal{O}_{X,p}$ is the maximal ideal.

The case where $\mathrm{char}(k) > 3$ or $\mathrm{char}(k) = 0$.

In this case, we have the following:

Let $P = (p, H)$ be a point of $\mathcal{E}$. The following two conditions are equivalent:

- **$P \in \mathcal{E}^{A_2}$,**
- **$P \in \mathcal{E}^{\mathrm{sm}}$, and the projection $\mathcal{E} \to \mathbb{P}$ is a closed immersion formally at P.**

Moreover, if these conditions are satisfied, then the curve $C_\Lambda = \pi_2^{-1}(\Lambda)$ is smooth at P, and $\pi_\Lambda : C_\Lambda \to \Lambda$ has a critical point of A_2-type at P; that is,

$$\pi_\Lambda^* u = a\, t^2 + b\, t^3 + (\text{terms of degree} \geq 4) \quad \text{and}$$
$$\pi_\Lambda^* v = c\, t^2 + d\, t^3 + (\text{terms of degree} \geq 4)$$

with $ad - bc \neq 0$ hold for a formal parameter system (u, v) of Λ at $\pi(P) = H$ and a formal parameter t of C_Λ at P.

By suitable choice of formal parameters, we have

$$\pi_\Lambda^* u = t^3, \quad \pi_\Lambda^* v = t^2,$$

and the plane curve $\pi_2(\mathcal{C}) \cap \Lambda \subset \Lambda$ is defined by $u^2 - v^3 = 0$ locally at $H \in \Lambda$.

The case where $\mathrm{char}(k) = 3$.

In this case, $P \in \mathcal{E}^{A_2}$ does not necessarily imply $P \in \mathcal{E}^{\mathrm{sm}}$. Our main results are as follows.

(I) Let $\varpi : \mathcal{E}^{\mathrm{sm}} \to \mathbb{P}$ be the projection. Then the kernel $\mathcal{K}$ of $d\varpi : T(\mathcal{E}^{\mathrm{sm}}) \to \varpi^* T(\mathbb{P})$ is a subbundle of $T(\mathcal{E}^{\mathrm{sm}})$ with rank 1. Hence ϖ factors as

$$\mathcal{E}^{\mathrm{sm}} \quad \xrightarrow{q} \quad (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}} \quad \xrightarrow{\tau} \quad \mathbb{P},$$

where $\mathcal{E}^{\mathrm{sm}} \to (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}}$ is the quotient morphism by $\mathcal{K}$.

(II) Suppose that P is a point of $\mathcal{E}^{\mathrm{sm}} \cap \mathcal{E}^{A_2}$. Then $\tau : (\mathcal{E}^{\mathrm{sm}})^{\mathcal{K}} \to \mathbb{P}$ is a closed immersion formally at $q(P)$. Moreover the curve C_Λ is smooth at P, and $\pi_\Lambda : C_\Lambda \to \Lambda$ has a critical point of E_6-type at P; i. e.,

$$\pi_\Lambda^* u \;=\; a\,t^3 + b\,t^4 + (\text{terms of degree} \geq 5) \quad \text{and}$$
$$\pi_\Lambda^* v \;=\; c\,t^3 + d\,t^4 + (\text{terms of degree} \geq 5)$$

with $ad - bc \neq 0$ hold.

By suitable choice of formal parameters, we have either

$$(\pi_\Lambda^* u = t^3,\ \pi_\Lambda^* v = t^4) \ \text{ or } \ (\pi_\Lambda^* u = t^3 + t^5,\ \pi_\Lambda^* v = t^4).$$

The plane curve $\pi_2(\mathcal{C}) \cap \Lambda \subset \Lambda$ is defined at $H \in \Lambda$ by either

$$x^4 + y^3 = 0 \ \text{ or } \ x^4 + y^3 + x^2 y^2 = 0.$$

In the case of a projective plane curve (i.e., the case where $(n, m) = (1, 2)$), the locus $\mathcal{E}^{\mathrm{sm}}$ is always empty. In this case, we have the following:

(III) Suppose that $(n, m) = (1, 2)$, and that the projection $\mathcal{C} \to \mathbb{P}$ is separable onto its image. (This assumption excludes the case of, for example, the Fermat curve of degree $3^\nu + 1$.)

Then $\dim \mathcal{E} = 0$. Let $P = (p, H)$ be a point of $\mathcal{E}$. Then the length of $\mathcal{O}_{\mathcal{E},P}$ is divisible by 3. If $P \in \mathcal{E}^{A_2}$ (that is, H is an ordinary flex tangent line to X at p), then, with appropriate choice of formal parameters, the formal completion of $\pi_2 : \mathcal{C} \to \mathbb{P}$ at P is given by

$$T_l \quad : \quad t \ \mapsto \ (\, t^{3l+1},\, t^3 + t^{3l+2}\,),$$

where $l := \operatorname{length} \mathcal{O}_{\mathcal{E},P}/3$.

§7. The case where $\mathrm{char}(k) = 2$ and $\dim X$ is even.

For simplicity, we assume that $|M|$ is so ample that the evaluation homomorphism

$$v_p^{[4]} : M \to \mathcal{L}_p / m_p^5 \mathcal{L}_p$$

is surjective at every point p of X.

Then $\mathcal{E}$ is an irreducible divisor of $\mathcal{C}$, and is written as $2\mathcal{R}$, where $\mathcal{R}$ is a reduced divisor of $\mathcal{C}$.

We denote by $\mathcal{R}^{\mathrm{sm}}$ the smooth locus of $\mathcal{R}$, and by $\varpi : \mathcal{R}^{\mathrm{sm}} \to \mathbb{P}$ the projection.
Then we have the following:

(I) The kernel $\mathcal{K}$ of $d\varpi : T(\mathcal{R}^{\mathrm{sm}}) \to \varpi^* T(\mathbb{P})$ is a subbundle of $T(\mathcal{R}^{\mathrm{sm}})$ with rank 2.
In particular, the projection ϖ factors through a finite inseparable morphism of degree 4.

(II) Let $P = (p, H)$ be a general point of $\mathcal{R}$.
Let $L \subset \mathbb{P}$ be a general linear subspace of dimension 3 containing Λ. We put $S_L := \pi_2^{-1}(L) \subset \mathcal{C}$.
Then S_L is smooth of dimension 2 at P, and C_Λ is a curve on S_L that has an ordinary cusp at P.
Let $\nu : \widetilde{C}_\Lambda \to C_\Lambda$ be the normalization of C_Λ at P, and let z be a formal parameter of $\widetilde{C}_\Lambda$ at the inverse image $P' \in \widetilde{C}_\Lambda$ of P. Then the formal completion at P' of $\pi_\Lambda \circ \nu : \widetilde{C}_\Lambda \to \Lambda$ is written as

$$(\pi_\Lambda \circ \nu)^* u \;=\; a\, z^4 \;+\; (\text{terms of degree} \geq 6) \quad \text{and}$$

$$(\pi_\Lambda \circ \nu)^* v \;=\; b\, z^4 \;+\; (\text{terms of degree} \geq 6)$$

for some $a, b \in k$, where (u, v) is a formal parameter system of Λ at H.

Hence the plane curve singularity of $\pi_2(\mathcal{C}) \cap \Lambda$ at H is *not* a rational double point any more.