Mordell-Weil groups of a certain K3 surface

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Contents

In the 1st part, we present some algorithms about Mordell-Weil groups of elliptic $K3$ surfaces.

In the 2nd part, we apply these algorithms to a **virtual** $K3$ surface of Picard number 26, and give a new method of constructing the Leech lattice.

An algorithm on a lattice

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle \quad \rangle \colon L \times L \to \mathbb{Z}.$$

A lattice L is *even* if $\langle v, v\rangle \in 2\mathbb{Z}$ for all $v \in L$. Let L be an even lattice.

- $r \in L$ is a *root* $\iff$ $\langle r, r\rangle$ is either 2 or -2.
- $r \in L$ is a (-2)-*vector* $\iff$ $\langle r, r\rangle = -2$.

A lattice L of rank $n > 1$ is said to be *hyperbolic* if the signature of $L \otimes \mathbb{R}$ is $(1, n-1)$. Let L be an even hyperbolic lattice. A *positive cone* of L is one of the two connected components of the space

$$\{\, x \in L \otimes \mathbb{R} \mid \langle x, x\rangle > 0 \,\}.$$

Let $\mathcal{P}$ be a positive cone of L. For $v \in L \otimes \mathbb{R}$ with $\langle v, v\rangle < 0$, we put

$$(v)^{\perp} := \{\, x \in \mathcal{P} \mid \langle x, v\rangle = 0 \,\},$$

which is a real hyperplane of $\mathcal{P}$.

A (-2)-vector $r \in L$ defines a reflection into the mirror $(r)^{\perp}$:

$$s_r \colon x \mapsto x + \langle x, r \rangle r.$$

Definition

- The *Weyl group* of L is the subgroup $W(L) = \langle s_r \rangle$ of $\mathrm{O}(L)$, where r runs through the set of all (-2)-vectors.
- A *standard fundamental domain of* $W(L)$ is the closure of a connected component of

$$\mathcal{P} \setminus \bigcup (r)^{\perp} \qquad (r \text{ runs through the set of } (-2)\text{-vectors}).$$

When L is the *Néron-Severi lattice* S_X of a $K3$ surface X (that is, the group of numerical equivalence classes of divisors of X with the intersection pairing), the **nef-and-big cone** $N_X \subset \mathcal{P}_X$ of X is a standard fundamental domain of $W(S_X)$. Here $\mathcal{P}_X \subset S_X \otimes \mathbb{R}$ is the positive cone of S_X containing an ample class.

Let $N \subset \mathcal{P}$ be a standard fundamental domain of $W(L)$.

Definition

We say that a (-2)-vector $r \in L$ *defines a wall of* N if

- $(r)^{\perp}$ is disjoint from the interior of N,
- $N \cap (r)^{\perp}$ contains a non-empty open subset of $(r)^{\perp}$, and
- $\langle r, x \rangle > 0$ for an interior point x of N.

It is an important task to enumerate (-2)-vectors defining walls of N. When $L = S_X$ and $N = N_X$, this is equivalent to calculate

$$\mathrm{Rats}(X) := \{\, [C] \in S_X \mid C \text{ is a smooth rational curve on } X \,\}.$$

We can carry out this task by Vinberg's algorithm. We have an **alternative approach** to this problem.

An alternative to Vinberg's algorithm

Let $v_1, v_2 \in L \otimes \mathbb{Q}$ be vectors in $\mathcal{P}$. We can calculate the finite set

$$\mathrm{Sep}(v_1, v_2) := \{\, r \in L \mid \langle r, v_1 \rangle > 0,\ \langle r, v_2 \rangle < 0,\ \langle r, r \rangle = -2 \,\}$$

of (-2)-vectors *separating* v_1 *and* v_2. We have an algorithm to calculate this set. (Details are omitted.)

An application to a $K3$ surface

Let $\boldsymbol{a} \in S_X$ be an ample class. Let $r \in S_X$ be a (-2)-vector such that $\langle \boldsymbol{a}, r \rangle > 0$. Then there is an effective divisor D of X such that $r = [D]$. We have $r \in \mathrm{Rats}(X)$ if and only if D is irreducible. We put

$$b := \boldsymbol{a} + (\langle \boldsymbol{a}, r \rangle / 2) r,$$

which is the point of $(r)^\perp$ such that the line segment $\overline{\boldsymbol{a}b}$ is perpendicular to $(r)^\perp$. Then

$$r \in \mathrm{Rats}(X) \iff (\ \mathrm{Roots}(\,[b]^\perp) = \{r, -r\}\ \text{ and }\ \mathrm{Sep}(b, \boldsymbol{a}) = \emptyset),$$

where $\mathrm{Roots}(\,[b]^\perp)$ is the set of (-2)-vectors orthogonal to b.

Mordell-Weil group

We work over an algebraically closed field k with $\mathrm{char}(k) \neq 2, 3$.

Let X be a $K3$ surface, and let

$$\phi\colon X \to \mathbb{P}^1$$

be an elliptic fibration. Let

$$\eta = \mathrm{Spec}\, k(\mathbb{P}^1) \to \mathbb{P}^1$$

be the generic point of the base curve $\mathbb{P}^1$. Then the generic fiber

$$E_\eta := \phi^{-1}(\eta)$$

is a genus 1 curve defined over $k(\mathbb{P}^1)$, and the sections of ϕ are identified with the $k(\mathbb{P}^1)$-rational points of E_η. We assume that ϕ has a distinguished section

$$\zeta\colon \mathbb{P}^1 \to X,$$

that is, ϕ is a *Jacobian fibration*.

The curve E_η is an elliptic curve with the origin being the $k(\mathbb{P}^1)$-rational point corresponding to ζ, and the set

$$\mathrm{MW}_\phi := \mathrm{MW}(X, \phi, \zeta)$$

of sections of ϕ has a structure of the abelian group with $\zeta = 0$. This group MW_ϕ is called the *Mordell-Weil group*.

The group MW_ϕ acts on E_η via the translation on E_η:

$$x \mapsto x +_E \sigma \qquad (x \in E_\eta,\ \ \sigma \in \mathrm{MW}_\phi),$$

where $+_E$ denotes the addition in the elliptic curve E_η. Since X is minimal, this automorphism of E_η gives an automorphism of X:

$$\mathrm{MW}_\phi \hookrightarrow \mathrm{Aut}(X).$$

Since $\mathrm{Aut}(X)$ acts on the lattice S_X, we obtain a homomorphism

$$\mathrm{MW}_\phi \to \mathrm{Aut}(X) \to \mathrm{O}(S_X).$$

Let $f \in S_X$ be the class of a fiber of ϕ, and $z = [\zeta] \in S_X$ the class of the image of ζ. We show that we can calculate the homomorphism

$$\mathrm{MW}_\phi \to \mathrm{Aut}(X) \to \mathrm{O}(S_X)$$

from the classes f, z and an ample class $\boldsymbol{a} \in S_X$ by using only lattice-theoretic computation. We explain this algorithm.

The classes f and z generate a unimodular hyperbolic plane U_ϕ in S_X:

$$U_\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad f = (1,0), \quad z = (-1,1).$$

Since U_ϕ is unimodular, we have an orthogonal direct-sum decomposition

$$S_X = U_\phi \oplus W_\phi.$$

Since W_ϕ is negative-definite, we can calculate

$$\mathrm{Roots}(W_\phi) = \{\, r \in W_\phi \mid \langle r, r \rangle = -2 \,\}.$$

Hence we can compute

$$\Theta_\phi := \mathrm{Roots}(W_\phi) \cap \mathrm{Rats}(X)$$

by the ample class $\boldsymbol{a}$. Then Θ_ϕ is equal to the set of classes of smooth rational curves that are contracted to points by ϕ and are disjoint from ζ.

Let Σ_ϕ be the sublattice of W_ϕ generated by $\mathrm{Roots}(W_\phi)$, and τ_ϕ the ADE-type of $\mathrm{Roots}(W_\phi)$.

The vectors in Θ_ϕ form a basis of Σ_ϕ, and their dual graph is the Dynkin diagram of type τ_ϕ.

Let

$$\Theta_\phi = \Theta_1 \sqcup \cdots \sqcup \Theta_n$$

be the decomposition according to the decomposition of the Dynkin diagram into connected components. Then $\{\Theta_1, \dots, \Theta_n\}$ is naturally in one-to-one correspondence with the set

$$\{\, p \in \mathbb{P}^1 \mid \phi^{-1}(p) \text{ is reducible} \,\} = \{p_1, \dots, p_n\}.$$

We investigate reducible fibers $\phi^*(p_\nu)$. We put

$$\rho(\nu) := \mathrm{Card}(\Theta_\nu), \quad \tau_\nu := \text{the ADE-type of } \Theta_\nu,$$

and let $\Sigma_\nu \subset \Sigma_\phi$ be the sublattice generated by Θ_ν. We have $\tau_\phi = \tau_1 + \cdots + \tau_n$, and

$$\Sigma_\phi = \Sigma_1 \oplus \cdots \oplus \Sigma_n.$$

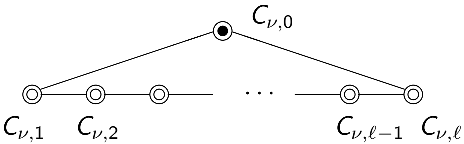
The fiber $\phi^{-1}(p_\nu)$ consists of $\rho(\nu) + 1$ smooth rational curves

$$C_{\nu,0},\ C_{\nu,1},\ \dots,\ C_{\nu,\rho(\nu)}$$

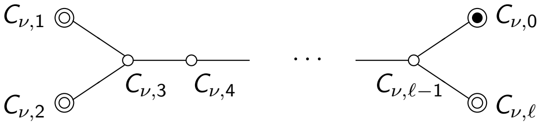
such that $\Theta_\nu = \{[C_{\nu,1}], \dots, [C_{\nu,\rho(\nu)}]\}$ and that $C_{\nu,0}$ intersects the zero section ζ. The dual graph of

$$\widetilde{\Theta}_\nu := \{[C_{\nu,0}]\} \cup \Theta_\nu$$

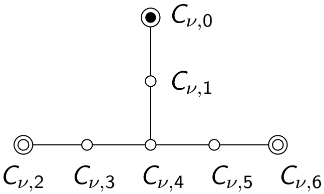
is the *affine* Dynkin diagram of type τ_ν.



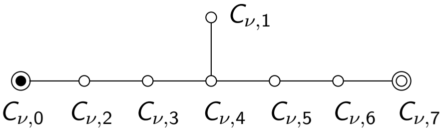
A fiber of type A_ℓ



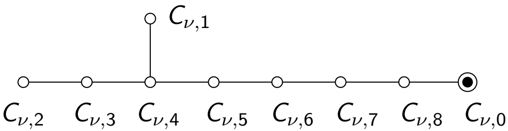
A fiber of type D_ℓ



A fiber of type E_6



A fiber of type E_7



A fiber of type E_8

$C_{\nu,0}$ is indicated by ◉ , and
$C_{\nu,j}$ for $j \in J_\nu - \{0\}$ is indicated by ◎ .

The divisor $\phi^*(p_\nu)$ is written as

$$\phi^*(p_\nu) = \sum_{j=0}^{\rho(\nu)} m_{\nu,j} C_{\nu,j} \qquad (m_{\nu,j} \in \mathbb{Z}_{>0}),$$

where the coefficients $m_{\nu,j}$ are well known. We put

$$J_\nu := \{\, j \mid m_{\nu,j} = 1 \,\}.$$

We have $0 \in J_\nu$. Let $\phi^*(p_\nu)^\sharp$ denote the smooth part of the divisor $\phi^*(p_\nu)$:

$$\phi^*(p_\nu)^\sharp = \bigcup_{j \in J_\nu} C^\circ_{\nu,j},$$

where $C^\circ_{\nu,j}$ is $C_{\nu,j}$ minus the intersection points with other components of $\phi^{-1}(p_\nu)$. Taking the limit of the group structures of general fibers of ϕ, we can equip $\phi^*(p_\nu)^\sharp$ with a structure of the abelian Lie group. Then J_ν has a natural structure of the abelian group, as the set of connected components of $\phi^*(p_\nu)^\sharp$. The index $0 \in J_\nu$ is the zero.

τ_ν	J_ν	Group structure
A_ℓ	$\{0, 1, \ldots, \ell\}$	cyclic group $\mathbb{Z}/(\ell+1)\mathbb{Z}$ generated by $1 \in J_\nu$
D_ℓ $(\ell : \text{even})$	$\{0, 1, 2, \ell\}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
D_ℓ $(\ell : \text{odd})$	$\{0, 1, 2, \ell\}$	cyclic group $\mathbb{Z}/4\mathbb{Z}$ generated by $1 \in J_\nu$ with $\ell \in J_\nu$ being of order 2
E_6	$\{0, 2, 6\}$	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\{0, 7\}$	$\mathbb{Z}/2\mathbb{Z}$
E_8	$\{0\}$	trivial

Table: Group structure of J_ν

The following observation is the key for our method. Let $\Sigma_\nu^\vee$ be the dual lattice of Σ_ν, and let $\gamma_{\nu,1}, \dots, \gamma_{\nu,\rho(\nu)}$ be the basis of $\Sigma_\nu^\vee$ dual to the basis $[C_{\nu,1}], \dots, [C_{\nu,\rho(\nu)}]$ of Σ_ν. We also put

$$\gamma_{\nu,0} := 0 \quad \in \quad \Sigma_\nu^\vee.$$

Lemma

The map $j \mapsto \gamma_{\nu,j} \bmod \Sigma_\nu$ *gives an isomorphism*

$$J_\nu \cong \Sigma_\nu^\vee / \Sigma_\nu$$

of abelian groups.

Hence, for any $x \in \Sigma_\nu^\vee$, there exists a unique $j \in J_\nu$ such that x and $\gamma_{\nu,j}$ are equivalent modulo Σ_ν.

Definition

The sublattice $U_\phi \oplus \Sigma_\phi$ of S_X is called the *trivial sublattice*.

Theorem

Let $[\ ]\colon \mathrm{MW}_\phi \to \mathrm{Rats}(X)$ *denote the mapping that associates to each section* $\sigma \in \mathrm{MW}_\phi$ *the class* $[\sigma] \in \mathrm{Rats}(X)$ *of the image of* σ. *Then the composite*

$$\mathrm{MW}_\phi \quad \xrightarrow{[\ ]} \quad \mathrm{Rats}(X) \quad \hookrightarrow \quad S_X \quad \twoheadrightarrow \quad S_X/(U_\phi \oplus \Sigma_\phi)$$

is an isomorphism of abelian groups. □

This holds, not only for $K3$ surfaces, but also for elliptic surfaces in general.

For a vector $v \in S_X$, let $s(v) \in \mathrm{MW}_\phi$ be the section corresponding to v mod $(U_\phi \oplus \Sigma_\phi)$ via $\mathrm{MW}_\phi \cong S_X/(U_\phi \oplus \Sigma_\phi)$. We will calculate

$$[s(v)] \in \mathrm{Rats}(X).$$

1. $\langle [s(v)], [s(v)] \rangle = -2$ and $\langle [s(v)], f \rangle = 1$. Hence, by the orthogonal direct-sum decomposition $S_X = U_\phi \oplus W_\phi$, we have $[s(v)] = tf + z + w$, where $w \in W_\phi$ and $t = -\langle w, w \rangle / 2$.
2. $[s(v)] \equiv v$ mod $U_\phi \oplus \Sigma_\phi$. In particular, for each $\nu = 1, \dots, n$, we have

$$([s(v)] - v)|_\nu \in \Sigma_\nu.$$

3. For each $\nu = 1, \dots, n$, there exists a unique index $j(v) \in J_\nu$ such that $[s(v)]|_\nu = \gamma_{\nu, j(v)}$. This $j(v)$ is calculated by $v|_\nu$ mod $\Sigma_\nu = \overline{\gamma}_{\nu, j(v)}$.

These data are enough to compute $[s(v)]$.

Next, we explain how to calculate the isometry $g := g(s(v)) \in \mathrm{O}(S_X)$ induced by $s(v) \in \mathrm{MW}_\phi$. Let $m = \dim(\mathrm{MW}_\phi \otimes \mathbb{Q})$ be the Mordell-Weil rank of ϕ. We choose vectors $u_1, \dots, u_m \in S_X$ such that their images by

$$S_X \to (S_X/(U_\phi \oplus \Sigma_\phi)) \otimes \mathbb{Q}$$

form a basis of $\mathrm{MW}_\phi \otimes \mathbb{Q}$. Then $S_X \otimes \mathbb{Q}$ is spanned by

$$f, \;\; z = [s(0)], \;\; [s(u_1)], \dots, [s(u_m)], \text{ and the vectors in } \Theta_\nu \;\, (\nu = 1, \dots, n).$$

Therefore it is enough to calculate the images of these vectors by $g := g(s(v))$. It is obvious that

$$\begin{aligned} f^g &= f, \\ z^g &= [s(v)], \\ [s(u_\mu)]^g &= [s(u_\mu + v)] \;\text{ for }\; \mu = 1, \dots, m. \end{aligned}$$

Hence it remains only to calculate the image by g of the classes in Θ_ν. This is computed from the action of J_ν on $\phi^*(p_\nu)$.

An Example

Let $\overline{X}$ be the double cover of $\mathbb{P}^2$ defined by

$$w^2 = f(x, y, z)^2 + g(x, y, z)^3,$$

where f and g are general homogeneous polynomials on $\mathbb{P}^2$ of degree 3 and 2, respectively, and X the minimal resolution of $\overline{X}$.
The singularities $\overline{X}$ consist of $6A_2$, and the rank of S_X is 13. Looking for Jacobian fibrations of X and calculating thier Mordell-Weil groups, we obtain the following:

Theorem

The automorphism group $\mathrm{Aut}(X)$ *of* X *is generated by* 463 *involutions associated with double coverings* $X \to \mathbb{P}^2$ *and* 360 *elements of infinite order in Mordell-Weil groups of Jacobian fibrations of* X*.*

Here, by a *double covering*, we mean a generically finite morphism of degree 2.

Construction of the Leech lattice

Definition

An even unimodular **negative-definite** lattice of rank 24 is called a *Niemeier lattice*.

(**Caution**) We employ the sign convention opposite of the usual one.

Theorem (Niemeier)

Up to isomorphism, there exist exactly 24 *Niemeier lattices.*
One of them contains no roots. This lattice is called the Leech lattice and denoted by Λ.
Each of the other 23 *lattices* N *contains a sublattice* N_{roots} *of finite index generated by roots.*

In this talk, we mean by an N-*lattice* a Niemeier lattice that is not isomorphic to Λ.

We present methods to construct the Leech lattice from N-lattices using an idea coming from the theory of elliptic $K3$ surfaces.

no.	τ_N	N/N_{roots}	h
1	$24\,A_1$	$(\mathbb{Z}/2\mathbb{Z})^{12}$	2
2	$A_{11}+D_7+E_6$	$\mathbb{Z}/12\mathbb{Z}$	12
3	$2\,A_{12}$	$\mathbb{Z}/13\mathbb{Z}$	13
4	$A_{15}+D_9$	$\mathbb{Z}/8\mathbb{Z}$	16
5	$A_{17}+E_7$	$\mathbb{Z}/6\mathbb{Z}$	18
6	$12\,A_2$	$(\mathbb{Z}/3\mathbb{Z})^6$	3
7	A_{24}	$\mathbb{Z}/5\mathbb{Z}$	25
8	$8\,A_3$	$(\mathbb{Z}/4\mathbb{Z})^4$	4
9	$6\,A_4$	$(\mathbb{Z}/5\mathbb{Z})^3$	5
10	$4\,A_5+D_4$	$\mathbb{Z}/2\mathbb{Z}\times(\mathbb{Z}/6\mathbb{Z})^2$	6
11	$4\,A_6$	$(\mathbb{Z}/7\mathbb{Z})^2$	7
12	$2\,A_7+2\,D_5$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$	8
13	$3\,A_8$	$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}$	9
14	$2\,A_9+D_6$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/10\mathbb{Z}$	10
15	$D_{10}+2\,E_7$	$(\mathbb{Z}/2\mathbb{Z})^2$	18
16	$2\,D_{12}$	$(\mathbb{Z}/2\mathbb{Z})^2$	22
17	$D_{16}+E_8$	$\mathbb{Z}/2\mathbb{Z}$	30
18	D_{24}	$\mathbb{Z}/2\mathbb{Z}$	46
19	$6\,D_4$	$(\mathbb{Z}/2\mathbb{Z})^6$	6
20	$4\,D_6$	$(\mathbb{Z}/2\mathbb{Z})^4$	10
21	$3\,D_8$	$(\mathbb{Z}/2\mathbb{Z})^3$	14
22	$4\,E_6$	$(\mathbb{Z}/3\mathbb{Z})^2$	12
23	$3\,E_8$	0	30
24	none	$\mathbb{Z}^{24}$	

τ_N is the *ADE*-type of the roots in N, and
h is the Coxeter numebr of N.

Table: Niemeier lattices

Let L_{26} denote an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. For any $\mathbb{N}$-lattice N, we have

$$L_{26} \cong U \oplus \Lambda \cong U \oplus N,$$

where U is the hyperbolic plane $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If we write an isomorphism $U \oplus \Lambda \cong U \oplus N$ **explicitly**, we obtain a construction of Λ from N.

We choose a positive cone $\mathcal{P}_{26}$ of L_{26}.

Definition

A vector $\mathbf{w} \in L_{26}$ is called a *Weyl vector* if

- $\mathbf{w}$ is a non-zero primitive vector contained in $\overline{\mathcal{P}}_{26}$,
- $\langle \mathbf{w}, \mathbf{w} \rangle = 0$ (hence $\mathbb{Z}\mathbf{w} \subset (\mathbb{Z}\mathbf{w})^{\perp}$) and,
- $(\mathbb{Z}\mathbf{w})^{\perp}/\mathbb{Z}\mathbf{w}$ is isomorphic Λ.

Hence a Weyl vector is written as $\mathbf{w} = (1, 0, 0)$ via some $L_{26} \cong U \oplus \Lambda$.

Definition

Let $\mathbf{w}$ be a Weyl vector. A (-2)-vector $r \in L_{26}$ is said to be a *Leech root* with respect to $\mathbf{w}$ if $\langle \mathbf{w}, r \rangle = 1$. We then put

$$\mathbf{C}(\mathbf{w}) := \{\, x \in \mathcal{P}_{26} \mid \langle x, r \rangle \geq 0 \ \text{ for all Leech roots } r \text{ with respect to } \mathbf{w} \,\}.$$

Theorem (Conway)

(1) The mapping $\mathbf{w} \mapsto \mathbf{C}(\mathbf{w})$ gives a bijection from the set of Weyl vectors to the set of standard fundamental domains of $W(L_{26})$.
(2) Let $\mathbf{w}$ be a Weyl vector. The mapping $r \mapsto \mathbf{C}(\mathbf{w}) \cap (r)^{\perp}$ gives a bijection from the set of Leech roots with respect to $\mathbf{w}$ to the set of walls of the chamber $\mathbf{C}(\mathbf{w})$.

Definition

We call a standard fundamental domain of $W(L_{26})$ a *Conway chamber*.

Let $\mathbb{X} = \mathbb{X}_{26}$ be a $K3$ surface such that $S_{\mathbb{X}} \cong L_{26}$.

Warning

There are no such $K3$ surfaces.

We use this virtual $K3$ surface $\mathbb{X}$ *heuristically*.

Via $S_{\mathbb{X}} \cong L_{26}$, the nef-and-big cone $N_{\mathbb{X}}$ of $\mathbb{X}$ is a Conway chamber, and hence there exists a Weyl vector $\mathbf{w}_0$ such that

$$N_{\mathbb{X}} = \mathbf{C}(\mathbf{w}_0).$$

This $\mathbf{w}_0 \in S_{\mathbb{X}}$ is the class of a fiber of an elliptic fibration

$$\Phi \colon \mathbb{X} \to \mathbb{P}^1.$$

By Conway's theorem, we see that

- every fiber of Φ is irreducible (because $\Lambda_{\mathrm{roots}} = 0$),
- $\mathrm{MW}(\Phi) \cong \Lambda \cong \mathbb{Z}^{24}$, and
- every smooth rational curve on $\mathbb{X}$ is a section of Φ (because $\mathrm{Rats}(\mathbb{X})$ is the set of Leech roots).

If we find a Leech root $r \in \mathrm{Rats}(\mathbb{X})$, then the orthogonal complement $U(\mathbf{w}_0, r)^\perp$ of the sublattice $U(\mathbf{w}_0, r) \subset S_\mathbb{X}$ generated by $\mathbf{w}_0$ and r is isomorphisc to Λ.

Let N be an N-lattice. We start from

$$S_\mathbb{X} = U_N \oplus N \cong L_{26},$$

where the hyperbolic lattice U_N is generated by the class $f_N = (1, 0)$ of a fiber of a Jacobian fibration

$$\Phi_N \colon \mathbb{X} \to \mathbb{P}^1$$

and the class $z_N = (-1, 1) \in \mathrm{Rats}(\mathbb{X})$ of the zero section of Φ_N. Then we see that

- the ADE-type of reducible fibers of Φ_N is the ADE-type τ_N of N_{roots},
- $\mathrm{MW}(\Phi_N) \cong N/N_{\mathrm{roots}}$, which is a finite abelian group.

We calculate the set $\Theta = \mathrm{Roots}(N) \cap \mathrm{Rats}(\mathbb{X})$ of classes $[C]$ of smooth rational curves C in fibers of Φ_N that are disjoint from z_N.

From the classes $r = [C] \in \Theta$ and $r = z_N$, we determine the Weyl vector $\mathbf{w}$ such that $N_{\mathbb{X}} = \mathbf{C}(\mathbf{w})$ by solving the linear equations

$$\langle \mathbf{w}, r \rangle = 1.$$

Let $\rho \in N$ be the vector such that

$$\langle r, \rho \rangle = 1 \quad \text{for all} \quad r \in \Theta.$$

We have the *Coxeter number* h such that $\langle \rho, \rho \rangle = -2h(h+1)$. Then we have

$$\mathbf{w} = (h+1, h, \rho) \in U_N \oplus N.$$

From this $\mathbf{w}$ and various classes $z \in \mathrm{Rats}(\mathbb{X})$, we obtain

$$\Lambda \;\cong\; U(\mathbf{w}, z)^{\perp} \subset S_{\mathbb{X}}.$$

By the projection, we obtain a linear homomorphism $\Lambda \to N$, from which we get a recipe to construct the Leech lattice Λ from the N-lattice N.

Example

We consider the case $N = 3E_8$. Let $\lambda\colon N \to \mathbb{Z}$ be defined by

$$\lambda(v) := \langle \rho, v \rangle.$$

We put

$$N_0 := \{\, v \in N \mid \lambda(v) \equiv 0 \bmod 61 \,\}.$$

Then the $\mathbb{Z}$-module N_0 together with the quadratic fotm

$$v \;\;\mapsto\;\; \langle v, v \rangle + \frac{2}{61^2}\,\lambda(v)^2$$

is isomorphisc to Λ.

Thank you very much for listening!