Supersingular $K3$ surfaces in characteristic 5

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30 January, 2011, Naruto

Let X be a smooth minimal complete surface defined over a field k.

We say that X is a $K3$ surface if

$$\Omega_X^2 \cong \mathcal{O}_X \quad \text{and} \quad h^1(\mathcal{O}_X) = 0.$$

Examples of $K3$ surfaces:

- $X \to \mathbb{P}^2$: a double cover branched along a smooth curve of degree 6.
- $X \subset \mathbb{P}^3$: a smooth quartic surface.
- $X \subset \mathbb{P}^4$: a smooth complete intersection of degree $(2,3)$.
- $X \subset \mathbb{P}^5$: a smooth complete intersection of degree $(2,2,2)$.
- $X \to A/\langle i \rangle$: the minimal resolution of the quotient of an abelian surface A by the involution $i(x) = -x$, where $\mathrm{char}\, k \neq 2$.

The Néron-Severi lattice $\mathrm{NS}(X)$ of X is the group of numerical equivalence classes of divisors on X with the intersection form.

Definition

A $K3$ surface X is *supersingular* (in the sense of Shioda) if the rank of the Néron-Severi lattice $\mathrm{NS}(X)$ is equal to $B_2(X) = 22$.

Supersingular $K3$ surfaces exist only in positive characteristics.

Examples of supersingular $K3$ surfaces:

- $X \to \mathbb{P}^2$: the double cover branched along the Fermat sextic $x^6 + y^6 + z^6 = 0$ in characteristic 5.
- $X \subset \mathbb{P}^3$: the Fermat quartic surface $w^4 + x^4 + y^4 + z^4 = 0$ in characteristic 3.

If X is a supersingular $K3$ surface, then $\mathrm{NS}(X)$ is an even lattice of rank 22 with signature $(1, 21)$.

Theorem (Artin, Rudakov-Shafarevich)

Let X be a supersingular $K3$ surface in characteristic $p > 0$. Then there exists a positive integer $\sigma \leq 10$ such that

$$\mathrm{NS}^{\vee}(X)/\mathrm{NS}(X) \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus 2\sigma}.$$

Definition

The positive integer σ is called the *Artin invariant* of X.

Theorem (Ogus, Rudakov-Shafarevich)

For any prime p, a supersingular $K3$ surface in characteristic p with the Artin invariant 1 is unique up to isomorphisms.

In fact, for a supersingular elliptic curve E in characteristic $p \neq 2$, the minimal resolution $X \to (E \times E)/\langle i \rangle$ of the quotient of $E \times E$ by the involution $i(x) = -x$ is the supersingular $K3$ surface with the Artin invariant 1.

In this talk, we exhibit projective models of the supersingular $K3$ surface with the Artin invariant 1 in characteristic 5.

From now on, we work over an algebraically closed field k of characteristic 5.

For a polynomial $f \in k[x]$ of degree ≤ 6, let $B_f \subset \mathbb{P}^2$ denote the projective plane curve of degree 6 whose affine part is defined by

$$y^5 - f(x) = 0.$$

(If $\deg f < 6$, we add the line at infinity so that $\deg B_f$ is always 6.)

Theorem (Pho-S.)

If B_f has only simple singularities (ADE-singularities), then the minimal resolution $X_f \to Y_f$ of the double cover $Y_f \to \mathbb{P}^2$ branched along B_f is supersingular with Artin invariant ≤ 3.

Conversely, for any supersingular $K3$ surface X with Artin invariant ≤ 3, there is a polynomial f such that $X \cong X_f$.

Let $\omega \in \mathbb{F}_{25}$ be a root of $\omega^2 + \omega + 1 = 0$.

Theorem

The Artin invariant of X_f is 1 if and only if $B_f \subset \mathbb{P}^2$ is projectively isomorphic to one of the following. We put $f(x) = x^2(x-1)^2 g(x)$.

No.	g	$\mathrm{Sing}(B_f)$
1	$x(x-1)$	$2E_8 + A_4$
2	x	$A_9 + E_8 + A_4$
3	$x(x-2)$	$E_8 + 3A_4$
4	1	$A_9 + 3A_4$
5	$x + 2\omega + 3$	$A_9 + 3A_4$
6	$x^2 - x + 2$	$5A_4$
7	$(x+1)(x+3)$	$5A_4$
8	$x^2 - \omega x + \omega$	$5A_4$
$\bar{8}$	$x^2 - \bar{\omega} x + \bar{\omega}$	$5A_4$

These 9 models are not projectively isomorphic.

We have a similar description for the locus

$$\{ f \mid \text{the Artin invariant of } X_f \text{ is } 2 \},$$

and hence we can calculate the Artin invariant of X_f from the polynomial f.

Problem

The supersingular $K3$ surface $X_F \to \mathbb{P}^2$ branched along

$$x^6 + y^6 + z^6 = 0$$

is also with Artin invariant 1. Write the birational maps between X_F and the 9 projective models.