Computer-aided calculations in the study of K3 surfaces

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The purpose of this talk is to demonstrate, on concrete examples, how far we can go in the study of $K3$ surfaces with the lattice theory and a help of a computer.

Definition

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle \colon L \times L \to \mathbb{Z}.$$

Let L be a lattice of rank n. If we choose a basis $v_1, \ldots, v_n$ of the free $\mathbb{Z}$-module L, then the bilinear form $\langle\ ,\ \rangle \colon L \times L \to \mathbb{Z}$ is expressed by the *Gram matrix*

$$G_L := (\langle v_i, v_j \rangle)_{1 \le i,j \le n}.$$

We will use a Gram matrix to express a lattice in the computer.

By a *quadratic triple* of n-variables, we mean a triple $[Q, \ell, c]$, where

- Q is an $n \times n$ symmetric matrix with entries in $\mathbb{Q}$,
- ℓ is a column vector of length n with entries in $\mathbb{Q}$, and
- c is a rational number.

An element of $\mathbb{R}^n$ is written as a row vector

$$\boldsymbol{x} = [x_1, \dots, x_n] \quad \in \quad \mathbb{R}^n.$$

The *inhomogeneous quadratic function* $q_{QT} : \mathbb{Q}^n \to \mathbb{Q}$ associated with a quadratic triple $QT = [Q, \ell, c]$ is defined by

$$q_{QT}(\boldsymbol{x}) := \boldsymbol{x}\, Q\, {}^t\boldsymbol{x} + 2\, \boldsymbol{x}\, \ell + c.$$

We say that $QT = [Q, \ell, c]$ is *negative* if the symmetric matrix Q is negative-definite.

Algorithm

Let $QT = [Q, \ell, c]$ be a negative quadratic triple of n-variables. Then we can compute the finite set

$$E(QT) := \{\ \mathbf{x} \in \mathbb{Z}^n\ \mid\ q_{QT}(\mathbf{x}) \geq 0\ \}$$

of integer points in the compact subspace $\{\mathbf{x} \in \mathbb{R}^n \,|\, q_{QT}(\mathbf{x}) \geq 0\}$ of $\mathbb{R}^n$.

Remark

This algorithm can be made much faster if you use the technique of the lattice reduction basis (LLL-basis) due to Lenstra-Lenstra-Lovász. See the standard textbook of the computational number theory; for example,

Cohen. A course in computational algebraic number theory. GTM 138. Springer (2000).

Definition

A lattice L of rank n is *hyperbolic* if the signature of the real quadratic space $L \otimes \mathbb{R}$ is $(1, n-1)$ (that is, the Gram matrix G_L has exactly one positive eigenvalue).

Suppose that L is a hyperbolic lattice. Then the space

$$\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}$$

has two connected components. A *positive cone* of L is one of the two connected components.

Let L be a hyperbolic lattice, and let $\mathcal{P}$ be a positive cone of L.

Algorithm

Let h be a vector in $\mathcal{P} \cap L$. Then, for given integers a and b, we can compute the finite set

$$\{\, x \in L \mid \langle h, x\rangle = a,\ \ \langle x, x\rangle = b \,\}.$$

Algorithm

Let h, h' be vectors of $\mathcal{P} \cap L$. Then, for a negative integer d, we can compute the finite set of all vectors x of L that satisfy

- $\langle h, x\rangle > 0$, $\langle h', x\rangle < 0$ *and*
- $\langle x, x\rangle = d$

(that is, the set of vectors $x \in L$ of square norm $d < 0$ that separate h and h').

Definition

A lattice L is *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for any $x \in L$.

Definition

Let L be a lattice. The *orthogonal group* $\mathrm{O}(L)$ of L is the group of $g\colon L \xrightarrow{\sim} L$ that satisfies $\langle x, y\rangle = \langle x^g, y^g\rangle$ for any $x, y \in L$.

Let L be an even hyperbolic lattice, and let $\mathcal{P}$ be a positive cone.

- Let $\mathrm{O}^+(L)$ denote the stabilizer subgroup of $\mathcal{P}$ in $\mathrm{O}(L)$.
- A vector $r \in L$ with $\langle r, r\rangle = -2$ defines a *reflection*

  $$s_r\colon x \mapsto x + \langle x, r\rangle r.$$

  We have $s_r \in \mathrm{O}^+(L)$. Let $W(L)$ denote the subgroup of $\mathrm{O}^+(L)$ generated by all the reflections s_r.

Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$.
For a vector $r \in L$ with $\langle r, r \rangle = -2$, we put

$$(r)^{\perp} := \{ \, x \in \mathcal{P} \mid \langle x, r \rangle = 0 \, \}.$$

Then s_r is the reflection into this real hyperplane.
A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \ \setminus \ \bigcup_{r} (r)^{\perp}.$$

All standard fundamental domains are congruent to each other.
The cone $\mathcal{P}$ is tessellated by standard fundamental domains.
Let D be a standard fundamental domain. We put

$$\mathrm{Aut}(D) := \{ \, g \in \mathrm{O}^{+}(L) \mid D^{g} = D \, \}.$$

Then $\mathrm{O}^{+}(L)$ is the semi-direct product of $W(L)$ and $\mathrm{Aut}(D)$.

Example

Let $\mathbb{L}_{26}$ be an even *unimodular* hyperbolic lattice of rank 26, which is unique up to isomorphism. Let $\mathbb{D}$ be a standard fundamental domain of the action of $W(\mathbb{L}_{26})$.

Theorem (Conway)

The walls of $\mathbb{D}$ correspond bijectively to the vectors of the Leech lattice, and $\mathrm{Aut}(\mathbb{D})$ is isomorphic to the group of affine isometries of the Leech lattice.

Remark

The even hyperbolic lattices with finite $\mathrm{Aut}(D)$ have been classified by Nikulin and Vinberg. Such lattices have rank ≤ 19.

Let h, h' be vectors of $\mathcal{P} \cap L$. Let D be a standard fundamental domain containing h.

Using the algorithm that calculates the set of vectors of square norm $d = -2$ separating h and h', we can determine whether h' is contained in D or not.

More precisely, we can calculate a sequence $r_1, \ldots, r_N$ of vectors of square norm -2 such that the product

$$s_1 \cdots s_N$$

of reflections s_i with respect to r_i maps h' to D.

K3 means "Kummer, Kähler and Kodaira", named by André Weil (1958) after K2 at Karakorum (8611 m).
$K3$ surfaces are the 2-dimensional analogue of the elliptic curves.
$K3$ surfaces are 2-dimensional Calabi-Yau manifolds.

Definition

A smooth projective surface X defined over an algebraically closed field is called a $K3$ *surface* if

- $H^1(X, \mathcal{O}_X) = 0$, and
- the line bundle K_X of regular 2 forms is trivial.

Example

A smooth surface in the projective space $\mathbb{P}^3$ is a $K3$ surface if and only if it is of degree 4. In particular, the Fermat quartic surface

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

over a field of characteristic $\neq 2$ is a $K3$ surface.

Let X be a $K3$ surface. Then we have the intersection pairing on the group of divisors (or line bundles) on X.

Lemma

Let $\mathcal{L}$ and $\mathcal{L}'$ be line bundles on X. Then $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic if and only if

$$\deg \mathcal{L}|C = \deg \mathcal{L}'|C$$

for any curve C on X (that is, the numerical equivalence class is equal to the isomorphism class for line bundles on a $K3$ surface).

Definition

The Néron-Severi lattice S_X of X is the lattice of numerical equivalence classes of line bundles on X. Its rank ρ_X is called the *Picard number* of X

Proposition

The Néron-Severi lattice S_X of a $K3$ surface X is an even hyperbolic lattice of rank ≤ 20 or 22. The case $\rho_X = 22$ occurs only when the base field is of positive characteristic.

Definition

- A complex $K3$ surface is *singular* if its Picard number is 20.
- A $K3$ surface is *supersingular* if its Picard number is 22.

Example

Let X be the Fermat quartic surface $x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$ defined over a field of characteristic $p \neq 2$. Then

$$\rho_X := \begin{cases} 20 & \text{if } p = 0 \text{ or } p \equiv 1 \bmod 4, \\ 22 & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

$$\begin{bmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & \vdots & & & & & & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2
\end{bmatrix}$$

S_X of the complex Fermat quartic (discriminant -64)

$$
\begin{bmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
 & & & & & & \vdots & & & & & & & & & & & & & & & \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2
\end{bmatrix}
$$

S_X of the Fermat quartic in characteristic 3 (discriminant -9)

In general, it is difficult to calculate a Gram matrix of the Néron-Severi lattice of a $K3$ surface.

In the two example above, we had known the rank and the discriminant of S_X beforehand. Using this information, we search for curves on X whose classes generate S_X. It turns out that the classes of *lines* on $X \subset \mathbb{P}^3$ generate S_X.

Remark

Over $\mathbb{C}$, the Fermat quartic contains 48 *lines.*
Over the field of characteristic 3*, it contains* 112 *lines.*

The basis are the classes of lines.

Let X be a $K3$ surface. Since X is a projective surface, we have a very ample class $h \in S_X$ (that is, h is the class of a hyperplane section of an embedding $X \hookrightarrow \mathbb{P}^N$). We choose the connected component $\mathcal{P}_X$ of $\{x \in S_X \mid \langle x, x\rangle > 0\}$ that contains h.

Definition

The *nef cone* $N(X)$ of X is the cone

$$\{\, x \in \mathcal{P}_X \;\mid\; \langle x, [C]\rangle \geq 0 \text{ for any curve } C \text{ on } X \,\},$$

where $[C] \in S_X$ is the class of a curve $C \subset X$.

Proposition

The nef cone $N(X)$ of X is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$.

Lemma

A curve C on a $K3$ surface is a smooth rational curve if and only if its self-intersection number is -2.

Proposition

A hyperplane $(r)^{\perp}$ of $\mathcal{P}_X$ with $\langle r, r\rangle = -2$ is a boundary wall of $N(X)$ if and only if r or $-r$ is the class of a smooth rational curve.

Let $\mathrm{Aut}(X)$ denote the automorphism group of X. Since the action of $\mathrm{Aut}(X)$ on S_X preserves the nef cone, we have a natural homomorphism

$$\mathrm{Aut}(X) \to \mathrm{Aut}(N(X)).$$

The following is a corollary of the Torelli theorem (Piatetski-Shapiro and Shafarevich for complex $K3$ surfaces, Ogus for supersingular $K3$ surfaces).

Theorem

Suppose that X is defined over $\mathbb{C}$, or X is supersingular. Then the kernel of the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(N(X))$ is finite, and its image is of finite index.

Recall that, by Nikulin-Vinberg classification, an even hyperbolic lattices with finite $\mathrm{Aut}(D)$ must be of rank ≤ 19.

Corollary

If X is singular or supersingular, then $\mathrm{Aut}(X)$ is infinite.

A smooth quartic surface containing 56 lines

The following theorem is due to B. Segre (1943), Rams-Schütt (2015), Degtyarev, Itenberg and Sertöz (preprint).

Theorem

The number of lines lying on a complex smooth quartic surface is either in $\{64, 60, 56, 54\}$ *or* ≤ 52.

- The maximum number 64 is attained by the Schur quartic.
- The defining equations of smooth quartics containing 60 lines have been obtained by Schütt.
- There are possibly three smooth quartics containing 56 lines. Their defining equations are not known.

Note that the complex Fermat quartic surface

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

contains only 48 lines.

By the theory of Shioda-Inose on the classification of singular $K3$ surfaces (complex $K3$ surfaces with Picard number 20), we know that one of the smooth quartics containing 56 lines, which we denote by X_{56}, is isomorphic (as a complex surface) to the Fermat quartic, which we denote by X_{48}.

We know the Néron-Severi lattice S_{48} of $X_{48} \cong X_{56}$.

Theorem

We put

$$\zeta := \exp(2\pi\sqrt{-1}/8), \quad A := -1 - 2\zeta - 2\zeta^3, \quad B := 3 + A,$$

and

$$\begin{aligned} \Psi \;\; := \;\; & y_1^3 y_2 + y_1 y_2^3 + y_3^3 y_4 + y_3 y_4^3 \\ & + (y_1 y_4 + y_2 y_3)(A(y_1 y_3 + y_2 y_4) + B(y_1 y_2 - y_3 y_4)). \end{aligned}$$

Then the surface X_{56} defined by $\Psi = 0$ is smooth, contains exactly 56 lines, and is isomorphic to the Fermat quartic surface X_{48}.

The isomorphism $X_{48} \xrightarrow{\sim} X_{56}$ is explicitly given by

$$(x_1 : x_2 : x_3 : x_4) \mapsto (y_1 : y_2 : y_3 : y_4) = (f_1 : f_2 : f_3 : f_4)$$

where

$$
\begin{aligned}
f_1 &= \left(1+\zeta-\zeta^3\right) x_1{}^3 + \left(\zeta+\zeta^2+\zeta^3\right) x_1{}^2x_3 + (1+\zeta)\, x_1{}^2x_4 + \left(-\zeta-\zeta^2-\zeta^3\right) x_1x_2{}^2 + \\
&\quad (-1-\zeta)\, x_1x_2x_3 + \left(\zeta+\zeta^2\right) x_1x_2x_4 - x_1x_3{}^2 + \left(\zeta+\zeta^2\right) x_1x_3x_4 - \zeta^3 x_1x_4{}^2 + \\
&\quad \left(1-\zeta^2-\zeta^3\right) x_2{}^2x_3 + \left(-\zeta-\zeta^2\right) x_2x_3{}^2 + \left(\zeta^2+\zeta^3\right) x_2x_3x_4 + \zeta^2 x_3{}^3 + x_3x_4{}^2 \\
f_2 &= x_1{}^3 - \zeta^2 x_1{}^2x_3 + \left(-1+\zeta^3\right) x_1{}^2x_4 - \zeta^2 x_1x_2{}^2 + \left(1-\zeta^3\right) x_1x_2x_3 + (-1-\zeta)\, x_1x_2x_4 + \\
&\quad \left(1+\zeta-\zeta^3\right) x_1x_3{}^2 + \left(-\zeta^2-\zeta^3\right) x_1x_3x_4 + \left(-1-\zeta-\zeta^2\right) x_1x_4{}^2 + \zeta\, x_2{}^2x_3 + \\
&\quad \left(\zeta^2+\zeta^3\right) x_2x_3{}^2 + \left(1-\zeta^3\right) x_2x_3x_4 + \left(\zeta+\zeta^2+\zeta^3\right) x_3{}^3 + \left(1+\zeta-\zeta^3\right) x_3x_4{}^2 \\
f_3 &= \left(1+\zeta+\zeta^2\right) x_1{}^2x_2 + \left(\zeta+\zeta^2+\zeta^3\right) x_1{}^2x_4 + (-1-\zeta)\, x_1x_2x_3 + \left(\zeta+\zeta^2\right) x_1x_2x_4 + \\
&\quad \left(-\zeta-\zeta^2\right) x_1x_3x_4 + \left(\zeta^2+\zeta^3\right) x_1x_4{}^2 + \left(1-\zeta^2-\zeta^3\right) x_2{}^3 + \left(-\zeta-\zeta^2\right) x_2{}^2x_3 + \\
&\quad \left(1+\zeta+\zeta^2\right) x_2{}^2x_4 + \zeta^2 x_2x_3{}^2 + \left(-\zeta^2-\zeta^3\right) x_2x_3x_4 + \zeta^3 x_2x_4{}^2 + \zeta^3 x_3{}^2x_4 + \zeta\, x_4{}^3 \\
f_4 &= -\zeta\, x_1{}^2x_2 + x_1{}^2x_4 + \left(-1+\zeta^3\right) x_1x_2x_3 + (1+\zeta)\, x_1x_2x_4 + \left(-\zeta^2-\zeta^3\right) x_1x_3x_4 + \\
&\quad \left(-1+\zeta^3\right) x_1x_4{}^2 + \zeta^3 x_2{}^3 + (-1-\zeta)\, x_2{}^2x_3 + \zeta\, x_2{}^2x_4 + \left(-1-\zeta+\zeta^3\right) x_2x_3{}^2 + \\
&\quad \left(1-\zeta^3\right) x_2x_3x_4 + \left(-1+\zeta^2+\zeta^3\right) x_2x_4{}^2 + \left(1-\zeta^2-\zeta^3\right) x_3{}^2x_4 + \left(-1-\zeta-\zeta^2\right) x_4{}^3
\end{aligned}
$$

Let $h_{48} \in S_{48}$ be the class of a hyperplane section of the embedding $X_{48} \hookrightarrow \mathbb{P}^3$.

Proposition

A nef class $h \in S_{48}$ with $\langle h, h \rangle = 4$ is the class of a hyperplane section of some embedding $X_{48} \hookrightarrow \mathbb{P}^3$ if and only if the following hold:

(a) $\{ e \in S_{48} \mid \langle e, e \rangle = 0, \langle e, h \rangle = 1 \}$ *is empty,*

(b) $\{ e \in S_{48} \mid \langle e, e \rangle = 0, \langle e, h \rangle = 2 \}$ *is empty, and*

(c) $\{ r \in S_{48} \mid \langle r, r \rangle = -2, \langle r, h \rangle = 0 \}$ *is empty.*

If $h \in S_{48}$ satisfies them, then the set of classes of lines contained in the image X_h of the morphism $X_{48} \to \mathbb{P}^3$ induced by h is equal to

$$\mathcal{F}_h := \{ r \in S_X \mid \langle r, r \rangle = -2, \langle r, h \rangle = 1 \}.$$

The calculation

For each $d = 1, 2, 3, \ldots$, we make the following calculations:

- Compute the finite set

$$\mathcal{H}_d := \{\ h \in S_{48}\ \mid\ \langle h, h \rangle = 4,\ \langle h, h_{48} \rangle = d\ \}.$$

- For each $h \in \mathcal{H}_d$, we determine whether h is nef or not, by calculating the (-2)-vectors separating h and h_{48}.
- If h is nef, then we check the conditions (a), (b), (c).
- If h satisfies (a), (b), (c), then we calculate the set $\mathcal{F}_h$ of classes of lines contained in X_h.
- If $|\mathcal{F}_h| = 56$, then we calculate the global sections $f_1, \ldots, f_4$ of the corresponding line bundle. (Since S_{48} is generated by the classes of the 48 lines on X_{48}, h is a linear combination of some of these lines.)
- Calculate the linear relation Ψ of the quartic monomials of $f_1, \ldots, f_4$.

The automorphism group of the Fermat quartic in characteristic 3

Let L be an even hyperbolic lattice, and
let D be a standard fundamental domain of the action of $W(L)$.

Let $\mathbb{L}_{26}$ be the even unimodular hyperbolic lattice of rank 26, and let $\mathbb{D}$ be a standard fundamental domain of the action of $W(\mathbb{L}_{26})$. Recall that the structure of $\mathbb{D}$ has been already determined by Conway.

Suppose that L can be embedded primitively into $\mathbb{L}_{26}$. Then there exists an algorithm (Borcherds method) that calculates generators of $\mathrm{Aut}(D)$ from the structure of $\mathbb{D}$.

Let X denote the Fermat quartic surface in *characteristic* 3;

$$X : x_1^4 + x_2^4 + x_3^4 + x_4^4 = x_1\bar{x}_1 + x_2\bar{x}_2 + x_3\bar{x}_3 + x_4\bar{x}_4 = 0,$$

where $\bar{x} = x^3$ is the hermitian conjugate of $\mathbb{F}_9/\mathbb{F}_3$. Then the projective automorphism group

$$\mathrm{Aut}(X \subset \mathbb{P}^3) := \{\, \gamma \in \mathrm{PGL}_4 \mid \gamma(X) = X \,\}$$

is isomorphic to the finite group $\mathrm{PGU}_4(\mathbb{F}_9)$ of order $13, 063, 680$.

Theorem (Kondo and S.)

The full automorphism group $\mathrm{Aut}(X)$ *of* X *is generated by* $\mathrm{Aut}(X \subset \mathbb{P}^3)$ *and two involutions.*

Idea of the proof

The Néron-Severi lattice S_X of X can be embedded into $\mathbb{L}_{26}$ primitively. The tessellation by the chambers $\mathbb{D}^{\gamma}$ $(\gamma \in \mathrm{O}^+(\mathbb{L}_{26}))$ induces a tessellation of the positive cone of S_X, and the nef cone is a union of some of them. Investigation of this tessellation gives $\mathrm{Aut}(X)$.

Remark

The Borcherds method can be applied to the complex Fermat quartic. But the computation seems to be very huge and intractable. Hence the calculation of the full automorphism group of the complex Fermat quartic is still open.

Other applications

- We obtain automorphisms of irreducible Salem type on supersingular $K3$ surfaces in characteristic ≤ 7919. We conjecture that every supersingular $K3$ surface has an automorphism of irreducible Salem type.
- We can determine whether an even lattice of a given signature and a given discriminant form exists or not by a finite steps of computation (the genus theory of lattices). Combining this theory with the Torelli theorem for $K3$ surfaces, we can make the list of combinatorial data of complex elliptic $K3$ surfaces. Here, a combinatorial data is the pair of the ADE-type of singular fibers and the torsion part of the Mordell-Weil group.