Computation of automorphism groups of Enriques surfaces

(joint work with Simon Brandhorst)

Ichiro Shimada

Hiroshima University

ODTU-Bilkent Algebraic Geometry Seminar
2021 December 17

Enriques surface

For simplicity, we work over the the complex number field $\mathbb{C}$.

Definition

A projective smooth surface Y is an *Enriques surface* if it is the quotient $X/\langle\iota\rangle$ of a $K3$ surface X by a fixed-point free involution ι.

Enriques surfaces form an important piece in the Enriques-Kodaira classification of compact complex surfaces.

- The Kodaira dimension of Enriques surfaces is $\kappa = 0$.
- Enriques surfaces form an irreducible deformation family. In particular, they are diffeomorphic to each other. The dimension of the moduli is 10.
- $\pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$, $H^2(Y, \mathbb{Z}) = \mathbb{Z}^{10} \oplus (\mathbb{Z}/2\mathbb{Z})$ and $H^{2,0}(Y) = 0$.
- Let S_Y denote the lattice of numerical equivalence classes of divisors on Y. Then $S_Y = H^2(Y, \mathbb{Z})/\mathrm{torsion}$ is isomorphic to L_{10}.

Terminologies about lattices

- A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $\langle\, ,\, \rangle \colon L \times L \to \mathbb{Z}$.
- The automorphism group of a lattice L is denoted by $\mathrm{O}(L)$. The action is from the right: $v \mapsto v^g$ for $g \in \mathrm{O}(L)$.
- A lattice L is *unimodular* if $\det(\mathrm{Gram\ matrix}) = \pm 1$.
- A lattice L is *even* (or *of type* II) if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$.
- A lattice L of rank n is *hyperbolic* if the signature of $L \otimes \mathbb{R}$ is $(1, n-1)$.

We will mainly deal with even hyperbolic lattices.

- A *positive cone* $\mathcal{P}_L$ of a hyperbolic lattice L is one of the two connected components of

$$\{\, x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \,\}.$$

- A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r \rangle = -2$.

Terminologies about even hyperbolic lattices

Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$. We put

$$\mathrm{O}(L, \mathcal{P}) := \{\, g \in \mathrm{O}(L) \mid \mathcal{P}^g = \mathcal{P} \,\}.$$

We have $\mathrm{O}(L) = \mathrm{O}(L, \mathcal{P}) \times \{\pm 1\}$.
For a vector $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle < 0$, we put

$$(v)^\perp := \{\, x \in \mathcal{P} \mid \langle v, x \rangle = 0 \,\}.$$

A (-2)-vector $r \in L$ defines the reflection into the mirror $(r)^\perp$:

$$s_r \colon x \mapsto x + \langle x, r \rangle r.$$

Then s_r preserves the positive cone $\mathcal{P}$, that is, $s_r \in \mathrm{O}(L, \mathcal{P})$.
Let $W(L)$ denote the subgroup of $\mathrm{O}(L, \mathcal{P})$ generated by all reflections s_r with respect to (-2)-vectors r. We call $W(L)$ the *Weyl group of L*.
Note that $W(L)$ is a normal subgroup in $\mathrm{O}(L, \mathcal{P})$.

Standard fundamental domain

Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$.
A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure of a connected component of

$$\mathcal{P} \setminus \bigcup (r)^{\perp},$$

where r runs through the set of all (-2)-vectors. Then $W(L)$ acts on the set of standard fundamental domains simple-transitively.

We fix a standard fundamental domain N. We put

$$\mathrm{O}(L, N) := \{\, g \in \mathrm{O}(L) \mid N^{g} = N \,\}.$$

Then we have

$$\begin{aligned} W(L) &= \langle\, s_r \mid \text{the hyperplane } (r)^{\perp} \text{ is a wall of } N \,\rangle, \\ \mathrm{O}(L, \mathcal{P}) &= W(L) \rtimes \mathrm{O}(L, N). \end{aligned}$$

Even unimodular hyperbolic lattice

Theorem

For a positive integer n with $n \equiv 2 \bmod 8$, there exists an even unimodular hyperbolic lattice L_n of rank n. (A more standard notation is $\mathrm{II}_{1,n-1}$.) For each n, the lattice L_n is unique up to isomorphism.

We denote by U (instead of L_2) the *hyperbolic plane* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

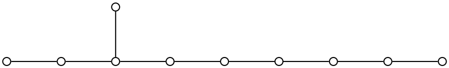
If Y is an Enriques surface, then the lattice S_Y of numerical equivalence classes of divisors on Y is even, unimodular (by Poincaré duality) and hyperbolic (by Hodge index theorem) of rank 10. Hence

$$S_Y \cong L_{10}.$$

We study the geometry of an Enriques surface Y by looking at the lattice $S_Y \cong L_{10}$. In particular, we study the automorphism group $\mathrm{Aut}(Y)$ of Y and its action on S_Y by means of lattice theory.

Vinberg chambers

The lattice L_{10} is generated by ten (-2)-vectors $r_1, \ldots, r_{10}$ whose dual graph is the Dynkin diagram of type E_{10}.



Theorem (Vinberg)

A standard fumdamental domain N of the action of $W(L_{10})$ is bounded by 10 hyperplanes $(r_1)^\perp, \ldots, (r_{10})^\perp$ defined by the ten (-2)-vectors $r_1, \ldots, r_{10}$

Since the graph E_{10} has only trivial symmetries, we have $\mathrm{O}(L_{10}, N) = \{\mathrm{id}\}$ and $\mathrm{O}(L_{10}, \mathcal{P}) = W(L_{10})$.

Definition

We call a standard fundamental domain of L_{10} a *Vinberg chamber*.

Conway chamber

The standard fumdamental domain of the action of $W(L_{26})$ was determined by Conway.

A negative-definite even unimodular lattice of rank 24 is called a *Niemeier lattice*. Niemeier showed that there exist exactly 24 isomorphism classes of Niemeier lattices, one of which is the famous *Leech lattice* Λ.
The Leech lattice is characterized as the unique Niemeier lattice that has no vectors r of $\langle r, r \rangle = -2$.

The lattice L_{26} is written as an orthogonal direct sum

$$U \oplus (\text{a Niemeier lattice}).$$

A vector $\mathbf{w} \in L_{26}$ is called a *Weyl vector* if $\mathbf{w}$ is written as $(1, 0, \mathbf{0})$ in a decomposition

$$L_{26} = U \oplus \Lambda.$$

Equivalently, a vector $\mathbf{w} \in L_{26}$ is a Weyl vector if and only if $\mathbf{w}$ is non-zero and primitive, $\langle \mathbf{w}, \mathbf{w} \rangle = 0$, and the Niemeier lattice $(\mathbb{Z}\mathbf{w})^{\perp}/\mathbb{Z}\mathbf{w}$ is isomorphic to the Leech lattice Λ.

We fix a positive cone $\mathcal{P}_{L_{26}} \subset L_{26} \otimes \mathbb{R}$. Let $\mathbf{w} \in L_{26}$ be a Weyl vector contained in the boundary $\partial\overline{\mathcal{P}}_{L_{26}}$ of $\mathcal{P}_{L_{26}}$. A (-2)-vector $r \in L_{26}$ is a *Leech root* with respect to $\mathbf{w}$ if $\langle \mathbf{w}, r \rangle = 1$. Under the decomposition $L_{26} = U \oplus \Lambda$ such that $\mathbf{w} = (1, 0, \mathbf{0})$, Leech roots are written as

$$r_\lambda := \left(-\frac{\lambda^2}{2} - 1, 1, \lambda \right), \quad \text{where } \lambda \in \Lambda.$$

Theorem (Conway)

There exists a bijection between the set of Weyl vectors and the set of standard fundamental domains of L_{26} in such a way that a standard fundamental domain N is bounded by hyperplanes $(r_\lambda)^\perp$ defined by Leech roots r_λ with respect to the corresponding Weyl vector $\mathbf{w}$.

Definition

We call a standard fundamental domain of L_{26} a *Conway chamber*.

The bijection between the walls $(r_\lambda)^\perp$ of a Conway chamber $\mathcal{C}$ and the vectors $\lambda \in \Lambda$ yields the following:

Corollary

Let $\mathcal{C}$ be a Conway chamber of L_{26}. Then the group

$$\mathrm{O}(L_{26}, \mathcal{C}) = \{\, g \in \mathrm{O}(L_{26}) \mid \mathcal{C}^g = \mathcal{C} \,\}$$

is the group Co_∞ of **affine** *isometries of Λ ($\mathrm{O}(\Lambda) +$* **translations** *).*

Borcherds method

Note that $\mathcal{P}_{L_{26}}$ is tessellated by Conway chambers. Borcherds contrived a method to study an even hyperbolic lattice (not necessarily unimodular) S with a positive cone $\mathcal{P}_S$ by embedding

$$S \hookrightarrow L_{26}$$

and pulling back the tessellation of $\mathcal{P}_{L_{26}}$ by Conway chambers to a tessellation of $\mathcal{P}_S$.

This method has been used in the computation of the automorphism groups of some $K3$ surfaces. For example, see

Shigeyuki Kondo and Ichiro Shimada
The automorphism group of a supersingular K3 surface with Artin invariant 1 in characteristic 3.
Int. Math. Res. Not. IMRN 2014, no. 7, 1885–1924.

Borcherds method for $L_{10}(2)$

Let $L_{10}(2)$ denote the lattice obtained from L_{10} by multiplying the bilinear form $L_{10} \times L_{10} \to \mathbb{Z}$ by 2. We have $\mathrm{O}(L_{10}(2)) = \mathrm{O}(L_{10})$ and $L_{10}(2)^\vee / L_{10}(2) \cong (\mathbb{Z}/2\mathbb{Z})^{10}$.

Theorem (S. and Brandhorst)

Up to the action of $\mathrm{O}(L_{10})$ and $\mathrm{O}(L_{26})$, there exist exactly 17 primitive embeddings of $L_{10}(2)$ into L_{26}.

$$\mathtt{12A}, \mathtt{12B}, \mathtt{20A}, \dots, \mathtt{20F}, \mathtt{40A}, \dots, \mathtt{40E}, \mathtt{96A}, \dots, \mathtt{96C}, \mathtt{infty}.$$

Recall that the positive cone $\mathcal{P}_{L_{26}}$ of L_{26} is tessellated by Conway chambers. Hence an embedding $\iota\colon L_{10}(2) \hookrightarrow L_{26}$ such that $\iota(\mathcal{P}_{L_{10}}) \subset \mathcal{P}_{L_{26}}$ induces a tessellation of $\mathcal{P}_{L_{10}}$ by **induced chambers**

$$\iota^{-1}(\mathcal{C}) = \mathcal{P}_{L_{10}} \cap \mathcal{C},$$

where $\mathcal{C}$ are Conway chambers such that $\iota^{-1}(\mathcal{C})$ contains a non-empty open subset of $\mathcal{P}_{L_{10}}$.

Theorem (S. and Brandhorst)

Except for the embedding of type `infty`, *the following hold.*

- *The induced chambers on $\mathcal{P}_{L_{10}}$ are isomorphic to each other under the action of $\mathrm{O}(L_{10}, \mathcal{P}_{L_{10}})$.*
- *Each induced chamber D is bounded by a finite number of walls $D \cap (r)^{\perp}$, and each wall $D \cap (r)^{\perp}$ is defined by a (-2)-vector r of L_{10}. (The name of the embedding indicates the number of walls.)*
- *The reflection s_r with respect to r maps D to the induced chamber adjacent to D across the wall $D \cap (r)^{\perp}$.*

By the second assertion, each induced chamber is tessellated by Vinberg chambers. The volume of an induced chamber is defined to be the number of Vinberg chambers contained in the induced chamber.

17 embeddings

No.	name	volume	$\lvert$aut$\rvert$	isom	NK
1	12A	269824	2^2		I
2	12B	12142080	$2^3 \cdot 3$		II
3	20A	64757760	$2^3 \cdot 3$		V
4	20B	145704960	2^6		III
5	20C	777093120	$2^3 \cdot 3 \cdot 5$	20D	VII
6	20D	777093120	$2^3 \cdot 3 \cdot 5$	20C	VII
7	20E	906608640	$2^3 \cdot 3 \cdot 5$		VI
8	20F	2039869440	$2^6 \cdot 5$		IV
9	40A	8159477760	$2^7 \cdot 3$		
10	40B	18650234880	$2^7 \cdot 3^2$	40C	
11	40C	18650234880	$2^7 \cdot 3^2$	40B	
12	40D	32637911040	$2^5 \cdot 3^2 \cdot 5$	40E	
13	40E	32637911040	$2^5 \cdot 3^2 \cdot 5$	40D	
14	96A	163189555200	$2^{13} \cdot 3$		
15	96B	652758220800	$2^{12} \cdot 3^3$	96C	
16	96C	652758220800	$2^{12} \cdot 3^3$	96B	
17	infty	∞			

Nikulin–Kondo classification

Nikulin (1984) and Kondo (1986) classified Enriques surfaces Y with finite automorphism group.

If $\mathrm{Aut}(Y)$ is finite, then Y contains only finite number of smooth rational curves. By the configuration of these smooth rational curves, Enriques surfaces Y with finite automorphism group are devided into 7 classes $\mathrm{I}, \mathrm{II}, \dots, \mathrm{VII}$.

Each smooth rational curve is represented by a (-2)-vector of $S_Y \cong L_{10}$.

These 7 configurations appear as the configurations of (-2)-vectors bounding the induced chambers of $\mathcal{P}_{L_{10}}$.

Application to the geometry of Enriques surfaces

Let Y be an Enriques surface with the universal covering $\pi\colon X \to Y$. Recall that $S_Y \cong L_{10}$. Let S_X be the lattice of numerical equivalence classes of divisors on the $K3$ surface X. Then S_X is an even hyperbolic lattice. Let $\mathcal{P}_X$ be the positive cone of S_X containing an ample class of X. We put

$$N_X := \{\, x \in \mathcal{P}_X \mid \langle x, C\rangle \geq 0 \ \text{for all curves } C \text{ on } X \,\},$$

and call it the *nef-and-big cone* of X.

Proposition

The cone N_X is a standard fundamental domain of the action of the Weyl group $W(S_X)$.

We define $\mathcal{P}_Y$ and N_Y similarly. Then the double covering $\pi\colon X \to Y$ induces a primitive embedding

$$S_Y(2) \hookrightarrow S_X,$$

which induces $\mathcal{P}_Y \hookrightarrow \mathcal{P}_X$, and we have $N_Y = \mathcal{P}_Y \cap N_X$.

Goal

Calculate the image G of $\mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)$, and the fundamental domain $N_Y / \mathrm{Aut}(Y)$ of the action of $\mathrm{Aut}(Y)$ on the cone N_Y.

For simplicity, we assume that the period $H^{2,0}(X)$ is general in the period domain of $T_X \otimes \mathbb{C}$, where T_X is the transcendental lattice of X, and hence, via Torelli theorem for $K3$ surfaces, the following holds.

Proposition

An isometry $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ belongs to G if and only if g extends to an isometry $\tilde{g}$ of S_X that preserves N_X and acts on the discriminant group $S_X^\vee / S_X$ of S_X as ± 1.

In particular, we can determine, for a given isometry $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$, whether $g \in G$ or not effectively.

An algorithm on a graph

We consider the following graph (V, E), on which G acts. Recall that N_Y is tessellated by Vinberg chambers.

$$\begin{aligned} V &:= \text{the set of Vinberg chambers } D \text{ contained in } N_Y, \\ E &:= \text{the set of pairs } \{D, D'\} \text{ of distinct Vinberg chambers in } N_Y \\ &\quad\;\; \text{such that } D \text{ and } D' \text{ share a common wall.} \end{aligned}$$

Our goal is to calculate

- a complete set of representatives of the orbits V/G, and
- a generating set of the group G.

The set V of vertices is infinite in general, but (V, E) and G have the following *local effectiveness* properties:

1. For any $v \in V$, the set

$$\mathrm{adj}(v) := \{\, v' \in V \mid \{v, v'\} \in E \,\}$$

is finite, and can be calculated effectively. Indeed, a wall $(r)^{\perp}$ of a Vinberg chamber $v \in V$ with $\langle r, r \rangle = -2$ is a wall of N_Y if and only if the image of r by $\pi^*\colon S_Y(2) \hookrightarrow S_X$ is a sum of (-2)-vectors of S_X.

2. For any $v, v' \in V$, we can determine effectively whether

$$T_G(v, v') := \{\, g \in G \mid v^g = v' \,\}$$

is empty or not, and when $T_G(v, v') \neq \emptyset$, we can calculate an element $g \in T_G(v, v')$. In the *current situation*, the set $\{\, g \in \mathrm{O}(S_Y) \mid v^g = v' \,\}$ is a singleton.

3. For any $v \in V$, the stabilizer subgroup $T_G(v, v)$ of v in G is finitely generated, and a finite set of generators of $T_G(v, v)$ can be calculated effectively. Indeed, in the *current situation*, $T_G(v, v)$ is always $\{1\}$.

Let $\sim$ denote the G-equivalence relation: $v \sim v' \iff T_G(v, v') \neq \emptyset$. Suppose that $V_0 \subset V$ is a non-empty finite subset with the following properties:

- If $v, v' \in V_0$ and $v \neq v'$, then $v \not\sim v'$.
- We put $\widetilde{V}_0 := \bigcup_{v_0' \in V_0} \mathrm{adj}(v_0')$. For each $v \in \widetilde{V}_0$, there is a vertex $v' \in V_0$ such that $v \sim v'$. Note that v' is unique for each $v \in \widetilde{V}_0$.

For each $v \in \widetilde{V}_0 - V_0$, we choose an element $h(v) \in T_G(v, v')$, where $v' \in V_0$ satisfies $v \sim v'$, and put $\mathcal{H} := \{\, h(v) \mid v \in \widetilde{V}_0 - V_0 \,\} \subset G$.

Proposition

Let v_0 be an element of V_0. The natural mapping

$$V_0 \hookrightarrow V \twoheadrightarrow V/{\sim} = V/G$$

is a bijection, and the group G is generated by $T_G(v_0, v_0) \cup \mathcal{H}$.

For the proof, the connectedness of (V, E) is crucial.

We can calculate V_0 and $\mathcal{H}$ by the following procedure. This procedure terminates if and only if $|V/G| < \infty$.

```
Initialize V_0 := [v_0], H := {}, and i := 0.
while i < |V_0| do
    Let v_i be the (i + 1)st entry of the list V_0.
    Let adj(v_i) be the set of vertices adjacent to v_i.
    for each vertex v' in adj(v_i) do
        Set flag := true.
        for each v'' in V_0 do
            if T_G(v', v'') ≠ ∅ then
                Add an element h of T_G(v', v'') to H.
                Replace flag by false.
                Break from the innermost for–loop.
        if flag = true then
            Append v' to the list V_0 as the last entry.
    Replace i by i + 1.
```

Thus we can calculate a complete set of representatives for V/G and a finite set of generators of G.

Note that the size $|V/G|$ can be regarded as a volume of the fundamental domain of the action of $\mathrm{Aut}(Y)$ on the cone N_Y (the volume measured by the number of Vinberg chambers). We define

$$\mathrm{vol}(N_Y/\operatorname{Aut}(Y)) := |V/G|.$$

This naive method does not work in practice, because the volume $|V_0| = |V/G|$ is very large, and the computation is too heavy.
We have an example due to Barth-Peters.

Aut of a generic Enriques surface

Let Y be a *generic* Enriques surface, that is, $S_Y(2) \cong S_X$ and $H^{2,0}(X)$ is very general in the period domain of $T_X \otimes \mathbb{C}$, where T_X is the transcendental lattice of the $K3$ surface X.

Then Y has no smooth rational curves. Therefore we have $N_Y = \mathcal{P}_Y$, and hence V is the set of *all* Vinberg chambers.

Theorem (Barth-Peters)

The fundamental domain of the action of $\mathrm{Aut}(Y)$ *on the cone* $N_Y = \mathcal{P}_Y$ *is a union of*

$$|\mathrm{O}(L_{10} \otimes \mathbb{F}_2)| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 = 46998591897600 \approx 47 \times 10^{12}$$

copies of Vinberg chambers.

Therefore we have to go through the `while`–loop about 47×10^{12} times.

We use the embedding $S_Y(2) \cong L_{10}(2) \hookrightarrow L_{26}$ that factors as

$$S_Y(2) \cong L_{10}(2) \quad \xrightarrow{\pi^*} \quad S_X \quad \longrightarrow \quad L_{26}.$$

We generalize the notion of induced chambers.

Definition

Let $M \hookrightarrow L$ be a primitive embedding of hyperbolic lattices, and let $\mathcal{P}_M \hookrightarrow \mathcal{P}_L$ be the induced embedding of positive cones. A closed subset D_M of $\mathcal{P}_M$ is an *L/M-chamber* if

- D_M contains a non-empty open subset of $\mathcal{P}_M$, and
- there exists a standard fundamental domain N_L of the action of $W(L)$ on $\mathcal{P}_L$ such that $D_M = \mathcal{P}_M \cap N_L$.

Remark

- *L/M-chambers D_M need not be isomorphic to each other.*
- *A standard fundamental domain N_M of the action of $W(M)$ on $\mathcal{P}_M$ is tessellated by L/M-chambers D_M.*

We consider

$$S_Y(2) \cong L_{10}(2) \quad \xrightarrow{\pi^*} \quad S_X \quad \longrightarrow \quad L_{26}.$$

We assume that the composite $S_Y(2) \hookrightarrow L_{26}$ is not of type `infty`.
Since N_X is a standard fundamental domain of the action of $W(S_X)$, the cone $N_Y = N_X \cap \mathcal{P}_Y$ is an $S_X/S_Y(2)$-chamber, and it is tessellated by $L_{26}/S_Y(2)$-chambers. (Recall that $L_{26}/S_Y(2)$-chambers are the induced chambers in the previous terminology.)

Since elements of

$$G = \mathrm{Im}(\mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y))$$

acts on the discriminant group $\cong (\mathbb{Z}/2\mathbb{Z})^{10}$ of $S_Y(2)$ as the identity (by the assumption on the period of X), the action of G on N_Y preserves this tessellation.

We replace the previous graph (V, E) with

$$V' := \text{the set of } L_{26}/S_Y(2)\text{-chambers contained in } N_Y,$$

and E' being the usual adjacency relation of the tessellation, and apply the same algorithm.

Since each $L_{26}/S_Y(2)$-chamber consists of large number of Vinberg chambers, the amount of the computation $|V'/G|$ becomes much smaller.

No.	name	volume	|aut|	isom	NK
...	...				
15	96B	652758220800	$2^{12} \cdot 3^3$	96C	
16	96C	652758220800	$2^{12} \cdot 3^3$	96B	

We have to compensate this reduction of the amount of computation by the calculation of the automorphism group $T_G(v, v)$ of an $L_{26}/S_Y(2)$-chamber. Nevertheless, we obtain a huge computational advantage, and the algorithm becomes **tractable** in many cases.

Main results in geometry

We need the notion of $(\tau, \bar{\tau})$-generic Enriques surfaces, where τ and $\bar{\tau}$ are ADE-types of the same rank.

Examples

- The generic Enriques surface of Barth-Peters is $(0, 0)$-generic.
- A general nodal Enriques surface is (A_1, A_1)-generic. More generally, if Y is an Enriques surface that is very general in the moduli of Enriques surfaces containing n disjoint smooth rational curves, then Y is (nA_1, nA_1)-generic.
- If Y is very general in the moduli of Enriques surfaces containing two smooth rational curves whose dual graph is $\circ\!\!-\!\!\!-\!\!\circ$, then Y is (A_2, A_2)-generic. We say that such an Enriques surface Y is *general cuspidal*.

There are 156 types $(\tau, \bar{\tau})$ for which $(\tau, \bar{\tau})$-generic Enriques surfaces exist.

Volume formula

We put $1_{\mathrm{BP}} := 469985918976000$. (BP stands for Barth-Peters.)

Theorem (S. and Brandhorst)

Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface. Then we have

$$\mathrm{vol}(N_Y / \mathrm{Aut}(Y)) = |\{\textit{Vinberg chambers in } N_Y\}/G| = \frac{c_{(\tau,\bar{\tau})}}{|W(R_\tau)|} \cdot 1_{\mathrm{BP}},$$

where $W(R_\tau)$ is the Weyl group of type τ, and $c_{(\tau,\bar{\tau})} \in \{1, 2\}$ is the number of numerically trivial automorphisms of Y, that is, the size of the kernel of $\mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)$.

Example

- If Y is generic, then $\mathrm{vol} = 1_{\mathrm{BP}}$. This is the definition of 1_{BP}.
- If Y is general nodal, then $\mathrm{vol} = 1_{\mathrm{BP}}/2$.
  If Y is general n-nodal, then $\mathrm{vol} = 1_{\mathrm{BP}}/2^n n!$ for $n \leq 8$.
- If Y is general cuspidal, then $\mathrm{vol} = 1_{\mathrm{BP}}/6$.

There are two good things about this formula.

- We can confirm the formula by computer.
- We have a proof that does not use computer.

We have geometric applications of the explicit computation of V/G.

First, we obtain a finite set of generators of
$G = \mathrm{Im}(\mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y))$.

Second, we can calculate the sets

$$\begin{aligned} \mathcal{R}(Y) &:= \text{the set of smooth rational curves on } Y\text{, and} \\ \mathcal{E}(Y) &:= \text{the set of elliptic fibrations } Y \to \mathbb{P}^1 \end{aligned}$$

modulo the action of $\mathrm{Aut}(Y)$.

Application to rational curves on Y

We put

$$\mathcal{R}(Y) := \text{the set of smooth rational curves on } Y.$$

Theorem

Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface. Suppose that $\operatorname{rank}(\tau) \leq 6$. Then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)|$ is equal to the number of connected components of the Dynkin graph of τ.

Example

- If Y is general nodal, then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)| = 1$. This had been proved by Cossec-Dolgachev.
- If Y is general n-nodal with $n \leq 6$, then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)| = n$.
- If Y is general cuspidal, then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)| = 1$.
- ...

Application to elliptic fibrations on Y

We put

$$\mathcal{E}(Y) := \text{the set of elliptic fibrations } Y \to \mathbb{P}^1.$$

Theorem (Barth-Peters)

Let Y be a generic Enriques surface. Then $|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 527$.

We generalize this theorem as follows:

Theorem

Let Y be a general nodal Enriques surface. Then

$$|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 136 + 255.$$

In the representatives of elements of $\mathcal{E}(Y)/\operatorname{Aut}(Y)$,
136 *elliptic fibrations have no reducible fibers, and*
255 *elliptic fibrations have one non-multiple reducible fiber of type A_1.*

Our preprints are available from:

Borcherds' method for Enriques surfaces
Simon Brandhorst, Ichiro Shimada
arXiv:1903.01087

Automorphism groups of certain Enriques surfaces
Simon Brandhorst, Ichiro Shimada
arXiv:2012.10622

Thank you very much for listening!