Zariski pairs and lattice theory

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2009.05 Antalya

Example

Consider two surfaces $S_\pm$ in $\mathbb{C}^3$ defined by

$$w^2(G(x,y) \pm \sqrt{5} \cdot H(x,y)) = 1, \quad \text{where}$$

$$\begin{aligned} G(x,y) &:= -9\,x^4 - 14\,x^3y + 58\,x^3 - 48\,x^2y^2 - 64\,x^2y \\ &\quad +10\,x^2 + 108\,xy^3 - 20\,xy^2 - 44\,y^5 + 10\,y^4, \\ H(x,y) &:= 5\,x^4 + 10\,x^3y - 30\,x^3 + 30\,x^2y^2 + \\ &\quad +20\,x^2y - 40\,xy^3 + 20\,y^5. \end{aligned}$$

Then S_+ and S_- are not homeomorphic.

Many examples of non-homeomorphic conjugate complex varieties are known since Serre (1964).
I hope the above is the most concrete and simplest example.

Plane curves

Let B and B' be reduced (possibly reducible) projective plane curves of the same degree. We assume that they have only simple singularities (the singularities without moduli):

$$
\begin{array}{lll}
A_n & x^{n+1} + y^2 = 0 & (n \geq 1) \\
D_n & x^{n-1} + xy^2 = 0 & (n \geq 4) \\
E_6 & x^4 + y^3 = 0 & \\
E_7 & x^3 y + y^3 = 0 & \\
E_8 & x^5 + y^3 = 0 &
\end{array}
$$

We denote by

$$
\begin{array}{rl}
R_B : & \text{the } ADE\text{-type of } \operatorname{Sing} B, \\
\operatorname{degs} B : & \text{the list of degrees of irred comps of } B.
\end{array}
$$

Two equivalence relations

B and B' are of the same config type and write $B \sim_{\mathrm{cfg}} B'$ if

- they have the same type of singularities; $R_B = R_{B'}$,
- the degrees of irred comps are same; $\operatorname{degs} B = \operatorname{degs} B'$,
- their intersection patterns are same.

We write $B \sim_{\mathrm{emb}} B'$ if there exists a homeomorphism

$$\psi : (\mathbb{P}^2, B) \xrightarrow{\simeq} (\mathbb{P}^2, B').$$

We have $B \sim_{\mathrm{emb}} B' \implies B \sim_{\mathrm{cfg}} B'$.

Example: For degree 6, we have

$$\# \text{ of config types} = 11159 < \# \text{ of emb-top types} = ?$$

Zariski pairs

A *Zariski pair* is a pair $[B, B']$ of projective plane curves of the same degree with only simple singularities such that

$$B \sim_{\mathrm{cfg}} B' \quad \text{but} \quad B \not\sim_{\mathrm{emb}} B'.$$

A *Zariski k-ple* is a collection of k plane curves such that any two of them form a Zariski pair.

Example (Zariski)

Let B be a plane curve of degree 6 defined by

$$f^3 + g^2 = 0, \quad \deg f = 2, \ \deg g = 3, \ \text{general}.$$

Then B is irreducible and has six cusps as its only singularities; $\mathrm{degs}\, B = [6]$, $R_B = 6A_2$. We have

$$\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/(2) * \mathbb{Z}/(3).$$

Zariski showed that there exists B' with $\mathrm{degs}\, B' = [6]$, $R_{B'} = 6A_2$ such that

$$\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/(2) \times \mathbb{Z}/(3).$$

Example of Zariski triple

We have three plane curves of degree 6

$$B_1 = C_1 + Q_1, \quad B_2 = C_2 + Q_2, \quad B_4 = C_4 + Q_4,$$

where Q_i is a quartic with one tacnode and C_i is a smooth conic tangent to Q_i at two points with multiplicity 4;

$$\operatorname{degs} B_i = [2, 4], \quad R_{B_i} = A_3 + 2A_7.$$

Let $E_i \to Q_i$ be the normalization of Q_i. Then E_i is of genus 1 and has four special points

$$p, q \;\text{ the pull-back of } A_3, \quad s, t \;\text{ the pull-back of } 2A_7.$$

Then the order of $[p + q - s - t]$ in $\mathrm{Pic}^0(E_i)$ is 1, 2 and 4 according to $i = 1, 2, 4$. Their emb-top types are different, and they form a Zariski triple.

Algebraic lattice-theoretic invariants

The idea is to consider the double covering

$$Y_B \longrightarrow \mathbb{P}^2$$

branching along the plane curve B, and the cup-product on the middle cohomology group

$$H^2(X_B, \mathbb{Z})$$

of the minimal resolution X_B of Y_B.

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$L \times L \to \mathbb{Z}.$$

A lattice L is canonically embedded into its *dual lattice*

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z})$$

as a submodule of finite index. The finite abelian group

$$D_L := L^\vee / L$$

is called the *discriminant group* of L. The $\mathbb{Z}$-valued symmetric bilinear form on L extends to a $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$, and it defines a finite quadratic form

$$q_L : D_L \to \mathbb{Q}/\mathbb{Z}, \;\; \bar{x} \mapsto x^2 \bmod \mathbb{Z}.$$

We call q_L the *discriminant form* of L.

Let B be a plane curve of even degree. We denote by

$$\rho_B : X_B \to Y_B \to \mathbb{P}^2$$

the composite of the minimal resolution and the double covering. Let $\mathcal{E}_B$ be the set of exceptional curves $X_B \to Y_B$, and let

$$\Sigma_B := \langle\, [E] \mid E \in \mathcal{E}_B \,\rangle \oplus \langle h_B \rangle \;\subset\; H^2(X_B)$$

be the sublattice generated by the classes $[E]$ of $E \in \mathcal{E}_B$ and the polarization class $h_B = [\rho_B^* \mathcal{O}_{\mathbb{P}^2}(1)]$.

$$B \sim_{\mathrm{cfg}} B' \;\;\Rightarrow\;\; \Sigma_B \cong \Sigma_{B'}.$$

We then denote by

$$\Lambda_B \;\subset\; H^2(X_B)$$

the primitive closure of Σ_B.

The topological property of the lattice invariants

Theorem

If $B \sim_{\mathrm{emb}} B'$, then we have $(D_{\Lambda_B}, q_{\Lambda_B}) \cong (D_{\Lambda_{B'}}, q_{\Lambda_{B'}})$.

Proof.

Let T_B denote the orthogonal complement of Λ_B in the *unimodular* lattice $H^2(X_B)$. Then we have

$$(D_{T_B}, q_{T_B}) \cong (D_{\Lambda_B}, -q_{\Lambda_B}).$$

On the other hand, T_B is a topological invariant of the open surface

$$U_B := \rho_B^{-1}(\mathbb{P}^2 \setminus B) \quad \subset \quad X_B.$$

In fact, we have

$$T_B := H^2(U_B)/\operatorname{Ker}.$$

If $B \sim_{\mathrm{emb}} B'$, then U_B and $U_{B'}$ are homeomorphic. □

We consider the finite abelian group

$$G(B) := \Lambda_B / \Sigma_B.$$

Corollary

If $B \sim_{\mathrm{cfg}} B'$ but $|G(B)| \neq |G(B')|$, then $B \not\sim_{\mathrm{emb}} B'$.

Indeed, $B \sim_{\mathrm{cfg}} B'$ implies $\Sigma_B \cong \Sigma_{B'}$, and hence their discriminant groups are isomorphic. Then $|G(B)| \neq |G(B')|$ implies that the discriminant groups of Λ_B and $\Lambda_{B'}$ have different orders.

This corollary produces many examples of Zariski pairs.

Example of Zariski triple again

For the Zariski triple

$$B_1 = C_1 + Q_1, \quad B_2 = C_2 + Q_2, \quad B_4 = C_4 + Q_4$$

described above, we have

$$G(B_1) \cong \mathbb{Z}/2\mathbb{Z}, \quad G(B_2) \cong \mathbb{Z}/4\mathbb{Z}, \quad G(B_4) \cong \mathbb{Z}/8\mathbb{Z}.$$

Hence they are topologically distinct.

Remark on even discriminant forms

A lattice L is said to be *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in L$.
If L is even, then the discriminant form

$$q_L : D_L \to \mathbb{Q}/\mathbb{Z},$$

is refined to

$$\tilde{q}_L : D_L \to \mathbb{Q}/2\mathbb{Z},$$

which we call the *even discriminant form* of L.

If $H^2(X_B)$ is an even unimodular lattice, that is, if $\deg B \equiv 2 \bmod 4$, then we can refine the above results to the even discriminant forms.

These invariants $(D_{\Lambda_B}, q_{\Lambda_B})$, $(D_{\Sigma_B}, q_{\Sigma_B})$ and $G(B)$ are algebraic in the following sense:

Definition

An invariant ϕ of plane curves is said to be *algebraic* if $\phi(B) = \phi(B^\sigma)$ holds for any B and $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Hence they cannot distinguish conjugate plane curves.

Remark that the configuration type is algebraic.

A topological lattice invariant

Recall that $T_B \subset H^2(X_B)$ is the orthogonal complement of Λ_B.

$T_B := \{t \in H^2(X_B) \,|\, (t, h_B) = 0 \text{ and } (t, [E]) = 0 \text{ for } \forall\ E \in \mathcal{E}\}.$

As we have said, T_B is a topological invariant of the open surface

$$U_B := \rho_B^{-1}(\mathbb{P}^2 \setminus B) \subset X_B.$$

We use the isomorphism class of T_B as an invariant of B.

T_B is not algebraic

We consider the plane curves B of degree 6. Then X_B is a $K3$ surface, and the total Milnor number μ_B of B is at most 19.

We say that B is a *maximizing sextic* if $\mu_B = 19$.

A $K3$ surface X is said to be a *singular $K3$ surface* if its Picard number is 20, that is, its transcendental lattice

$$T(X) := \mathrm{NS}(X)^{\perp}$$

is a positive-definite even lattice of rank 2.

If B is a maximizing sextic, then X_B is a singular $K3$ surface and $T(X_B) = T_B$.

X: a singular $K3$ surface

$T(X)$ is canonically oriented by

$$T(X) \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{0,2}(X).$$

Theorem (Shioda-Inose)

The map $X \mapsto T(X)$ is a bijection from the set of isom. classes of singular $K3$ surfaces to the set of isom. classes of oriented pos-definite even binary lattices.

Theorem (S. and Schütt)

Let X and X' be singular $K3$ surfaces defined over $\overline{\mathbb{Q}}$ such that $T(X)$ and $T(X')$ have the isomorphic (even) discriminant forms. Then $\exists\ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $X' \cong X^{\sigma}$.

Application to maximizing sextics

The set of positive-definite even lattices of rank 2 with a given discriminant form is called a *genus*.

Corollary

Let B be a maximizing sextic defined over $\overline{\mathbb{Q}}$. If the genus containing T_B contains more than one isomorphism class of lattices, then $\exists\ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $B \not\sim_{\mathrm{emb}} B^{\sigma}$.

Example revisited

We consider the configuration type of maximizing sextics $B = L + Q$ with

- $\deg L = 1$, $\deg Q = 5$,
- L and Q are tangent at one point with multiplicity 5 (A_9-singularity), and
- Q has one A_{10}-singular point.

Such maximizing sextics are projectively isomorphic to

$$z \cdot (G(x, y, z) \pm \sqrt{5} \cdot H(x, y, z)) = 0,$$

where $G(x, y, z)$ and $H(x, y, z)$ are homogenizations of the G and H in the first example.

The genus corresponding to $(D_{\Lambda_B}, -q_{\Lambda_B})$ (that is, the genus containing the transcendental lattice T_B) consists of

$$\begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \qquad \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

We can show that, if we choose $+\sqrt{5}$, then

$$T_{B_+} \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix},$$

while if we choose $-\sqrt{5}$, then

$$T_{B_-} \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

Question

$\pi_1(\mathbb{P}^2 \setminus B_-) \cong \pi_1(\mathbb{P}^2 \setminus B_+)$?

(Their profinite completions are isomorphic.)

Summary

Some lattice theoretic invariants of plane curves are defined.

They are a strong tool to study the topology of embeddings of plane curves into $\mathbb{P}^2$.

In particular, we can construct many Zariski pairs by means of them.

More important summary

We appreciate the excellent work of the organizers.

Thank you very much.