On supersingular varieties

Ichiro Shimada

Hiroshima University

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Let X be a smooth projective variety over $\mathbb{F}_q$.
The following are equivalent:

(i) There is a polynomial $N(t) \in \mathbb{Z}[t]$ such that

$$|X(\mathbb{F}_{q^\nu})| = N(q^\nu)$$

for all $\nu \in \mathbb{Z}_{>0}$.

(ii) The eigenvalues of the qth power Frobenius on the l-adic cohomology ring are powers of q by integers.

If these are satisfied, then $b_{2i-1}(X) = 0$ and

$$N(t) = \sum_{i=0}^{\dim X} b_{2i}(X)\, t^i.$$

We say that X is *Frobenius supersingular* if (i) and (ii) are satisfied.

If the cohomology ring of X is generated by the classes of algebraic cycles over $\mathbb{F}_q$, then X is Frobenius supersingular.

The converse is true if the Tate conjecture is assumed.

We have examples of Frobenius supersingular varieties of **non-negative Kodaira dimension**.

Theorem

The Fermat variety

$$X := \{x_0^{q+1} + \cdots + x_{2m+1}^{q+1} = 0\} \subset \mathbb{P}^{2m+1}$$

of dimension $2m$ and degree $q+1$ regarded as a variety over $\mathbb{F}_{q^2}$ is Frobenius supersingular.

This follows from

$$|X(\mathbb{F}_{q^2})| = 1 + q^2 + \cdots + q^{4m} + (b_{2m}(X) - 1)q^{2m}.$$

Problems on Frobenius supersingular varieties

- Construct non-trivial examples.
- Prove (or disprove) the unirationality.
- Present explicitly algebraic cycles that generate the cohomology ring.
- Investigate the lattice given by the intersection pairing of algebraic cycles.
- Produce dense lattices by the intersection pairing in small characteristics.

We discuss these problems
for the classical example of **Fermat varieties of degree** $q+1$, and
for the new example of **Frobenius incidence varieties**.

Unirationality and Supersingularity

A variety X is called *(purely-inseparably) unirational* if there is a dominant (purely-inseparable) rational map

$$\mathbb{P}^n \dashrightarrow X.$$

Theorem (Shioda)

Let S be a smooth projective surface defined over $k = \bar{k}$. If S is unirational, then the Picard number $\rho(S)$ is equal to $b_2(S)$; that is, S is *supersingular in the sense of Shioda*.

The converse is conjectured to be true for $K3$ surfaces.

Artin-Shioda conjecture

Every supersingular $K3$ surface S (in the sense of Shioda) is conjectured to be (purely-inseparably) unirational.

The discriminant of the Néron-Severi lattice $\mathrm{NS}(S)$ is $-p^{2\sigma(S)}$, where $\sigma(S)$ is a positive integer ≤ 10, which is called the *Artin invariant* of S.

The conjecture is confirmed to be true in the following cases:

- p odd and $\sigma(S) \leq 2$ (Ogus and Shioda):
- $p = 2$ (Rudakov and Shafarevich, S.-):
- $p = 3$ and $\sigma(S) \leq 6$
  (Rudakov and Shafarevich, S.- and De Qi Zhang):
- $p = 5$ and $\sigma(S) \leq 3$ (S.- and Pho Duc Tai).

Method: The structure theorem for $\mathrm{NS}(S)$ by Rudakov-Shafarevich.

Fermat variety of degree $q+1$

Unirationality of the Fermat variety

Theorem (Shioda-Katsura, S.-)

The Fermat variety X of degree $q+1$ and dimension $n \geq 2$ in characteristic $p > 0$ is purely-inseparably unirational, where $q = p^\nu$.

Indeed, X contains a linear subspace $\Lambda \subset \mathbb{P}^{n+1}$ of dimension $[n/2]$. The unirationality is proved by the projection from the center Λ.

Lattice

By a *quasi-lattice*, we mean a free $\mathbb{Z}$-module L of finite rank with a symmetric bilinear form

$$(\, , \,) : L \times L \to \mathbb{Z}.$$

If the symmetric bilinear form is non-degenerate, we say that L is a *lattice*.

If L is a quasi-lattice, then $L/L^\perp$ is a lattice, where

$$L^\perp := \{ \, x \in L \mid (x, y) = 0 \text{ for all } y \in L \, \}.$$

Lattices associated with the Fermat varieties

The Fermat variety

$$X := \{x_0^{q+1} + \cdots + x_{2m+1}^{q+1} = 0\} \quad \subset \quad \mathbb{P}^{2m+1}$$

of dimension $2m$ and degree $q+1$ contains many m-dimensional linear subspaces Λ_i. The number is

$$\prod_{\nu=0}^{m} (q^{2\nu+1} + 1).$$

Each of them is defined over $\mathbb{F}_{q^2}$.

Let $\widetilde{\mathcal{N}}(X) \subset A^m(X)$ be the $\mathbb{Z}$-module generated by the rational equivalence classes of Λ_i, where $A(X)$ is the Chow ring.

By the intersection pairing

$$\widetilde{\mathcal{N}}(X) \times \widetilde{\mathcal{N}}(X) \ \to \ \mathbb{Z},$$

we can consider $\widetilde{\mathcal{N}}(X)$ as a quasi-lattice.

Let $\mathcal{N}(X) := \widetilde{\mathcal{N}}(X)/\widetilde{\mathcal{N}}(X)^{\perp}$ be the associated lattice.

Theorem (Tate, S.-)

(1) The rank of $\mathcal{N}(X)$ is equal to $b_{2m}(X)$.
(2) The discriminant of $\mathcal{N}(X)$ is a power of p.

Corollary

The cycle map induces an isomorphism $\mathcal{N}(X) \otimes \mathbb{Q}_l \cong H^{2m}(X, \mathbb{Q}_l)$.

The assertion (2) is an analogue of the result that the discriminant of the Néron-Severi lattice $\mathrm{NS}(S)$ of a supersinglar $K3$ surface S is a power of p.

Let $h \in \mathcal{N}(X)$ be the numerical equivalence class of a linear plane section $X \cap \mathbb{P}^{m+1}$.

We put

$$\mathcal{N}_{\mathrm{prim}}(X) := \{\, x \in \mathcal{N}(X) \mid (x, h) = 0 \,\} = \langle h \rangle^{\perp}.$$

Theorem

The lattice $[-1]^m \mathcal{N}_{\mathrm{prim}}(X)$ is positive-definite.

Here $[-1]^m \mathcal{N}_{\mathrm{prim}}(X)$ is the lattice obtained from $\mathcal{N}_{\mathrm{prim}}(X)$ by changing the sign with $(-1)^m$.

Dense lattices

Let L be a positive-definite lattice of rank m.
The *minimal norm* of L is defined by

$$N_{\mathrm{min}}(L) := \min\{x^2 \mid x \in L, x \neq 0\},$$

and the *normalized center density* of L is defined by

$$\delta(L) := (\mathrm{disc}\, L)^{-1/2} \cdot (N_{\mathrm{min}}(L)/4)^{m/2}.$$

Minkowski and Hlawka proved in a non-constructive way that, for each m, there is a positive-definite lattice L of rank m with

$$\delta(L) \; > \; \mathrm{MH}(m) := \frac{\zeta(m)}{2^{m-1} V_m},$$

where V_m is the volume of the m-dimensional unit ball.

We say that a positive-definite lattice L of rank m is *dense* if

$$\delta(L) > \mathrm{MH}(m).$$

The intersection pairing of algebraic cycles in positive characteristic has been used to construct dense lattices.

For example, Elkies and Shioda constructed many dense lattices as Mordell-Weil lattices of elliptic surfaces in positive characteristics.

Dense lattices arising from Fermat varieties

Let X be the Fermat *cubic* variety of dimension $2m$ in characteristic 2.
Recall that X contains many m-dimensional linear subspaces Λ_i.

We consider the positive-definite lattice

$$\langle [\Lambda_i] - [\Lambda_j] \rangle \quad \subset \quad [-1]^m \mathcal{N}_{\mathrm{prim}}(X)$$

generated by the classes $[\Lambda_i] - [\Lambda_j]$. Their properties are as follows:

$\dim X$	rank	$N_{\min}$	$\log_2 \delta$	$\log_2 \mathrm{MH}$	name
2	6	2	$-3.792\ldots$	$-7.344\ldots$	E_6
4	22	4	$-1.792\ldots$	$-13.915\ldots$	Λ_{22}
6	86	8	$34.207\ldots$	$19.320\ldots$	$\mathcal{N}_{86}$

Frobenius incidence variety

We fix an n-dimensional linear space V over $\mathbb{F}_p$ with $n \geq 3$.

We denote by $G_{n,l} = G_n^{n-l}$ the Grassmannian variety of l-dimensional subspaces of V.

Let F be a field of characteristic p, and consider an F-rational linear subspace $L \in G_{n,l}(F)$ of V.

Let ϕ be the pth power Frobenius morphism of $G_{n,l}$. For a positive integer ν, we put

$$L^{(p^\nu)} := \phi^\nu(L).$$

Let l and c be positive integers such that $l + c < n$.

We denote by $\mathcal{I}^c_{n,l}$ the incidence subvariety of $G_{n,l} \times G^c_n$:

$$\mathcal{I}^c_{n,l}(F) = \{ \, (L, M) \in G_{n,l}(F) \times G^c_n(F) \mid L \subset M \, \}.$$

Let $r := p^a$ and $s := p^b$ be powers of p by positive integers. We define the **Frobenius incidence variety** $X^c_{n,l}$ by

$$X^c_{n,l} := (\phi^a \times \mathrm{id})^* \mathcal{I}^c_{n,l} \cap (\mathrm{id} \times \phi^b)^* \mathcal{I}^c_{n,l}.$$

Then $X^c_{n,l}$ is defined over $\mathbb{F}_p$, and we have

$$\begin{aligned} X^c_{n,l}(F) &= \{ \, (L, M) \in G_{n,l}(F) \times G^c_n(F) \mid L^{(r)} \subset M \ \text{and} \ L \subset M^{(s)} \, \} \\ &= \{ \, (L, M) \in G_{n,l}(F) \times G^c_n(F) \mid L + L^{(rs)} \subset M^{(s)} \, \} \\ &= \{ \, (L, M) \in G_{n,l}(F) \times G^c_n(F) \mid L^{(r)} \subset M \cap M^{(rs)} \, \}. \end{aligned}$$

Theorem

(1) The scheme $X^c_{n,l}$ is smooth and geometrically irreducible of dimension $(n-l-c)(l+c)$.

(2) If $X^c_{n,l}$ is regarded as a scheme over $\mathbb{F}_{rs}$, then $X^c_{n,l}$ is Frobenius supersingular.

The smoothness of $X^c_{n,l}$ is proved by computing the dimension of Zariski tangent spaces.

We prove the second assertion by counting the number of $\mathbb{F}_{(rs)^\nu}$-rational points of $X^c_{n,l}$.

We put

$$q := rs.$$

The main ingredient of the proof is the finite set

$$T_{l,d}(q,q^{\nu}) := \{\ L \in G_{n,l}(\mathbb{F}_{q^{\nu}})\ \mid\ \dim(L \cap L^{(q)}) = d\ \}.$$

When $l = d$, we have $T_{l,l}(q,q^{\nu}) = G_{n,l}(\mathbb{F}_q)$ for any ν.

For $d < l$, we calculate the cardinality of the set

$$\begin{aligned}\mathcal{P} &:= \{\ (L,M) \in G_{n,l}(\mathbb{F}_{q^{\nu}}) \times G_{n,2l-d}(\mathbb{F}_{q^{\nu}})\ \mid\ L + L^{(q)} \subset M\ \}\\ &= \{\ (L,M) \in G_{n,l}(\mathbb{F}_{q^{\nu}}) \times G_{n,2l-d}(\mathbb{F}_{q^{\nu}})\ \mid\ L^{(q)} \subset M \cap M^{(q)}\ \},\end{aligned}$$

in two ways using the projections
$\mathcal{P} \to G_{n,l}(\mathbb{F}_{q^{\nu}})$ and $\mathcal{P} \to G_{n,2l-d}(\mathbb{F}_{q^{\nu}})$.
Then we get

$$\begin{aligned}|\mathcal{P}| &= \sum_{t=d}^{l} |T_{l,t}(q,q^{\nu})| \cdot |G_{n-2l+t,t-d}(\mathbb{F}_{q^{\nu}})|\\ &= \sum_{u=l}^{2l-d} |T_{2l-d,u}(q,q^{\nu})| \cdot |G_{u,l}(\mathbb{F}_{q^{\nu}})|.\end{aligned}$$

By this equality, we obtain a recursive formula for $|T_{l,d}(q, q^\nu)|$.

Using the projection $X^c_{n,l}(\mathbb{F}_{q^\nu}) \to G_{n,l}(\mathbb{F}_{q^\nu})$, we obtain the following:

$$|X^c_{n,l}(\mathbb{F}_{q^\nu})| = \sum_{d=0}^{l} |T_{l,d}(q, q^\nu)| \cdot |G^c_{n-2l+d}(\mathbb{F}_{q^\nu})|.$$

By the recursive formula for $|T_{l,d}(q, q^\nu)|$, we prove that there is a monic polynomial $N^c_{n,l}(t)$ of degree $(l+c)(n-l-c)$ such that

$$|X^c_{n,l}(\mathbb{F}_{q^\nu})| = N^c_{n,l}(q^\nu).$$

Therefore $X^c_{n,l}$ is Frobenius supersingular.
Since $N^c_{n,l}(t)$ is monic, $X^c_{n,l}$ is geometrically irreducible.
Moreover we obtain the Betti numbers of $X^c_{n,l}$.

Example

Let $(x_1 : \cdots : x_n)$ and $(y_1 : \cdots : y_n)$ be homogeneous coordinates of $G_{n,1} = \mathbb{P}_*(V)$ and $G_n^1 = \mathbb{P}^*(V)$ that are dual to each other. Then $\mathcal{I}_{n,1}^1 = \{\sum x_i y_i = 0\}$, and hence $X_{n,1}^1$ is defined by

$$\begin{cases} x_1^r\, y_1 + \cdots + x_n^r\, y_n = 0, \\ x_1\, y_1^s + \cdots + x_n\, y_n^s = 0. \end{cases}$$

The Betti numbers of $X_{n,1}^1$ are as follows:

$$b_{2i} = b_{2(n-2)-2i} = \begin{cases} i+1 & \text{if } i < n-2, \\ n-2+(q^n-1)/(q-1) & \text{if } i = n-2. \end{cases}$$

When $r = s = 2$ (and hence $q = 4$), $X_{3,1}^1$ is the supersingular $K3$ surface with Artin invariant 1 (Mukai's model).

Example

The Betti numbers of $X^2_{7,2}$ are calculated as follows:

$$
\begin{array}{ll}
b_0 = b_{24}: & 1 \\
b_2 = b_{22}: & 2 \\
b_4 = b_{20}: & 5 \\
b_6 = b_{18}: & \quad q^6 + q^5 + q^4 + q^3 + q^2 + q \quad + 8 \\
b_8 = b_{16}: & 2\,(q^6 + q^5 + q^4 + q^3 + q^2 + q) + 12 \\
b_{10} = b_{14}: & 3\,(q^6 + q^5 + q^4 + q^3 + q^2 + q) + 14 \\
b_{12}: & q^{10} + q^9 + 2\,q^8 + 2\,q^7 + 6\,q^6 + \\
 & +6\,q^5 + 6\,q^4 + 5\,q^3 + 5\,q^2 + 4\,q + 16.
\end{array}
$$

Unirationality of $X^c_{n,l}$

Theorem

The Frobenius incidence variety $X^c_{n,l}$ is purely-inseparably unirational.

Idea of the proof for the case $2l + c \leq n$.

We define $\widetilde{X} \subset G_{n,l} \times G^c_n$ by

$$\widetilde{X}(F) = \{ (L, M) \mid L \subset M, \quad L^{(rs)} \subset M \}.$$

The projection $\widetilde{X} \to G_{n,l}$ is dominant. Using this projection, we can show that $\widetilde{X}$ is rational. The map $(L, M) \mapsto (L, M^{(s)})$ is a dominant morphism from $\widetilde{X}$ to $X^c_{n,l}$.

Algebraic cycles on $X^l_{n,l}$

Let Λ be an $\mathbb{F}_{rs}$-rational linear subspace of V such that $l \le \dim \Lambda \le n - c$. We define $\Sigma_\Lambda \subset G_{n,l} \times G^c_n$ by

$$\Sigma_\Lambda(F) := \{\, (L, M) \in G_{n,l}(F) \times G^c_n(F) \;\mid\; L \subset \Lambda \;\text{ and }\; \Lambda^{(r)} \subset M \,\}.$$

It follows from $\Lambda^{(rs)} = \Lambda$ that Σ_Λ is contained in $X^c_{n,l}$.

When $l = c$, we have $2 \dim \Sigma_\Lambda = \dim X^l_{n,l}$.

We can calculate the intersection numbers of these Σ_Λ on $X^l_{n,l}$.

We consider the case where $l = c = 1$:

$$X^1_{n,1} \subset \mathbb{P}_*(V) \times \mathbb{P}^*(V).$$

We put

$$\mathcal{H} := \mathrm{Im}(\ A^{n-2}(\mathbb{P}_*(V) \times \mathbb{P}^*(V)) \to A^{n-2}(X^1_{n,1})\ ).$$

By the intersection pairing, we can consider the submodule

$$\widetilde{\mathcal{N}}(X^1_{n,1}) := \mathcal{H} + \langle [\Sigma_\Lambda] \rangle \ \subset\ A^{n-2}(X^1_{n,1})$$

as a quasi-lattice. Let

$$\mathcal{N}(X^1_{n,1}) := \widetilde{\mathcal{N}}(X^1_{n,1}) / \widetilde{\mathcal{N}}(X^1_{n,1})^\perp$$

be the associated lattice, and put

$$\mathcal{N}_{\mathrm{prim}}(X^1_{n,1}) := \mathcal{H}^\perp \ \subset\ \mathcal{N}(X^1_{n,1}).$$

Theorem

(1) The rank of $\mathcal{N}(X^1_{n,1})$ is $b_{2(n-2)}(X^1_{n,1})$.
(2) The discriminant of $\mathcal{N}(X^1_{n,1})$ is a power of p.
(3) The lattice $[-1]^n\mathcal{N}_{\mathrm{prim}}(X^1_{n,1})$ is positive-definite.

Corollary

The cohomology ring of $X^1_{n,1}$ is generated by the classes of Σ_Λ and the image of $A(\mathbb{P}_*(V) \times \mathbb{P}^*(V)) \to A(X^1_{n,1})$.

Dense lattices of rank 84 and 85

Theorem

Suppose that $p = r = s = 2$. Then $\mathcal{N}_{\mathrm{prim}}(X^1_{4,1})$ is an even positive-definite lattice of rank 84, with discriminant $85 \cdot 2^{16}$, and with minimal norm 8.

In fact, $\mathcal{N}_{\mathrm{prim}}(X^1_{4,1})$ is a section of a larger lattice $\mathcal{M}_{\mathcal{C}}$ of rank

$$85 = |\mathbb{P}^3(\mathbb{F}_4)|$$

constructed by the projective geometry over $\mathbb{F}_4$ and a code over

$$R := \mathbb{Z}/8\mathbb{Z}.$$

We put

$$T := \mathbb{P}^3(\mathbb{F}_4).$$

For $S \subset T$, we denote by $v_S \in R^T$ and $\tilde{v}_S \in \mathbb{Z}^T$ the characteristic functions of S.

Let $\mathcal{C} \subset R^T$ be the submodule generated by

$$2^{2-k}(v_P - v_{P'}),$$

where P and P' are $\mathbb{F}_4$-rational linear subspaces of $\mathbb{P}^3$ of dimension k ($k = 0, 1, 2$), and let $\mathcal{M}_\mathcal{C}$ be the pull-back of $\mathcal{C}$ by $\mathbb{Z}^T \to R^T$.

We define a $\mathbb{Q}$-valued symmetric bilinear form on $\mathbb{Z}^T$ by

$$(\tilde{v}_{\{t\}}, \tilde{v}_{\{t'\}}) = \delta_{tt'}/4 \qquad (t, t' \in T).$$

Then $\mathcal{M}_\mathcal{C} \subset \mathbb{Z}^T$ is a lattice.

name	rank	disc	$N_{\min}$	$\log_2 \delta$	$\log_2 \mathrm{MH}$
$\mathcal{N}_{\mathrm{prim}}(X^1_{4,1})$	84	$85 \cdot 2^{16}$	8	30.795...	17.546...
$\mathcal{M}_\mathcal{C}$	85	2^{20}	8	32.5	18.429...
$\mathcal{N}_{86}$	86	$3 \cdot 2^{16}$	8	34.207...	19.320...

Thank you!