Conway Theory and K3 Surfaces

Ichiro Shimada

Hiroshima University

Algebraic Geometry and Related Topics,
Hanoi University of Science
2025 March 14

Abstract

Conway theory is a theory about a certain **hyperbolic lattice**.

Néron–Severi lattices of K3 surfaces are hyperbolic lattices.

Borcherds's method enables us to use Conway theory in the computation of the automorphism groups of K3 surfaces.

We explain this method without going into too much detail.

What can we do with Conway theory?

We illustrate, through an example, what we can achieve in the theory of K3 surfaces with Conway theory via Borcherds' method.

We consider the complex affine smooth surface defined by

$$\xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} = s,$$

where ξ_1, ξ_2, ξ_3 are coordinates of the 3-dimensional affine space $\mathbb{A}^3$ and $s \in \mathbb{C}$ is a parameter.

For simplicity, we always assume that $s \in \mathbb{C}$ is very general. The projective completion of this affine surface is a quartic surface in $\mathbb{P}^3$, and it has only rational double points as its singularities. By the minimal resolution, we arrive at a K3 surface. We call this K3 surface

an **Apéry-Fermi K3 surface**.

What can we do with Conway theory? (continued)

This K3 surface has been studied extensively by many researchers.
I will explain the origin of the names “Apéry” and “Fermi”.

The Picard-Fuchs equation of a family of Apéry-Fermi K3 surfaces is related to a recurrence relation associated with Apéry’s proof of the irrationality of $\zeta(3)$. (Beukers and Peters.)

The real part of the affine surface

$$X^\circ \ : \ \xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} = s$$

with $s \in \mathbb{R}$ is the Fermi surface of a toy model of a metal in the solid-state physics. (Peters and Stienstra.)

What can we do with Conway theory? (continued)

We study the automorphism group $\mathrm{Aut}(X)$ of the Apéry-Fermi K3 surface X. Note that the affine surface X° above has the group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3$ of order 48 as its symmetry. The full automorphism group $\mathrm{Aut}(X)$ is, however, infinite. Our result is as follows:

Theorem

The automorphism group $\mathrm{Aut}(X)$ is generated by a finite subgroup $\mathrm{Aut}(X, D_0)$ of order 16 isomorphic to the dihedral group, and eight additional automorphisms.

I will explain D_0 later.
Moreover, we can describe these generators explicitly

What is Conway theory?

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form

$$\langle\, ,\, \rangle \colon L \times L \to \mathbb{Z}.$$

We let the automorphism group $\mathrm{O}(L)$ of a lattice L (the orthogonal group of L) act on L from the right: $v \mapsto v^g$ for $v \in L$ and $g \in \mathrm{O}(L)$.
A lattice L is *even* if $\langle v, v\rangle \in 2\mathbb{Z}$ for all $v \in L$.
A lattice L is *unimodular* if the natural embedding $L \to \mathrm{Hom}(L, \mathbb{Z})$ given by $v \mapsto \langle -, v\rangle$ is an isomorphism.
A lattice L of rank $n > 1$ is *hyperbolic* if the signature of $L \otimes \mathbb{R}$ is $(1, n-1)$. (Caution: This sign convention is opposite of that of standard lattice theory. Ours is more suitable for algebraic geometry.)

Theorem

An even unimodular hyperbolic lattice of rank n exists if and only if $n \equiv 2 \pmod 8$ holds.

What is Conway theory? (continued)

We also have the uniqueness.

Theorem

For n such that $n \equiv 2 \pmod 8$, an even unimodular hyperbolic lattice $\mathbb{L}_n$ of rank n is unique up to isomorphism.

The lattice

$$\mathbb{L}_2 = U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is called a *hyperbolic plane*. We have

$$\mathbb{L}_{2+8k} \cong U \oplus E_8^{\oplus k},$$

where E_8 is the even unimodular *negative*-definite lattice of rank 8, which is unique up to isomorphism.

Vinberg determined $\mathrm{O}(\mathbb{L}_{10})$ and $\mathrm{O}(\mathbb{L}_{18})$.

Conway determined $\mathrm{O}(\mathbb{L}_{26})$.

How to describe $\mathrm{O}(L)$?

Let L be an even hyperbolic lattice. A *positive cone* of L is one of the two connected components of the space

$$\{\, x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \,\}.$$

Let $\mathcal{P}_L$ be a positive cone of L. Then we have

$$\mathrm{O}(L) = \mathrm{O}(L, \mathcal{P}_L) \times \{\pm 1\},$$

where $\mathrm{O}(L, \mathcal{P}_L) := \{\, g \in \mathrm{O}(L) \mid \mathcal{P}_L^g = \mathcal{P}_L \,\}$ is the stabilizer of $\mathcal{P}_L$.
For $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle < 0$, we define

$$(v)^\perp := \{\, x \in \mathcal{P}_L \mid \langle x, v \rangle = 0 \,\},$$

which is a real hyperplane of $\mathcal{P}_L$.
We say that $r \in L$ is a (-2)-*vector* if $\langle r, r \rangle = -2$. A (-2)-vector r defines a reflection

$$s_r : x \mapsto x + \langle x, r \rangle r \quad \in \quad \mathrm{O}(L, \mathcal{P}_L)$$

into the mirror $(r)^\perp$.

How to describe $\mathrm{O}(L)$? (continued)

The *Weyl group* $W(L)$ is the subgroup of $\mathrm{O}(L, \mathcal{P}_L)$ generated by all the reflections s_r associated with (-2)-vectors $r \in L$.

Definition

A *standard fundamental domain of the action of the Weyl group* $W(L)$ *on* $\mathcal{P}_L$ is the closure in $\mathcal{P}_L$ of a connected component of the complement

$$\mathcal{P}_L \setminus \bigcup (r)^\perp$$

of all mirrors $(r)^\perp$ associated with (-2)-vectors r.

Then the Weyl group $W(L)$ acts on the set of standard fundamental domains simple transitively.
Moreover $W(L)$ is generated by reflections with respect to the (-2)-vectors defining the walls of a standard fundamental domain F.

How to describe $\mathrm{O}(L)$? (continued)

Let F be a standard fundamental domain, and we put

$$\mathrm{O}(L, F) := \{ g \in \mathrm{O}(L) \mid F^g = F \}.$$

Then we have

$$\mathrm{O}(L, \mathcal{P}_L) = W(L) \rtimes \mathrm{O}(L, F).$$

Hence all we need to do is to describe the set of walls of F, and calculate the stabilizer subgroup $\mathrm{O}(L, F)$.

Vinberg showed that, for $L = \mathbb{L}_{10}$ and $L = \mathbb{L}_{18}$, the standard fundamental domain F has only finitely many walls: 10 walls and 19 walls, respectively.

For $L = \mathbb{L}_{26}$, F has infinitely many walls.

Conway's result

A *Niemeier lattice* is an even unimodular *negative*-definite lattice of rank 24. There exist exactly 24 isomorphism classes of Niemeier lattices. For any Niemeier lattice N (for example, $E_8^{\oplus 3}$), we have $\mathbb{L}_{26} \cong U \oplus N$.

The famous *Leech lattice* Λ is characterized as the unique Niemeier lattice (up to isomorphism) that has no (-2)-vectors. (Other Niemeier lattices are generated by (-2)-vectors up to finite index.)

Conway proved the following:

Theorem

We put

$$w_0 := (1, 0, \mathbf{0}) \ \in \ U \oplus \Lambda = \mathbb{L}_{26}.$$

Then the set

$$\{\, r \in \mathbb{L}_{26} \mid \langle r, r \rangle = -2, \langle w_0, r \rangle = 1 \,\}$$

of **Leech roots** *form the set of walls of a standard fundamental domain F.*

Conway's result (continued)

The Leech roots are parametrized by vectors $\lambda \in \Lambda$ as follows:

$$r_\lambda = (a(\lambda), 1, \lambda) \in U \oplus \Lambda,$$

where $2a(\lambda) + \lambda^2 = -2$. Hence we obtain the following:

Corollary

The group $\mathrm{O}(\mathbb{L}_{26}, F)$ is isomorphic to the group Co_∞ of affine isometries of the Leech lattice Λ.

Note that $\mathrm{Co}_0 := \mathrm{O}(\Lambda)$ is of order

$$8,315,553,613,086,720,000,$$

and its quotient by $\{\pm 1\}$ is a simple group. The group Co_∞ is the semidirect product

$$\Lambda \rtimes \mathrm{Co}_0.$$

The Néron–Severi lattice of a $K3$ surface

Let X be a $K3$ surface. For simplicity, we suppose that X defined over $\mathbb{C}$, but the theory below can be applied to supersingular $K3$ surfaces in positive characteristics.

The *Néron–Severi lattice*

$$S_X := H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$$

is the group of numerical equivalence classes of divisors of X with the intersection pairing. We assume that $\operatorname{rank} S_X > 1$. Then S_X is an even hyperbolic lattice. Let

$$\mathcal{P}_X \subset S_X \otimes \mathbb{R}$$

be the positive cone of S_X containing an ample class.

The Néron–Severi lattice of a $K3$ surface (continued)

By Riemann-Roch, we see that the **nef-and-big cone**

$$N_X := \{\, x \in \mathcal{P}_X \mid \langle x, C \rangle \geq 0 \ \text{ for any curve } \ C \,\}$$

is a standard fundamental domain of $W(S_X)$, or is the standard fundamental domain of $W(S_X)$ that contains an ample class. Note that the class of a smooth rational curve C is a (-2)-vector:

$$\langle C, C \rangle = -2.$$

We have a more precise description of N_X:

$$N_X = \{\, x \in \mathcal{P}_X \mid \langle x, C \rangle \geq 0 \ \text{ for any smooth rational curve } \ C \,\}$$

and a (-2)-vector $r \in S_X$ is the class of a smooth rational curve if and only if the hyperplane $(r)^{\perp}$ defines a wall of N_X and r is *outward* from N_X.

We have a natural homomorphism

$$\mathrm{Aut}(X) \to \mathrm{O}(S_X, N_X).$$

Torelli theorem for complex $K3$ surfaces implies that

- the kernel of $\mathrm{Aut}(X) \to \mathrm{O}(S_X, N_X)$ is finite, and
- the image is of finite index.

The kernel and cokernel are described by the **period** of X, and are easily handled. Hence the essential part of the calculation of $\mathrm{Aut}(X)$ is to calculate $\mathrm{O}(S_X, N_X)$

Remark

Usually X contains infinitely many smooth rational curves. In fact, $\mathrm{Aut}(X)$ is finite if and only if X contains only finitely many smooth rational curves, and such $K3$ surfaces have been classified by Kondo.

Remark

Conway theory can be regarded as a study of the nef-and-big cone of a **non-existing (← Caution!)** *K3 surface $\mathbb{X}$ with Néron–Severi lattice $\mathbb{L}_{26}$. We can say $\mathrm{Aut}(\mathbb{X}) \cong \mathrm{Co}_\infty$.*

Borcherds' method

An idea due to Borcherds is that we embed S_X into $\mathbb{L}_{26}$ primitively and to analyze N_X by Conway theory.

We fix a primitive embedding

$$S_X \hookrightarrow \mathbb{L}_{26}.$$

Let $\mathcal{P}_{26}$ be the positive cone of $\mathbb{L}_{26}$ such that

$$\mathcal{P}_X = (S_X \otimes \mathbb{R}) \cap \mathcal{P}_{26}.$$

Recall that $\mathcal{P}_{26}$ is tessellated by standard fundamental domains of $W(\mathbb{L}_{26})$, which we will call *Conway chambers*.

Definition

An *induced chamber* is a closed subset D of $\mathcal{P}_X$ containing a non-empty open subset of $\mathcal{P}_X$ such that D is of the form $\mathcal{P}_X \cap D_L$, where D_L is a Conway chamber.

Borcherds' method (continued)

Observation 1.

Since $\mathcal{P}_{26}$ is tessellated by Conway chambers, the cone $\mathcal{P}_X$ is also tessellated by induced chambers. Since every (-2)-vector of S_X is also a (-2)-vector of $\mathbb{L}_{26}$, each wall of the standard fundamental domain N_X is the intersection of $\mathcal{P}_X$ with a wall of a Conway chamber. Hence N_X is tessellated by induced chambers.

Observation 2.

In general, the induced chambers are not congruent to each other. However the finiteness theorem of Siegel implies that the number of congruence classes of induced chambers is finite. Therefore the number of the $\mathrm{Aut}(X)$-congruence classes of induced chambers is also finite.

Observation 3.

We can show the following. Suppose that the orthogonal complement $(S_X \hookrightarrow \mathbb{L}_{26})^{\perp}$ contains a (-2)-vector. Then each induced chamber has only finitely many walls.

Borchers' method (continued)

We assume that $(S_X \hookrightarrow \mathbb{L}_{26})^\perp$ contains a (-2)-vector.
Starting from an induced chamber D_0 contained in N_X, we execute the following algorithm. We set

$$V := [D_0], \quad i := 1, \quad G := \{\ \}.$$

While the counter i is $\leq |V|$, we do the following:
Let D_i be the ith element of the list V. We calculate the adjacent induced chambers of D_i.
For each adjacent induced chamber D',

- if D' is not contained in N_X, then do nothing,
- if D' is contained in N_X and is $\mathrm{Aut}(X)$-congruent to some $D_j \in V$, add an element $g \in \mathrm{Aut}(X)$ such that $D_i^g = D_j$ to G, and
- if D' is contained in N_X and is not $\mathrm{Aut}(X)$-congruent to any $D_j \in V$, then add D_i to V.

Borchers' method (continued)

This algorithm terminates, because
each induced chamber has only finitely many adjacent induced chambers, and there exists only finitely many $\mathrm{Aut}(X)$-congruence classes of induced chambers.

When this algorithm terminates, we obtain a complete set V of representatives of $\mathrm{Aut}(X)$-congruence classes of induced chambers, and a finite subset G of $\mathrm{Aut}(X)$.

Proposition

The group $\mathrm{Aut}(X)$ is generated by G and the finite subgroup

$$\mathrm{Aut}(X, D_0) := \{ g \in \mathrm{Aut}(X) \mid D_0^g = D_0 \}.$$

Hence we obtain a finite generating set of $\mathrm{Aut}(X)$.

Apéry-Fermi K3 surface revisited

Recall that the Apéry-Fermi K3 surface is the K3 surface birational to the smooth affine surface

$$X^\circ \;:\; \xi_1 + \frac{1}{\xi_1} + \xi_2 + \frac{1}{\xi_2} + \xi_3 + \frac{1}{\xi_3} = s.$$

Let X be the Apéry-Fermi K3 surface. From the equation, we find 32 smooth rational curves on X. Then S_X is of rank 19 generated by the classes of these 32 smooth rational curves, and is isomorphic to

$$U \oplus E_8 \oplus E_8 \oplus \langle -12 \rangle,$$

where $\langle -12 \rangle$ is the lattice of rank 1 generated by a vector with square-norm -12.

From the polarization of degree 4 given by $X^\circ \hookrightarrow \mathbb{A}^3$ and the singularities of the projective completion of X°, we describe the nef-and-big cone N_X.

Apéry-Fermi K3 surface revisited (continued)

We embed S_X into $\mathbb{L}_{26}$ primitively in such a way that the orthogonal complement is isomorphic to the root lattice of type $D_5 + A_2$.
We then find an induced chamber $D_0 \subset N_X$. It turns out that D_0 has exactly 80 walls, and $\mathrm{Aut}(X, D_0)$ is isomorphic to a dihedral group of order 16. The action $\mathrm{Aut}(X, D_0)$ on D_0 decomposes the 80 walls in 10 orbits:

$$80 = 8 + 16 + 4 + 8 + 8 + 8 + 8 + 4 + 8 + 8.$$

Among the 80 walls, $8 + 16$ walls are walls of N_X; they are defined by (-2)-vectors in S_X. Hence the corresponding adjacent chambers are outside of N_X.

Apéry-Fermi K3 surface revisited (continued)

The other 56 walls are not walls of N_X; we cannot take the defining vector in S_X of square norm -2. Hence the corresponding adjacent chambers are inside of N_X.

These inner adjacent chambers are all $\mathrm{Aut}(X)$-congruent to D_0. This means that the algorithm described above terminates at $i = 1$. Hence $\mathrm{Aut}(X)$ is generated by $\mathrm{Aut}(X, D_0)$ and eight extra automorphisms, which correspond to the eight orbits of inner walls.

Thus we obtain the following:

Theorem

The automorphism group $\mathrm{Aut}(X)$ *is generated by a finite subgroup* $\mathrm{Aut}(X, D_0)$ *of order* 16 *isomorphic to the dihedral group, and eight additional automorphisms.*

Another example: a double plane of degree 6

As another application, we calculate the automorphism group of the complex $K3$ surface $X_{f,g}$ obtained as the minimal resolution of the double cover of $\mathbb{P}^2$ defined by

$$w^2 = f(x, y, z)^2 + g(x, y, z)^3, \tag{1}$$

where f and g are very general homogeneous polynomials on $\mathbb{P}^2$ of degree 3 and 2, respectively.

Remark

The plane curve $f(x, y, z)^2 + g(x, y, z)^3 = 0$ is called a torus sextic, and was investigated by Pho Duc Tai and Mutsuo Oka.

Another example: a double plane of degree 6 (continued)

The rank of S_X for $X = X_{f,g}$ is 13. We find a primitive embedding

$$S_X \hookrightarrow \mathbb{L}_{26}$$

such that, by the algorithm of Borcherds' method, we find **six** $\mathrm{Aut}(X)$-congruence classes of induced chambers.

Theorem

The automorphism group $\mathrm{Aut}(X_{f,g})$ *of* $X_{f,g}$ *is generated by* 463 *involutions associated with double coverings* $X_{f,g} \to \mathbb{P}^2$ *and* 360 *elements of infinite order in Mordell–Weil groups of Jacobian fibrations of* $X_{f,g}$*.*

The last remark

In many cases, the number of $\mathrm{Aut}(X)$-congruence classes of induced chambers is extremely large. In these cases, the algorithm of Borcherds' method does not terminate in practical time.

Thank you very much for your attention!