Automorphisms of K3 surfaces and Enriques surfaces

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2017 May Hakodate

We present a method to calculate the automorphism group of a K3 surface or an Enriques surface.

This method is based on the classical results of Vinberg (1973) and Conway (1983) on the standard fundamental domain of the action of the Weyl group for even unimodular hyperbolic lattices.

The first application of this method to K3 surfaces was given by Borcherds (1987, 1998) and Kondo (1998). Hence we call it **Borcherds-Kondo method**.

The new features in this talk are

- a generalization of Borcherds-Kondo method to the cases of non-simple Borcherds type,
- a practical algorithm to carry out Borcherds-Kondo method on a computer, and
- an application to Enriques surfaces.

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\ ,\ \rangle : L \times L \to \mathbb{Z}.$$

When a basis $b_1, \dots, b_n$ of L is given, the intersection form $\langle\ ,\ \rangle$ is expressed by the *Gram matrix* $(\langle b_i, b_j \rangle)_{i,j=1,\dots,n}$.

Let L be a lattice. We put

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}), \quad L_\mathbb{Q} := L \otimes \mathbb{Q}, \quad L_\mathbb{R} := L \otimes \mathbb{R}.$$

Then we have natural inclusions $L \hookrightarrow L^\vee \hookrightarrow L_\mathbb{Q} \hookrightarrow L_\mathbb{R}$. The *discriminant group* of L is defined to be $L^\vee / L$.

A lattice L is *unimodular* if $L^\vee = L$.

A lattice L of rank n is

- *hyperbolic* if the signature of $L_\mathbb{R}$ is $(1, n-1)$,
- *negative-definite* if the signature of $L_\mathbb{R}$ is $(0, n)$.

A lattice L is *even* if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$.

A sublattice $L' \subset L$ of L is *primitive* if L/L' is torsion-free.

K3 surface
Let X be an algebraic K3 surface. Then the lattice S_X of numerical equivalence classes $[D]$ of divisors D on X is an even hyperbolic lattice.
Over $\mathbb{C}$, the lattice S_X is a primitive sublattice of the even *unimodular* lattice $H^2(X, \mathbb{Z})$ of rank 22 and signature $(3, 19)$, which is unique up to isomorphism.

Enriques surface
Let Y be a complex Enriques surface. Then the lattice S_Y of numerical equivalence classes $[D]$ of divisors D on Y is an even unimodular hyperbolic lattice L_{10} of rank 10, which is unique up to isomorphism. In fact, the lattice S_Y is isomorphic to $H^2(Y, \mathbb{Z})/(\mathrm{torsion})$ with the cup product.

The Weyl group of a hyperbolic lattice

Let L be an even hyperbolic lattice. Then $\{x \in L_{\mathbb{R}} \mid \langle x, x\rangle > 0\}$ has two connected components. We choose one of them, denote it by $\mathcal{P}$, and call it a *positive cone*. Let $\mathrm{O}^+(L)$ denote the stabilizer subgroup of $\mathcal{P}$ in the group $\mathrm{O}(L)$ of isometries of L. We have

$$\mathrm{O}(L) = \mathrm{O}^+(L) \times \{\pm 1\}.$$

For a non-zero vector $v \in L_{\mathbb{R}}$, we put

$$(v)^{\perp} := \{\ x \in \mathcal{P} \ \mid\ \langle x, v\rangle = 0\ \}.$$

Then $(v)^{\perp} \neq \emptyset$ if and only if $\langle v, v\rangle < 0$. Consider a closed subset

$$\mathcal{C} = \{\ x \in \mathcal{P} \ \mid\ \langle x, v_i\rangle \geq 0 \ \text{ for all } \ v_i\ \}$$

of $\mathcal{P}$ defined by a set of vectors v_i of $L_{\mathbb{R}}$. Suppose that $\mathcal{C} \subset \mathcal{P}$ has an interior point. A hyperplane $(v)^{\perp}$ of $\mathcal{P}$ is said to *define a wall* $(v)^{\perp} \cap \mathcal{C}$ of the cone $\mathcal{C}$ if $(v)^{\perp}$ is disjoint from the interior of $\mathcal{C}$ and $(v)^{\perp} \cap \mathcal{C}$ contains a non-empty open subset of $(v)^{\perp}$.

A vector $r \in L$ is a *root* if $\langle r, r \rangle = -2$. Let r be a root. Then r defines a *reflection*

$$s_r \colon x \mapsto x + \langle x, r \rangle r$$

into the mirror $(r)^\perp$. We have $s_r \in \mathrm{O}^+(L)$. Let $W(L)$ denote the subgroup of $\mathrm{O}^+(L)$ generated by all the reflections s_r. We call $W(L)$ the *Weyl group* of L. A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \ \setminus \ \bigcup (r)^\perp.$$

Then $W(L)$ acts on the set of standard fundamental domains simple-transitively. Let Δ be a standard fundamental domain. We put

$$\mathrm{Aut}(\Delta) := \{\, g \in \mathrm{O}^+(L) \mid \Delta^g = \Delta \,\}.$$

Then we have a splitting exact sequence

$$1 \to W(L) \to \mathrm{O}^+(L) \to \mathrm{Aut}(\Delta) \to 1.$$

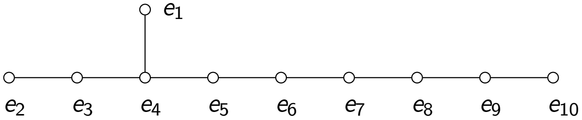
By looking at the walls of Δ, we can find a presentation of $W(L)$.

Even unimodular hyperbolic lattices

First remark that an even unimodular hyperbolic lattice L_n of rank n exists if and only if $n \equiv 2 \bmod 8$, and that L_n is unique up to isomorphism for each n. The lattice $U := L_2$ has the Gram matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Vinberg's result. Let $e_1, \ldots, e_{10}$ be roots that form the following Dynkin diagram $T_{2,3,7}$. (The roots e_i and e_j are connected if $\langle e_i, e_j \rangle = 1$, and not connected if $\langle e_i, e_j \rangle = 0$.)



Then the lattice L_{10} generated by $e_1, \ldots, e_{10}$ is an even unimodular hyperbolic lattice of rank 10.

Let $\mathcal{P}_{10}$ be the positive cone of L_{10} containing the vector

$$w := [115, 76, 153, 231, 195, 160, 126, 93, 61, 30],$$

which is characterized by $\langle w, e_i \rangle = 1 \quad (i = 1, \dots, 10)$.

Theorem (Vinberg). The set

$$\Delta := \{\, x \in \mathcal{P}_{10} \mid \langle x, e_i \rangle \geq 0 \,\}$$

is a standard fundamental domain of the action of $W(L_{10})$ on $\mathcal{P}_{10}$, and each $(e_i)^{\perp} \cap \Delta$ is a wall of the chamber Δ for $i = 1, \dots, 10$.

Since $\mathrm{Aut}(\Delta) = \{1\}$, we have $\mathrm{O}^+(L_{10}) = W(L_{10})$. Moreover, we see that $W(L_{10})$ is generated by the 10 reflections s_{e_i}, and that the defining relations for these generators can be obtained from the Dynkin diagram $T_{2,3,7}$; that is, $s_{e_i}^2 = 1$ for $i = 1, \dots, 10$, and

$$\begin{aligned} (s_{e_i} s_{e_j})^2 &= 1 \qquad \text{if } e_i \text{ and } e_j \text{ are not connected,} \\ (s_{e_i} s_{e_j})^3 &= 1 \qquad \text{if } e_i \text{ and } e_j \text{ are connected.} \end{aligned}$$

We skip L_{18}, which was also studied by Vinberg.

Conway's result

Let Λ be the negative-definite Leech lattice; that is, Λ is the even unimodular negative-definite lattice of rank 24 with no roots, which is unique up to isomorphism. Then

$$L_{26} := U \oplus \Lambda$$

is an even unimodular hyperbolic lattice of rank 26. A vector of L_{26} is written as (a, b, λ), where $(a, b) \in U$ and $\lambda \in \Lambda$. We put

$$w := (1, 0, 0),$$

and let $\mathcal{P}_{26}$ the positive cone such that $w \in \overline{\mathcal{P}_{26}}$. For each $\lambda \in \Lambda$, we have a root

$$r_\lambda := (-(\lambda^2 + 2)/2, 1, \lambda) \quad \in \quad L_{26}.$$

It is easy to see that

$$\{ \, r \in L_{26} \mid \langle r, r \rangle = -2, \langle r, w \rangle = 1 \, \} = \{ \, r_\lambda \mid \lambda \in \Lambda \, \}.$$

Theorem (Conway). The set

$$\Delta := \{\, x \in \mathcal{P}_{26} \;\mid\; \langle x, r_\lambda \rangle \geq 0 \;\text{ for all }\; \lambda \in \Lambda \,\}$$

is a standard fundamental domain of the action of $W(L_{26})$ on $\mathcal{P}_{26}$, and each root r_λ defines a wall of Δ.

It is easy to see that $\mathrm{Aut}(\Delta)$ isomorphic to the group $\cdot\infty$ of affine isometries of Λ. Hence we have

$$\mathrm{O}^+(L_{26}) = W(L_{26}) \rtimes \cdot\infty,$$

and that the Weyl group $W(L_{26})$ is generated by the reflections s_{r_λ}.

Definition. We call a standard fundamental domain of $W(L_{10})$ a *Vinberg chamber*, and a standard fundamental domain of $W(L_{26})$ a *Conway chamber*.

Discriminant form

We need the theory of discriminant forms due to Nikulin (1979). Suppose that L is even. Since $L^\vee \subset L_{\mathbb{Q}}$, the dual lattice $L^\vee$ has a natural $\mathbb{Q}$-valued non-degenerate symmetric bilinear form.
The *discriminant form*

$$q_L \,:\, L^\vee/L \to \mathbb{Q}/2\mathbb{Z}$$

is defined by $q_L(x \bmod L) := \langle x, x\rangle \bmod 2\mathbb{Z}$ for $x \in L^\vee$. We have a natural homomorphism

$$\mathrm{O}(L) \to \mathrm{O}(q_L)$$

to the automorphism group $\mathrm{O}(q_L)$ of the finite quadratic form q_L.

An *overlattice* M is a $\mathbb{Z}$-module such that $L \subset M \subset L^\vee$, and that the $\mathbb{Q}$-valued form on $L^\vee$ takes values in $\mathbb{Z}$ on M.

By definition, the correspondence

$$M \mapsto M/L$$

is a bijection from the set of even overlattices of L to the set of isotropic subgroups of q_L.

Hence we obtain the following:

Proposition. Let S and T be even lattices. If H is an even *unimodular* overlattice of the orthogonal direct sum $S \oplus T$ such that $S \subset H$ and $T \subset H$ are primitive, then

$$H/(S \oplus T) \subset S^\vee/S \oplus T^\vee/T$$

is the graph of an isomorphism $(S^\vee/S, q_S) \cong (T^\vee/T, -q_T)$.

Aut($K3$)

For simplicity, we work over $\mathbb{C}$. (By using Torelli theorem due to Ogus (1983), we can develop a similar method for supersingular $K3$ surfaces in positive characteristics.)

Let X be an algebraic $K3$ surface. Let $\mathcal{P}_X$ be the positive cone of S_X containing an ample class. We denote the nef-and-big cone of X by

$$N(X) := \{\, x \in \mathcal{P}_X \mid \langle x, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } X \,\}.$$

Proposition.
The cone $N(X)$ is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$. The correspondence $C \mapsto N(X) \cap ([C])^{\perp}$ gives a bijection from the set of smooth rational curves C on X to the set of walls of the cone $N(X)$.

We have a natural homomorphism

$$\mathrm{Aut}(X) \to \mathrm{Aut}(N(X)).$$

We investigate the image and the kernel of this homomorphism.

Let T_X denote the orthogonal complement of S_X in $H^2(X, \mathbb{Z})$, and let ω_X be a generator of $H^{2,0}(X) \subset T_X \otimes \mathbb{C}$. We put

$$\mathrm{O}^{\omega}(T_X) := \{\, g \in \mathrm{O}(T_X) \mid \omega_X^g \in \mathbb{C}\,\omega_X \,\}.$$

Since $H_X := H^2(X, \mathbb{Z})$ is unimodular and both of $S_X \subset H_X$ and $T_X \subset H_X$ are primitive, we obtain an isomorphism

$$\sigma_X : q_{S_X} \xrightarrow{\sim} -q_{T_X}.$$

We denote by

$$\sigma_{X*} : \mathrm{O}(q_{S_X}) \xrightarrow{\sim} \mathrm{O}(q_{T_X})$$

the isomorphism induced by σ_X.

Then an isometry $g \in \mathrm{O}(S_X)$ extends to an isometry of $H^2(X, \mathbb{Z})$ preserving the Hodge structure if and only if

$$\textbf{[period condition]} \qquad \sigma_{X*}(\eta_S(g)) \in \eta_T(\mathrm{O}^\omega(T_X)),$$

where $\eta_S \colon \mathrm{O}(S_X) \to \mathrm{O}(q_{S_X})$ and $\eta_T \colon \mathrm{O}(T_X) \to \mathrm{O}(q_{T_X})$ are the natural homomorphisms.

Example. Suppose that $\mathrm{rank}\, T_X \geq 3$, and that $\omega_X \in T_X$ is very general. Then we have $\mathrm{O}^\omega(T_X) = \{\pm 1\}$. Hence the period condition is equal to

$$\eta_S(g) = \pm 1.$$

We put

$$\Gamma_X := \{\, g \in \mathrm{O}^+(S_X) \mid \sigma_{X*}(\eta_S(g)) \in \eta_T(\mathrm{O}^\omega(T_X)) \,\}.$$

Note that Γ_X is of finite index in $\mathrm{O}^+(S_X)$.

By Torelli theorem for complex $K3$ surfaces due to Piatetski-Shapiro and Shafarevich (1971), we have the following:

Theorem. (1) The image of the natural homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(N(X))$ is $\mathrm{Aut}(N(X)) \cap \Gamma_X$.
(2) The kernel of $\mathrm{Aut}(X) \to \mathrm{Aut}(N(X))$ is isomorphic to

$$\{\, g \in \mathrm{O}^{\omega}(T_X) \mid \eta_T(g) = 1 \,\}.$$

Hence, if we know $\mathrm{Aut}(N(X)) \cap \Gamma_X$, then we can calculate $\mathrm{Aut}(X)$.

Note that $\mathrm{Aut}(N(X)) = \mathrm{O}^+(S_X)/W(S_X)$ and Γ_X are defined from S_X by purely lattice-theoretic terms.

Borcherds-Kondo method

Let S be an even hyperbolic lattice with a positive cone $\mathcal{P}_S$, and let N be a standard fundamental domain of the action of $W(S)$ on $\mathcal{P}_S$. Let $\Gamma \subset \mathrm{O}^+(S)$ be a subgroup of finite index.

Borcherds-Kondo method calculates the group

$$\mathrm{Aut}(N) \cap \Gamma$$

by embedding S into L_{10}, L_{18}, or L_{26} primitively. We explain the case L_{26}.

Recall that Δ is the Conway chamber defined above by $w = (1, 0, 0)$. Then the positive cone $\mathcal{P}_{26}$ is tessellated by Conway chambers:

$$\mathcal{P}_{26} = \bigcup_{\gamma \in W(L_{26})} \Delta^{\gamma}.$$

For a Conway chamber Δ^γ, we call w^γ the *Weyl vector* of Δ^γ. Then Δ^γ is bounded by the mirrors associated with roots in

$$\{\, r \in L_{26} \mid \langle r, r\rangle = -2, \langle r, w^\gamma\rangle = 1 \,\} = \{\, r_\lambda^\gamma \mid \lambda \in \Lambda \,\}.$$

Suppose that we have a primitive embedding

$$S \hookrightarrow L_{26}$$

that maps $\mathcal{P}_S$ to $\mathcal{P}_{26}$. Then $\mathcal{P}_S = \mathcal{P}_{26} \cap (S \otimes \mathbb{R})$, and hence

$$\mathcal{P}_S = \bigcup_{\gamma \in W(L_{26})} D_\gamma, \quad \text{where } D_\gamma := \mathcal{P}_S \cap \Delta^\gamma.$$

We say that $D_\gamma = \mathcal{P}_S \cap \Delta^\gamma$ is an *induced chamber* if D_γ contains a non-empty open subset of $\mathcal{P}_S$. Let $\mathcal{I}$ denote the set of induced chambers. Then we have

$$\mathcal{P}_S = \bigcup_{D \in \mathcal{I}} D.$$

Since a root of S is a root of L_{26}, there exists a subset $\mathcal{I}_N$ of $\mathcal{I}$ such that

$$N = \bigcup_{D \in \mathcal{I}_N} D.$$

We say that two induced chambers D and D' is Γ-*equivalent* if $D' = D^\gamma$ for some $\gamma \in \Gamma$. Since $\Gamma \subset \mathrm{O}^+(S)$ is of finite index,

Proposition. The number of Γ-equivalence classes of induced chambers is finite.

We make the following:

Assumption. The orthogonal complement of S in L_{26} cannot be embedded into the Leech lattice Λ.

Proposition. Let $D = \mathcal{P}_S \cap \Delta^\gamma$ be an induced chamber. Then D has only a finite number of walls, and these walls can be calculated effectively from the Weyl vector w^γ of the Conway chamber Δ^γ.

Algorithm

We choose an induced chamber $D_0 \subset N$, and set

$$\mathcal{D} := [D_0], \quad \mathcal{G} := \{\}.$$

When the algorithm terminates, $\mathcal{D} = [D_0, \dots, D_m]$ is a complete set of representatives of Γ-equivalence classes of induced chambers in N, and $\mathcal{G} = \{g_1, \dots, g_l\}$ is a set of generators of $\mathrm{Aut}(N) \cap \Gamma$.

Starting from $i = 0$, we do the following while $i + 1 \leq |\mathcal{D}|$.

1. We calculate the finite set of walls of D_i, and the finite group $\mathrm{Aut}(D_i) \cap \Gamma$. We append $\mathrm{Aut}(D_i) \cap \Gamma$ to $\mathcal{G}$.
2. For each wall $(v)^\perp \cap D_i$ of D_i that is *not* a wall of N, we calculate a Weyl vector w' of a Conway chamber Δ' such that $D' := \mathcal{P}_S \cap \Delta'$ is an induced chamber adjacent to D_i across the wall $(v)^\perp \cap D_i$. If $D' = D_j^\gamma$ for some $D_j \in \mathcal{D}$ and $\gamma \in \Gamma$, then we add γ to $\mathcal{G}$. If there are no such pairs of D_j and γ, we append D' to $\mathcal{D}$.
3. Increment i by $+1$.

This method has been applied to various $K3$ surfaces.

- Kondo (1998): a generic Jacobian Kummer surface $/\mathbb{C}$.
- Kondo-Keum (2001): Kummer surfaces of product type $/\mathbb{C}$.
- Dolgachev-Keum (2002): a quartic Hessians surface $/\mathbb{C}$.
- Dolgachev-Kondo (2003): a supersingular K3 surface in characteristic 2 with Artin invariant 1.
- Kondo-S. (2012): the Fermat quartic in characteristic 3.
- Ujikawa (2013): the singular K3 surface whose transcendental lattice is of discriminant 7.
- S. (2015): several singular K3 surfaces whose transcendental lattices are of relatively small discriminants.

Remark. Vinberg (1983) had determined the automorphism group of the singular K3 surface whose transcendental lattice is of discriminant 3 and 4 by different method.

Example: Quartic Hessian surface

We review the work of Dolgachev-Keum (2002) on a quartic Hessian surface over $\mathbb{C}$.

Let $F = F(x_1, \dots, x_4)$ be a general cubic homogeneous polynomial, and let $\overline{X}$ be the quartic surface in $\mathbb{P}^3$ defined by

$$\det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right) = 0.$$

Then $\overline{X}$ has ten ordinary nodes p_α as its only singularities, and contains exactly ten lines ℓ_β.

The minimal resolution X of $\overline{X}$ is a K3 surface with $\operatorname{rank} S_X = 16$. Let E_α be the exceptional curve over p_α, and L_β the strict transform of ℓ_β. Then S_X is generated by the classes $[E_\alpha]$ and $[L_\beta]$. We have $\mathrm{O}^\omega(T_X) = \{\pm 1\}$, and the natural homomorphism

$$\mathrm{Aut}(X) \to \mathrm{Aut}(N(X)) \subset \mathrm{O}^+(S_X)$$

is injective.

A Gram matrix of S_X with respect to a certain basis:

$$\begin{bmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix}$$

A Gram matrix of L_{26}:

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -8 & -4 & -4 & -4 & -4 & -4 & -4 & -2 & -4 & -4 & -4 & -2 & -4 & -2 & -2 & -2 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 3 \\
0 & 0 & -4 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & -2 & -1 & -1 & -2 & -1 & 0 & 0 & 1 \\
0 & 0 & -4 & -2 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -1 & -2 & -2 & -1 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & -4 & -2 & -2 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & -2 & -2 & -1 & -2 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & -4 & -2 & -2 & -2 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & 0 & 0 & 1 \\
0 & 0 & -4 & -2 & -2 & -2 & -2 & -4 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -2 & -1 & -1 & -2 & -1 & -2 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -4 & -2 & -2 & -2 & -2 & -2 & -4 & -2 & -2 & -2 & -2 & -1 & -2 & -1 & -2 & -1 & -2 & -1 & -1 & -2 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -4 & -1 & -1 & -1 & -2 & -1 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & 0 & 0 & -1 \\
0 & 0 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & -1 & 1 \\
0 & 0 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -4 & -2 & -2 & -2 & -2 & -1 & -1 & -2 & -2 & -1 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -2 & -4 & -2 & -2 & -1 & -2 & -1 & -2 & -1 & -2 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & -2 & -2 & -2 & -2 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & -4 & -1 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & -1 & -1 & -1 \\
0 & 0 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -1 & -4 & -2 & -2 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
0 & 0 & -2 & -2 & -1 & -1 & -2 & -2 & -1 & -2 & -2 & -2 & -1 & -2 & -2 & -4 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & -2 & -1 & -1 \\
0 & 0 & -2 & -1 & -2 & -1 & -2 & -1 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & -4 & -2 & -1 & -2 & -2 & -2 & -2 & -1 & -2 & -1 \\
0 & 0 & -2 & -1 & -1 & -2 & -2 & -1 & -1 & -2 & -2 & -1 & -1 & -2 & -2 & -2 & -2 & -4 & -1 & -2 & -2 & -2 & -2 & -1 & -1 & -1 \\
0 & 0 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -1 & -2 & -1 & -1 & -1 & -4 & -2 & -2 & -2 & -1 & -1 & -1 & 1 \\
0 & 0 & -2 & -1 & -2 & -1 & -2 & -1 & -1 & -2 & -2 & -2 & -1 & -2 & -1 & -2 & -2 & -2 & -2 & -4 & -2 & -2 & -2 & -2 & -1 & -1 \\
0 & 0 & -2 & -1 & -1 & -2 & -2 & -2 & -1 & -2 & -2 & -1 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & -2 & -4 & -2 & -2 & -1 & -2 & -1 \\
0 & 0 & -2 & -2 & -1 & -1 & -2 & -1 & -2 & -2 & -2 & -1 & -1 & -2 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -4 & -2 & -1 & -1 & -1 \\
0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & -2 & -1 & 0 & 0 & -2 & -1 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -4 & -2 & -2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & -2 & -1 & -1 & -1 & -2 & -1 & -1 & -2 & -4 & -2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -2 & -1 & -1 & -1 & -2 & -1 & -2 & -2 & -4 & -2 \\
0 & 0 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -2 & -2 & -2 & -4
\end{bmatrix}
$$

We embed S_X into L_{26} primitively by $v \mapsto vM$, where M is as follows:

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -6 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & 6 & -2 & -2 & -2 & -1 & -2 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 2 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 5 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 2 \\
1 & 1 & -3 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -3 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\
1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\
1 & 1 & 3 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & -1 & 2 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

The orthogonal complement of $S_X \hookrightarrow L_{26}$ has a root, and hence we can apply our algorithm. The output is as follows.

Theorem (Dolgachev-Keum). There exists only one Γ_X-equivalence class of induced chambers. Let $D_X \subset N(X)$ be the induced chamber.

- The group $\mathrm{Aut}(D_X)$ is of order 240, and the group $\mathrm{Aut}(D_X) \cap \Gamma_X$ is of order 2. The non-trivial element of $\mathrm{Aut}(D_X) \cap \Gamma_X$ is an Enriques involution $\varepsilon \colon X \to X$.
- The number of walls of D_X is $20 + 10 + 24 + 30 = 84$, among which 20 walls are walls of $N(X)$ and they are defined by the roots $[E_\alpha]$ and $[L_\beta]$, whereas the other $10 + 24 + 30$ walls are *not* walls of $N(X)$.

Therefore $\mathrm{Aut}(X)$ is generated by ε and $10 + 24 + 30$ isometries, each of which is an involution that maps D_X to an induced chamber in $N(X)$ adjacent to D_X across one of the $10 + 24 + 30$ walls.

The geometric realization of generators are also obtained.

The Enriques involution $\varepsilon\colon X \to X$ is classically known. Recall that X is a minimal resolution of the quartic surface defined by the Hessian of a general cubic polynomial $F(x_1, \dots, x_4)$. Since F is general, there exist complex numbers $\lambda_1, \dots, \lambda_5$ such that the cubic surface $F = 0$ is written in the Sylvester form

$$\lambda_1 z_1^3 + \cdots + \lambda_5 z_5^3 = z_1 + \cdots + z_5 = 0$$

in $\mathbb{P}^4$. Then $\overline{X}$ is isomorphic

$$\frac{1}{\lambda_1 z_1} + \cdots + \frac{1}{\lambda_5 z_5} = z_1 + \cdots + z_5 = 0.$$

The involution given by

$$z_i \mapsto \frac{1}{\lambda_i z_i} \qquad (i = 1, \dots, 5)$$

induces the Enriques involution $\varepsilon\colon X \to X$.

The Enriques involution ε is given by the following matrix.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-2 & 1 & 1 & -2 & -2 & 1 & 0 & 0 & 3 & 0 & 2 & -1 & -1 & 2 & 2 & -3 \\
-1 & 1 & 0 & -1 & -2 & 0 & 1 & 0 & 2 & 0 & 1 & -1 & 0 & 1 & 2 & -2 \\
-1 & 0 & 1 & -2 & -1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & -1 & 2 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 1 & -1 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

The matrix representations for the other $10 + 24 + 30$ involutions are also available.

Remark

In all the cases where Borcherds-Kondo method was carried out *manually*, the number of Γ_X-equivalence classes of induced chambers is 1. We call such a case *of simple Borcherds type*.

Examples of non-simple Borcherds type (S. (2015)).
Let X be the complex $K3$ surface with $\operatorname{rank} S_X = 20$ and

$$T_X = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}.$$

Then the number of Γ_X-equivalence classes of induced chambers is 1098. We have obtained a set $\mathcal{G}$ of generators of $\operatorname{Aut}(X)$ with cardinality 767.

When X is the complex Fermat quartic surface, we have $\operatorname{rank} S_X = 20$ and

$$T_X = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

Then the number of Γ_X-equivalence classes is > 100000.

Aut(Enriques)

We work over $\mathbb{C}$.
For a lattice L, let $L(2)$ denote the lattice obtained from L by multiplying the intersection form $\langle\ ,\ \rangle$ by 2.

An *Enriques surface* is a smooth projective surface Y with $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$ whose universal cover X is a $K3$ surface.
Let Y be an Enriques surface. Then $H^2(Y, \mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z}$, and since $h^{2,0}(Y) = 0$, we have

$$S_Y = H^2(Y, \mathbb{Z})/(\mathrm{torsion}) \cong L_{10}.$$

Let $\pi\colon X \to Y$ be the universal covering, and let $\varepsilon \in \mathrm{Aut}(X)$ be the deck-transformation. For a group G and an element $g \in G$, we denote by $Z_G(g)$ the centralizer of g in G. We have a natural isomorphism

$$\mathrm{Aut}(Y) \cong Z_{\mathrm{Aut}(X)}(\varepsilon)/\langle\varepsilon\rangle.$$

We put

$$S_X^+ := \{v \in S_X \mid v^\varepsilon = v\}, \quad S_X^- := \{v \in S_X \mid v^\varepsilon = -v\}.$$

The pull-back $\pi^* \colon S_Y \to S_X$ induces an isomorphism of lattices

$$\pi^* : S_Y(2) \xrightarrow{\sim} S_X^+.$$

Let $\mathcal{P}_Y$ be the positive cone of S_Y containing an ample class. We put

$$N(Y) := \{ \, y \in \mathcal{P}_Y \mid \langle y, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } Y \, \}.$$

By the isomorphism $\pi^* \colon S_Y(2) \xrightarrow{\sim} S_X^+$, we regard S_Y as a $\mathbb{Z}$-submodule of S_X. In particular, we have

$$\mathcal{P}_Y = (S_Y \otimes \mathbb{R}) \cap \mathcal{P}_X, \quad N(Y) = (S_Y \otimes \mathbb{R}) \cap N(X) = \mathcal{P}_Y \cap N(X).$$

Suppose that $\mathrm{Aut}(X)$ is determined by a primitive embedding $S_X \hookrightarrow L_{26}$. Then, composing this embedding with $\pi^*\colon S_Y(2) \hookrightarrow S_X$, we have a primitive embedding

$$S_Y(2) \hookrightarrow L_{26},$$

and hence we obtain

$$N(Y) = \bigcup D',$$

where D' are induced chambers; that is, D' is a closed subset of $\mathcal{P}_Y$ with non-empty interior and of the form $(S_Y \otimes \mathbb{R}) \cap (\text{a Conway chamber})$, or equivalently, $(S_Y \otimes \mathbb{R}) \cap (\text{an induced chamber in } \mathcal{P}_X)$.
By this decomposition, we can calculate $\mathrm{Aut}(Y)$.

Difficulty.
The cone $N(Y)$ is not a standard fundamental domain of $W(S_Y)$. Hence we need an extra work to determine whether a wall of an induced chamber is a wall of $N(Y)$ or not.

A generic Enriques surface

Suppose that Y is a *generic* Enriques surface. In this case, we have $S_Y(2) = S_X$. Since there exist no smooth rational curves on Y, we have $N(Y) = \mathcal{P}_Y = \mathcal{P}_X = N(X)$.

Barth and Peters (1983) and Nikulin (1984) proved the following:

Theorem. Suppose that Y is *generic*. Then

$$\mathrm{Aut}(Y) \to \mathrm{O}^+(S_Y)$$

is injective, and its image is equal to the kernel of the mod 2 reduction homomorphism

$$\mathrm{O}^+(S_Y) \to \mathrm{O}(S_Y \otimes \mathbb{F}_2).$$

Corollary. Since $\mathrm{O}^+(S_Y) \to \mathrm{O}(S_Y \otimes \mathbb{F}_2) \cong \mathrm{GO}^+_{10}(\mathbb{F}_2)$ is surjective, we have

$$[\mathrm{O}^+(S_Y) : \mathrm{Aut}(Y)] = |\mathrm{GO}^+_{10}(\mathbb{F}_2)| = 46998591897600.$$

Enriques surface associated with the quartic Hessian

Let X be the minimal resolution of the quartic surface

$$\overline{X} \;:\; \mathrm{Hessian}(F) = 0,$$

where $F(x_1, \dots, x_4)$ is a general cubic homogeneous polynomial. Recall that X has an Enriques involution $\varepsilon : X \to X$. We consider the Enriques surface

$$Y := X/\langle \varepsilon \rangle.$$

Recall that $\mathrm{Aut}(X)$ is generated by ε and $10 + 24 + 30$ involutions. The 10 involutions ι_α among them come from the projections

$$\overline{X} \to \mathbb{P}^2$$

with the center being the 10 nodes p_α of $\overline{X}$. It was observed by Dolgachev-Keum that these involutions commute with ε. In particular, each $\iota_\alpha \in Z_{\mathrm{Aut}(X)}(\varepsilon)$ defines an involution $j_\alpha \in \mathrm{Aut}(Y)$.

Theorem 1. The natural homomorphism $\mathrm{Aut}(Y) \to \mathrm{Aut}(N(Y)) \subset \mathrm{O}^+(S_Y)$ is injective.

Theorem 2. $\mathrm{Aut}(Y)$ is generated by j_α. The following relations form a set of defining relations of $\mathrm{Aut}(Y)$ with respect to these generators j_α;

- $j_\alpha^2 = \mathrm{id}$ for each node p_α,
- $(j_\alpha j_{\alpha'} j_{\alpha''})^2 = \mathrm{id}$ for each triple $(p_\alpha, p_{\alpha'}, p_{\alpha''})$ of distinct three nodes on a line $\subset \overline{X}$, and
- $(j_\alpha j_{\alpha'})^2 = \mathrm{id}$ for each pair $(p_\alpha, p_{\alpha'})$ of distinct nodes such that the line $\overline{p_\alpha p_{\alpha'}} \subset \mathbb{P}^3$ is not contained in $\overline{X}$.

Remark. Mukai and Ohashi have also proved that the 10 involutions j_α generate $\mathrm{Aut}(Y)$.

Theorem 2 (continued).
There exists a fundamental domain D_Y of the action of $\mathrm{Aut}(Y)$ on $N(Y)$ with the following properties.

- D_Y is bounded by $10+10$ hyperplanes $(\bar{u}_\alpha)^\perp$ and $(\bar{v}_\alpha)^\perp$, where $\bar{u}_\alpha$ and $\bar{v}_\alpha$ are roots of S_Y.
- For each α, we have $\bar{u}_\alpha = [\pi(E_\alpha)] = [\pi(L_{\bar{\alpha}})]$. Hence $(\bar{u}_\alpha)^\perp$ is a wall of $N(Y)$.
- For each α, the root $\bar{v}_\alpha$ is *not* the class of a smooth rational curve on Y, and $j_\alpha \in \mathrm{Aut}(Y)$ maps D_Y to the chamber adjacent to D_Y across the wall $D_Y \cap (\bar{v}_\alpha)^\perp$ of D_Y.
- D_Y is a union of

$$2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 31 = 906608640$$

  Vinberg chambers.

The situation is follows.

$$\begin{array}{lcl} & & \mathrm{O}^+(S_Y) = \mathrm{O}^+(L_{10}) \\ \text{index } 906608640 & \cup & \\ & & G_{QH} \quad \supset \quad \mathrm{Aut}(Y) \\ \text{index } 51840 & \cup & \\ & & \mathrm{Aut}(Y_{\mathrm{generic}}) = \mathrm{Ker}(\mathrm{O}^+(L_{10}) \to \mathrm{O}(L_{10} \otimes \mathbb{F}_2)) \end{array}$$

By specialization from a generic Enriques surface Y_{generic} to Y, the period condition is relaxed and $\mathrm{Aut}(Y_{\mathrm{generic}})$ becomes a larger group G_{QH} with $10+10$ generators. But the presence of smooth rational curves $\pi(E_\alpha)$ prevents 10 generators (reflections with respect to $\bar{u}_\alpha = [\pi(E_\alpha)]$) from entering into $\mathrm{Aut}(Y)$.

Remark. We have

$$G / \mathrm{Aut}(Y_{\mathrm{generic}}) \cong W(E_6) \cong \mathrm{GO}_6^-(2).$$

By looking at the fundamental domain D_Y, we can classify various geometric objects (elliptic fibrations, ...) on Y modulo $\mathrm{Aut}(Y)$.

Idea of the proof.

We describe the walls and faces of D_Y explicitly.

- The relations among the generators j_α correspond to the 8-dimensional faces of D_Y.
- The elliptic fibrations correspond to the 1-dimensional faces contained in $\partial\overline{\mathcal{P}}_Y \setminus \mathcal{P}_Y$.
- ...

The main tool is *Linear Programming*.

Let V be a finite dimensional vector space over $\mathbb{Q}$, and put

$$V^\vee := \mathrm{Hom}(V, \mathbb{Q}).$$

For a non-zero $f \in V^\vee$, we put

$$(f)^\perp := \{\, x \in V \otimes \mathbb{R} \mid f(x) = 0 \,\}.$$

Let $f_1, \dots, f_m$ be non-zero elements of $V^\vee$ such that $(f_1)^\perp, \dots, (f_m)^\perp$ are distinct.
We consider the chamber

$$C := \{\, x \in V \otimes \mathbb{R} \mid f_i(x) \geq 0 \ \text{ for all } \ i \,\}.$$

Then $(f_i)^\perp$ is a wall of C if and only if the solution of the following problem of linear programing is unbounded to $-\infty$:

$$\begin{aligned} &\text{minimize } f_i(x) \\ &\text{subject to the constraints } f_j(x) \geq 0 \text{ for all } j \neq i. \end{aligned}$$

We cut the Dolgachev-Keum chamber $D_X \subset S_X \otimes \mathbb{R}$ by

$$S_Y \otimes \mathbb{R} := \{\, x \in S_X \otimes \mathbb{R} \mid x^\varepsilon = x \,\},$$

and investigate the walls and faces of $D_Y := D_X \cap (S_Y \otimes \mathbb{R})$ by applying the linear-programming method iteratively. (A wall of a wall of D_Y is an 8-dimensional face of D_Y, a wall of an 8-dimensional face of D_Y is a 7-dimensional face of D_Y, ...)

An application: generators of $\mathrm{Aut}(Y_{\mathbf{generic}})$

We have the following equalities:

$$\begin{aligned}\mathrm{Aut}(Y_{\mathrm{generic}}) &= \mathrm{Ker}(\rho\colon \mathrm{O}^+(L_{10}) \twoheadrightarrow \mathrm{GO}^+_{10}(2)) \\ &= \mathrm{Ker}(\rho|G_{QH}\colon G_{QH} \twoheadrightarrow \mathrm{GO}^-_6(2)).\end{aligned}$$

Since we know finite sets of generators for $\mathrm{O}^+(L_{10})$ and for G_{QH}, we can obtain a finite set of generators of $\mathrm{Aut}(Y_{\mathrm{generic}})$ by the Reidemeister-Schreier method. Since

$$|\mathrm{GO}^+_{10}(2)| = 46998591897600$$

is very large, however, making use of the first equality is not practical. On the other hand, since

$$|\mathrm{GO}^-_6(2)| = 51840$$

is much smaller, we have managed to obtain a finite set of generators of $\mathrm{Aut}(Y_{\mathrm{generic}})$ in a reasonable computation time by means of the second equality.