Real line arrangements and vanishing cycles

Ichiro Shimada

Hiroshima University

ODTU-Bilkent Algebraic Geometry Seminars
2024 April 19

Introduction

We investigate the classical topology of a smooth complex algebraic variety.

We give an *explicit* description of a second homology group (with its intersection form) of a smooth surface X birational to a double cover of a complex affine plane branched along a nodal *real* line arrangement.

"Being explicit" is important. For example, in "numerical algebraic geometry", people calculate the periods of algebraic varieties by numerical integration and the lattice basis reduction algorithm (the LLL method). Here we need an explicit description of topological cycles.

Using the real structure of the arrangement, we define certain topological 2-cycles on X. These cycles are similar with the vanishing cycles of Lefschetz. We then compute their intersection numbers.

The surface X and topological 2-cycles on X

Let $\mathbb{A}^2(\mathbb{R})$ be a real affine plane. A *nodal real line arrangement* is an arrangement

$$\mathcal{A} := \{\ell_1(\mathbb{R}), \dots, \ell_N(\mathbb{R})\}$$

of real lines on $\mathbb{A}^2(\mathbb{R})$ such that, for any point $P \in \mathbb{A}^2(\mathbb{R})$, there are at most two lines in $\mathcal{A}$ passing though P.

Let $\mathcal{A}$ be a nodal real line arrangement. We consider its complexification

$$\mathcal{A}_{\mathbb{C}} := \{\ell_1(\mathbb{C}), \dots, \ell_N(\mathbb{C})\},$$

which is an arrangement of complex affine lines in the complex affine plane $\mathbb{A}^2(\mathbb{C})$. We put

$$B(\mathbb{R}) := \bigcup_{i=1}^{N} \ell_i(\mathbb{R}), \quad B(\mathbb{C}) := \bigcup_{i=1}^{N} \ell_i(\mathbb{C}).$$

Then $B(\mathbb{C})$ is a complex affine plane curve of degree N whose singular locus consists of only ordinary nodes.

We will investigate the topology of a smooth surface X defined by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y(\mathbb{C}) & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}). \end{array}$$

Here the surfaces and the arrows are deined as follows.

- $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ is the double covering whose branch locus is $B(\mathbb{C})$. If $\ell_i(\mathbb{R}) \in \mathcal{A}$ is defined by a real affine linear form $\lambda_i(x, y)$, then W is defined in $\mathbb{C} \times \mathbb{A}^2(\mathbb{C})$ by

$$w^2 = \prod_{i=1}^{N} \lambda_i(x, y).$$

- Then W has $|\operatorname{Sing} B(\mathbb{C})|$ ordinary nodes as its singularities. The arrow $\rho\colon X \to W$ is the minimal resolution.
- $\beta\colon Y(\mathbb{C}) \to \mathbb{A}^2(\mathbb{C})$ is the blowing up at the singular points of $B(\mathbb{C})$.
- $\phi\colon X \to Y(\mathbb{C})$ is the double covering whose branch locus is the strict transform of $B(\mathbb{C})$ by β.

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y(\mathbb{C}) & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}) \end{array}$$

For a subset S of $\mathbb{A}^2(\mathbb{C})$, we put

$$S^\bullet := S \setminus (S \cap \mathrm{Sing}(B(\mathbb{C}))), \quad \beta^\sharp S := \text{the closure of } \beta^{-1}(S^\bullet) \text{ in } Y(\mathbb{C}).$$

We call $\beta^\sharp S$ the *strict transform* of S by β. We then put

$$Y(\mathbb{R}) := \beta^\sharp \mathbb{A}^2(\mathbb{R}).$$

Definition

A *chamber* is the closure in $\mathbb{A}^2(\mathbb{R})$ of a connected component of $\mathbb{A}^2(\mathbb{R}) \setminus B(\mathbb{R})$. Let $\boldsymbol{Ch}_{\mathrm{b}}$ be the set of *bounded* chambers.

For $C \in \boldsymbol{Ch}_{\mathrm{b}}$, we have $\beta^\sharp C \subset Y(\mathbb{R})$. We put

$$\mathrm{Vert}(C) := C \cap \mathrm{Sing}\, B(\mathbb{C}).$$

For $P \in \mathrm{Sing}\, B(\mathbb{C})$, let $E_P \subset Y(\mathbb{C})$ denote the exceptional (-1)-curve of β over P. Then $E_P(\mathbb{R}) := E_P \cap Y(\mathbb{R})$ is a circle $\mathbb{S}^1$ on the 2-sphere E_P.

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y(\mathbb{C}) & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}) \end{array}$$

We put

$$D_P := \phi^{-1}(E_P) = (\pi \circ \rho)^{-1}(P) \subset X,$$

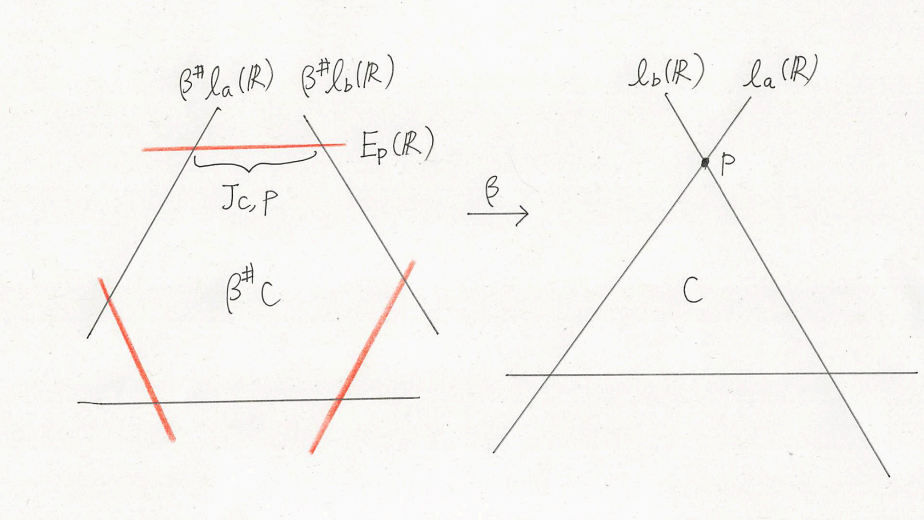
which is a smooth rational curve on X with self-intersection number -2. If $P \in \operatorname{Sing} B(\mathbb{C})$ is the intersection point of $\ell_a(\mathbb{R})$ and $\ell_b(\mathbb{R})$, then

$$\phi|D_P \colon D_P \to E_P$$

is the double covering branching at the intersection points of E_P and the strict transforms $\beta^\sharp \ell_a(\mathbb{C})$ and $\beta^\sharp \ell_b(\mathbb{C})$. Then

$$J_{C,P} := \beta^\sharp C \cap E_P$$

is a part of the circle $E_P(\mathbb{R}) = Y(\mathbb{R}) \cap E_P$ connecting these two branch points of $\phi|D_P \colon D_P \to E_P$.


$\beta^\# l_a(\mathbb{R})$
$\beta^\# l_b(\mathbb{R})$
$E_p(\mathbb{R})$
$J_{C,p}$
$\beta^\# C$
β
$l_b(\mathbb{R})$
$l_a(\mathbb{R})$
p
C

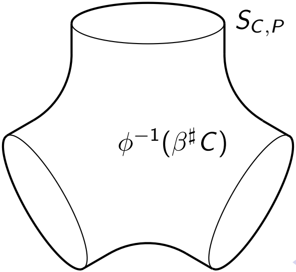
$$\boxed{\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi\downarrow & & \downarrow\pi \\ Y(\mathbb{C}) & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}) \end{array}}$$

Therefore

$$S_{C,P} := \phi^{-1}(\beta^\sharp C) \cap D_P = \phi^{-1}(J_{C,P})$$

is a circle on $D_P \cong \mathbb{P}^1$. The space $\phi^{-1}(\beta^\sharp C)$ is homeomorphic to a 2-sphere minus a union of disjoint open discs, and we have

$$\partial\,\phi^{-1}(\beta^\sharp C) = \bigsqcup_{P \in \mathrm{Vert}(C)} S_{C,P}.$$



Let γ_C be an orientation of $\phi^{-1}(\beta^\sharp C)$. We put

$$\Delta(C, \gamma_C) := \text{the topological 2-chain } \phi^{-1}(\beta^\sharp C) \text{ oriented by } \gamma_C.$$

Note that each boundary component $S_{C,P} \cong \mathbb{S}^1$ of $\Delta(C, \gamma_C)$ divides the 2-sphere $D_P \cong \mathbb{P}^1$ into the union of two closed hemispheres. The two hemispheres with their complex structures induce opposite orientations on $S_{C,P}$.

Definition

The *capping hemisphere* for γ_C at P is the closed hemisphere $H_{C,\gamma_C,P}$ on D_P with $\partial H_{C,\gamma_C,P} = S_{C,P}$ such that the orientation on $S_{C,P}$ induced by the complex structure of $H_{C,\gamma_C,P}$ is opposite to the orientation induced by the orientation γ_C of $\Delta(C, \gamma_C)$.

We have

$$H_{C,\gamma_C,P} \cup H_{C,-\gamma_C,P} = D_P, \qquad H_{C,\gamma_C,P} \cap H_{C,-\gamma_C,P} = S_{C,P}.$$

Then

$$\Sigma(C, \gamma_C) := \Delta(C, \gamma_C) \ \cup \ \bigsqcup_{P \in \mathrm{Vert}(C)} H_{C, \gamma_C, P}$$

is a topological 2-cycle homeomorphic to a 2-sphere. Here the orientations on $H_{C, \gamma_C, P}$ are defined by the complex structure.

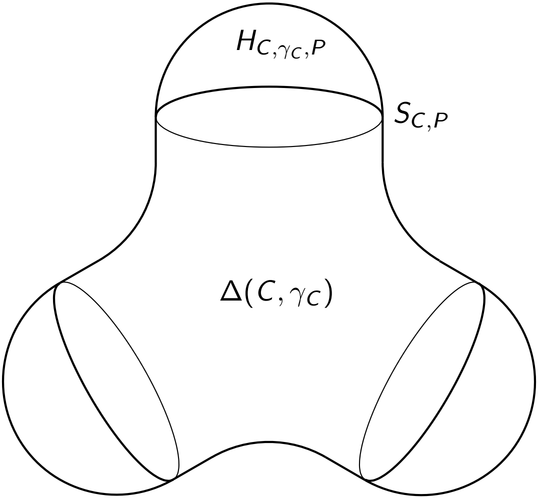
Definition

The 2-cycle $\Sigma(C, \gamma_C)$ is called a *vanishing cycle* over the bounded chamber C. Its homology class $[\Sigma(C, \gamma_C)] \in H_2(X, \mathbb{Z})$ is also called a vanishing cycle over C.

By definition, we have

$$[\Sigma(C, \gamma_C)] + [\Sigma(C, -\gamma_C)] = \sum_{P \in \mathrm{Vert}(C)} [D_P]$$

in $H_2(X, \mathbb{Z})$.



The cycle $\Sigma(C, \gamma_C)$

Main results

Let

$$\mathcal{A} := \{\ell_1(\mathbb{R}), \dots, \ell_N(\mathbb{R})\}$$

be a nodal arrangement of real lines on $\mathbb{A}^2(\mathbb{R})$, and we consider the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y(\mathbb{C}) & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}) \end{array}$$

Theorem

The $\mathbb{Z}$-module $H_2(X, \mathbb{Z})$ is free of rank

$$|\boldsymbol{Ch}_{\mathrm{b}}| + |\operatorname{Sing} B(\mathbb{C})|.$$

The homology classes $[\Sigma(C, \gamma_C)]$, where C runs through $\boldsymbol{Ch}_{\mathrm{b}}$, and the homology classes $[D_P]$, where P runs through $\operatorname{Sing} B(\mathbb{C})$, form its basis.

Main results (continued)

We calculate the intersection form of $H_2(X, \mathbb{Z})$.

Theorem

Let C and C' be distinct bounded chambers.

1. *If $P \in \mathrm{Vert}(C)$, then we have*

$$\langle\ [\Sigma(C, \gamma_C)],\ [D_P]\ \rangle = -1.$$

2. *If $C \cap C'$ consists of a single point, then*

$$\langle\ [\Sigma(C, \gamma_C)],\ [\Sigma(C', \gamma_{C'})]\ \rangle = \begin{cases} 0 & \text{if } \gamma_C \text{ and } \gamma_{C'} \text{ are compatible,} \\ -1 & \text{if } -\gamma_C \text{ and } \gamma_{C'} \text{ are compatible.} \end{cases}$$

The definition of being "compatible" is easy, but we have to prepare some notation. So we omit the definition.

Main results (continued)

Definition

We say that a line $\ell_i(\mathbb{R}) \in \mathcal{A}$ defines an *edge* $C \cap \ell_i(\mathbb{R})$ of a bounded chamber C if $C \cap \ell_i(\mathbb{R})$ contains a non-empty open subset of $\ell_i(\mathbb{R})$.

Theorem

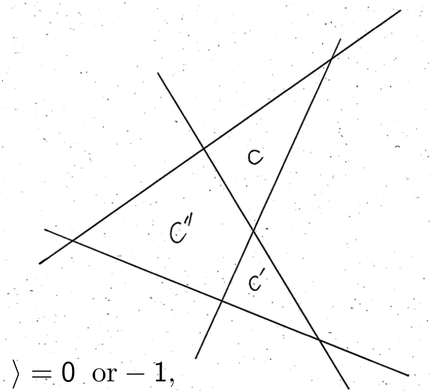
3. *If C and C' share a common edge, then*

$$\langle\ [\Sigma(C, \gamma_C)],\ [\Sigma(C', \gamma_{C'})]\ \rangle = -1.$$

4. *The self-intersection number of $[\Sigma(C, \gamma_C)]$ is*

$$\langle\ [\Sigma(C, \gamma_C)],\ [\Sigma(C, \gamma_C)]\ \rangle = -2.$$

Since the cycles $[\Sigma(C, \gamma_C)]$ are represented by 2-spheres with self-intersection number -2 and they generate the transcendental part of $H_2(X, \mathbb{Z})$, we call them *vanishing cycles*.



$$\langle\ \Sigma(C), \Sigma(C')\ \rangle = 0 \ \text{ or} -1,$$
$$\langle\ \Sigma(C), \Sigma(C'')\ \rangle = \langle\ \Sigma(C'), \Sigma(C'')\ \rangle = -1,$$
$$\langle\ \Sigma(C), \Sigma(C)\ \rangle = \langle\ \Sigma(C'), \Sigma(C')\ \rangle = \langle\ \Sigma(C''), \Sigma(C'')\ \rangle = -2.$$

We have a system of orientations $\{\,\gamma_C \mid C \in \boldsymbol{Ch}_{\mathrm{b}}\,\}$ such that γ_C and $\gamma_{C'}$ are compatible whenever C and C' have a single vertex in common. Hence we can calculate $\langle\ \rangle$. Note that $\langle\ \rangle$ is degenerate in general.

Checking the formulas

Let us check the compatibility of these formulas with $\gamma_C \mapsto -\gamma_C$.

If $P \in \mathrm{Vert}(C)$, then

$$\begin{aligned}
&\langle\ [\Sigma(C,-\gamma_C)],\ [D_P]\ \rangle \\
=&\left\langle \sum_{P' \in \mathrm{Vert}(C)} [D_{P'}] - [\Sigma(C,\gamma_C)],\ \ [D_P] \right\rangle \\
=&[D_P]^2 - (-1) = -2 - (-1) = -1.
\end{aligned}$$

If C and C' share a common edge $P_0 P_1$, then

$$\begin{aligned}
&\langle\ [\Sigma(C,-\gamma_C)],\ [\Sigma(C',\gamma_{C'})]\ \rangle \\
=&\left\langle \sum_{P' \in \mathrm{Vert}(C)} [D_{P'}] - [\Sigma(C,\gamma_C)],\ \ [\Sigma(C',\gamma_{C'})] \right\rangle \\
=&\langle\ [D_{P_0}],\ [\Sigma(C',\gamma_{C'})]\ \rangle + \langle\ [D_{P_1}],\ [\Sigma(C',\gamma_{C'})]\ \rangle - (-1) \\
=&(-1) + (-1) - (-1) = -1.
\end{aligned}$$

Checking the formulas (continued)

We also check the self-intersection number is invariant under $\gamma_C \mapsto -\gamma_C$.

If $|\mathrm{Vert}(C)| = n$, then

$$
\begin{aligned}
&\langle\ [\Sigma(C, -\gamma_C)],\ [\Sigma(C, -\gamma_C)]\ \rangle \\
=&\left\langle \sum_{P\in\mathrm{Vert}(C)} [D_P] - [\Sigma(C, \gamma_C)],\ \sum_{P\in\mathrm{Vert}(C)} [D_P] - [\Sigma(C, \gamma_C)] \right\rangle \\
=&\sum_{P\in\mathrm{Vert}(C)} [D_P]^2 - n(-1) - n(-1) + \langle\ [\Sigma(C, \gamma_C)],\ [\Sigma(C, \gamma_C)]\ \rangle \\
=&-2n + n + n + (-2) = -2.
\end{aligned}
$$

A lemma on simply branched coverings

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y(\mathbb{C}) & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}) \end{array}$$

The theorem that the classes $[D_P]$ of exceptional curves and the vanishing cycles $[\Sigma(C, \gamma_C)]$ form a basis $H_2(X, \mathbb{Z})$ follows from the following two propositions.

Proposition

The kernel of the homomorphism $H_2(X, \mathbb{Z}) \to H_2(W, \mathbb{Z})$ induced by the minimal resolution ρ is freely generated by $[D_P]$, where $P \in \operatorname{Sing} B(\mathbb{C})$.

Proposition

We put $\mathcal{C} := \bigcup_{C \in \boldsymbol{Ch}_{\mathrm{b}}} C \ \subset \ \mathbb{A}^2(\mathbb{R})$. Then the inclusion $\pi^{-1}(\mathcal{C}) \hookrightarrow W$ induces a homotopy equivalence.

To prove the second proposition, we prove a general lemma, which may be useful in other situations.

Definition

Let d be a positive integer. Let S and T be topological spaces, and $\varpi\colon S \to T$ a continuous map. Let V be a closed subspace of T. We say that $\varpi\colon S \to T$ is a *simply branched covering of degree d with branch locus V* if the following conditions hold:

(a) Every fiber of ϖ is discrete.

(b) We put

$$S_V := \varpi^{-1}(V), \quad S_{T\setminus V} := \varpi^{-1}(T \setminus V).$$

Then the restriction $\varpi|S_V\colon S_V \to V$ to S_V is a homeomorphism, and the restriction $\varpi|S_{T\setminus V}\colon S_{T\setminus V} \to T \setminus V$ to $S_{T\setminus V}$ is a usual covering map of degree d. (We do *not* assume that $S_{T\setminus V}$ is connected.)

(c) For any $P \in S_V$ and any open neighborhood $U_{P,S}$ of P in S, there exists an open neighborhood $U_{\varpi(P),T}$ of $\varpi(P)$ in T such that $\varpi^{-1}(U_{\varpi(P),T})$ is contained in $U_{P,S}$.

The usual cyclic branched covering of complex manifolds is a simply branched covering.

For simply branched coverings, we have the following lemma of *liftability* of a strong deformation retraction.

Lemma

Let $\varpi\colon S \to T$ *be a simply branched covering with the branch locus* V*. Let* R *be a closed subspace of* T*, and let*

$$F\colon T \times I \to T$$

be a strong deformation retraction of T *onto* R*, where* $I := [0, 1] \subset \mathbb{R}$*. Suppose the following: For any* $Q \in T$*, if there exists a value* $u_0 \in I$ *such that* $F(Q, u_0) \in V$*, then* $F(Q, u) \in V$ *holds for any* $u \in [u_0, 1]$*. Then there exists a strong deformation retraction*

$$\widetilde{F}\colon S \times I \to S$$

of S *onto* $\varpi^{-1}(R)$ *that is a lift of* $F\colon T \times I \to T$*.* □

$$\begin{array}{ccc} X & \xrightarrow{\rho} & W \\ \phi \downarrow & & \downarrow \pi \\ Y(\mathbb{C}) & \xrightarrow[\beta]{} & \mathbb{A}^2(\mathbb{C}) \end{array}$$

We prove that

$$\pi^{-1}(\mathcal{C}) = \bigcup_{C \in \boldsymbol{Ch}_{\mathrm{b}}} \pi^{-1}(C)$$

is a strong deformation retract of W as follows.
We make strong deformation retractions of $\mathbb{A}^2(\mathbb{C})$ onto $\mathbb{A}^2(\mathbb{R})$ and of $\mathbb{A}^2(\mathbb{R})$ onto $\mathcal{C}$ with the property required in Lemma with respect to $\pi\colon W \to \mathbb{A}^2(\mathbb{C})$ and its restriction to $\pi^{-1}(\mathbb{A}^2(\mathbb{R})) \subset W$.
We then lift them to strong deformation retractions of W onto $\pi^{-1}(\mathbb{A}^2(\mathbb{R}))$ and of $\pi^{-1}(\mathbb{A}^2(\mathbb{R}))$ onto $\pi^{-1}(\mathcal{C})$, respectively.

Examples

We consider a real projective plane $\mathbb{P}^2(\mathbb{R})$ containing the real affine plane $\mathbb{A}^2(\mathbb{R})$ as an affine part. We put

$$\tilde{\ell}_\infty(\mathbb{R}) := \mathbb{P}^2(\mathbb{R}) \setminus \mathbb{A}^2(\mathbb{R}).$$

We take the closure of each $\ell_i(\mathbb{R}) \in \mathcal{A}$ in $\mathbb{P}^2(\mathbb{R})$ and define the arrangement $\widetilde{\mathcal{A}}$ of real projective lines by

$$\widetilde{\mathcal{A}} := \begin{cases} \mathcal{A} \cup \{\tilde{\ell}_\infty(\mathbb{R})\} & \text{if } N = |\mathcal{A}| \text{ is odd,} \\ \mathcal{A} & \text{if } N \text{ is even.} \end{cases}$$

Let $\widetilde{B}(\mathbb{C})$ be the union of the complex projective lines in the complexification $\widetilde{\mathcal{A}}_{\mathbb{C}}$ of $\widetilde{\mathcal{A}}$.

Since $\deg \widetilde{B}(\mathbb{C}) = |\widetilde{\mathcal{A}}|$ is even, we have a double covering

$$\tilde{\pi}\colon \widetilde{W} \to \mathbb{P}^2(\mathbb{C})$$

whose branch locus is equal to $\widetilde{B}(\mathbb{C})$. Let

$$\tilde{\rho}\colon \widetilde{X} \to \widetilde{W}$$

be the minimal resolution. We put

$$\Lambda_\infty := \tilde{\rho}^{-1}(\tilde{\pi}^{-1}(\tilde{\ell}_\infty(\mathbb{C}))) \quad \subset \quad \widetilde{X}.$$

Then we have

$$X = \widetilde{X} \setminus \Lambda_\infty.$$

The inclusion $\iota\colon X \hookrightarrow \widetilde{X}$ induces a natural homomorphism

$$\iota_*\colon H_2(X, \mathbb{Z}) \to H_2(\widetilde{X}, \mathbb{Z}),$$

which preserves the intersection form.

For simplicity, we assume that $H_2(\widetilde{X}, \mathbb{Z})$ is torsion free. Since $\widetilde{X}$ is compact, the intersection form on $H_2(\widetilde{X}, \mathbb{Z})$ is non-degenerate, and hence is a unimodular lattice by the intersection pairing. Let

$$H_\infty \subset H_2(\widetilde{X}, \mathbb{Z})$$

be the submodule generated by the classes of irreducible components of $\Lambda_\infty := \tilde{\rho}^{-1}(\tilde{\pi}^{-1}(\tilde{\ell}_\infty(\mathbb{C})))$. Then the image of

$$\iota_* \colon H_2(X, \mathbb{Z}) \to H_2(\widetilde{X}, \mathbb{Z})$$

is equal to the orthogonal complement $H_\infty^\perp$ of H_∞ in $H_2(\widetilde{X}, \mathbb{Z})$. The kernel of ι_* is equal to

$$\operatorname{Ker} \langle \ \rangle := \{ x \in H_2(X, \mathbb{Z}) \mid \langle x, y \rangle = 0 \ \text{ for any } \ y \in H_2(X, \mathbb{Z}) \}.$$

Therefore we can calculate the sublattice $H_\infty^\perp \subset H_2(\widetilde{X}, \mathbb{Z})$ from $\langle \ \rangle$ on $H_2(X, \mathbb{Z})$.

Comparing the lattice $H_\infty^\perp$ with the lattice $H_2(X, \mathbb{Z}) / \operatorname{Ker} \langle \ \rangle$, we can check the validity of our formulas of intersection numbers on $H_2(X, \mathbb{Z})$.

We consider the case where, for any line ℓ on $\mathbb{A}^2(\mathbb{R})$, at most two lines in $\mathcal{A}$ is parallel to ℓ. Then $\widetilde{B}(\mathbb{C})$ has only a_1 or d_4 singular points. (The singular point of type d_4 may appear on $\tilde{\ell}_\infty$.) Hence, by the theorem of the simultaneous resolution of rational double points on surfaces, the minimal resolution $\widetilde{X}$ is diffeomorphic to the double cover of $\mathbb{P}^2(\mathbb{C})$ branched along a *smooth* curve of degree $|\widetilde{\mathcal{A}}|$. In particular, we can calculate the rank and the signature of the unimodular lattice $H_2(\widetilde{X}, \mathbb{Z})$.

For example, we consider the case where $|\widetilde{\mathcal{A}}| = 6$. Then $\widetilde{X}$ is a $K3$ surface. Therefore $H_2(\widetilde{X}, \mathbb{Z})$ is an even unimodular lattice of rank $b_2(\widetilde{X}) = 22$ with signature $(3, 19)$.

Suppose that $|\mathcal{A}| = 6$ and that no pair of lines of $\mathcal{A}$ is parallel. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 10, \quad |\operatorname{Sing} B(\mathbb{C})| = 15,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 25. On the other hand, Λ_∞ is irreducible and H_∞ is of rank 1 with signature $(1, 0)$ and with discriminant group $\mathbb{Z}/2\mathbb{Z}$. Hence $H_\infty^\perp$ is of rank 21 with signature $(2, 19)$ and with discriminant group $\mathbb{Z}/2\mathbb{Z}$.

For randomly generated such arrangements, we checked that $H_2(X, \mathbb{Z}) / \operatorname{Ker}\langle\ \rangle$ has expected invariants (rank, signature, discriminant group).

Remark that there are several combinatorial structures of nodal arrangements of 6 real lines with no parallel pairs. For example, the numbers of n-gons in $\boldsymbol{Ch}_{\mathrm{b}}$ can vary as follows:

n	3	4	5	6
	4	4	2	0
	4	5	1	0
	4	5	0	1
	4	6	0	0
	5	3	2	0
	5	4	1	0

n	3	4	5	6
	5	4	0	1
	6	2	2	0
	6	3	1	0
	6	3	0	1
	7	0	3	0

.

Suppose that $|\mathcal{A}| = 6$ and that $\mathcal{A}$ consists of three pairs of parallel lines. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 7, \quad |\operatorname{Sing} B(\mathbb{C})| = 12,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 19. On the other hand, H_∞ is of rank 5 with signature $(1, 4)$ and discriminant group $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$. Hence $H_\infty^\perp$ is of rank 17 with signature $(2, 15)$ and with discriminant group $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$.

For randomly generated such arrangements, we checked that $H_2(X, \mathbb{Z})/\operatorname{Ker}\langle\ \rangle$ has expected invariants

Suppose that $|\mathcal{A}| = 24$ and that $\mathcal{A}$ contains exactly 10 pairs of parallel liens. We have

$$|\boldsymbol{Ch}_{\mathrm{b}}| = 243, \quad |\operatorname{Sing} B(\mathbb{C})| = 266,$$

and hence $H_2(X, \mathbb{Z})$ is of rank 509. On the other hand, the unimodular lattice $H_2(\widetilde{X}, \mathbb{Z})$ is of rank 508 with signature $(111, 397)$, and the sublattice H_∞ is of rank 11 with signature $(1, 10)$ and

$$\operatorname{disc}(H_\infty) \cong (\mathbb{Z}/2\mathbb{Z})^{11}.$$

Hence $H_\infty^\perp$ is of rank 497 with signature $(110, 387)$ and $\operatorname{disc}(H_\infty^\perp) \cong (\mathbb{Z}/2\mathbb{Z})^{11}$.
For randomly generated such arrangements, we checked that $H_2(X, \mathbb{Z})/\operatorname{Ker}\langle\ \rangle$ has expected invariants.

The preprint is avairable from `arXiv:2401.15929`.

Thank you very much for listening!