On the supersingular $K3$ surface in characteristic 5 with Artin invariant 1

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Introduction

A $K3$ surface is called *supersingular* if its Picard number is 22.
Let Y be a supersingular $K3$ surface in characteristic $p > 0$.
Let S_Y be its Néron-Severi lattice, and put $S_Y^\vee := \mathrm{Hom}(S_Y, \mathbb{Z})$.
The intersection form on S_Y yields $S_Y \hookrightarrow S_Y^\vee$.

Artin proved that

$$S_Y^\vee / S_Y \cong (\mathbb{Z}/p\mathbb{Z})^{2\sigma},$$

where σ is an integer such that $1 \le \sigma \le 10$,
which is called the *Artin invariant* of Y.

Ogus and Rudakov-Shafarevich proved that
a supersingular $K3$ surface with Artin invariant 1 in characteristic p is unique up to isomorphisms.

We consider *the* supersingular $K3$ surface X in characteristic 5 with Artin invariant 1.

We work in characteristic 5.

Let B_F be the Fermat sextic curve (or the Hermitian curve) in $\mathbb{P}^2$:

$$x^6 + y^6 + z^6 = 0 \quad (x\bar{x} + y\bar{y} + z\bar{z} = 0).$$

Let $\pi_F : X \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along B_F:

$$X : w^2 = x^6 + y^6 + z^6.$$

Then X is a supersingular $K3$ surface in characteristic 5 with Artin invariant 1

Proof.

Let P be an $\mathbb{F}_{25}$-rational point of B_F,
and ℓ_P the tangent line to B_F at P.
Then ℓ_P intersects B_F at P with multiplicity 6,
and hence $\pi_F^{-1}(\ell_P)$ splits into two smooth rational curves.
Since $|B_F(\mathbb{F}_{25})| = 126$,
we obtain 252 smooth rational curves on X.
Calculating the intersection numbers of these 252 smooth rational curves, we see that
their classes span a lattice of rank 22 (hence X is supersingular)
with discriminant -25 (hence $\sigma = 1$). □

In fact, the lattice S_X is generated by appropriately chosen 22 curves among these 252 curves.

Corollary

Every class of S_X is represented by a divisor defined over $\mathbb{F}_{25}$.

Corollary

Every projective model of X can be defined over $\mathbb{F}_{25}$.

Remark

Schütt proved the above results for supersingular $K3$ surfaces of Artin invariant 1 in any characteristics.

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$$

Problem: Find distinct projective models of X (especially of degree 2) as many as possible.

We put

$$\mathcal{P}_2 := \{\ h \in S_X \ \mid\ h \text{ is a polarization of degree } 2\ \},$$

that is, $h \in S_X$ belongs to $\mathcal{P}_2$ if and only if the line bundle $\mathcal{L} \to X$ corresponding to h gives a double covering $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$.
Let B_h be the branch curve of $\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^2$.

For $h, h' \in \mathcal{P}_2$, we say $h \sim h'$ if there exists $g \in \mathrm{Aut}(X)$ such that $g^*(h) = h'$, or equivalently, there exists $\phi \in \mathrm{PGL}_3(k)$ such that $\phi(B_{h'}) = B_h$.

Problem: Describe $\mathcal{P}_2/\sim$.

The lattice S_X is characterized as the unique even hyperbolic lattice of rank 22 with $S_X^\vee/S_X \cong (\mathbb{Z}/5\mathbb{Z})^2$.
Therefore we can obtain a list of combinatorial data of these B_h by lattice theoretic method, which was initiated by Yang.

We try to find defining equations of these B_h, and understand their relations.

- Naive method.
  Projective models of the supersingular K3 surface with Artin invariant 1 in characteristic 5. J. Algebra 403 (2014), 273-299.
- Specialization from $\sigma = 3$ (joint work with Pho Duc Tai).
  Unirationality of certain supersingular K3 surfaces in characteristic 5. Manuscripta Math. 121 (2006), no. 4, 425–435.
- Ballico-Hefez curve (joint work with Hoang Thanh Hoai).
  On Ballico-Hefez curves and associated supersingular surfaces, to appear in Kodai Math. J.
- Borcherds' method (joint work with T. Katsura and S. Kondo).
  On the supersingular K3 surface in characteristic 5 with Artin invariant , preprint, arXiv:1312.0687

Naive method

Classification by relative degrees with respect to h_F.

We have the polarization $h_F \in \mathcal{P}_2$ that gives the Fermat double sextic plane model $\pi_F : X \to \mathbb{P}^2$:

$$h_F = [1, 1, 0, \dots, 0].$$

We have

$$\mathrm{Aut}(X, h_F) = \mathrm{PGU}_3(\mathbb{F}_{25}).2,$$

which is of order 756000.

For $a \in \mathbb{Z}_{>0}$, we put

$$\mathcal{P}_2(a) := \{ \, h \in \mathcal{P}_2 \mid \langle h_F, h \rangle = a \, \}.$$

For any $a \in \mathbb{Z}_{>0}$, the set

$$\mathcal{V}_2(a) := \{\ h \in S_X \ |\ h^2 = 2,\ \ \langle h_F, h \rangle = a\ \}$$

is finite.

Then $h \in \mathcal{V}_2(a)$ belongs to $\mathcal{P}_2(a)$ if h is nef and not of the form

$$2 \cdot f + z, \ \ \text{with}\ \ f^2 = 0, z^2 = -2, \langle f, z \rangle = 1.$$

The vector $h \in \mathcal{V}_2$ is nef if and only if there are no vectors $r \in S_X$ such that

$$r^2 = -2,\ \ \langle h_F, r \rangle > 0,\ \ \langle h, r \rangle < 0.$$

Thus we can calculate $\mathcal{P}_2(a)$ for a given $a \in \mathbb{Z}_{>0}$.

We have calculated $\mathcal{P}_2(a)$ for $a \leq 5$.
Their union consists of $146, 945, 851$ vectors.

From the defining ideals of the 22 lines on X_F we have chosen as a basis of S_X,
we can calculate the defining equations of B_h for each h, and hence we can determine whether $h \sim h'$ or not.

Under $\sim$, they are decomposed into 65 equivalence classes.

0: $\mathrm{Sing} = 0$: $\mathrm{N} = 13051$: $\mathrm{h} = [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$:
$x^6 + y^6 + 1$

1: $\mathrm{Sing} = 6A_1$: $\mathrm{N} = 5607000$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1]$:
$x^6 + 3\,x^5y + x^4y^2 + 2\,x^3y^3 + y^6 + 3\,x^4 + 3\,x^2y^2 + xy^3 + 3\,xy + 2\,y^2 + 4$

2: $\mathrm{Sing} = 7A_1$: $\mathrm{N} = 6678000$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]$:
$x^6 + 2\,x^4y^2 + x^2y^4 + x^2y^3 + 2\,y^5 + x^4 + 2\,y^4 + 2\,x^2y + 2\,y^3 + 3\,y^2 + 3\,y + 2$

3: $\mathrm{Sing} = 3A_1 + 2A_2$: $\mathrm{N} = 2268000$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0]$:
$x^6 + 3\,x^3y^3 + y^6 + 3\,x^3y + 2\,y^2 + 2$

4: $\mathrm{Sing} = 8A_1$: $\mathrm{N} = 2457000$: $\mathrm{h} = [0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0]$:
$x^6 + 3\,x^4y^2 + x^2y^4 + 4\,x^2y^3 + 4\,y^5 + x^4 + 2\,x^2y^2 + 3\,y^4 + 2\,x^2y + 4\,x^2 + y^2 + 4\,y$

5: $\mathrm{Sing} = 8A_1$: $\mathrm{N} = 2268000$: $\mathrm{h} = [0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0]$:
$x^4y^2 + x^2y^4 + 2\,x^4 + 4\,x^2y^2 + y^4 + x^2 + 4\,y^2 + 4$

6: $\mathrm{Sing} = 6A_1 + A_2$: $\mathrm{N} = 1512000$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0]$:
$x^6 + 4\,x^4y^2 + 2\,x^2y^4 + 2\,x^2y + y^3 + 4$

7: $\mathrm{Sing} = 6A_1 + A_2$: $\mathrm{N} = 4914000$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1]$:
$\sqrt{2}x^3y^3 + \left(1 + 3\sqrt{2}\right)x^2y^4 + x^4 + \left(2 + 2\sqrt{2}\right)x^3y + \left(1 + 4\sqrt{2}\right)x^2y^2 + xy^3 + \left(2 + 2\sqrt{2}\right)y^4 + \sqrt{2}x^2 + \left(1 + 3\sqrt{2}\right)xy$

11: $\mathrm{Sing} = 9A_1$: $\mathrm{N} = 84000$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, -1, 0, 0, 0]$:
$x^6 + 4\,x^3y^3 + 4\,y^6 + x^4 + 4\,xy^3 + 3\,x^2 + 4$

24: $\mathrm{Sing} = 5A_1 + 2A_2$: $\mathrm{N} = 378000$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0]$:
$x^3y^3 + x^4 + x^2y^2 + y^4 + xy$

32: $\mathrm{Sing} = 10A_1$: $\mathrm{N} = 226800$: $\mathrm{h} = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1]$:
$x^6 + 2\,x^4y + y^5 + 4\,x^2y^2 + y^3 + 4\,x^2 + 4\,y$

33: $\mathrm{Sing} = 10A_1$: $\mathrm{N} = 756000$: $\mathrm{h} = [0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0]$:
$x^6 + x^4y^2 + 3\,x^3y^3 + 3\,x^2y^4 + 2\,y^6 + x^2y^2 + 4\,xy + 4$

$\vdots$

Remark. Up to $\langle h, h_F \rangle \leq 5$, only A_1 and A_2 appear as singularities of B_h.

Specialization from $\sigma = 3$

For a polynomial $f \in k[x]$ of degree ≤ 6, let $B_f \subset \mathbb{P}^2$ be the projective plane curve of degree 6 whose affine part is

$$y^5 - f(x) = 0.$$

(If $\deg f < 6$, we add the line at infinity.)

Remark If f is general of degree 6, then $\mathrm{Sing}(B_f)$ is $5A_4$.

Theorem
If B_f has only ADE-singularities, then the minimal resolution $W_f \to Y_f$ of the double cover $Y_f \to \mathbb{P}^2$ branched along B_f is supersingular with Artin invariant ≤ 3.

Conversely, for any supersingular $K3$ surface W with Artin invariant ≤ 3, there is a polynomial f such that $W \cong W_f$.

Let $\omega \in \mathbb{F}_{25}$ be a root of $\omega^2 + \omega + 1 = 0$.

Theorem

The Artin invariant of W_f is 1 if and only if $B_f \subset \mathbb{P}^2$ is projectively isomorphic to one of the following. We put $f(x) = x^2(x-1)^2 g(x)$.

No.	g	$\mathrm{Sing}(B_f)$
1	$x(x-1)$	$2E_8 + A_4$
2	x	$A_9 + E_8 + A_4$
3	$x(x-2)$	$E_8 + 3A_4$
4	1	$A_9 + 3A_4$
5	$x + 2\omega + 3$	$A_9 + 3A_4$
6	$x^2 - x + 2$	$5A_4$
7	$(x+1)(x+3)$	$5A_4$
8	$x^2 - \omega x + \omega$	$5A_4$
$\bar{8}$	$x^2 - \bar{\omega} x + \bar{\omega}$	$5A_4$

These 9 models are not projectively isomorphic.

Ballico-Hefez curve (joint work with Hoang Thanh Hoai)

Let $k = \bar{k}$ be of characteristic p, and q a power of p.
A *Ballico-Hefez curve* B is a projective plane curve defined by

$$x^{\frac{1}{q+1}} + y^{\frac{1}{q+1}} + z^{\frac{1}{q+1}} = 0.$$

More precisely, B is the image of $x + y + z = 0$ by the morphism

$$[x : y : z] \mapsto [x^{q+1} : y^{q+1} : z^{q+1}].$$

Then B has the following properties:

- of degree $q+1$ with $(q^2-q)/2$ ordinary nodes as its only singularities,
- the dual curve $B^\vee$ is of degree 2,
- the natural morphism $C(B) \to B^\vee$ has inseparable degree q, where $C(B) \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ is the conormal variety of B.

Ballico and Hefez proved the following.

Theorem

Let $D \subset \mathbb{P}^2$ be an irreducible singular curve of degree $q+1$ such that $D^\vee$ is of degree > 1 and the natural morphism $C(D) \to D^\vee$ has inseparable degree q. Then D is projectively isomorphic to the Ballico-Hefez curve.

Proposition

When p is odd, B is defined by

$$2(x^q y + xy^q) - z^{q+1} - (z^2 - 4yx)^{\frac{q+1}{2}} = 0.$$

Proposition

Let d be a divisor of $q+1$. Then the cyclic cover S of $\mathbb{P}^2$ of degree d branched along B is unirational and hence is supersingular.

Proposition

Suppose that $p = q = 5$ and $d = 2$. Then S is the supersingular $K3$ surface X in characteristic 5 with Artin invariant 1 with $10A_1$.

Borcherds' method (joint work with Katsura and Kondo)

The lattice S_X can be embedded primitively
into an even unimodular hyperbolic lattice L of rank 26,
which is unique up to isomorphisms.
The chamber decomposition of the positive cone of L into standard fundamental domains of the Weyl group $W(L)$ was determined by Conway.
The tessellation by Conway chambers induces a chamber decomposition of the positive cone of S_X,
and the nef cone of X is a union of induced chambers.

In an attempt to determine $\mathrm{Aut}(X)$, we have investigated several induced chambers in the nef cone of X, and obtained the following polarizations with big automorphism groups.

Theorem

(1) There exist 300 polarizations h_1 with the following properties.
$h_1^2 = 60$, $\langle h_F, h_1 \rangle = 15$.
$\mathrm{Aut}(X, h_1) \cong \mathfrak{A}_8$.
The minimal degree of curves on (X, h_1) is 5,
(X, h_1) contains exactly 168 smooth rational curves of degree 5, on which $\mathrm{Aut}(X, h_1)$ acts transitively.

Under suitable definition of adjacency relation, these 300 polarizations form 6 Hoffman-Singleton graphs.

(2) There exist 15700 polarizations h_2 with the following properties.
$h_2^2 = 80$, $\langle h_F, h_2 \rangle = 40$.
$\mathrm{Aut}(X, h_2) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/3\mathbb{Z} \times \mathfrak{S}_4)$ (order 1152).
The minimal degree of curves on (X, h_2) is 5,
and (X, h_2) contains exactly 96 smooth rational curves of degree 5, which decompose into two orbits under the action of $\mathrm{Aut}(X, h_2)$.
These 96 curves form six (16_6)-configurations.