Computation of automorphism groups of $K3$ and Enriques surfaces

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Terminologies about lattices

- A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $\langle\, ,\, \rangle \colon L \times L \to \mathbb{Z}$.
- The automorphism group of L is denoted by $\mathrm{O}(L)$. The action is from the right: $v \mapsto v^g$ for $g \in \mathrm{O}(L)$.
- A lattice L is *unimodular* if $\det(\mathrm{Gram\ matrix}) = \pm 1$.
- A lattice L is *even* (or *of type* II) if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$.
- A lattice L of rank n is *hyperbolic* if the signature of $L \otimes \mathbb{R}$ is $(1, n-1)$.

We will mainly deal with even hyperbolic lattices.

- A *positive cone* of a hyperbolic lattice L is one of the two connected components of

$$\{\, x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \,\}.$$

- A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r \rangle = -2$.

Terminologies about even hyperbolic lattices

Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$. We put

$$\mathrm{O}(L, \mathcal{P}) := \{\, g \in \mathrm{O}(L) \mid \mathcal{P}^g = \mathcal{P} \,\}.$$

We have $\mathrm{O}(L) = \mathrm{O}(L, \mathcal{P}) \times \{\pm 1\}$.
For a vector $v \in L \otimes \mathbb{Q}$ with $\langle v, v \rangle < 0$, we put

$$(v)^{\perp} := \{\, x \in \mathcal{P} \mid \langle v, x \rangle = 0 \,\}.$$

A (-2)-vector $r \in L$ defines the reflection into the mirror $(r)^{\perp}$:

$$s_r \colon x \mapsto x + \langle x, r \rangle r.$$

Let $W(L)$ denote the subgroup of $\mathrm{O}(L, \mathcal{P})$ generated by all reflections s_r with respect to (-2)-vectors r.
Note that $W(L)$ is a normal subgroup in $\mathrm{O}(L, \mathcal{P})$.

Standard fundamental domain

A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure of a connected component of

$$\mathcal{P} \setminus \bigcup (r)^{\perp},$$

where r runs through the set of all (-2)-vectors.

Then $W(L)$ acts on the set of standard fundamental domains simple-transitively. Let N be a standard fundamental domain. We put

$$\mathrm{O}(L, N) := \{\, g \in \mathrm{O}(L) \mid N^g = N \,\}.$$

Then we have

$$\begin{aligned} W(L) &= \langle\, s_r \mid \text{the hyperplane } (r)^{\perp} \text{ bounds } N \,\rangle, \\ \mathrm{O}(L, \mathcal{P}) &= W(L) \rtimes \mathrm{O}(L, N). \end{aligned}$$

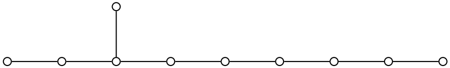
Even unimodular hyperbolic lattice

Theorem

For $n \in \mathbb{Z}_{>0}$ with $n \equiv 2 \bmod 8$, there exists an even unimodular hyperbolic lattice L_n of rank n. (A more standard notation is $\mathrm{II}_{1,n-1}$.) For each n, the lattice L_n is unique up to isomorphism.

We denote by U (instead of L_2) the *hyperbolic plane* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Example by Vinberg. A standard fumdamental domain of the action of $W(L_{10})$ is bounded by 10 hyperplanes $(r_1)^\perp, \dots, (r_{10})^\perp$ defined by (-2)-vectors $r_1, \dots, r_{10}$ that form the dual graph below. Since this graph has no non-trivial symmetries, we have $\mathrm{O}(L_{10}, \mathcal{P}) = W(L_{10})$.



The lattice of a non-singular projective surface

For simplicity, we work over $\mathbb{C}$.
For a non-singular projective surface Z, we denote by S_Z the lattice of numerical equivalence classes of divisors on Z. The rank of S_Z is the *Picard number* of Z. Then S_Z is hyperbolic by Hodge index theorem.

If Z is a $K3$ surface, then S_Z is even.
If Z is an Enriques surface, then S_Z is isomorphic to L_{10}.

Let $\mathcal{P}_Z$ be the positive cone containing an ample class of Z. We put

$$N_Z := \{\, x \in \mathcal{P}_Z \mid \langle x, C \rangle \geq 0 \ \text{ for all curves } C \text{ on } Z \,\}.$$

Plenty of information about geometry of a $K3$ surface or an Enriques surface is provided by the lattice S_Z.

Geometry of $K3$ surfaces

Suppose that X is a complex $K3$ surface.

Theorem

The nef cone N_X is a standard fundamental domain of the action of $W(S_X)$ on $\mathcal{P}_X$. The walls of N_X are the hyperplanes defined by the classes of smooth rational curves on X.

Theorem

The natural homomorphism $\mathrm{Aut}(X) \to \mathrm{O}(S_X, N_X)$ is an isomorphism up to finite kernel and finite cokernel.

The kernel and the cokernel can be calculated by looking at the action on the discriminant group of S_X and the period $H^{2,0}(X)$.

Algorithms for $K3$ surfaces

Suppose that we have an ample class $a \in S_X$. Then a is an interior point of the nef cone N_X.

- We can determine whether a given vector $v \in \mathcal{P}_X \cap S_X$ is nef or not by calculating the finite set

$$\{\, r \in S_X \mid \langle r, r \rangle = -2, \langle r, a \rangle > 0, \langle r, v \rangle < 0 \,\}.$$

- Let $r \in S_X$ be a (-2)-vector such that

$$d := \langle r, a \rangle > 0,$$

so that r is the class of an effective divisor D. Then D is irreducible if and only if $\langle r, C' \rangle \geq 0$ for any smooth rational curve C' with $\langle C', a \rangle < d$. Hence we can determine whether r is the class of a smooth rational curve or not by induction on d.

We can enumerate

- all classes f of fibers of elltptic fibrations with $\langle f, a\rangle \le d$,
- all polarizations $h_2 \in S_X$ of degree $\langle h_2, h_2\rangle = 2$ with $\langle h_2, a\rangle \le d$, and the matrix representations on S_X of involutions associated with the double covers $X \to \mathbb{P}^2$,
- $\dots$.

A "$K3$ surface" $\mathbb{X}_{26}$ with Picard number 26

Let $\mathbb{X}_{26}$ be a "$K3$ surface" such that $S_{\mathbb{X}_{26}}$ is isomorphic to the even unimodular hyperbolic lattice L_{26} of rank 26. We can state theorems on the lattice L_{26} as theorems on the geometry of this *non-existing* $K3$ surface $\mathbb{X}_{26}$.

A negative-definite even unimodular lattice of rank 24 is called a *Niemeier lattice*. Niemeier showed that there exist exactly 24 isomorphism classes of Niemeier lattices, one of which is the famous *Leech lattice* Λ.

The lattice L_{26} is written as

$$U \oplus (\text{a Niemeier lattice}).$$

A vector $\mathbf{w} \in L_{26}$ is called a *Weyl vector* if $\mathbf{w}$ is written as $(1, 0, \mathbf{0})$ in a decomposition

$$L_{26} = U \oplus \Lambda.$$

A (-2)-vector $r \in L_{26}$ is a *Leech root* with respect to $\mathbf{w}$ if $\langle \mathbf{w}, r \rangle = 1$. Under the expression $L_{26} = U \oplus \Lambda$ such that $\mathbf{w} = (1, 0, \mathbf{0})$, Leech roots are written as

$$\left(-\frac{\lambda^2}{2} - 1, 1, \lambda \right), \quad \text{where } \lambda \in \Lambda.$$

Theorem (Conway (1983))

The nef cone $N_{\mathbb{X}_{26}}$ of $\mathbb{X}_{26}$ is bounded by hyperplanes defined by Leech roots with respect to a Weyl vector.

Corollary

The group $\mathrm{O}(S_{\mathbb{X}_{26}}, N_{\mathbb{X}_{26}})$ is the group Co_∞ of **affine** *isometries of Λ ($\mathrm{O}(\Lambda) +$* **translations** *).*

Elliptic fibrations

For a $K3$ surface X, we put $\partial\overline{\mathcal{P}}_X := \overline{\mathcal{P}}_X \setminus \mathcal{P}_X$.

Theorem

The elliptic fibrations of a $K3$ surface X are in one-to-one correspondence with the rays in $\partial\overline{\mathcal{P}}_X \cap \overline{N}_X$.

The classification of Niemerer lattices can also be regarded as the classification of elliptic fibrations on $\mathbb{X}_{26}$.

Theorem

Up to the action of Co_∞, there exist exactly 24 rays in $\partial\overline{\mathcal{P}}_{\mathbb{X}_{26}} \cap \overline{N}_{\mathbb{X}_{26}}$. Each of them gives the orthogonal decomposition $L_{26} = U \oplus N$, where N is a Niemeier lattice.

Borchers' method

We call standard fundamental domains of the action of $W(L_{26})$ on $\mathcal{P}(L_{26})$ *Conway chambers*. The positive cone $\mathcal{P}(L_{26})$ is tessellated by Conway chambers $\mathcal{C}$.

Let X be a $K3$ surface. Suppose that we have a primitive embedding $S_X \hookrightarrow L_{26}$, and hence $\mathcal{P}_X$ is a subspace of $\mathcal{P}(L_{26})$.

An *induced chamber* is a closed subset D of $\mathcal{P}_X$ that has an interior point and is obtained as the intersection $\mathcal{P}_X \cap \mathcal{C}$ of $\mathcal{P}_X$ and a Conway chamber $\mathcal{C}$. The tessellation of $\mathcal{P}(L_{26})$ by Conway chambers $\mathcal{C}$ induces a tessellation of $\mathcal{P}_X$ by these induced chambers $D = \mathcal{P}_X \cap \mathcal{C}$.

We assume the following mild assumption:

The orthogonal complement of S_X in L_{26} contains a (-2)-vector.

Then any induced chamber of $\mathcal{P}_X$ has only finite number of walls.

Since N_X is bounded by walls $(r)^\perp$ of (-2)-vectors r, and a (-2)-vector r of S_X is a (-2)-vector of L_{26}, the nef cone N_X is tessellated by induced chambers.

Definition

We say that the induced tessellation of $\mathcal{P}_X$ is *simple* if the induced chambers are congruent to each other by the action of $\mathrm{O}(S_X, \mathcal{P}_X)$.

When the induced tessellation is simple, we can calculate the shape of N_X and hence $\mathrm{Aut}(X)$.

This method, which was contrived by Borcherds (1987), is regarded as a calculation of $\mathrm{Aut}(X)$ by a generalization of "the $K3$ surface" $\mathbb{X}_{26}$ to X, that is, we regard the embedding

$$S_X \hookrightarrow L_{26} = S_{\mathbb{X}_{26}}$$

as the embedding induced by a "specialization" of X to $\mathbb{X}_{26}$.

$\mathrm{Aut}(X)$ for many $K3$ surfaces X have been calculated by this method.

Example by Kondo (1999)

Let

$$X := \mathrm{Km}(\mathrm{Jac}(C))$$

be the Kummer surface of the Jacobian variety $\mathrm{Jac}(C)$ of a general genus 2 curve

$$C\colon y^2 = (x-\lambda_1)\cdots(x-\lambda_6).$$

Then S_X is of rank 17, and we have a primitive embedding $S_X \hookrightarrow L_{26}$ such that $\mathcal{P}_X$ is *simply* tessellated by induced chambers.
An induced chamber $D \subset N_X$ has $32+60+32+192$ walls.
The 32 walls are defined by the classes of smooth rational curves: the 32 lines on the $(2,2,2)$-complete intersection model $X_{2,2,2}$ of X.

$$\begin{aligned} x_1^2 + \quad x_2^2 + \quad x_3^2 + \quad x_4^2 + \quad x_5^2 + \quad x_6^2 &= 0, \\ \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 x_5^2 + \lambda_6 x_6^2 &= 0, \\ \lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + \lambda_3^2 x_3^2 + \lambda_4^2 x_4^2 + \lambda_5^2 x_5^2 + \lambda_6^2 x_6^2 &= 0. \end{aligned}$$

The group

$$\mathrm{Aut}(X, D) := \{ g \in \mathrm{Aut}(X) \mid D^g = D \}$$

is the projective automorphism group $\mathrm{Aut}(X_{2,2,2})$ of $X_{2,2,2} \subset \mathbb{P}^5$, which is isomorphisc to $(\mathbb{Z}/2\mathbb{Z})^5$.
For each of the other $60 + 32 + 192$ walls w, there exists an involution $g_w \in \mathrm{Aut}(X)$ that maps D to the induced chamber adjacent to D across the wall w.

Theorem

The automorphism group $\mathrm{Aut}(X)$ *is generated by* $\mathrm{Aut}(X_{2,2,2}) \cong (\mathbb{Z}/2\mathbb{Z})^5$ *and* $60 + 32 + 192$ *involutions* g_w.

60 involutions: Hutchinson-Göpel involutions (Enriques involutions).
32 involutions: projections from a node on a quartic surface model.
192 involutions: Hutchinson-Weber involutions (Enriques involutions).

Example by Dolgachev-S. arXiv:1908.05390

Borcherds' method is suitable for the analysis of the change of $\mathrm{Aut}(X)$ under generalization/specialization of $K3$ surfaces.

The surface $X = \mathrm{Km}(\mathrm{Jac}(C))$ has a quartic surface model with 16 ordinary nodes (Kummer quartic). We generalize X to a $K3$ surface X' that has a quartic surface model with 15 ordinary nodes. This X' is related to the line congruence of type $(2, 3)$ in $\mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3)$.

From Kondo's embedding $S_X \hookrightarrow L_{26}$, we obtain $S_{X'} \hookrightarrow L_{26}$, which induces a simple tessellation of $\mathcal{P}_{X'}$.

Theorem

The automorphism group of X' is generated by

$$6 + 45 + 6 + 15 + 120 + 72$$

automorphisms, each of which is described explicitly and geometrically.

Remark. We also obtained a set of defining relations of $\mathrm{Aut}(X')$ with respect to these generators.

Remark. For every complex $K3$ surface X, we can embed S_X into L_{26} primitively. Usually, however, the induced tessellation on $\mathcal{P}_X$ is not simple. For example, we observed that, when X is the Fermat quartic surface X_{FQ}, there exist more than 10^5 types of induced chambers, and hence the calculation of $\mathrm{Aut}(X_{\mathrm{FQ}})$ by Borcherds' method is very difficult.

The last remark is **NOT** the case for the calculation of Aut of Enriques surfaces, as will be seen below.

Enriques involution

An involution ε of a $K3$ surface X is called an *Enriques involution* if ε is fixed-point free, or equivalently, $Y := X/\langle\varepsilon\rangle$ is an Enriques surface.

Let $\pi\colon X \to Y$ be the universal covering of Y. Then we obtain a primitive embedding

$$\pi^*\colon S_Y(2) \cong L_{10}(2) \hookrightarrow S_X,$$

where $S_Y(2)$ is the lattice with the same $\mathbb{Z}$-module as S_Y and with the intersection form being that of S_Y multiplied by 2.

Theorem

An involution ε of a $K3$ surface X is an Enriques involution if and only if the fixed sublattice $\{\, v \in S_X \mid v^{\varepsilon} = v \,\}$ of S_X is isomorphic to $L_{10}(2)$, and its orthogonal complement contains no (-2)-vectors.

Enriques involutions on $\mathbb{X}_{26}$

We have classified all Enriques involutions on the "$K3$ surface" $\mathbb{X}_{26}$. This is a joint work with S. Brandhorst (arXiv:1903.01087).

Theorem

Up to the action of $\mathrm{O}(L_{10})$ *and* $\mathrm{O}(L_{26})$*, there exist exactly* 17 *primitive embeddings of* $L_{10}(2)$ *into* L_{26}*.*

$$\mathtt{12A}, \mathtt{12B}, \mathtt{20A}, \dots, \mathtt{20F}, \mathtt{40A}, \dots, \mathtt{40E}, \mathtt{96A}, \dots, \mathtt{96C}, \mathtt{infty}.$$

Among them, only one (the one named as `infty`*) satisfies the condition that the orthogonal complement contains no* (-2)*-vectors.*

No.	name	volume	|aut|	isom	NK
1	12A	269824	2^2		I
2	12B	12142080	$2^3 \cdot 3$		II
3	20A	64757760	$2^3 \cdot 3$		V
4	20B	145704960	2^6		III
5	20C	777093120	$2^3 \cdot 3 \cdot 5$	20D	VII
6	20D	777093120	$2^3 \cdot 3 \cdot 5$	20C	VII
7	20E	906608640	$2^3 \cdot 3 \cdot 5$		VI
8	20F	2039869440	$2^6 \cdot 5$		IV
9	40A	8159477760	$2^7 \cdot 3$		
10	40B	18650234880	$2^7 \cdot 3^2$	40C	
11	40C	18650234880	$2^7 \cdot 3^2$	40B	
12	40D	32637911040	$2^5 \cdot 3^2 \cdot 5$	40E	
13	40E	32637911040	$2^5 \cdot 3^2 \cdot 5$	40D	
14	96A	163189555200	$2^{13} \cdot 3$		
15	96B	652758220800	$2^{12} \cdot 3^3$	96C	
16	96C	652758220800	$2^{12} \cdot 3^3$	96B	
17	infty	∞			

Borchers method for Enriques surface

Recall that $\mathcal{P}(L_{26})$ is tessellated by Conway chambers. A primitive embedding $L_{10}(2) \hookrightarrow L_{26}$ induces a tessellation of $\mathcal{P}(L_{10})$.

The following theorem is very useful in the calculation of $\mathrm{Aut}(Y)$.

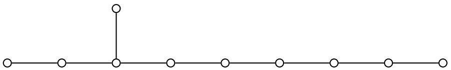
Theorem

Except for the embedding of type `infty`, *the following hold.*

- *The induced tessellation on $\mathcal{P}(L_{10})$ is* **simple**.
- *Each induced chamber D is bounded by a wall $D \cap (r)^\perp$ perpendicular to a (-2)-vector r. (The name of the embedding indicates the number of walls.)*
- *The reflection s_r maps D to the induced chamber adjacent to D across the wall $D \cap (r)^\perp$.*

Remark. Nikulin (1984) and Kondo (1986) classified Enriques surfaces Y with finite automorphism group. If $\mathrm{Aut}(Y)$ is finite, then Y contains only finite number of smooth rational curves. By the configuration of these smooth rational curves, Enriques surfaces Y with finite automorphism group are devided into 7 classes $\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}$.
These 7 configurations appear as the configurations of (-2)-vectors bounding the induced chambers of $\mathcal{P}(L_{10})$.

Remark. Recall that the standard fundamental domain Δ of the action of $W(L_{10})$ on $\mathcal{P}(L_{10})$ is bounded by 10 walls with the dual graph below.



Each induced chamber is a union of copies of Δ.

The induced chambers are much bigger than Δ, and hence we need only small number of copies of chambers to describe N_Y.

For example, let Y be a *generic* Enriques surface. We have $N_Y = \mathcal{P}_Y$. By Barth-Peters (1983), the fundamental domain $\mathcal{F}$ of the action of $\mathrm{Aut}(Y)$ on $N_Y = \mathcal{P}_Y$ is a union of

$$|\mathrm{O}(L_{10} \otimes \mathbb{F}_2)| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 = 46998591897600$$

copies of Δ. If we use the embedding 96C, we can express $\mathcal{F}$ as a union of

$$\frac{46998591897600}{652758220800} = 72$$

copies of induced chambers.

Jacobian Kummer revisited

Ohashi (2009) classified the conjugacy classes of Enriques involutions in $\mathrm{Aut}(\mathrm{Km}(\mathrm{Jac}(C)))$. There exist exactly $6+10+15$ conjugacy classes of Enriques involutions, where
6 are Hutchinson-Weber $\Longrightarrow$ `20E`,
15 are Hutchinson-Göpel $\Longrightarrow$ `40A`,
10 are in $\mathrm{Aut}(X_{2,2,2}) \Longrightarrow$ `40C`.
Let $\sigma \in \mathrm{Aut}(X_{2,2,2})$ be an Enriques involution, and let Y be the corresponding Enriques surface. We have a canonical isomorphism

$$\mathrm{Aut}(Y) \cong \mathrm{Cen}(\sigma)/\langle\sigma\rangle.$$

The centralizer $\mathrm{Cen}(\sigma)$ of σ is generated by $\mathrm{Aut}(X_{2,2,2})$ and 24 Hutchinson-Göpel involutions.

Summary

We can calculate many geometric data of $K3$ surfaces and Enriques surfaces by means of the "$K3$ surface" $\mathbb{X}_{26}$ of Picard number 26.

Thank you for the attention!