Toppology and arithmetic of maximizing sextics

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- By a lattice, we mean a finitely generated free $\mathbb{Z}$-module Λ equipped with a non-degenerate symmetric bilinear form
$$\Lambda \times \Lambda \quad \to \quad \mathbb{Z}.$$
- A lattice Λ is said to be *even* if $(v, v) \in 2\mathbb{Z}$ for any $v \in \Lambda$.

§1. Singular $K3$ surfaces

A smooth projective surface X is called a $K3$ surface if $K_X \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$.

For a $K3$ surface X defined over a field k, we denote by $\mathrm{NS}(X)$ the Néron-Severi lattice of $X \otimes \bar{k}$, where $\bar{k}$ is the algebraic closure of k; that is, $\mathrm{NS}(X)$ is the lattice of numerical equivalence classes of divisors on $X \otimes \bar{k}$ with the intersection pairing.

Definition. A $K3$ surface X defined over a field of characteristic 0 is said to be *singular* if $\mathrm{rank}(\mathrm{NS}(X))$ attains the possible maximum 20.

Shioda and Inose showed that every singular $K3$ surface X is defined over a number field F.

Let X be a singular $K3$ surface defined over a number field F. We denote by $\mathrm{Emb}(F, \mathbb{C})$ the set of embeddings of F into $\mathbb{C}$, and investigate the transcendental lattice

$$T(X^\sigma) := (\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X^\sigma, \mathbb{Z}))^\perp$$

for each embedding $\sigma \in \mathrm{Emb}(F, \mathbb{C})$, where X^σ is the complex $K3$ surface $X \otimes_{F,\sigma} \mathbb{C}$. Note that each $T(X^\sigma)$ is a positive-definite even lattice of rank 2.

We put

$$\mathcal{M} := \left\{ \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \;\middle|\; \begin{array}{l} a, b, c \in \mathbb{Z},\ a > 0,\ c > 0, \\ 4ac - b^2 > 0 \end{array} \right\},$$

on which $g \in GL_2(\mathbb{Z})$ acts by $M \mapsto {}^t gMg$. We denote the set of isomorphism classes of even, positive-definite lattices (resp. *oriented* lattices) of rank 2 by

$$\mathcal{L} := \mathcal{M}/\ GL_2(\mathbb{Z}) \quad (\text{resp.} \quad \widetilde{\mathcal{L}} := \mathcal{M}/\ SL_2(\mathbb{Z})\ ).$$

Let S be a complex singular $K3$ surface. By the Hodge decomposition $T(S) \otimes \mathbb{C} = \mathrm{H}^{2,0}(S) \oplus \mathrm{H}^{0,2}(S)$, we can define a canonical orientation on $T(S)$. We denote by $\widetilde{T}(S)$ the *oriented* transcendental lattice of S.

Theorem (Shioda and Inose). The map $S \mapsto \widetilde{T}(S)$ induces a bijection from the set of isomorphism classes of complex singular $K3$ surfaces to the set $\widetilde{\mathcal{L}}$.

Definition. Let Λ be an even lattice. Then Λ is canonically embedded into

$$\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z})$$

as a subgroup of finite index, and we have a natural symmetric bilinear form

$$\Lambda^\vee \times \Lambda^\vee \to \mathbb{Q}$$

that extends the symmetric bilinear form on Λ. The finite abelian group

$$D_\Lambda := \Lambda^\vee / \Lambda,$$

together with the natural quadratic form

$$q_\Lambda : D_\Lambda \to \mathbb{Q}/2\mathbb{Z}$$

is called the *discriminant form of* Λ.

Proposition. Suppose that an even lattice M is embedded into an even unimodular lattice L primitively. Let N denote the orthogonal complement of M in L. Then we have

$$(D_M, q_M) \cong (D_N, -q_N)$$

Definition. Two lattices

$$\lambda : \Lambda \times \Lambda \to \mathbb{Z} \quad \text{and} \quad \lambda' : \Lambda' \times \Lambda' \to \mathbb{Z}$$

are said to be *in the same genus* if

$$\lambda \otimes \mathbb{Z}_p \ : \ \Lambda \otimes \mathbb{Z}_p \times \Lambda \otimes \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{and}$$
$$\lambda' \otimes \mathbb{Z}_p \ : \ \Lambda' \otimes \mathbb{Z}_p \times \Lambda' \otimes \mathbb{Z}_p \to \mathbb{Z}_p$$

are isomorphic for any p including $p = \infty$, where $\mathbb{Z}_\infty = \mathbb{R}$.

We have the following:

Theorem (Nikulin). Two even lattices of the same rank are in the same genus if and only if they have the same signature and their discriminant forms are isomorphic.

Proposition. Let X be a singular $K3$ surface defined over a number field F. For $\sigma, \sigma' \in \mathrm{Emb}(F, \mathbb{C})$, the lattices $T(X^\sigma)$ and $T(X^{\sigma'})$ are in the same genus.

This follows from Nikulin's theorem. We have

$$\mathrm{NS}(X) \;\cong\; \mathrm{NS}(X^\sigma) \;\cong\; \mathrm{NS}(X^{\sigma'}).$$

Since $\mathrm{H}^2(X^\sigma, \mathbb{Z})$ is unimodular, the discriminant form of $T(X^\sigma)$ is isomorphic to (-1) times the discriminant form of $\mathrm{NS}(X^\sigma)$:

$$(D_{T(X^\sigma)}, q_{T(X^\sigma)}) \;\cong\; (D_{\mathrm{NS}(X^\sigma)}, -q_{\mathrm{NS}(X^\sigma)}).$$

The same holds for $T(X^{\sigma'})$. Hence $T(X^\sigma)$ and $T(X^{\sigma'})$ have the isomorphic discriminant forms.

Theorem (S.- and Schütt). Let $\mathcal{G} \subset \mathcal{L}$ be a genus of even positive-definite lattices of rank 2, and let $\widetilde{\mathcal{G}} \subset \widetilde{\mathcal{L}}$ be the pull-back of $\mathcal{G}$ by the natural projection $\widetilde{\mathcal{L}} \to \mathcal{L}$. Then there exists a singular $K3$ surface X defined over a number field F such that the set

$$\{\, \widetilde{T}(X^\sigma) \mid \sigma \in \mathrm{Emb}(F, \mathbb{C}) \,\} \subset \widetilde{\mathcal{L}}$$

coincides with the oriented genus $\widetilde{\mathcal{G}}$.

We denote by $\mathrm{Emb}(\mathbb{C}, \mathbb{C})$ the set of embeddings $\sigma : \mathbb{C} \hookrightarrow \mathbb{C}$ of the complex number field $\mathbb{C}$ into itself. Two complex varieties X and X' are said to be *conjugate* if there exists $\sigma \in \mathrm{Emb}(\mathbb{C}, \mathbb{C})$ such that X' is isomorphic over $\mathbb{C}$ to

$$X^\sigma := X \otimes_{\mathbb{C},\sigma} \mathbb{C}.$$

Corollary. Let X and X' be complex singular $K3$ surfaces. If their transcendental lattices are in the same genus, then they are conjugate.

§2. Non-homeomorphic conjugate varieties

It is obvious from the definition that conjugate varieties are homeomorphic in Zariski topology.

Problem. How about in the classical complex topology?

Example. The betti numbers of a smooth projective complex variety X are "algebraic", that is,

$$b_i(X) = b_i(X^\sigma) \quad \text{for any } \sigma \in \mathrm{Emb}(\mathbb{C}, \mathbb{C}),$$

in virtue of the theory of étale cohomology groups.

We have the following:

Example (Serre (1964)). There exist conjugate smooth projective varieties X and X^σ such that their topological fundamental groups are *not* isomorphic:

$$\pi_1(X) \not\cong \pi_1(X^\sigma).$$

In particular, X and X^σ are not homotopically equivalent.

Grothendieck's dessins d'enfant (1984).
Let $f : C \longrightarrow \mathbb{P}^1$ be a finite covering defined over $\overline{\mathbb{Q}}$ branching only at the three points $0, 1, \infty \in \mathbb{P}^1$. For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, consider the conjugate covering

$$f^\sigma : C^\sigma \longrightarrow \mathbb{P}^1.$$

Then f and f^σ are topologically distinct in general.

Belyi's theorem asserts that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of topological types of the covering of $\mathbb{P}^1$ branching only at $0, 1, \infty$ is faithful.

Other examples of non-homeomorphic conjugate varieties.

- Abelson: Topologically distinct conjugate varieties with finite fundamental group.
  Topology 13 (1974).
- Artal Bartolo, Carmona Ruber, Cogolludo Agustín: Effective invariants of braid monodromy.
  Trans. Amer. Math. Soc. 359 (2007).
- S.-: On arithmetic Zariski pairs in degree 6.
  arXiv:math/0611596
- S.-: Non-homeomorphic conjugate complex varieties.
  arXiv:math/0701115
- Easton, Vakil: Absolute Galois acts faithfully on the components of the moduli space of surfaces: A Belyi-type theorem in higher dimension. arXiv:0704.3231
- Bauer, Catanese, Grunewald: The absolute Galois group acts faithfully on the connected components of the moduli space of surfaces of general type. arXiv:0706.1466
- F. Charles: Conjugate varieties with distinct real cohomology algebras. arXiv:0706.3674

Let V be an oriented topological manifold of real dimension 4. We put

$$\mathrm{H}_2(V) := \mathrm{H}_2(V, \mathbb{Z})/\text{torsion},$$

and let

$$\iota_V \,:\, \mathrm{H}_2(V) \times \mathrm{H}_2(V) \,\to\, \mathbb{Z}$$

be the intersection pairing. We then put

$$J_\infty(V) \;:=\; \bigcap_K \mathrm{Im}(\mathrm{H}_2(V \setminus K) \to \mathrm{H}_2(V)),$$

where K runs through the set of compact subsets of V, and set

$$\widetilde{B}_V := \mathrm{H}_2(V)/J_\infty(V) \quad \text{and} \quad B_V := (\widetilde{B}_V)/\text{torsion}.$$

Since any topological cycle is compact, the intersection pairing ι_V induces a symmetric bilinear form

$$\beta_V \,:\, B_V \times B_V \,\to\, \mathbb{Z}.$$

It is obvious that the isomorphism class of (B_V, β_V) is a topological invariant of V.

Theorem. Let X be a complex smooth projective surface, and let $C_1, \ldots, C_n$ be irreducible curves on X. We put

$$V := X \setminus \bigcup C_i.$$

Suppose that the classes $[C_1], \ldots, [C_n]$ span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Then (B_V, β_V) is isomorphic to the transcendental lattice

$$T(X) := (\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X))^{\perp}/\text{torsion}.$$

Hence $T(X)$ is a topological invariant of the open surface $V \subset X$.

Construction of examples.

Let T_1 and T_2 be even positive-definite lattices of rank 2 that are in the same genus but not isomorphic. We have a singular $K3$ surface X defined over a number field F, and embeddings $\sigma_1, \sigma_2 \in \mathrm{Emb}(F, \mathbb{C})$ such that

$$T(X^{\sigma_1}) \cong T_1 \quad \text{and} \quad T(X^{\sigma_2}) \cong T_2.$$

Let $C_1, \dots, C_n$ be irreducible curves on X whose classes span $\mathrm{NS}(X) \otimes \mathbb{Q}$. Enlarging F, we can assume that

$$V := X \setminus \bigcup C_i.$$

is defined over F. Then the conjugate open varieties

$$V^{\sigma_1} \quad \text{and} \quad V^{\sigma_2}$$

are not homeomorphic.

§3. Maximizing sextic

Definition. (1) A complex plane curve $C \subset \mathbb{P}^2$ of degree 6 is called a *maximizing sextic* if C has only simple singularities and its total Milnor number is 19.
(2) Two complex projective plane curves C and C' are said to be *conjugate* if there exists $\sigma \in \mathrm{Emb}(\mathbb{C}, \mathbb{C})$ such that $C^\sigma \subset \mathbb{P}^2$ is projectively equivalent to $C' \subset \mathbb{P}^2$.

If C is a maximizing sextic, the minimal resolution $X_C \to Y_C$ of the double cover $Y_C \to \mathbb{P}^2$ of $\mathbb{P}^2$ branching exactly along C is a complex singular $K3$ surface. We put

$$T[C] := T(X_C), \quad \text{and} \quad \widetilde{T}[C] := \widetilde{T}(X_C).$$

Theorem. Let C be a maximizing sextic, and let $\widetilde{T}'$ be an oriented lattice such that its underlying (non-oriented) lattice is in the same genus, but not isomorphic, with $T[C]$. Then there is a maximizing sextic C' such that $\widetilde{T}[C'] \cong \widetilde{T}'$, and that C and C' are conjugate, but $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic.

Definition. A pair of complex projective plane curves C and C' is called an *arithmetic Zariski pair* if they are conjugate but $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic.

Remark. The first example of an arithmetic Zariski pair was discovered by Artal, Carmona and Cogolludo in degree 12 by means of completely different method.

Using Torelli theorem, we can make a complete list of the ADE-types of maximizing sextics and their oriented transcendental lattices. Thus we obtain the following complete list of arithmetic Zariski pairs of maximizing sextics.

In the table below, $L[2a, b, 2c]$ denotes the lattice of rank 2 given by the matrix

$$\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

1	$E_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
2	$E_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
3	$E_6 + D_5 + A_6 + A_2$	$L[12,0,42]$,	$L[6,0,84]$
4	$E_6 + A_{10} + A_3$	$L[12,0,22]$,	$L[4,0,66]$
5	$E_6 + A_{10} + A_2 + A_1$	$L[18,6,24]$,	$L[6,0,66]$
6	$E_6 + A_7 + A_4 + A_2$	$L[24,0,30]$,	$L[6,0,120]$
7	$E_6 + A_6 + A_4 + A_2 + A_1$	$L[30,0,42]$,	$L[18,6,72]$
8	$D_8 + A_{10} + A_1$	$L[6,2,8]$,	$L[2,0,22]$
9	$D_8 + A_6 + A_4 + A_1$	$L[8,2,18]$,	$L[2,0,70]$
10	$D_7 + A_{12}$	$L[6,2,18]$,	$L[2,0,52]$
11	$D_7 + A_8 + A_4$	$L[18,0,20]$,	$L[2,0,180]$
12	$D_5 + A_{10} + A_4$	$L[20,0,22]$,	$L[12,4,38]$
13	$D_5 + A_6 + A_5 + A_2 + A_1$	$L[12,0,42]$,	$L[6,0,84]$
14	$D_5 + A_6 + 2A_4$	$L[20,0,70]$,	$L[10,0,140]$
15	$A_{18} + A_1$	$L[8,2,10]$,	$L[2,0,38]$
16	$A_{16} + A_3$	$L[4,0,34]$,	$L[2,0,68]$
17	$A_{16} + A_2 + A_1$	$L[10,4,22]$,	$L[6,0,34]$
18	$A_{13} + A_4 + 2A_1$	$L[8,2,18]$,	$L[2,0,70]$
19	$A_{12} + A_6 + A_1$	$L[8,2,46]$,	$L[2,0,182]$
20	$A_{12} + A_5 + 2A_1$	$L[12,6,16]$,	$L[4,2,40]$
21	$A_{12} + A_4 + A_2 + A_1$	$L[24,6,34]$,	$L[6,0,130]$
22	$A_{10} + A_9$	$L[10,0,22]$,	$L[2,0,110]$
23	$A_{10} + A_9$	$L[8,3,8]$,	$L[2,1,28]$
24	$A_{10} + A_8 + A_1$	$L[18,0,22]$,	$L[10,2,40]$
25	$A_{10} + A_7 + A_2$	$L[22,0,24]$,	$L[6,0,88]$
26	$A_{10} + A_7 + 2A_1$	$L[10,2,18]$,	$L[2,0,88]$
27	$A_{10} + A_6 + A_2 + A_1$	$L[22,0,42]$,	$L[16,2,58]$
28	$A_{10} + A_5 + A_3 + A_1$	$L[12,0,22]$,	$L[4,0,66]$
29	$A_{10} + 2A_4 + A_1$	$L[30,10,40]$,	$L[10,0,110]$
30	$A_{10} + A_4 + 2A_2 + A_1$	$L[30,0,66]$,	$L[6,0,330]$
31	$A_8 + A_6 + A_4 + A_1$	$L[22,4,58]$,	$L[18,0,70]$
32	$A_7 + A_6 + A_4 + A_2$	$L[24,0,70]$,	$L[6,0,280]$
33	$A_7 + A_6 + A_4 + 2A_1$	$L[18,4,32]$,	$L[2,0,280]$
34	$A_7 + A_5 + A_4 + A_2 + A_1$	$L[24,0,30]$,	$L[6,0,120]$

§4. Maximizing sextics of type $A_{10} + A_9$

There are 4 connected components in the moduli space of maximizing sextics of type

$$A_{10} + A_9.$$

Two of them have irreducible members, and their oriented transcendental lattices are

$$\begin{bmatrix} 10 & 0 \\ 0 & 22 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 110 \end{bmatrix}.$$

The other two have reducible members (a line and an irreducible quintic), and their oriented transcendental lattices are

$$\begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}.$$

We will consider these reducible members.

Arima has found a defining equation over $\mathbb{Q}(\sqrt{5})$:

$$C_{\pm} \ : \ z \cdot (G(x,y,z) \pm \sqrt{5} \cdot H(x,y,z)) = 0, \quad \text{where}$$

$$\begin{aligned} G(x,y,z) \ &:= \ -9\,x^4 z - 14\,x^3 yz + 58\,x^3 z^2 - 48\,x^2 y^2 z - \\ &\quad -64\,x^2 yz^2 + 10\,x^2 z^3 + +108\,xy^3 z - \\ &\quad -20\,xy^2 z^2 - 44\,y^5 + 10\,y^4 z, \\ H(x,y,z) \ &:= \ 5\,x^4 z + 10\,x^3 yz - 30\,x^3 z^2 + 30\,x^2 y^2 z + \\ &\quad +20\,x^2 yz^2 - 40\,xy^3 z + 20\,y^5. \end{aligned}$$

We have two possibilities:

$$T[C_+] \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix} \quad \text{and} \quad T[C_-] \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad \text{or}$$

$$T[C_+] \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix} \quad \text{and} \quad T[C_-] \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

Problem. Which is the case?

Remark. This problem cannot be solved by any algebraic methods.

Calculating the vanishing cycle associated with a pencil of genus 2 curves on the $K3$ surfaces $X_{C_\pm}$ coming from the pencil of lines on $\mathbb{P}^2$, we obtain the following:

Proposition.

$$T[C_+] \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T[C_-] \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

Problem.

$$\pi_1(\mathbb{P}^2 \setminus C_+) \cong \pi_1(\mathbb{P}^2 \setminus C_-)?$$