BORCHERDS METHOD AND AUTOMORPHISM GROUPS OF K3 SURFACES

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1. Introduction

In this note, we explain Borcherds method to calculate the automorphism group of a certain chamber in a hyperbolic space associated with an even hyperbolic lattice, and its application to the study of the automorphism groups of $K3$ surfaces. We then present some examples of our computations. See the preprint [18] for details.

2. Borcherds method

First we fix some terminologies and notation. Let S be a lattice; that is, S is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle : S \times S \to \mathbb{Z}.$$

We say that S is *hyperbolic* if $S \otimes \mathbb{R}$ is of signature $(1, n-1)$. A *positive cone* of a hyperbolic lattice S is one of the two connected components of

$$\{\ x \in S \otimes \mathbb{R}\ \mid\ x^2 > 0\ \}.$$

Let $\mathcal{P}(S)$ be a positive cone of a hyperbolic lattice S. The stabilizer subgroup in $\mathrm{O}(S)$ of $\mathcal{P}(S)$ is denoted by $\mathrm{O}^+(S)$. We say that S is *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in S$. Suppose that S is even. A *root* is a vector $r \in S$ such that $r^2 = -2$. Each root $r \in S$ defines a reflection

$$s_r : x \mapsto x + \langle x, r\rangle r.$$

We denote by $W(S)$ the subgroup of $\mathrm{O}^+(S)$ generated by all the reflections s_r with respect to the roots. Then $W(S)$ is a normal subgroup of $\mathrm{O}^+(S)$, and $W(S)$ acts on $\mathcal{P}(S)$. For $v \in S \otimes \mathbb{R}$ with $v^2 < 0$, we put

$$(v)^\perp := \{\ x \in \mathcal{P}(S)\ \mid\ \langle x, v\rangle = 0\ \}.$$

Let N be the closure in $\mathcal{P}(S)$ of a connected component of

$$\mathcal{P}(S) \setminus \bigcup_{r^2=-2} (r)^\perp,$$

and we consider its automorphism group

$$\mathrm{Aut}(N) := \{ \ g \in \mathrm{O}^+(S) \ \mid \ N^g = N \ \}.$$

(We let $\mathrm{O}(S)$ act on $S \otimes \mathbb{R}$ *from the right*.) Then N is a standard fundamental domain of the action of $W(S)$ on $\mathcal{P}(S)$, and $\mathrm{O}^+(S)$ is the semi-direct product $W(S) \rtimes \mathrm{Aut}(N)$. Let G be a subgroup of $\mathrm{O}^+(S)$ with finite index. Borcherds method [1, 2] is a method to calculate a finite set of generators of

$$\mathrm{Aut}(N) \cap G$$

by embedding S into an even hyperbolic unimodular lattice of rank $n = 10, 18$ or 26 primitively.

Remark 2.1. The lattices for which $\mathrm{Aut}(N)$ is finite are classified by Nikulin [11, 12] and Vinberg [23]. Therefore we will be concerned with the cases where $\mathrm{Aut}(N)$ is infinite.

Borcherds method is based on the theory of Weyl vectors due to Conway [3]. Let L_n denote the even hyperbolic unimodular lattice of rank $n = 10, 18$ or 26. Then L_n is unique up to isomorphisms. Let $\mathcal{D}$ be the closure in $\mathcal{P}(L_n)$ of a connected component of

$$\mathcal{P}(L_n) \setminus \bigcup_{r^2=-2} (r)^\perp,$$

which is a standard fundamental domain of the action of $W(L_n)$ on $\mathcal{P}(L_n)$. We call $\mathcal{D}$ a *Conway chamber*. We say that a vector $w \in L_n$ is a *Weyl vector of* $\mathcal{D}$ if

$$\{ \ (r)^\perp \ \mid \ r^2 = -2, \ \ \langle w, r \rangle = 1 \ \}$$

is the set of walls of $\mathcal{D}$.

Theorem 2.2 (Conway [3])**.** *A Weyl vector exists.*

In fact, Conway [3] gave an explicit description of Weyl vectors.

Example 2.3. Let U denote the hyperbolic plane with a Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let Λ be the *negative-definite* Leech lattice. Then we have $L_{26} \cong U \oplus \Lambda$. Under this isomorphism, we denote vectors of L_{26} by (x, y, λ), where $(x, y) \in U$ and $\lambda \in \Lambda$. Then $w_0 := (1, 0, 0)$ is a Weyl vector of a Conway chamber $\mathcal{D}_0$. The set of walls of $\mathcal{D}_0$ is equal to $\{(r)^\perp \,|\, r \in \mathcal{R}_0\}$, where

$$\mathcal{R}_0 := \{ \ (-1 - \lambda^2/2, 1, \lambda) \ \mid \ \lambda \in \Lambda \ \}.$$

Hence $\mathrm{Aut}(\mathcal{D}_0) \subset \mathrm{O}^+(L_{26})$ is isomorphic to the Conway group Co_∞.

Suppose that we are given the following objects:

- an even hyperbolic lattice S of rank < 26,
- a subgroup $G \subset \mathrm{O}^+(S)$ of finite index, and
- a standard fundamental domain N of the action of $W(S)$ on $\mathcal{P}(S)$.

We assume that S is embedded in L_n primitively, and that any element of G can be extended to an isometry of L_n. (In the actual application to the study of $K3$ surfaces, the second condition can be easily checked by the theory of discriminant forms.) Moreover, when $n = 26$, we further assume that the orthogonal complement R of S in L_{26} cannot be embedded into Λ. (This condition is satisfied if R has a vector of square norm -2.)

A Conway chamber $\mathcal{D}$ is said to be *S-nondegenerate* if $D := \mathcal{D} \cap \mathcal{P}(S)$ contains a non-empty open subset of $\mathcal{P}(S)$. In this case, we say that D is an *induced chamber*. Since $\mathcal{P}(L_n)$ is tiled by Conway chambers, $\mathcal{P}(S)$ is tiled by induced chambers. Moreover, since a root of S is a root of L_n, the given standard fundamental domain N in $\mathcal{P}(S)$ is a union of induced chambers. Two induced chambers D and D' are said to be *G-congruent* if there exists $g \in G$ such that $D' = D^g$.

Proposition 2.4. *The number of G-congruence classes of induced chambers is finite.*

Proposition 2.5. *The number of walls of an induced chamber $D = \mathcal{D} \cap \mathcal{P}(S)$ is finite, and we can calculate the set of walls of D from the Weyl vector of $\mathcal{D}$.*

Hence $\mathrm{Aut}(D) \cap G = \{g \in G \mid D^g = D\}$ is finite for any induced chamber D. Moreover, for two induced chambers D and D', we can determine whether D and D' are G-congruent or not.

Borcherds method makes a complete list $\mathbb{D}$ of representatives of all G-congruence classes of induced chambers contained in N. We start from an induced chamber D_0 contained in N, set $\Gamma := \{\}$ and $\mathbb{D} := [D_0]$, and proceed as follows. For an induced chamber $D_i \in \mathbb{D} = [D_0, \dots, D_k]$, we calculate the set of walls of D_i and the finite group $\mathrm{Aut}(D_i) \cap G$. We append a set of generators of $\mathrm{Aut}(D_i) \cap G$ to Γ. For each wall $(v)^\perp$ of D_i that is not a wall of N, we calculate the induced chamber D' adjacent to D_i along $(v)^\perp$, and determine whether D' is G-congruent to some $D_j \in \mathbb{D}$. If there are no such D_j, then we set $D_{k+1} := D'$ and append it to $\mathbb{D}$ as a representative of a new G-congruence class. If there exist $D_j \in \mathbb{D}$ and $h \in G$ such that $D' = D_j^h$, then we append h to Γ. We repeat this process until we reach the end of the list $\mathbb{D}$. By Proposition 2, this algorithm terminates.

Then the group $\mathrm{Aut}(N) \cap G$ is generated by the elements in the finite set Γ. Moreover, for each $D \in \mathbb{D}$, let $F(D) \subset D$ be a fundamental domain of the action of the finite group $\mathrm{Aut}(D) \cap G$ on D. Then their union $\bigcup F(D)$ is a fundamental domain of the action of $\mathrm{Aut}(N) \cap G$ on N.

3. The automorphism group of a $K3$ surface

Let X be a complex algebraic $K3$ surface, or a supersingular $K3$ surface in odd characteristic. In virtue of the Torelli-type theorem due to Piatetski-Shapiro and Shafarevich [15] and Ogus [13, 14], we can study $\mathrm{Aut}(X)$ by the Néron-Severi lattice S_X of X. Using Borcherds method, we will obtain a finite set of generators of the image the natural homomorphism

$$\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{O}(S_X).$$

For simplicity, we concentrate upon a complex algebraic $K3$ surface X. Then we have

$$S_X := \{ \, [D] \in H^2(X, \mathbb{Z}) \;\mid\; D \text{ is a divisor of } X \, \}.$$

Note that S_X is an even hyperbolic lattice. Let $\mathcal{P}(S_X)$ be the positive cone of S_X containing an ample class, and we put

$$N(X) := \{ \, x \in \mathcal{P}(S_X) \;\mid\; \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } X \, \}.$$

Then $N(X)$ is bounded by $([C])^{\perp}$, where C runs through the set of smooth rational curves on X. Since $C^2 = -2$ for any smooth rational curve C on X, the domain $N(X)$ is a standard fundamental domain of the action of the Weyl group $W(S_X)$ on $\mathcal{P}(S_X)$. (See [16], for example.) By Torelli theorem due to Piatetski-Shapiro and Shafarevich [15], the natural homomorphism φ_X has only finite kernel. Let G_ω denote the subgroup of $\mathrm{O}^+(S_X)$ consisting of elements $g \in \mathrm{O}^+(S_X)$ that lift to a Hodge isometry of $H^2(X, \mathbb{Z})$. Note that G_ω is of finite index in $\mathrm{O}^+(S_X)$. Then we have

$$\mathrm{Im}\, \varphi_X := \mathrm{Aut}(N(X)) \cap G_\omega.$$

Therefore, applying Borcherds method, we can calculate a finite set of generators of $\mathrm{Im}\, \varphi_X$.

Example 3.1. The first application was done by Kondo [9]. Let C be a generic genus 2 curve, and let $\mathrm{Jac}(C)$ be the Jacobian variety of C. We consider the Kummer surface

$$X := \mathrm{Km}(\mathrm{Jac}(C))$$

associated with $\mathrm{Jac}(C)$; that is, X is the minimal resolution of the quotient $\mathrm{Jac}(C)/\langle \iota \rangle$ with 16 ordinary nodes, where ι is the inversion $x \mapsto -x$ of $\mathrm{Jac}(C)$.

Let $p_0, \dots, p_5$ be the Weierstrass points of C, and let Θ_0 be the image of

$$C \hookrightarrow \mathrm{Jac}(C) = \mathrm{Pic}^0(C)$$

given by $p \mapsto [p - p_0]$. For a 2-torsion point t of $\mathrm{Jac}(C)$, let Θ_t denote the translate of Θ_0 by t. Then $\Theta_t/\langle \iota \rangle$ is a rational curve passing through exactly 6 points of the 16 ordinary nodes of $\mathrm{Jac}(C)/\langle \iota \rangle$. Let D_t be the strict transform of $\Theta_t/\langle \iota \rangle$ by the minimal resolution $X \to \mathrm{Jac}(C)/\langle \iota \rangle$, and let E_t be the exceptional curve over the node of $\mathrm{Jac}(C)/\langle \iota \rangle$ corresponding to t. Since we have assumed that C is generic, these $32 = 16 + 16$ curves $\{D_t, C_t\}$ on X generate the Néron-Severi lattice S_X of X. We have $\mathrm{rank}(S_X) = 17$ and $\mathrm{disc}(S_X) = 64$. On the other hand, the subgroup G_ω is of index 32 in $\mathrm{O}^+(S_X)$.

We embed S_X into $L_{26} = U \oplus \Lambda$, where U is the hyperbolic plane and Λ is the Leech lattice. Then, at the end of the Borcherds method, we have $\mathbb{D} = \{D_0\}$, and $|\,\mathrm{Aut}(D_0) \cap G_\omega| = 32$. The induced chamber D_0 has 316 walls, which are decomposed by the action of $\mathrm{Aut}(D_0) \cap G_\omega$ into 23 orbits as

$$316 = 32 \times 1 + 4 \times 15 + 32 \times 7 \quad (23 = 1 + 15 + 7).$$

The first orbit consists of 32 walls of $N(X)$, and corresponds to the set $\{D_t, C_t\}$ of smooth rational curves on X. From the other 22 orbits, we obtain extra automorphisms. Hence the image of $\varphi_X \colon \mathrm{Aut}(X) \to \mathrm{O}(S_X)$ is generated by the finite group $\mathrm{Aut}(D_0) \cap G_\omega$ and those 22 extra automorphisms.

Since this work, automorphism groups of the following $K3$ surfaces have been determined by this method;

- the supersingular $K3$ surface in characteristic 2 with Artin invariant 1 by Dolgachev and Kondo [4],
- complex Kummer surfaces of product type by Keum and Kondo [8],
- the Hessian quartic surface by Dolgachev and Keum [5],
- the singular $K3$ surface X with $\mathrm{disc}\, T_X = 7$ by Ujikawa [21], where T_X is the transcendental lattice of X, and
- the supersingular $K3$ surface in characteristic 3 with Artin invariant 1 by Kondo and Shimada [10].

The classical result of Vinberg [22] can be also treated by this method.

However, in all these cases, there exists only one G-congruence classes, and the computation is very easy. In fact, Borcherds [1, Lemma 5.1] proved the following:

Lemma 3.2 (Borcherds)**.** *If the orthogonal complement R of S in L_{26} is a root lattice, then any two induced chambers are $\mathrm{O}^+(S)$-congruent.*

4. New Examples

We have written Borcherds method using the `C` library `gmp` [6], and carried out the computation in some cases with many G-congruence classes. It turns out that, in the case where the orthogonal complement R of S in L_{26} is *not* a root lattice, the number of G-congruence classes of induced chambers can be very large.

Our main algorithm contains sub-algorithms that calculate the set of walls of a given induced chamber, compute the adjacent induced chamber along a given wall, and determine whether an induced chamber is G-congruent to another induced chamber. In these algorithms, we use methods given in our previous paper [17]. In order to calculate the set of walls of an induced chamber, we had to employ the standard algorithm of linear programming.

Example 4.1. Let X be a $K3$ surface with $\mathrm{rank}(S_X) = 20$ and $\mathrm{disc}(S_X) = 11$. Then the transcendental lattice T_X of X has a Gram matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix},$$

and X is unique up to isomorphisms by the theorem of Shioda and Inose [20]. We embed S_X into $L_{26} = U \oplus E_8 \oplus E_8 \oplus E_8$. Then we have $|\mathbb{D}| = 1098$. The domain $\bigcup D$ has 719 walls, among which 347 are walls of $N(X)$. In particular, the action of $\mathrm{Aut}(X)$ on the set of smooth rational curves on X has at most 347 orbits. The output Γ consists of 789 elements.

Example 4.2. Let X be a $K3$ surface with $\mathrm{rank}(S_X) = 20$ and $\mathrm{disc}(S_X) = 15$, which is unique up to isomorphisms. Then we have $|\mathbb{D}| = 2051$. The output Γ consists of 1098 elements.

Example 4.3. Let X be a $K3$ surface with $\mathrm{rank}(S_X) = 20$ and $\mathrm{disc}(S_X) = 16$, which is unique up to isomorphisms. Then we have $|\mathbb{D}| = 4538$. The output Γ consists of 3308 elements.

See the author's web page [19] for the numerical outputs of the computation of these three cases.

When $\mathrm{rank}\, S_X$ is small, we can embed S_X into L_{10}.

Example 4.4. Let X be a $K3$ surface whose Néron-Severi lattice S_X has a Gram matrix

$$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 4 & 2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix},$$

and whose period is sufficiently generic. We embed S_X into $L_{10} = U \oplus E_8$. Then we have $|\mathbb{D}| = 504$. The output Γ consists of 7 elements.

Example 5. Let k be an integer > 1. Let X be a $K3$ surface whose Néron-Severi lattice S_X has a Gram matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2k \end{bmatrix},$$

and whose period is sufficiently generic. This $K3$ surface X has an elliptic fibration $\phi : X \to \mathbb{P}^1$ with a zero section. We can assume that the vector $[1, 0, 0] \in S_X$ is the class f_ϕ of a fiber of ϕ and that the vector $[0, 1, 0] \in S_X$ is the class z_ϕ of the zero section of ϕ. Since $k > 1$, the Mordell-Weil group MW_ϕ of $\phi : X \to \mathbb{P}^1$ is of rank 1. Therefore $\mathrm{Aut}(X)$ contains a subgroup $\mathrm{MW}_\phi \rtimes \langle \iota_X \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ generated by the translations by MW_ϕ and the inversion ι_X of $\phi : X \to \mathbb{P}^1$. This subgroup is generated by the two involutions

$$h_1 := \iota_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad h_2 := \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & -1 \\ 2k & 0 & -1 \end{bmatrix}.$$

The norm of $[1, x, y] \in S_X \otimes \mathbb{R}$ is $2x - 2x^2 - 2ky^2$. Hence, by the map $[1, x, y] \mapsto (x, y)$, the hyperbolic plane associated with S_X is identified with

$$H_X := \{\ (x, y) \in \mathbb{R}^2 \ \mid\ (x - 1/2)^2 + (\sqrt{k}y)^2 < 1/4\ \}.$$

The vector f_ϕ corresponds to the point $(0, 0)$ of $\overline{H}_X$, and the hyperplane $(z_\phi)^\perp$ is given by $x = 1/2$.

Suppose that $2k = -18$. The union

$$F := \bigcup_{D \in \mathbb{D}} D$$

is depicted in Figure 4.2 using H_X. For each $D \in \mathbb{D}$, we have $\mathrm{Aut}(D) \cap G_\omega = \{1\}$, and hence F is the fundamental domain of the action of $\mathrm{Aut}(X)$ on $N(X)$. The

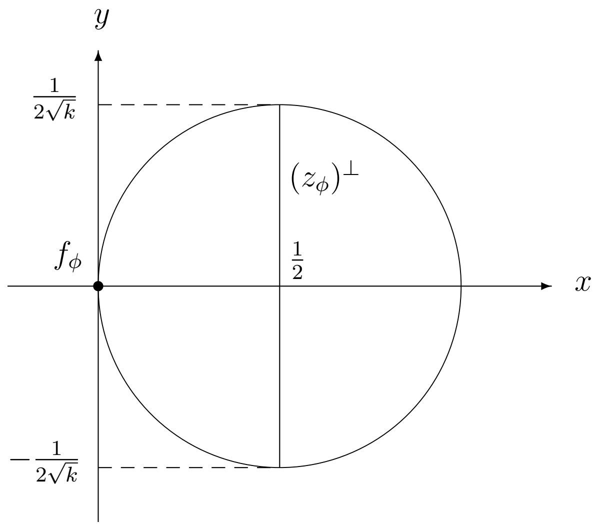

FIGURE 4.1. H_X

domain F has 4 walls, two of which are walls of $N(X)$ and is depicted by thick lines, while the other two walls correspond to the two automorphisms h_1 and h_2.

Suppose that $2k = -20$. Then F is depicted in Figure 4.3. For each $D \in \mathbb{D}$, we have $\mathrm{Aut}(D) \cap G_\omega = \{1\}$, and hence F is the fundamental domain of the action of $\mathrm{Aut}(X)$ on $N(X)$. The domain F has 5 walls, two of which are walls of $N(X)$, while the other 3 walls correspond to the automorphisms h_1 and h_2 and an extra automorphism

$$h_3 := \begin{bmatrix} 121 & 40 & -18 \\ 120 & 41 & -18 \\ 1080 & 360 & -161 \end{bmatrix}.$$

See the author's web page [19] for more examples of this type.

5. INTRACTABLE EXAMPLES

We applied our algorithm to the following $K3$ surfaces.

(1) The complex Fermat quartic surface $X \subset \mathbb{P}^3$. The Picard number of X is 20, and a Gram matrix of the transcendental lattice is

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

Note that X contains 48 lines. We can calculate a Gram matrix of S_X, because S_X is generated by the classes of 20 lines on X.

(2) The double plane $\pi : X \to \mathbb{P}^2$ branched along the Fermat curve $B \subset \mathbb{P}^2$ of degree 6 in characteristic 5. This $K3$ surface X is supersingular with Artin

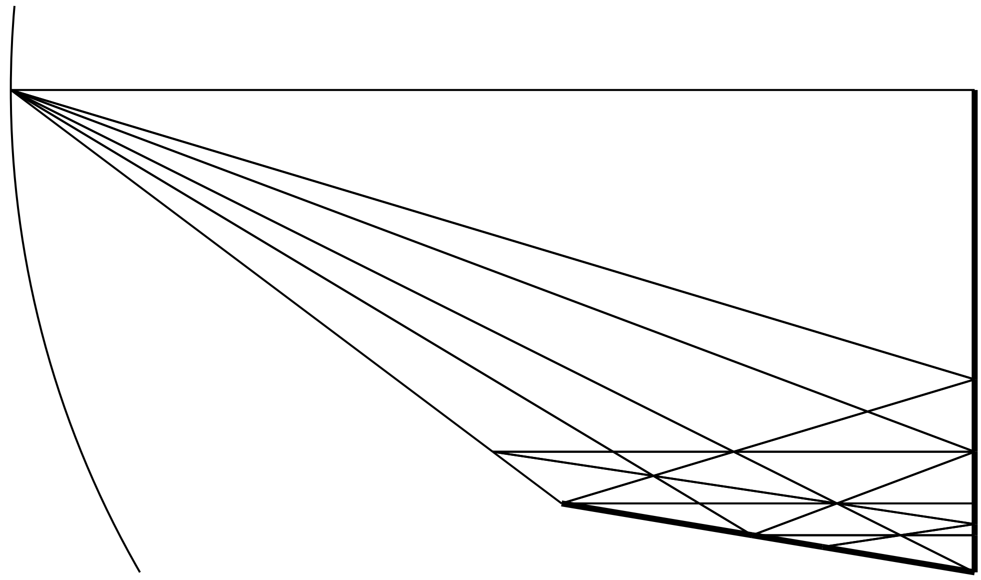

FIGURE 4.2. F for the case $-2k = -18$

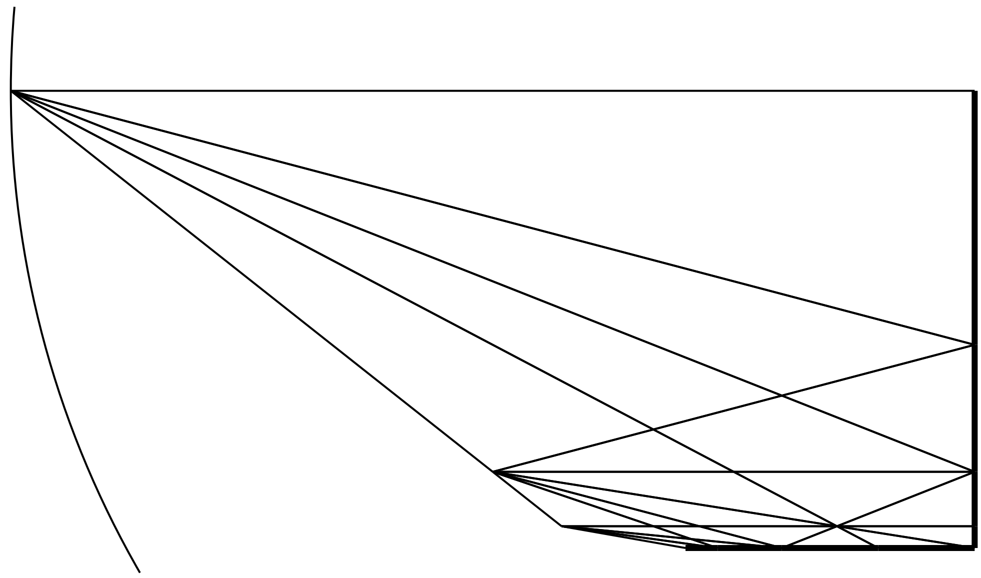

FIGURE 4.3. F for the case $-2k = -20$

invariant 1, and contains 252 smooth rational curves that are mapped to lines on $\mathbb{P}^2$ isomorphically by π. The lattice S_X is generated by the classes of 22 curves among them. Thus we can calculate a Gram matrix of S_X.

The computation for these two cases did not terminate in a reasonable time, because there are too many G-congruence classes of induced chambers. However, we obtained many interesting automorphisms of these $K3$ surfaces. For the supersingular case (2), see the preprint [7].

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