The automorphism group of an Apéry-Fermi $K3$ surface

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2024 September 6

Apéry-Fermi $K3$ surface

We consider the complex affine surface defined by

$$\xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} = s,$$

where ξ_1, ξ_2, ξ_3 are coordinates of the 3-dimensional affine space $\mathbb{A}^3$ and $s \in \mathbb{C}$ is a parameter.

For simplicity, we always assume that

the value of the parameter $s \in \mathbb{C}$ is very general.

Putting the terms over a common denominator, we obtain a quartic polynomial in the numerator. Hence the projective completion of this smooth affine surface is a quartic surface in $\mathbb{P}^3$, and it has only rational double points as its singularities. By the minimal desingularization, we arrive at a K3 surface. We call this K3 surface the **Apéry-Fermi K3 surface**.

Apéry

The “Apéry” in the Apéry-Fermi K3 surface is the mathematician who proved in 1979 that the zeta value at 3

$$\zeta(3) = 1.20206\ldots$$

is irrational. This number is now called the Apéry constant. His proof consists of very acrobatic arguments and miraculous equalities.

Then, in the same year, Frits Beukers gave a very short and simple proof based on Apéry’s arguments.

In 1984, Beukers and Peters constructed a one-dimensional family of K3 surfaces whose Picard-Fuchs equation is the differential equation related to Apéry’s proof of irrationality of $\zeta(3)$. The general member of this family is the Apéry-Fermi K3 surface.

Fermi

The other name in the Apéry-Fermi K3 surface, "Fermi", is the famous physicist in the first half of the 20th century. He is famous for very many things in particle physics and statistical mechanics. For example, the term "fermion" comes from Fermi.

He studied electrons moving in a crystal, a periodic potential. Since electrons are fermions, at most two electrons can occupy a single state. So the occupied states at the zero temperature form a certain body in the momentum space, and the boundary surface of this body is called the **Fermi surface**. This surface explains the various physical properties of *metals*.

> *Few people would define a metal as "a solid with a Fermi surface." This may nevertheless be the most meaningful definition of a metal one can give today; . . .*
>
> *A. R. Mackintosh*

Fermi surface

The affine surface

$$\xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} = s$$

is the Fermi surface of the very simple toy model of electrons moving in a crystal.

In 1989, Peters and Stienstra studied this surface and revealed that this surface is birational to the K3 surface Beukers and Peters discovered in the relation with Apéry's proof. Peters and Stienstra calculated the Néron–Severi lattice of the surface. They also studied the Picard-Fuchs equation and the monodromy with respect to the parameter s.

Hence the name "Apéry-Fermi K3 surface".

Previous research

There exist many works on this beautiful K3 surface.

- In 1996, Dolgachev introduced the notion of *lattice polarized K3 surfaces*. Peters and Stienstra showed that the Néron–Severi lattice S_X of X is isomorphic to the lattice M_6, where

$$M_6 := U \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -12 \rangle \ \cong \ S_X.$$

  Here, U is a hyperbolic plane, $E_8(-1)$ is the negative-definite root lattice of type E_8, and $\langle -12 \rangle$ is the lattice of rank 1 generated by a vector with square-norm -12. Hence the Apéry-Fermi K3 surface is an M_6-lattice polarized K3 surface. Dolgachev determined, among other things, the coarse moduli space of Apéry-Fermi K3 surfaces.
- In 2004, Hosono and others used Apéry-Fermi K3 surfaces in the study of the autoequivalences of derived category of its Fourier-Mukai partner, a K3 surface with Picard number 1 and of degree 12.

Previous research (continued)

- The smooth 3-fold birational to

$$\xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} + \xi_4 + \frac{1}{\xi_4} = 0$$

  is a rigid Calabi-Yau 3-fold, and its modularity was studied by van Geemen and Nygaard (1995), Verrill (2000), and Ahlgren and Ono (2000).
- In 2007, Dardanelli and van Geemen presented the Apéry-Fermi K3 surfaces as the Hessian quartics of certain cubic surfaces.
- In 2015, Mukai and Ohashi found another birational model of the Apéry-Fermi $K3$ surface; the symmetric quartic surface $Y_t \subset \mathbb{P}^3$ defined by

$$(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)^2 = t\, x_1x_2x_3x_4,$$

  where $(x_1 : x_2 : x_3 : x_4)$ is a homogeneous coordinate system of $\mathbb{P}^3$ and $t \in \mathbb{C}$ is a parameter.

Previous research (continued)

- In 2019, Festi and van Straten gives an account on the relation of the Apéry-Fermi K3 surface with quantum electrodynamics, a member of the Apéry-Fermi pencil appears in the calculation of the Feynman integrals of Bhabha scattering

$$e^- + e^+ \quad \to \quad e^- + e^+.$$

- In 2020, Bertin and Lecacheux determined all the Jacobian fibrations of the Apéry-Fermi $K3$ surface by Kneser–Nishiyama method, and studied the quotient by the 2-torsions.
- ....

Main result

We study the automorphism group $\mathrm{Aut}(X)$ of the Apéry-Fermi K3 surface X. Note that the affine surface

$$X^\circ \ : \ \xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} = s$$

has the group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3$ of order 48 as its symmetry. But the full automorphism group $\mathrm{Aut}(X)$ is infinite.

Our main result is as follows:

Theorem

The automorphism group $\mathrm{Aut}(X)$ is generated by a finite subgroup $\mathrm{Aut}(X, D_0)$ of order 16 isomorphic to the dihedral group, and eight extra automorphisms.

We explain the contents of this theorem more in detail.

Main result (continued)

Theorem

$$\mathrm{Aut}(X) = \langle \mathrm{Aut}(X, D_0), \text{eight extra automorphisms} \rangle.$$

The D_0 in $\mathrm{Aut}(X, D_0)$ is a finite polyhedral cone in the nef cone N_X of X:

$$D_0 \ \subset \ N_X \ \subset \ S_X \otimes \mathbb{R} \ \cong \ \mathbb{R}^{1,18}.$$

The nef cone N_X has infinitely many walls, but the cone D_0 has exactly 80 walls, and $\mathrm{Aut}(X, D_0)$ is the stabilizer subgroup in $\mathrm{Aut}(X)$ of D_0. The nef cone N_X is *tessellated* by the images of D_0 by the action of $\mathrm{Aut}(X)$, that is, we have

$$N_X = \bigcup_{g \in \mathrm{Aut}(X)} D_0^g,$$

and if $D_0^g \neq D_0^{g'}$, then $D_0^g \cap D_0^{g'}$ is contained in a linear subspace of $S_X \otimes \mathbb{R}$ of codimension ≥ 1.

Main result (continued)

Hence a fundamental domain of the action of $\mathrm{Aut}(X, D_0)$ on D_0 is a fundamental domain of the action of $\mathrm{Aut}(X)$ on N_X.

As a corollary, we obtain the following:

Corollary

- *The automorphism group $\mathrm{Aut}(X)$ acts on the set $\mathcal{C}(A_1)$ of smooth rational curves transitively.*
- *Let $\mathcal{C}(2A_1)$ be the set of pairs of disjoint smooth rational curves. Then the action of $\mathrm{Aut}(X)$ on $\mathcal{C}(2A_1)$ decomposes this set into two orbits.*
- *Let $\mathcal{C}(A_2)$ be the set of pairs of smooth rational curves intersecting at one point transversely. Then $\mathrm{Aut}(X)$ acts on $\mathcal{C}(A_2)$ transitively.*

We also have some more results in the same vein.

Method

We prove our results by the following procedure.

1. Describe the Néron–Severi lattice S_X and the transcendental lattice T_X of X. This task has been already done by Peters and Stienstra. It follows that $\mathrm{Aut}(X)$ acts on S_X faithfully, and hence we can regard $\mathrm{Aut}(X)$ as a subgroup of the orthogonal group $\mathrm{O}(S_X)$.
2. Apply Borcherds' method to obtain a finite set of generators of $\mathrm{Aut}(X)$ *lattice theoretically*, namely, we obtain a set of 19×19 matrices in $\mathrm{O}(S_X)$ that generate the subgroup $\mathrm{Aut}(X)$. Borcherds' method goes as follows:
    - a. Embed S_X into the even unimodular lattice L_{26} of rank 26 and signature $(1, 25)$.
    - b. Calculate the tessellation of N_X induced by the tessellation of the positive cone of L_{26} by Conway chambers.

   It turns out that, for the Apéry-Fermi K3 surface X, the induced tessellation is simple.
3. Find *geometric* of generators of $\mathrm{Aut}(X)$.

Smooth rational curves

We make the projective completion $\overline{X}$ of the affine surface

$$X^\circ \ : \ \ \xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} = s$$

in $\mathbb{P}^6$ by the following. Let $\overline{X}$ be defined in $\mathbb{P}^6$ by

$$\begin{aligned} &u_1 + u_2 + u_3 + v_1 + v_2 + v_3 = s\,w, \\ &u_1 v_1 - w^2 = u_2 v_2 - w^2 = u_3 v_3 - w^2 = 0, \end{aligned}$$

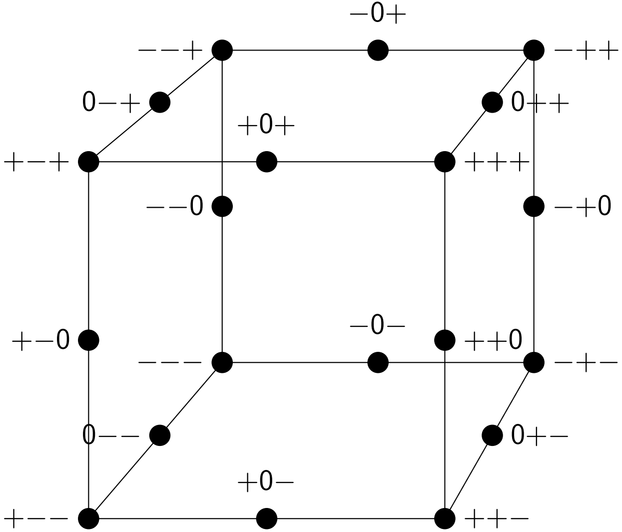
where $(w : u_1 : u_2 : u_3 : v_1 : v_2 : v_3)$ is a homogeneous coordinate system of $\mathbb{P}^6$ such that we have

$$\xi_i = u_i/w = w/v_i.$$

Then $\overline{X}$ has 12 ordinary nodes on the hyperplane $H_\infty = \{w = 0\}$ of $\mathbb{P}^6$. On the other hand, the intersection $H_\infty \cap \overline{X}$ consists on 8 lines. Thus we obtain $12 + 8$ smooth rational curves $L_{\gamma_1\gamma_2\gamma_3}$ on the minimal desingularization X of $\overline{X}$, where $\gamma_1, \gamma_2, \gamma_3 \in \{-, 0, +\}$.

Smooth rational curves (continued)

These lines form the following dual graph.



Smooth rational curves (continued)

We need some more smooth rational curves to generate S_X.

Let $\sigma, \sigma^{-1} \in \mathbb{C}$ be the roots of the equation $\xi + 1/\xi = s$. For $k \in \{1, 2, 3\}$, the intersection of the affine surface $X^\circ \subset \mathbb{A}^3$ and the plane

$$\xi_k = \sigma^{\pm 1}$$

is a union of two curves:

$$\xi_i + \frac{1}{\xi_i} + \xi_j + \frac{1}{\xi_j} = \frac{(\xi_i + \xi_j)(\xi_i \xi_j + 1)}{\xi_i \xi_j} = 0,$$

where $\{i, j, k\} = \{1, 2, 3\}$. Let $M^\circ_{k\alpha\beta}$ be the curve on $X^\circ \subset \mathbb{A}^3$ obtained in this way, where $\alpha, \beta \in \{+, -\}$ are defined in a suitable way. Taking the strict transform of the closure of $M^\circ_{k\alpha\beta}$, we obtain 12 smooth rational curves $M_{k\alpha\beta}$ on X.

Néron–Severi lattice S_X

Then Peters and Stienstra proved the following:

Theorem

The classes of these $(12+8)+12$ *smooth rational curves on* X *span the Néron–Severi lattice* S_X *of* X*, and* S_X *is isomorphic to the lattice*

$$M_6 := U \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -12 \rangle.$$

We can choose a basis of S_X in such a way that the Gram matrix is

$$\left[\begin{array}{cc|c|c|c} 0 & 1 & & & \\ 1 & 0 & & & \\ \hline & & E_8(-1) & & \\ \hline & & & E_8(-1) & \\ \hline & & & & -12 \end{array}\right],$$

where

$$E_8(-1) = \begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix},$$

and that the classes of the 32 smooth rational curves are given as

$$\begin{array}{lcl}
L_{---} & : & [0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \\
L_{--0} & : & [0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0] \\
L_{--+} & : & [0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \\
L_{-0-} & : & [4,3,-8,-5,\dots,-4,-8,-12,-10,-8,-6,-3,-1] \\
L_{-0+} & : & [0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \\
 & & \dots \\
 & & \dots \\
M_{3-+} & : & [0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0] \\
M_{3+-} & : & [0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0] \\
M_{3++} & : & [12,11,-24,-16,\dots,-48,-40,-30,-20,-10,-3]
\end{array}$$

Lattice theoretic algorithms

Our description of S_X is **so explicit** that we can do the following things by automated lattice theoretic calculations.

- For a given vector $v \in S_X$, we can determine whether v is nef (resp. ample) or not.
- For a given vector $r \in S_X$ with $\langle r, r \rangle = -2$, we can determine whether r is the class of a smooth rational curve C.
- Suppose that $h \in S_X$ is nef and $\langle h, h \rangle > 0$. We can make the list of the classes $[C]$ of smooth rational curves C that contract in the projective model (X, h). In particular, we can calculate the type of the rational double points on (X, h).
- For an isometry $g \in \mathrm{O}(S_X)$, we can determine whether $g \in \mathrm{Aut}(X)$ or not.
- Suppose that the classes (f, z) of a fiber and a zero section of a Jacobian fibration $\phi\colon X \to \mathbb{P}^1$ are given. We can calculate the Mordell-Weil group of $\phi\colon X \to \mathbb{P}^1$ and its action on S_X.
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Chambers and tessellations

We fix terminologies and notation about lattice theory and hyperbolic spaces. Let L be an even lattice of signature $(1, l-1)$ with $l \geq 2$. A *positive cone* of L is one of the two connected components of the space

$$\{\, x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \,\}.$$

We fix a positive cone $\mathcal{P}_L$, and denote the autochronous subgroup of $\mathrm{O}(L)$ by

$$\mathrm{O}(L, \mathcal{P}_L) := \{\, g \in \mathrm{O}(L) \mid \mathcal{P}_L^g = \mathcal{P}_L \,\}.$$

We then put

$$\mathcal{R}_L := \{\, r \in L \mid \langle r, r \rangle = -2 \,\}.$$

For $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle < 0$, let $(v)^\perp$ be the real hyperplane in $\mathcal{P}_L$ defined by

$$(v)^\perp := \{\, x \in \mathcal{P}_L \mid \langle x, v \rangle = 0 \,\}.$$

Chambers and tessellations (continued)

The *Weyl group* $W(L)$ is the subgroup of $\mathrm{O}(L, \mathcal{P}_L)$ generated by reflections

$$x \mapsto x + \langle x, r \rangle r$$

into the mirrors $(r)^\perp$ defined by vectors $r \in \mathcal{R}_L$.

Definition

A *standard fundamental domain of the action of the Weyl group* $W(L)$ *on* $\mathcal{P}_L$ is the closure in $\mathcal{P}_L$ of a connected component of the complement

$$\mathcal{P}_L \ \setminus \ \bigcup_{r \in \mathcal{R}_L} (r)^\perp$$

of all mirrors $(r)^\perp$ associated with vectors $r \in \mathcal{R}_L$.

Then $W(L)$ acts on the set of standard fundamental domains simple-transitively.

Induced tessellation

Let M be a primitive sublattice of L with signature $(1, m-1)$ with $m \geq 2$:

$$M \hookrightarrow L.$$

Let $\mathcal{P}_M$ be the positive cone $(M \otimes \mathbb{R}) \cap \mathcal{P}_L$ of M.

Definition

An *L/M-chamber* is a closed subset D of $\mathcal{P}_M$ containing a non-empty open subset of $\mathcal{P}_M$ such that D is of the form $\mathcal{P}_M \cap D_L$, where D_L is a standard fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$.

The positive cone $\mathcal{P}_M$ is then tessellated by L/M-chambers.

According to this terminology, we can rephrase "standard fundamental domain of the action of $W(L)$ on $\mathcal{P}_L$" by "L/L-chamber".

Example

Let $\mathcal{P}_X$ be the positive cone of S_X containing an ample class. The nef cone

$$N_X := \{\, x \in \mathcal{P}_X \mid \langle x, C\rangle \geq 0 \text{ for all curves } C \text{ on } X \,\}$$

of X is an S_X/S_X-chamber, and its walls are in one-to-one correspondence with the smooth rational curves of X.

Example

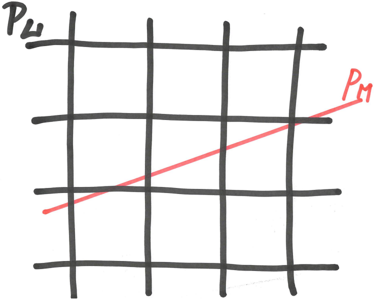
Let L_{26} be an even unimodular lattice of rank 26 and signature $(1, 25)$, which is unique up to isomorphism. Conway (1983) determined the shape of L_{26}/L_{26}-chambers. Hence we call an L_{26}/L_{26}-chamber a *Conway chamber*. Conway showed that

- the walls of a Conway chamber are in one-to-one correspondence with the vectors of the Leech lattice Λ, and
- $\mathrm{O}(L_{26}, \mathcal{P}_{26})$ is isomorphic to the group of *affine* isometries of Λ.

Induced tessellation (continued)

The L/M-chambers make a tessellation of $\mathcal{P}_M$, and each M/M-chamber is also tessellated by L/M-chambers.

In general, L/M-chambers are not congruent to each other. We say that the tessellation by L/M-chambers is *simple* if all L/M-chambers are congruent to each other. (In the figure below, the left one is not simple, because the squares in $\mathcal{P}_L$ cut out from $\mathcal{P}_M$ line segments of different lengths, whereas the right one is simple.)



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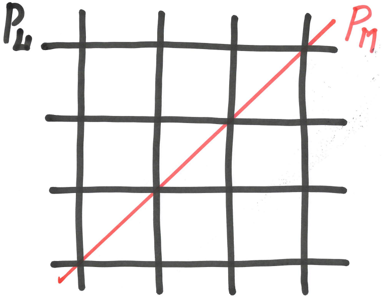


Borchers method

We embed S_X into L_{26} primitively, and consider the tessellation of the S_X/S_X-chamber N_X by L_{26}/S_X-chambers.

The crucial point of the whole work is that I found a primitive embedding $S_X \hookrightarrow L_{26}$ such that the tessellation by L_{26}/S_X-chambers is simple.

We obtained an L_{26}/S_X-chamber

$$D_0 \subset N_X$$

that has exactly 80 walls, and all other L_{26}/S_X-chambers are congruent to D_0. The stabilizer subgroup

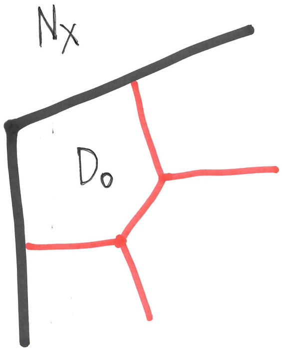
$$\mathrm{Aut}(X, D_0) := \{\, g \in \mathrm{Aut}(X) \mid D_0^g = D_0 \,\}$$

is a dihedral group of order 16, and its action on D_0 decomposes the 80 walls in 10 orbits:

$$80 = 8 + 16 + 4 + 8 + 8 + 8 + 8 + 4 + 8 + 8.$$

Borcherds method (continued)

A wall $D_0 \cap (v)^{\perp}$ of D_0 is *outer* if the L_{26}/S_X-chamber adjacent to D_0 across the wall $D_0 \cap (v)^{\perp}$ is outside of N_X. Otherwise the wall is called *inner*. A wall w of D_0 is outer if and only if there exists a smooth rational curve C such that $w = D_0 \cap (C)^{\perp}$.



BorchNerds method (continued)

Among the

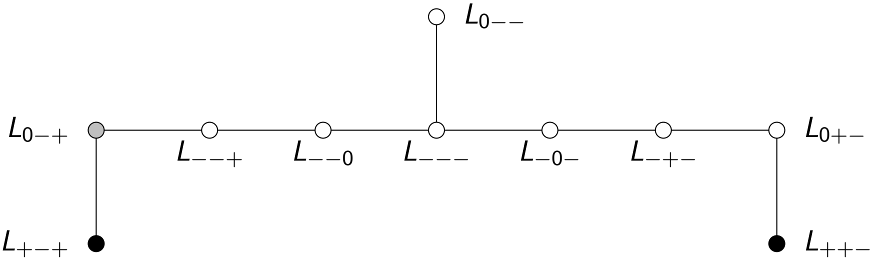
$$80 = 8 + 16 + 4 + 8 + 8 + 8 + 8 + 4 + 8 + 8$$

walls of D_0, the $8 + 16$ walls in the first and the second orbits are outer.

For each of the other 8 orbits $o_3, \dots, o_{10}$ of inner walls, we find an automorphism $g(o_i)$ that maps D_0 to the L_{26}/S_X-chamber adjacent to D_0 across a wall in the orbit o_i, that is, the L_{26}/S_X-chamber $D_0^{g(o_i)}$ shares a wall $w \in o_i$ with D_0.

Then $\mathrm{Aut}(X)$ is generated by $\mathrm{Aut}(X, D_0)$ and the eight automorphisms $g(o_3), \dots, g(o_{10})$.

The next task is to realize these automorphisms geometrically. We take $g(o_4)$ as an example. Consider the configuration



The seven white nodes with the gray node form an affine Dynkin diagram of type E_7. Therefore we obtain a Jacobian fibration $\phi\colon X \to \mathbb{P}^1$ with the zero section L_{+-+}. The Mordell–Weil group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and its non-trivial element is given by the section L_{++-}. The translation by this non-trivial torsion section serves as $g(o_4)$.

Looking at the faces of the finite polyhedral cone D_0, we can prove many geometric properties of X. For example, we obtain a set of defining relations of $\mathrm{Aut}(X)$ with respect to a certain set of generators.

Thank you very much for listening!