A smooth quartic surface containing 56 lines

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2016 May Istanbul

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The preprint is available from: arXiv:1604.06265

Main Result

The complex Fermat quartic surface

$$X_{48} : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

in $\mathbf{P}^3$ contains exactly 48 lines. We show that X_{48} has another smooth quartic surface $X_{56} \subset \mathbb{P}^3$ as a projective model. This new quartic surface X_{56} contains 56 lines, and hence X_{48} and X_{56} are not projectively isomorphic.

Theorem

We put $\zeta := \exp(2\pi\sqrt{-1}/8)$, *and*

$$A := -1 - 2\zeta - 2\zeta^3, \quad B := 3 + A.$$

Then the surface X_{56} *in* $\mathbb{P}^3$ *defined by*

$$y_1^3 y_2 + y_1 y_2^3 + y_3^3 y_4 + y_3 y_4^3 \\ + (y_1 y_4 + y_2 y_3)(\, A\,(y_1 y_3 + y_2 y_4) + B\,(y_1 y_2 - y_3 y_4)\,) = 0$$

is smooth, and contains exactly 56 *lines. Moreover, as an abstract variety,* X_{56} *is isomorphic to the Fermat quartic* X_{48}.

The rational map $\mathbf{P}^3 \cdots \to \mathbb{P}^3$ given by

$$[x_1 : x_2 : x_3 : x_4] \;\mapsto\; [y_1 : y_2 : y_3 : y_4] = [f_1 : f_2 : f_3 : f_4]$$

induces an isomorphism $\Phi\colon X_{48} \xrightarrow{\sim} X_{56}$, where

$$
\begin{aligned}
f_1 &= \left(1+\zeta-\zeta^3\right) x_1{}^3 + \left(\zeta+\zeta^2+\zeta^3\right) x_1{}^2 x_3 + (1+\zeta)\, x_1{}^2 x_4 + \left(-\zeta-\zeta^2-\zeta^3\right) x_1 x_2{}^2 + \\
&\quad (-1-\zeta)\, x_1 x_2 x_3 + \left(\zeta+\zeta^2\right) x_1 x_2 x_4 - x_1 x_3{}^2 + \left(\zeta+\zeta^2\right) x_1 x_3 x_4 - \zeta^3 x_1 x_4{}^2 + \\
&\quad \left(1-\zeta^2-\zeta^3\right) x_2{}^2 x_3 + \left(-\zeta-\zeta^2\right) x_2 x_3{}^2 + \left(\zeta^2+\zeta^3\right) x_2 x_3 x_4 + \zeta^2 x_3{}^3 + x_3 x_4{}^2 \\
f_2 &= x_1{}^3 - \zeta^2 x_1{}^2 x_3 + \left(-1+\zeta^3\right) x_1{}^2 x_4 - \zeta^2 x_1 x_2{}^2 + \left(1-\zeta^3\right) x_1 x_2 x_3 + (-1-\zeta)\, x_1 x_2 x_4 + \\
&\quad \left(1+\zeta-\zeta^3\right) x_1 x_3{}^2 + \left(-\zeta^2-\zeta^3\right) x_1 x_3 x_4 + \left(-1-\zeta-\zeta^2\right) x_1 x_4{}^2 + \zeta\, x_2{}^2 x_3 + \\
&\quad \left(\zeta^2+\zeta^3\right) x_2 x_3{}^2 + \left(1-\zeta^3\right) x_2 x_3 x_4 + \left(\zeta+\zeta^2+\zeta^3\right) x_3{}^3 + \left(1+\zeta-\zeta^3\right) x_3 x_4{}^2 \\
f_3 &= \left(1+\zeta+\zeta^2\right) x_1{}^2 x_2 + \left(\zeta+\zeta^2+\zeta^3\right) x_1{}^2 x_4 + (-1-\zeta)\, x_1 x_2 x_3 + \left(\zeta+\zeta^2\right) x_1 x_2 x_4 + \\
&\quad \left(-\zeta-\zeta^2\right) x_1 x_3 x_4 + \left(\zeta^2+\zeta^3\right) x_1 x_4{}^2 + \left(1-\zeta^2-\zeta^3\right) x_2{}^3 + \left(-\zeta-\zeta^2\right) x_2{}^2 x_3 + \\
&\quad \left(1+\zeta+\zeta^2\right) x_2{}^2 x_4 + \zeta^2 x_2 x_3{}^2 + \left(-\zeta^2-\zeta^3\right) x_2 x_3 x_4 + \zeta^3 x_2 x_4{}^2 + \zeta^3 x_3{}^2 x_4 + \zeta\, x_4{}^3 \\
f_4 &= -\zeta\, x_1{}^2 x_2 + x_1{}^2 x_4 + \left(-1+\zeta^3\right) x_1 x_2 x_3 + (1+\zeta)\, x_1 x_2 x_4 + \left(-\zeta^2-\zeta^3\right) x_1 x_3 x_4 + \\
&\quad \left(-1+\zeta^3\right) x_1 x_4{}^2 + \zeta^3 x_2{}^3 + (-1-\zeta)\, x_2{}^2 x_3 + \zeta\, x_2{}^2 x_4 + \left(-1-\zeta+\zeta^3\right) x_2 x_3{}^2 + \\
&\quad \left(1-\zeta^3\right) x_2 x_3 x_4 + \left(-1+\zeta^2+\zeta^3\right) x_2 x_4{}^2 + \left(1-\zeta^2-\zeta^3\right) x_3{}^2 x_4 + \left(-1-\zeta-\zeta^2\right) x_4{}^3
\end{aligned}
$$

Our method is very computational. In this talk, by using this pair of quartics surfaces as an example, we demonstrate how far we can go in the study of $K3$ surfaces with computer-aided calculation in the lattice theory.

We present a few computer-programs that are quite useful in the study of $K3$ surfaces.

Motivation

The following theorem is due to B. Segre (1943), Rams-Schütt (2015), Degtyarev, Itenberg and Sertöz (arXiv:1601.04238).

Theorem

The number of lines lying on a complex smooth quartic surface is either in $\{64, 60, 56, 54\}$ *or* ≤ 52.

- The maximum number 64 is attained by the Schur quartic.
- The defining equations of smooth quartics containing 60 lines have been obtained by Schütt.
- There are at least three smooth quartics containing 56 lines. But their defining equations have not been known.

For a complex $K3$ surface X, we denote the *Néron-Severi lattice* of X by

$$S_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X),$$

that is, S_X is the lattice of cohomology classes of divisors on X with the intersection pairing. This is a lattice of signature $(1, \rho_X - 1)$. Its rank ρ_X is called the *Picard number* of X.

We then denote by

$$T_X := (S_X \hookrightarrow H^2(X, \mathbb{Z}))^\perp$$

the *transcendental lattice* of X. This is a lattice of signature $(2, 20 - \rho_X)$.

Degtyarev, Itenberg and Sertöz calculated the transcendental lattices of smooth quartics containing 56 lines. All of them have $\rho_X = 20$, and one of them has

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix},$$

as the transcendental lattice, which is isomorphic to that of the Fermat quartic X_{48}. Hence these two $K3$ surfaces must be isomorphic by the following:

Theorem (Shioda and Inose)

Let X and X' be $K3$ surfaces with $\rho_X = 20$. If T_X and $T_{X'}$ are $SL_2(\mathbb{Z})$-equivalent, then X and X' are isomorphic.

Our goal is to find a defining equation of X_{56} and to exhibit an isomorphism $X_{48} \xrightarrow{\sim} X_{56}$. (Since $X_{48} \cong X_{56}$ has an infinite automorphism group, there exist infinitely many isomorphisms.)

Algorithms in the lattice theory

Definition

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle \colon L \times L \to \mathbb{Z}.$$

Let L be a lattice of rank n. If we choose a basis $v_1, \ldots, v_n$ of the free $\mathbb{Z}$-module L, then the bilinear form $\langle\ ,\ \rangle \colon L \times L \to \mathbb{Z}$ is expressed by the *Gram matrix*

$$G_L := (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n}.$$

We will use a Gram matrix to express a lattice in the computer.

By a *quadratic triple* of n-variables, we mean a triple $[Q, \ell, c]$, where

- Q is an $n \times n$ symmetric matrix with entries in $\mathbb{Q}$,
- ℓ is a column vector of length n with entries in $\mathbb{Q}$, and
- c is a rational number.

An element of $\mathbb{R}^n$ is written as a row vector

$$\boldsymbol{x} = [x_1, \dots, x_n] \in \mathbb{R}^n.$$

The *inhomogeneous quadratic function* $q_{QT} : \mathbb{Q}^n \to \mathbb{Q}$ associated with a quadratic triple $QT = [Q, \ell, c]$ is defined by

$$q_{QT}(\boldsymbol{x}) := \boldsymbol{x}\, Q\, {}^t\boldsymbol{x} + 2\, \boldsymbol{x}\, \ell + c.$$

We say that $QT = [Q, \ell, c]$ is *negative* if the symmetric matrix Q is negative-definite.

Algorithm

Let $QT = [Q, \ell, c]$ be a negative quadratic triple of n-variables. Then we can compute the finite set

$$E(QT) := \{ \boldsymbol{x} \in \mathbb{Z}^n \mid q_{QT}(\boldsymbol{x}) \geq 0 \}$$

of integer points in the n-dimensional ellipsoid

$$\mathcal{E}_n := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, q_{QT}(\boldsymbol{x}) \geq 0\} \quad \subset \quad \mathbb{R}^n.$$

Method: The image of the projection of $\mathcal{E}_n$ to a hyperplane $x_n = 0$ is an $(n-1)$-dimensional ellipsoid.

Remark

This algorithm can be made much faster if we use the lattice reduction basis (LLL-basis) due to Lenstra-Lenstra-Lovász.

Applications of the basic algorithm

Definition

A lattice L of rank n is *hyperbolic* if the signature of the real quadratic space $L \otimes \mathbb{R}$ is $(1, n-1)$.

By Hodge index theorem, the Néron-Severi lattice of a smooth algebraic surface is hyperbolic.

Suppose that L is a hyperbolic lattice. Then the space

$$\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}$$

has two connected components. A *positive cone* of L is one of the two connected components.

Let L be a hyperbolic lattice, and let $\mathcal{P}$ be a positive cone of L.

Algorithm

Let h be a vector in $\mathcal{P} \cap L$. Then, for given integers a and b, we can compute the finite set

$$\{ x \in L \mid \langle h, x \rangle = a, \ \langle x, x \rangle = b \}.$$

Algorithm

Let h, h' be vectors of $\mathcal{P} \cap L$. Then, for a negative integer d, we can compute the finite set of all vectors x of L that satisfy

- $\langle h, x \rangle > 0$, $\langle h', x \rangle < 0$ *and*
- $\langle x, x \rangle = d$,

(that is, we can calculate the set of vectors $x \in L$ of square norm $d < 0$ that separate h and h').

Definition

A lattice L is *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for any $x \in L$.

Definition

Let L be a lattice. The *orthogonal group* $\mathrm{O}(L)$ of L is the group of $g\colon L \xrightarrow{\sim} L$ that satisfies $\langle x, y\rangle = \langle x^g, y^g\rangle$ for any $x, y \in L$.

Let L be an even hyperbolic lattice. We fix a positive cone $\mathcal{P}$.

- Let $\mathrm{O}^+(L)$ denote the stabilizer subgroup of $\mathcal{P}$ in $\mathrm{O}(L)$.
- A vector $r \in L$ with $\langle r, r\rangle = -2$ defines a *reflection*

$$s_r\colon x \mapsto x + \langle x, r\rangle r.$$

  We have $s_r \in \mathrm{O}^+(L)$. Let $W(L)$ denote the subgroup of $\mathrm{O}^+(L)$ generated by all the reflections s_r.

Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$.
For a vector $r \in L$ with $\langle r, r \rangle = -2$, we put

$$(r)^{\perp} := \{ \, x \in \mathcal{P} \mid \langle x, r \rangle = 0 \, \}.$$

Then s_r is the reflection into this real hyperplane.
A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \ \setminus \ \bigcup_r (r)^{\perp}.$$

All standard fundamental domains are congruent to each other.
The cone $\mathcal{P}$ is tessellated by standard fundamental domains.

Let v and v' be two points in

$$L \ \cap \ (\mathcal{P} \ \setminus \ \bigcup_{r} (r)^{\perp}).$$

Then, by calculating the set of (-2)-vectors separating v and v', we can determine whether v and v' are in the same fundamental domain of $W(L)$ or not.

Application to the $K3$ surface X_{48}

Let X be an *algebraic* $K3$ surface, so that we have a very ample class $h \in S_X$. We choose the connected component $\mathcal{P}_X$ of $\{x \in S_X \mid \langle x, x \rangle > 0\}$ that contains h.

Definition

The *nef cone* $N(X)$ of X is the cone

$$\{\, x \in \mathcal{P}_X \;\mid\; \langle x, [C] \rangle \geq 0 \text{ for any curve } C \text{ on } X \,\},$$

where $[C] \in S_X$ is the class of a curve $C \subset X$.

Proposition

The nef cone $N(X)$ of X is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$.

Recall that $\zeta := \exp(2\pi\sqrt{-1}/8)$. The 48 lines on X_{48} are given by

$$x_1 + \zeta^{\mu} x_i = 0, \quad x_j + \zeta^{\nu} x_k = 0,$$

where μ and ν are positive odd integers ≤ 7, and i, j, k are integers such that $j < k$ and $\{1, i, j, k\} = \{1, 2, 3, 4\}$. We can calculate the intersection numbers of these lines.

We know that $S_{48} := S_{X_{48}}$ is of rank 20 and with discriminant 64. If we choose 20 lines from the 48 lines appropriately, they form a intersection matrix of discriminant 64, and hence they form a basis of S_{48}.

We fix such a list of 20 lines as a basis once and for all, so that every vector of S_{48} is expressed as a vector of length 20 with integer entries from now on.

$$
\begin{bmatrix}
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & \vdots & & & & & & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2
\end{bmatrix}
$$

The Gram matrix of the Néron-Severi lattice S_{48} of X_{48}

Let $h_{48} \in S_{48}$ be the class of a hyperplane section of the embedding $X_{48} \hookrightarrow \mathbb{P}^3$.

Proposition

A class $h \in S_{48}$ with $\langle h, h \rangle = 4$ is the class of a hyperplane section of some embedding $X_{48} \hookrightarrow \mathbb{P}^3$ if and only if the following hold:

(a) $\langle h, h_{48} \rangle > 0$,

(b) *there exist no* (-2)*-vectors separating* h *and* h_{48}*,*

(c) $\{ e \in S_{48} \mid \langle e, e \rangle = 0, \ \langle e, h \rangle = 1 \}$ *is empty*

(d) $\{ e \in S_{48} \mid \langle e, e \rangle = 0, \ \langle e, h \rangle = 2 \}$ *is empty, and*

(e) $\{ r \in S_{48} \mid \langle r, r \rangle = -2, \ \langle r, h \rangle = 0 \}$ *is empty.*

Conditions (a) and (b) mean that h is nef.
Condition (c) means that the complete linear system of the line bundle corresponding to h is fixed-point free, and hence induces a morphism $\Phi_h \colon X_{48} \to \mathbb{P}^3$. Condition (d) means that Φ_h is not hyperelliptic, and (e) means that the image X_h of Φ_h is smooth.

Proposition

If $h \in S_{48}$ satisfies these conditions, then the set of classes of lines contained in the image X_h of Φ_h is equal to

$$\mathcal{F}_h := \{ \, r \in S_X \;|\; \langle r, r \rangle = -2, \; \langle r, h \rangle = 1 \, \}.$$

For each $d = 1, 2, 3, \ldots,$ we make the following calculations:

- Compute the finite set

$$\mathcal{H}_d := \{ \, h \in S_{48} \;|\; \langle h, h \rangle = 4, \; \langle h, h_{48} \rangle = d \, \}.$$

- For each $h \in \mathcal{H}_d$, we determine whether h satisfies the conditions (a)–(e).
- If h satisfies (a)–(e), then we calculate the set $\mathcal{F}_h$ of classes of lines contained in X_h.
- If $|\mathcal{F}_h| = 56$, it means that we have found a polarization that induces an isomorphism to a smooth quartic surface containing 56 lines.

The group G_{48} of the *projective* automorphisms of $X_{48} \subset \mathbf{P}^3$ is of order $1536 = 24 \times 4^3$, and it acts on each $\mathcal{H}_d$. We have

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \emptyset, \quad \mathcal{H}_4 = \{h_{48}\}, \quad |\mathcal{H}_5| = 48, \quad |\mathcal{H}_6| = 48264.$$

The action of G_{48} on $\mathcal{H}_5$ is transitive, and no vectors in $\mathcal{H}_5$ are nef. The action of G_{48} decomposes $\mathcal{H}_6$ into 60 orbits. Among the vectors in $\mathcal{H}_6$,

- 792 vectors in 5 orbits are not nef,
- 792 vectors in other 5 orbits are nef, fixed-component free, but define hyperelliptic morphism,
- 46296 vectors in 48 orbits are nef, fixed-component free, define non-hyperelliptic morphism, but the images are singular (one node, two nodes, one cusp, ...) , and
- the remaining 384 vectors in 2 orbits are very ample, and the images contain exactly 56 lines.

Theorem

If $h \in S_X$ is a very ample polarization of degree 4 with relative degree $\langle h, h_{48} \rangle = 6$, then h is an X_{56}-polarization.

Theorem

There exist exactly 384 X_{56}-polarizations of relative degree 6. Under the action of G_{48}, they are decomposed into two orbits.

Remark

These two orbits of X_{56}-polarizations are conjugate under the action of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

Remark

Recently, Oguiso (arXiv: 1602.04588) showed that, if a $K3$ surface X has two very ample classes h_1, h_2 with $\langle h_1, h_1 \rangle = \langle h_2, h_2 \rangle = 4$ and $\langle h_1, h_2 \rangle = 6$, then the associated two smooth quartic surfaces X_1 and X_2 are not projectively isomorphic.

A defining equation of X_{56}

Let h_{56} be one of the 384 X_{56}-polarizations of relative degree 6, and let $\mathcal{L}_{56}$ be a line bundle whose class is h_{56}. Then we can find six lines $\ell_1, \ldots, \ell_6$ on X_{48} such that

$$h_{56} = 3h_{48} - [\ell_1] - \cdots - [\ell_6].$$

Hence the space of the global sections of $\mathcal{L}_{56}$ is identified with the space of homogeneous cubic polynomials in $x_1, \ldots, x_4$ that vanish along the lines $\ell_1, \ldots, \ell_6$.

Let $f_1, \ldots, f_4$ be a basis of this space. Calculating the linear dependence of the polynomials

$$f_1^{n_1}\, f_2^{n_2}\, f_3^{n_3}\, f_4^{n_4} \quad (n_1 + \cdots + n_4 = 4)$$

in the degree 12 part of the homogeneous ring

$$\mathbb{C}[x_1, \ldots, x_4]/(x_1^4 + x_2^4 + x_3^4 + x_4^4),$$

we obtain a defining equation of X_{56}.

Of course, the equation depends on the choice of the basis $f_1, \dots, f_4$. The equation

$$y_1^3 y_2 + y_1 y_2^3 + y_3^3 y_4 + y_3 y_4^3 \\ + (y_1 y_4 + y_2 y_3)(\, A\, (y_1 y_3 + y_2 y_4) + B\, (y_1 y_2 - y_3 y_4)\,) = 0$$

is the shortest one among the eqs I found.

Reductions at primes

- The reduction of the model of $X_{56} \subset \mathbb{P}^3$ over $\mathbb{Z}[\zeta]$ at primes P remains smooth except when P lies over 2 or 3, and each of these smooth reductions contains exactly 56 lines.
- There are two primes of $\mathbb{Z}[\zeta]$ over 3. The reduction at one of them is singular, whereas the reduction at the other gives us a smooth surface projectively isomorphic to $X_{112} := X_{48} \otimes \mathbb{F}_9$, which contains 112 lines. (See Degtyarev, Lines in supersingular quartics, arXiv:1604.05836).

We present some other applications of our algorithms.

The full automorphism groups

The standard fundamental domain of the even unimodular hyperbolic lattice L_{26} of rank 26 is completely described by Conway.

We embed S_X in L_{26}. Then the tessellation in L_{26} by the Conway domains induces a tessellation of the nef cone of X by cones with finite number of faces. (Borcherds-Kondo method).

Using this method, we calculate sets of generators of the *full* automorphism groups of several $K3$ surfaces (including X_{112}) and some Enriques surfaces.

Remark

Currently, a set of generators of the full automorphism group of $X_{48} \cong X_{56}$ is not yet known.

An experiment on automorphisms of supersingular $K3$ surfaces

A $K3$ surface is *supersingular* if $\rho_X = 22$. Supersingular $K3$ surface exist only in positive characteristics.

An automorphism of a supersingular $K3$ surface is *of irreducible Salem type* if its action on S_X has an irreducible characteristic polynomial that is not cyclotomic. Such an automorphism is important because it never lifts to characteristic 0.

We can find many automorphisms of a given $K3$ surface X by our algorithms, just by searching for the degree 2 polarization $X \to \mathbb{P}^2$.

An experiment suggests that every supersingular $K3$ surface in odd characteristic has an automorphism of irreducible Salem type.

This conjecture is confirmed for odd characteristics $p \leq 7919$.