Computation of the nef cone and the automorphism group of an Enriques surface

(joint work with Simon Brandhorst)

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An algorithm on a graph

We start with an easy general algorithm.

Let (V, E) be a simple non-oriented *connected* graph, where

- V is the set of vertices and,
- E is the set of edges, which is a set of non-ordered pairs of distinct elements of V (no orientation, no loops, no multiple edges).

The set V may be infinite, but we assume the following
local effectiveness property:
For any $v \in V$, the set

$$\mathrm{adj}(v) := \{\, v' \in V \mid \{v, v'\} \in E \,\}$$

is finite, and can be calculated effectively.

Suppose that a group G (possibly infinite) acts on the graph (V, E) from the right. Our goal is to calculate

- a complete set of representatives of the orbits V/G, and
- a generating set of the group G.

Again we assume the following local effectiveness properties on G:

1. For any $v, v' \in V$, we can determine effectively whether
$$T_G(v, v') := \{ g \in G \mid v^g = v' \}$$
is empty or not, and when $T_G(v, v') \neq \emptyset$, we can calculate an element $g \in T_G(v, v')$.
2. For any $v \in V$, the stabilizer subgroup $T_G(v, v)$ of v in G is finitely generated, and a finite set of generators of $T_G(v, v)$ can be calculated effectively.

Let $\sim$ denote the G-equivalence relation: $v \sim v' \iff T_G(v, v') \neq \emptyset$. Suppose that $V_0 \subset V$ is a non-empty finite subset with the following properties:

- If $v, v' \in V_0$ and $v \neq v'$, then $v \not\sim v'$.
- We put $\widetilde{V}_0 := \bigcup_{v_0' \in V_0} \mathrm{adj}(v_0')$. For each $v \in \widetilde{V}_0$, there is a vertex $v' \in V_0$ such that $v \sim v'$. Note that v' is unique for each $v \in \widetilde{V}_0$.

For each $v \in \widetilde{V}_0 - V_0$, we choose an element $h(v) \in T_G(v, v')$, where $v' \in V_0$ satisfies $v \sim v'$, and put $\mathcal{H} := \{\, h(v) \mid v \in \widetilde{V}_0 - V_0 \,\} \subset G$.

Proposition

Let v_0 be an element of V_0. The natural mapping

$$V_0 \hookrightarrow V \twoheadrightarrow V/{\sim} = V/G$$

is a bijection, and the group G is generated by $T_G(v_0, v_0) \cup \mathcal{H}$.

For the proof, the connectedness of (V, E) is crucial.

We can calculate V_0 and $\mathcal{H}$ by the following procedure. This procedure terminates if and only if $|V/G| < \infty$.

Initialize $V_0 := [v_0]$, $\mathcal{H} := \{\}$, and $i := 0$.
while $i < |V_0|$ **do**
  Let v_i be the $(i+1)$st entry of the list V_0.
  Let $\mathrm{adj}(v_i)$ be the set of vertices adjacent to v_i.
  for each vertex v' in $\mathrm{adj}(v_i)$ **do**
    Set `flag` $:= \mathrm{true}$.
    for each v'' in V_0 **do**
      if $T_G(v', v'') \neq \emptyset$ **then**
        Add an element h of $T_G(v', v'')$ to $\mathcal{H}$.
        Replace `flag` by false.
        Break from the innermost for–loop.
    if `flag` $= \mathrm{true}$ **then**
      Append v' to the list V_0 as the last entry.
  Replace i by $i + 1$.

Terminologies about hyperbolic lattices

By a lattice, we mean a $\mathbb{Z}$-lattice. We deal with even hyperbolic lattices, that is, even lattices L with signature $(1, \operatorname{rank} L - 1)$.
A *positive cone* $\mathcal{P}$ of a hyperbolic lattice L is one of the two connected components of $\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}$. Then $\mathcal{P}/\mathbb{R}_{>0}$ is a model of the hyperbolic space.

A vector $r \in L$ is called a (-2)-*vector* if $\langle r, r \rangle = -2$.

Let L be an even hyperbolic lattice with a positive cone $\mathcal{P}$. We put

$$\mathrm{O}(L, \mathcal{P}) := \{ g \in \mathrm{O}(L) \mid \mathcal{P}^g = \mathcal{P} \}.$$

For a vector $v \in L \otimes \mathbb{Q}$ with $\langle v, v \rangle < 0$, we put

$$(v)^{\perp} := \{ x \in \mathcal{P} \mid \langle v, x \rangle = 0 \}.$$

A (-2)-vector $r \in L$ defines the reflection into the mirror $(r)^{\perp}$:

$$s_r \colon x \mapsto x + \langle x, r \rangle r.$$

The *Weyl group* $W(L)$ is defined by

$$W(L) := \langle\, s_r \mid r \text{ is a } (-2)\text{-vector} \,\rangle \quad \triangleleft \quad \mathrm{O}(L, \mathcal{P}).$$

A *standard fundamental domain* of the action of $W(L)$ on $\mathcal{P}$ is the closure in $\mathcal{P}$ of a connected component of

$$\mathcal{P} \setminus \bigcup (r)^{\perp},$$

where r runs through the set of all (-2)-vectors.

Then $W(L)$ acts on the set of standard fundamental domains simple-transitively. Let N be a standard fundamental domain. We put

$$\mathrm{O}(L, N) := \{\, g \in \mathrm{O}(L, \mathcal{P}) \mid N^g = N \,\}.$$

Then we have

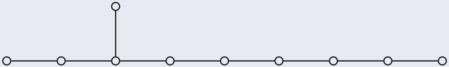
$$\begin{aligned} W(L) &= \langle\, s_r \mid \text{the hyperplane } (r)^{\perp} \text{ bounds } N \,\rangle, \\ \mathrm{O}(L, \mathcal{P}) &= W(L) \rtimes \mathrm{O}(L, N). \end{aligned}$$

Vinberg chamber

We put $L_{10} :=$ an even unimodular hyperbolic lattice of rank 10.
Note that L_{10} is unique up to isomorphism ($\cong U \oplus E_8$).

Theorem (Vinberg)

A standard fumdamental domain of the action of $W(L_{10})$ is bounded by 10 hyperplanes $(r_1)^\perp, \dots, (r_{10})^\perp$ defined by (-2)-vectors $r_1, \dots, r_{10}$ that form the dual graph below. Since this graph has no non-trivial symmetries, we have $\mathrm{O}(L_{10}, \mathcal{P}) = W(L_{10})$.



We call a standard fumdamental domain of the action of $W(L_{10})$ a **Vinberg chamber**. The positive cone $\mathcal{P}$ of L_{10} is tessellated by Vinberg chambers, in such a way that each Vinberg chamber has 10 adjacent Vinberg chambers.

Application to Enriques surfaces

For simplicity, we work over $\mathbb{C}$.

For a non-singular projective surface Z, we denote by S_Z the lattice of numerical equivalence classes of divisors on Z.

Suppose that Y is an Enriques surface. Then we have

$$S_Y \cong L_{10}.$$

Let $\mathcal{P}_Y$ be the positive cone containing an ample class of Y. Then we have a natural homomorphism

$$\rho\colon \operatorname{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y).$$

The *nef-and-big cone* of Y is defined by

$$N_Y := \{\, x \in \mathcal{P}_Y \mid \langle x, C\rangle \ge 0 \text{ for all curves } C \text{ on } Y \,\}.$$

Goal

Calculate the image G of $\rho\colon \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)$, and the fundamental domain $N_Y / \mathrm{Aut}(Y)$ of the action of $\mathrm{Aut}(Y)$ on the cone N_Y.

It is well-known that N_Y is bounder by hyperplanes $(C)^\perp$, where C are smooth rational curves on Y, and $\langle C, C \rangle = -2$ for a smooth rational curve C. Therefore N_Y is a union of Vinverg chambers of $S_Y \cong L_{10}$, that is, the cone N_Y is tessellated by Vinberg chambers.

We apply the general algorithm to the following:

$$\begin{array}{lcl} V & := & \text{the set of Vinberg chambers } D \text{ contained in } N_Y, \\ E & := & \text{the set of pairs } \{D, D'\} \text{ of distinct Vinberg chambers in } N_Y \\ & & \text{such that } D \text{ and } D' \text{ share a common wall,} \\ G & := & \mathrm{Im}(\rho\colon \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)). \end{array}$$

These data (V, E) and G have the local effectiveness properties, under certain assumptions.

Let $X \to Y$ be the universal covering of Y. Then X is a $K3$ surface, and we have a primitive embedding

$$S_Y(2) \hookrightarrow S_X.$$

Let $\mathcal{P}_X \subset S_X \otimes \mathbb{R}$ be the positive cone containing an ample class and $N_X \subset \mathcal{P}_X$ the nef-and-big cone of X. We regard $\mathcal{P}_Y$ as a subspace of $\mathcal{P}_X$. Then we have

$$N_Y = N_X \cap \mathcal{P}_Y.$$

Let $a \in S_Y$ be an ample class of Y. Then a is regarded as an ample class of X by $S_Y(2) \hookrightarrow S_X$. By Riemann-Roch, we have the following:

Proposition

The cone N_X is equal to the standard fundamental domain of the action of the Weyl group $W(S_X)$ on $\mathcal{P}_X$ containg the ample class a.

Hence a vector $v \in S_X \cap \mathcal{P}_X$ belongs to N_X if and only if the *set of separating* (-2)*-vectors*

$$\mathcal{S}_X(a, v) := \{\, r \in S_X \mid \langle r, r \rangle = -2, \;\; \langle r, a \rangle \cdot \langle r, v \rangle < 0 \,\}$$

is empty. We have an algorithm to calculate this set.

A Vinberg chamber D' is contained in N_Y if and only if $\mathcal{S}_X(a, v) = \emptyset$ for an interior point v of D'. Hence we can determine whether $D' \in V$ or not. In particular, for $D \in V$, we can determine which of the 10 Vinberg chambers D' adjacent to D belong to V, that is, we can calculate $\mathrm{adj}(D)$.

Hence the local effectiveness for (V, E) holds.

Suppose that $\operatorname{rank} S_X < 20$ and that the period ω of X is general enough so that

$$\{ g \in \mathrm{O}(T_X) \mid \omega^g \in \mathbb{C}\omega \} = \{\pm 1\},$$

where T_X is the transcendental lattice of X. If D, D' are Vinberg chambers in N_Y, then there exists a unique element $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ such that $D^g = D'$, because $\mathrm{O}(L_{10}, \mathcal{P}) = W(L_{10})$ acts on the set of Vinberg chambers simple-transitively. By Torelli theorem for $K3$ surfaces, we have the following:

Proposition

An isometry $g \in \mathrm{O}(S_Y, \mathcal{P}_Y)$ belongs to $G = \mathrm{Im}(\mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y))$ if and only if g lifts to an isometry $\tilde{g}$ of S_X that preserves N_X and acts as ± 1 on the discriminant group of S_X.

Hence the local effectiveness for G holds, provided that we know the embedding $S_Y(2) \hookrightarrow S_X$ explicitly.

Thus we can apply the general algorithm, and calculate a complete set of representatives for V/G and a finite set of generators of G.

Note that the size $|V/G|$ can be regarded as a volume of the fundamental domain of the action of $\mathrm{Aut}(Y)$ on the cone N_Y (the volume measured by the number of Vinberg chambers). We define

$$\mathrm{vol}(N_Y/\mathrm{Aut}(Y)) := |V/G|.$$

This naive method does not work in general, because the computation is too heavy.
We have an example due to Barth-Peters (1983).

Let Y be a *generic* Enriques surface. Since Y has no smooth rational curves, we have $N_Y = \mathcal{P}_Y$, and hence V is the set of *all* Vinberg chambers.

Theorem (Barth-Peters (1983))

The fundamental domain of the action of $\mathrm{Aut}(Y)$ *on the cone* $N_Y = \mathcal{P}_Y$ *is a union of*

$$|\mathrm{O}(L_{10} \otimes \mathbb{F}_2)| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31 = 46998591897600 \approx 47 \times 10^{12}$$

copies of Vinberg chambers.

Therefore we have to go through the `while`–loop about 47×10^{12} times.

To overcome this difficulty, we employ *Borcherds' method*. This is the technical core of our computation. For details, see

> Borcherds' Method for Enriques Surfaces
> Simon Brandhorst, Ichiro Shimada:
> arXiv:1903.01087

Main results

We need the notion of $(\tau, \bar{\tau})$-generic Enriques surfaces, where τ and $\bar{\tau}$ are ADE-types of the same rank.

Examples

- The generic Enriques surface of Barth-Peters is $(0, 0)$-generic.
- A general nodal Enriques surface is (A_1, A_1)-generic. More generally, if Y is an Enriques surface that is very general in the moduli of Enriques surfaces containing n disjoint smooth rational curves, then Y is (nA_1, nA_1)-generic.
- If Y is very general in the moduli of Enriques surfaces containing two smooth rational curves whose dual graph is $\circ\!\!-\!\!\!-\!\!\circ$, then Y is (A_2, A_2)-generic. We say that such an Enriques surface Y is *general cuspidal*.

There are 156 types $(\tau, \bar{\tau})$ for which $(\tau, \bar{\tau})$-generic Enriques surfaces exist.

Volume formula

We put $1_{\mathrm{BP}} := 46998591897600$. ($\mathrm{BP}$ stands for Barth-Peters.)

Theorem

Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface. Then we have

$$\mathrm{vol}(N_Y / \mathrm{Aut}(Y)) = |V/G| = \frac{c_{(\tau,\bar{\tau})}}{|W(R_\tau)|} \cdot 1_{\mathrm{BP}},$$

where $W(R_\tau)$ is the Weyl group of type τ, and $c_{(\tau,\bar{\tau})} \in \{1, 2\}$ is the number of numerically trivial automorphisms of Y, that is, the size of the kernel of $\rho\colon \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y)$.

Example

- If Y is generic, then $|V/G| = 1_{\mathrm{BP}}$. This is the definition of 1_{BP}.
- If Y is general nodal, then $|V/G| = 1_{\mathrm{BP}}/2$.
  If Y is general n-nodal, then $|V/G| = 1_{\mathrm{BP}}/2^n n!$ for $n \leq 8$.
- If Y is general cuspidal, then $|V/G| = 1_{\mathrm{BP}}/6$.

There are two good things about this formula.

- We have a proof that **does not use computer**.
- We can make an explicit list of representatives of V/G, and hence we can confirm the formula by computer.

We have geometric applications of the explicit computation of V/G.

First, we obtain a finite set of generators of
$G = \mathrm{Im}(\rho\colon \mathrm{Aut}(Y) \to \mathrm{O}(S_Y, \mathcal{P}_Y))$.

Second, we can calculate the sets

$$\begin{array}{rcl} \mathcal{R}(Y) & := & \text{the set of smooth rational curves on } Y, \text{ and} \\ \mathcal{E}(Y) & := & \text{the set of elliptic fibrations } Y \to \mathbb{P}^1 \end{array}$$

modulo the action of $\mathrm{Aut}(Y)$.

Geometric applications

Application to rational curves on Y.

We put $\mathcal{R}(Y) :=$ the set of smooth rational curves on Y.

Theorem

Let Y be a $(\tau, \bar{\tau})$-generic Enriques surface. Suppose that $\mathrm{rank}(\tau) \leq 6$. Then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)|$ is equal to the number of connected components of the Dynkin graph of τ.

Example

- If Y is general nodal, then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)| = 1$. This had been proved by Cossec-Dolgachev.
- If Y is general n-nodal with $n \leq 6$, then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)| = n$.
- If Y is general cuspidal, then $|\mathcal{R}(Y)/\operatorname{Aut}(Y)| = 1$.
- ...

Application to elliptic fibrations on Y.

We put

$$\mathcal{E}(Y) := \text{the set of elliptic fibrations } Y \to \mathbb{P}^1.$$

Theorem (Barth-Peters)

Let Y be a generic Enriques surface. Then $|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 527$.

We generalize this theorem as follows:

Theorem

Let Y be a general nodal Enriques surface. Then

$$|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 136 + 255.$$

In the representatives of elements of $\mathcal{E}(Y)/\operatorname{Aut}(Y)$,
136 elliptic fibrations have no reducible fibers, and
255 elliptic fibrations have one non-multiple reducible fiber of type A_1.

Theorem

Let Y be a general 2-nodal Enriques surface. Then

$$|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 36 + 1 + 128 + 126;$$

36 *elliptic fibrations have no reducible fiber,*
1 *elliptic fibrations have one multiple reducible fiber of type* A_1,
128 *elliptic fibrations have one non-multiple reducible fiber of type* A_1,
126 *elliptic fibrations have one non-multiple reducible fiber of type* A_2.

Theorem

Let Y be a general cuspidal Enriques surface. Then

$$|\mathcal{E}(Y)/\operatorname{Aut}(Y)| = 136 + 119;$$

136 *elliptic fibrations have one non-multiple reducible fiber of type* A_1, *and*
119 *elliptic fibrations have one non-multiple reducible fiber of type* A_2.

. . . .

The definition of $(\tau, \bar{\tau})$-generic Enriques surfaces

For an ADE-lattice R, let $\tau(R)$ denote the ADE-type of R. Let R be an ADE-sublattice of L_{10}, and $\overline{R}$ the primitive closure of R in L_{10}. Then $\overline{R}$ is also an ADE-sublattice of L_{10}.

Proposition

(1) *Let R' be another ADE-sublattice of L_{10} with the primitive closure $\overline{R}'$. Then R and R' are in the same orbit under the action of $\mathrm{O}(L_{10}, \mathcal{P})$ if and only if*

$$(\tau(R), \tau(\overline{R})) = (\tau(R'), \tau(\overline{R}')).$$

(2) *There exist exactly* 184 *pairs $(\tau, \bar{\tau})$ of ADE-types that are equal to $(\tau(R), \tau(\overline{R}))$ of an ADE-sublattice R of L_{10}.*

1	(A_1, A_1)	7	$(4A_1, 4A_1)$	115	$(D_5 + A_2 + A_1, D_5 + A_2 + A_1)$
2	$(2A_1, 2A_1)$	8	$(4A_1, D_4)$		$\dots$
	$\dots$		$\dots$	184	(D_9, D_9)

Let R be an ADE-sublattice of L_{10}. We denote by $\iota_R \colon R \hookrightarrow L_{10}$ the inclusion. We define M_R to be the $\mathbb{Z}$-submodule of $(L_{10}(2) \oplus R(2)) \otimes \mathbb{Q}$ generated by $L_{10}(2)$ and $(\iota_R(v), \pm v)/2 \in (L_{10} \oplus R) \otimes \mathbb{Q}$, where v runs through R. By definition, M_R is an even hyperbolic lattice with a chosen primitive embedding

$$\varpi_R \colon L_{10}(2) \hookrightarrow M_R.$$

Let Y be an Enriques surface with the universal covering $\pi \colon X \to Y$. Then the étale double covering π induces a primitive embedding

$$\pi^* \colon S_Y(2) \hookrightarrow S_X.$$

Let $(\tau, \bar{\tau})$ be one of the 184 pairs in the previous proposition, and let R be an ADE-sublattice of L_{10} with $(\tau(R), \tau(\overline{R})) = (\tau, \bar{\tau})$.

Definition

An Enriques surface Y is said to be $(\tau, \bar{\tau})$-*generic* if the following conditions are satisfied.

1. $$\mathrm{O}(T_X, \omega) := \{\, g \in \mathrm{O}(T_X) \mid \omega^g \in \mathbb{C}\omega \,\} = \{\pm 1\}.$$
2. There exist isometries $g\colon L_{10} \xrightarrow{\sim} S_Y$ and $\tilde{g}\colon M_R \xrightarrow{\sim} S_X$ that make the following commutative diagram

$$\begin{array}{ccc} L_{10}(2) & \overset{\varpi_R}{\hookrightarrow} & M_R \\ g \downarrow\wr & & \tilde{g} \downarrow\wr \\ S_Y(2) & \overset{\pi^*}{\hookrightarrow} & S_X. \end{array}$$

Among 184 types, 156 types $(\tau, \bar{\tau})$ appear as $(\tau, \bar{\tau})$-generic Enriques surfaces.

Our preprint is available from:

Automorphism groups of certain Enriques surfaces
Simon Brandhorst, Ichiro Shimada
arXiv:2012.10622

Thank you very much for listening!