Del Pezzo surfaces of degree one and examples of Zariski multiples

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Zariski multiples

The notion of **Zariski multiples** concerns the topological methods in classical complex projective geometry.

The knot theory studies the embedding topology of finite unions of circles (links) in S^3.

We study complex projective plane curves $C \subset \mathbb{P}^2$ (possibly reducible) and investigate their embedding topology in $\mathbb{P}^2$, that is, we explore an analogue of knot theory within the framework of algebraic geometry.

Example

In the 1930s, in his study of equisingular families of plane curves, Zariski found a pair of plane curves C_1 and C_2 of degree 6 such that

1. each of C_1 and C_2 has six ordinary cusps as its only singularities, but
2. $\pi_1(\mathbb{P}^2 \setminus C_1) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, whereas $\pi_1(\mathbb{P}^2 \setminus C_2) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, where $*$ denotes the free product.

Zariski multiples

In 1994, Artal Bartolo revisited Zariski's work, and formulated the notion of *Zariski pairs*.

In this talk, we work over $\mathbb{C}$, and by a plane curve, we mean a complex reduced, possibly reducible, projective plane curve.

Definition

We say that a pair of plane curves C_1 and C_2 of the same degree form a *Zariski pair* if

1. there is a tubular neighborhood T_i of C_i in $\mathbb{P}^2$ for $i = 1$ and 2 such that (T_1, C_1) is homeomorphic to (T_2, C_2), but
2. $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.

The first condition can be rephrased as "C_1 and C_2 have the same combinatorial type of singularities", and the second condition means that they differ in their embedding topology.

Definition

A collection of N plane curves of the same degree is called a *Zariski N-tuple* if any two in the collection form a Zariski pair.

Since this seminal definition by Artal Bartolo (1994), many Zariski multiples have been constructed using a wide variety of methods.

The notion of Zariski multiples itself has diversified, and many variants have been introduced; for example, arithmetic Zariski multiples, π_1-Zariski multiples, Alexander Zariski multiples, ...

In short, the construction of Zariski multiples has served as a good testing ground for various techniques in the study of embedding topology of plane curves.

Our result

In this talk, I will explain a construction of a Zariski N-tuple of plane curves of degree $127 = 1 + 6 + 2 \times 60$, where

$$N > 2.77 \times 10^{26}.$$

This lower bound, 2.77×10^{26}, is larger than Avogadro's number.
It is approximately equal to the number of $\mathrm{H_2O}$ molecules contained in about 8 liters of water.

Our method is very elementary,
and is based on the classical theory of del Pezzo surfaces and the Weyl groups $W(E_8)$.

So we start by reviewing the well-known properties of Pezzo surfaces.

Picard lattice of del Pezzo surfaces of degree $1, 2, 3$

Definition

A smooth projective surface X is called a *del Pezzo surface* of degree d if its anti-canonical class $\alpha_X := [-K_X]$ is ample and of self-intersection number $\langle \alpha_X, \alpha_X \rangle = d$.

We consider only the cases where $d = 1, 2, 3$, and put

$$n := 9 - d.$$

The Picard lattice $\mathrm{Pic}(X)$ of a del Pezzo surface X of degree d is of rank $n + 1$, and is canonically isomorphic to $H^2(X, \mathbb{Z})$. The surface X is obtained by a blowing-up

$$\beta \colon X \to \mathbf{P}^2$$

of $\mathbf{P}^2$ at distinct n points, and $\mathrm{Pic}(X)$ has a basis $h, e_1, \dots, e_n$, where h is the class of the pullback of a line, and $e_1, \dots, e_n$ are the classes of exceptional curves.

With respect to this basis, the Gram matrix of $\mathrm{Pic}(X)$ is

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix},$$

and the anti-canonical class $\alpha_X \in \mathrm{Pic}(X) = H^2(X, \mathbb{Z})$ is written as

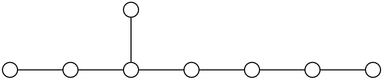
$$\alpha_X = (3, -1, \dots, -1) = 3h - e_1 - \cdots - e_n.$$

Then we can easily prove that its orthogonal complement

$$R(X) := (\alpha_X)^\perp$$

in $\mathrm{Pic}(X)$ is a *negative-definite* root lattice of Dynkin type type E_n, namely, $R(X)$ has a basis $r_1, \dots, r_n$ consisting of (-2)-vectors whose dual graph is the ordinary Dynkin diagram of type E_n.

Below is the Dynkin diagram of type E_8:



Let $W(R(X)) \cong W(E_n)$ be the Weyl group of the lattice $R(X)$, that is, the subgroup of the orthogonal group $\mathrm{O}(R(X))$ of $R(X)$ generated by reflections with respect to (-2)-vectors.

The group $W(E_n)$ has a generating set of n reflections with the defining relations given by the Dynkin diagram of type E_n.

d	n	$W(E_n)$	order
3	6	$W(E_6) \cong \mathrm{U}_4(2).2$	51840
2	7	$W(E_7) \cong 2 \times \mathrm{O}_7(2)$	2903040
1	8	$W(E_8) \cong 2.\mathrm{O}_8^+(2).2$	696729600

We have

$$\mathrm{O}(R(X)) = \begin{cases} W(E_6).2 & \text{if } d = 3, \\ W(E_7) & \text{if } d = 2, \\ W(E_8) & \text{if } d = 1. \end{cases}$$

We put

$$\mathrm{O}(\mathrm{Pic}(X), \alpha_X) := \{ g \in \mathrm{O}(\mathrm{Pic}(X)) \mid \alpha_X^g = \alpha_X \}.$$

The following is a purely lattice-theoretic result,
and is proved by the theory of discriminant forms of even lattices.

Proposition

The image of the natural injective homomorphism

$$\mathrm{O}(\mathrm{Pic}(X), \alpha_X) \hookrightarrow \mathrm{O}(R(X))$$

is equal to $W(R(X)) \subset \mathrm{O}(R(X))$.

Monodromy

Let $f\colon \mathcal{X} \to \mathcal{U}$ be a family of del Pezzo surfaces of degree d over a smooth and irreducible parameter space $\mathcal{U}$.
For a point u of $\mathcal{U}$, we put $X_u := f^{-1}(u)$ and $\alpha_u := \alpha_{X_u}$.
We choose a base point $b \in \mathcal{U}$. The local system

$$R^2 f_* \mathbb{Z} \ \to \ \mathcal{U}$$

is a family of the lattices $\mathrm{Pic}(X_u) \cong H^2(X_u, \mathbb{Z})$ with a section $u \mapsto \alpha_u$.
Hence we obtain a monodromy homomorphism

$$\Phi \ : \ \pi_1(\mathcal{U}, b) \ \to \ \mathrm{O}(\mathrm{Pic}(X_b), \alpha_b) \cong W(R(X_b)).$$

We investigate the surjectivity of the monodromy homomorphism Φ.

Remark

We have the classical theory of the monodromy on vasnishing cycles of *ADE*-singularities due to Brieskorn, and the theory of Mordell-Weil lattices of type E_n with large Galois groups due to Shioda.
We pursue an alternative approach.

Lines on a del Pezzo surface

A smooth rational curve ℓ on X is called a *line* if $\langle \ell, \alpha_X \rangle = 1$.
The set of lines in X is identified with

$$L(X) := \{ \lambda \in \mathrm{Pic}(X) \mid \langle \lambda, \lambda \rangle = -1,\ \langle \lambda, \alpha_X \rangle = 1 \},$$

and hence we can enumerate all elements of $L(X)$ explicitly.
The numbers of lines are

$$|L(X)| = \begin{cases} 27 & \text{for } d = 3, \\ 56 & \text{for } d = 2, \\ 240 & \text{for } d = 1. \end{cases}$$

Let $L^{[n]}(X)$ denote the set of all ordered n-tuples

$$\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_n]$$

of lines such that $\langle \lambda_i, \lambda_j \rangle = 0$ for any i, j with $i \neq j$.

We have the following purely lattice-theoretic result, which can be verified by a brute-force computation for $n = 6, 7, 8$.

Proposition

The natural action of $\mathrm{O}(\mathrm{Pic}(X), \alpha_X) \cong W(R(X))$ on $L^{[n]}(X)$ is free and transitive.

For $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_n] \in L^{[n]}(X)$, we have a birational morphism

$$\beta_{\boldsymbol{\lambda}} : X \to \boldsymbol{P}(X/\boldsymbol{\lambda})$$

to a projective plane that is the contraction of the lines $\ell_1, \dots, \ell_n$ whose classes are $\lambda_1, \dots, \lambda_n$. The variety of ordered n points on $\boldsymbol{P}^2$ that can be the centers of $\beta_{\boldsymbol{\lambda}}$ for some X and some $\boldsymbol{\lambda} \in L^{[n]}(X)$ via some isomorphism $\boldsymbol{P}(X/\boldsymbol{\lambda}) \cong \boldsymbol{P}^2$ is irreducible, since it is a Zariski open subset of $(\boldsymbol{P}^2)^n$.

Using this fact and the above lattice theoretic proposition, we can prove the surjectivity of $\Phi \colon \pi_1(\mathcal{U}, b) \to W(E_n)$ for many cases.

Family of cubic surfaces

Let X be a del Pezzo surface of degree $d = 3$. Then $|\alpha_X|$ embeds X in $\mathbb{P}^3$ as a smooth cubic surface. Conversely, every smooth cubic surface is the anti-canonical model of a del Pezzo surface of degree 3.

We fix a projective space $\mathbb{P}^3$, and consider the family $\mathcal{X} \to \mathcal{U}$ of smooth cubic surfaces, where $\mathcal{U}$ is the Zariski open subset of $|\mathcal{O}_{\mathbb{P}^3}(3)| \cong \mathbb{P}^{19}$ parameterizing all smooth cubic surfaces.

The following reproduces the result of Harris (1979) on the Galois group of 27 lines in a smooth cubic surface.

Proposition

For the family $\mathcal{X} \to \mathcal{U}$ of smooth cubic surfaces, the monodromy homomorphism Φ is surjective onto the Weyl group $W(R(X_b))$ of type E_6.

Family of quartic double planes

Let X be a del Pezzo surface of degree $d = 2$. Then $|\alpha_X|$ realizes X as a double cover $X \to \mathbb{P}^2$ branching along a smooth quartic curve. Conversely, every double plane branching along a smooth quartic curve is the anti-canonical model of a del Pezzo surface of $d = 2$. The 56 lines in X are mapped to the 28 bitangents of the branch curve by two-to-one map.

Let $\mathcal{U}$ be the Zariski open subset of $|\mathcal{O}_{\mathbb{P}^2}(4)| \cong \mathbb{P}^{14}$ that parameterizes all smooth quartic curves, and we consider the family $\mathcal{X} \to \mathcal{U}$ of smooth quartic double planes.

Proposition

For the family $\mathcal{X} \to \mathcal{U}$ of smooth quartic double planes, the monodromy Φ is surjective onto the Weyl group $W(R(X_b))$ of type E_7.

A result concerning the Galois group of the 28 bitangents of a smooth quartic curve had been proved in Harris (1979).

Bi-anti-canonical models of del Pezzo surfaces of degree 1

Let X be a del Pezzo surface of degree $d = 1$. Then the linear system $|2\alpha_X|$ of bi-anti-canonical divisors gives rise to a double covering

$$X \to Q \subset \mathbb{P}^3$$

of a singular quadric surface Q of rank 3 (a quadric cone) that branches along $B \cup \{V\}$, where B is a smooth member of $|\mathcal{O}_Q(3)|$ and $V \in Q$ is the vertex. Conversely, for a quadric cone Q with the vertex $V \in Q$ and a smooth member $B \in |\mathcal{O}_Q(3)|$, the double cover $X \to Q$ branching along $B \cup \{V\}$ is the $|2\alpha_X|$-model of a del Pezzo surface X of $d = 1$.

We fix a quadric cone $Q \subset \mathbb{P}^3$. Let $\mathcal{U}$ be the Zariski open subset of $|\mathcal{O}_Q(3)| \cong \mathbb{P}^{15}$ parameterizing all smooth members, and consider the family $\mathcal{X} \to \mathcal{U}$ of $|2\alpha_X|$-models of del Pezzo surfaces with $d = 1$.

Proposition

The monodromy homomorphism Φ for this family $\mathcal{X} \to \mathcal{U}$ is surjective onto the Weyl group $W(R(X_b))$ of type E_8.

From now on, we consider only the case $d = 1$. Note that

$$\mathrm{Pic}(X_b) = \mathbb{Z}\alpha_b \oplus R(X_b).$$

Definition

The deck-transformation of the bi-anti-canonical model $X_b \to Q$ is called the *Bertini involution*.

Let $B_b \in |\mathcal{O}_Q(3)|$ be the branch curve corresponding to b.

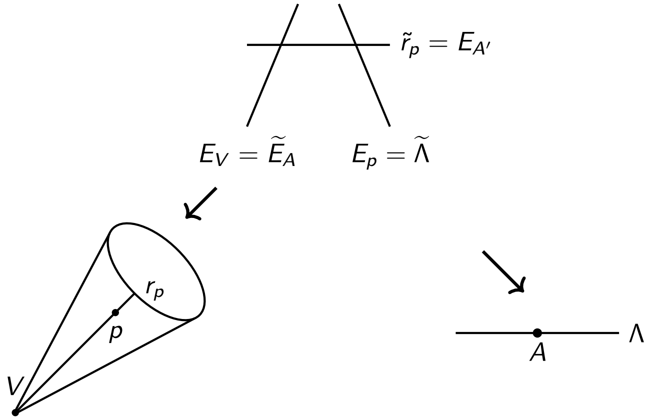
Definition

A plane section $H \cap Q$ of Q, where H is a linear plane in $\mathbb{P}^3$, is called a *tangent plane section for* B_b if $V \notin H$ and the local intersection number at each intersection point of H and B_b is 2.

Proposition

There are exactly 120 *tangent plane sections for* B_b*, and the double cover* $X_b \to Q$ *maps the* 240 *lines in* X_b *to the tangent plane sections of* B_b *by two-to-one mapping.*

The quadric cone $Q \subset \mathbb{P}^3$ is ruled by lines passing through the vertex V. We choose a smooth point $p \in Q \setminus \{V\}$. Then the projection from p induces a birational map $\pi_p \colon Q \dashrightarrow \mathbb{P}^2$. We describe π_p in detail.



This birational map π_p defines a line $\Lambda \subset \mathbb{P}^2$ and a point $A \in \Lambda$. We call this pair (A, Λ) a *frame*.

The line $r_p \subset Q$ is contracted to the point A. Every member of the ruling of Q other than r_p is mapped to a line passing through A.

A plane section $H \cap Q$ not containing V and p is mapped to a smooth conic that is passing through A and is tangent to Λ at A.

The branch curve $B_b \subset Q$ is mapped to a sextic curve on $\mathbb{P}^2$ that has a singular point at A. We investigate the image of B_b and its 120 tangent plane sections more closely.

A combinatorial type σ_k of plane curves

Definition

A germ $(C, \mathbf{0})$ of isolated plane curve singularity is a *t_m-singularity* if C consists of m smooth local branches and each pair of the local branches has intersection number 2.

Definition

Let $C \subset \mathbb{P}^2$ be a plane curve with a t_m-singularity at $A \in C$. The common tangent line $\Lambda \subset \mathbb{P}^2$ to the local branches of C at A is called the *tangent line to C at A*.

Definition

A plane curve C of degree 6 is called a *t_3-sextic* if $\mathrm{Sing}(C)$ consists of a single point A, and $A \in C$ is a t_3-singular point.

A t_3-sextic with the singular point A and the tangent line Λ at A is called a t_3-sextic *in the frame* (A, Λ).

Lemma

For a fixed (A, Λ), the variety $\mathcal{T}$ parameterizing all t_3-sextics in the frame (A, Λ) is a Zariski open subset of $\mathbb{P}^{15}$.

For a point $t \in \mathcal{T}$, let $C_t \subset \mathbb{P}^2$ denote the corresponding t_3-sextic in the frame (A, Λ).

Now we consider the frame (A, Λ) determined by the birational projection $\pi_p \colon Q \dashrightarrow \mathbb{P}^2$. Recall that the parameter space $\mathcal{U} \subset |\mathcal{O}_Q(3)|$ of all branch curves B_u of $X \to Q$ is of dimension 15.

Proposition

For a general $u \in \mathcal{U}$, the image of $B_u \subset Q$ by $\pi_p \colon Q \dashrightarrow \mathbb{P}^2$ is a t_3-sextic in the frame (A, Λ).
For a general $t \in \mathcal{T}$, the image of C_t by the inverse of the birational map $\pi_p \colon Q \dashrightarrow \mathbb{P}^2$ is B_u for some $u \in \mathcal{U}$.

Thus, π_p induces a birational map between the 15-dimensional varieties $\mathcal{U}$ and $\mathcal{T}$.

Let $u \in \mathcal{U}$ be a general point, and let C_t be the t_3-sextic in the frame (A, Λ) corresponding to B_u.

Definition

A smooth conic Γ is said to be a *special tangent conic* of C_t if the following hold;

- the conic Γ passes through A, and $C_t + \Gamma$ has t_4-singularity at A, and
- at every intersection point of C_t and Γ other than A, the intersection multiplicity is 2.

Proposition

The t_3-sextic C_t has exactly 120 *special tangent conics, and they are the images of* 120 *special tangent plane sections of B_u.*

For a special tangent conic Γ, we put

$$\mathrm{Tac}\,(\Gamma) := \mathrm{Sing}(C_t + \Gamma) \setminus \{A\}.$$

Recall that $u \in \mathcal{U}$ is a general point and hence the corresponding t is a *general* point of $\mathcal{T}$

Proposition

The special tangent conics $\Gamma_1, \dots, \Gamma_{120}$ of C_t are in a general position, in the sense that

- *any two of $\Gamma_1, \dots, \Gamma_{120}$ have local intersection number 2 at A,*
- *each $\mathrm{Tac}\,(\Gamma_i)$ consists of 3 tacnodes of $C_t + \Gamma_i$,*
- *the sets $\mathrm{Tac}\,(\Gamma_1), \dots, \mathrm{Tac}\,(\Gamma_{120})$ are disjoint to each other, and*
- *the singular points of the union $C_t + \Gamma_1 + \cdots + \Gamma_{120}$ other than A and the tacnodes in $\mathrm{Tac}\,(\Gamma_1), \dots, \mathrm{Tac}\,(\Gamma_{120})$ are ordinary nodes.*

For a subset Σ of $\{\Gamma_1, \dots, \Gamma_{120}\}$, we consider the plane curve

$$D(\Sigma) := \Lambda + C_t + \sum_{\Gamma_i \in \Sigma} \Gamma_i.$$

Corollary

If $|\Sigma| = |\Sigma'|$, *then* $D(\Sigma)$ *and* $D(\Sigma)$ *have the same combinatorial type of singularities.*

Definition

The combinatorial type of singularities of $D(\Sigma)$ is denoted by σ_k, where $k = |\Sigma|$.

The curve of combinatorial type σ_k is of degree $7 + 2k$, and its singularities consist of one t_{4+k}-singular point, $3k$ tacnodes, and $k(k-1)$ ordinary nodes.

For a fixed C_t, there are

$$\binom{120}{k}$$

curves $D(\Sigma)$ with combinatorial type of singularities σ_k. We classify these curves by their embedding topology into $\mathbb{P}^2$.

Embedding topology into $\mathbb{P}^2$

We have bijections

$$
\begin{aligned}
& \{\text{special tangent conics of } C_t\} \\
\cong\ & \{\text{special tangent plane sections of } B_u\} \\
\cong\ & \{\text{lines on } X_u\}/\langle\text{Bertini involution}\rangle \\
\cong\ & \{(-2)\text{-vectors in } R(X_u)\}/\langle\pm 1\rangle,
\end{aligned}
$$

where the first $\cong$ comes from the projection $\pi_p\colon Q \dashrightarrow \mathbb{P}^2$, the second $\cong$ comes from the double covering $X_u \to Q$, and the third $\cong$ comes from $\operatorname{Pic}(X_u) = \mathbb{Z}\alpha_X \oplus R(X_u)$.

We put

$$\overline{\Delta}(X_u) := \{(-2)\text{-vectors in } R(X_u)\}/\langle\pm 1\rangle.$$

Then, by these bijections, the set of all choices Σ of k special tangent conics of C_t is identified with

$$\binom{\overline{\Delta}(X_u)}{k}.$$

Main theorem

Recall that the monodromy action of $\pi_1(\mathcal{U}, u)$ on $R(X_u)$ factors though a surjection $\pi_1(\mathcal{U}, u) \to W(R(X_u))$, and hence its action on $\overline{\Delta}(X_u)$ factors though a surjection

$$\pi_1(\mathcal{U}, u) \to \overline{W}(R(X_u)) = W(R(X_u))/\langle \pm 1 \rangle.$$

Our main theorem is as follows:

Theorem

Let Σ and Σ' be two choices of k special tangent conics of C_t corresponding to

$$s, s' \in \binom{\overline{\Delta}(X_u)}{k}.$$

Then $D(\Sigma)$ and $D(\Sigma')$ have the same embedding topology into $\mathbb{P}^2$ if and only if s and s' are in the same orbit under the action of $\overline{W}(R(X_u))$.

A rough idea of the proof

The homemorphism type of $(\mathbb{P}^2, D(\Sigma))$ determines the lattice theoretic data of the corresponding point s up to the action of the Aut of lattice structure (that is, $W(E_8)$). Essentially, the lattice theoretic data is the homology of the double plane of $\mathbb{P}^2$ branched along C_t and the classes of pull-backes of its sepcial tangent conics.

Conversely, if s and s' have the same lattice theoretic data, then they are connected by a geometric monodromy, because of the surjectivity of

$$\pi_1(\mathcal{U}) \to W(E_8).$$

It is obvious that if s and s' are connected by geometric monodromy, then $(\mathbb{P}^2, D(\Sigma))$ and $(\mathbb{P}^2, D(\Sigma'))$ are homeomorphic, because $D(\Sigma)$ can be deformed continuously to $D(\Sigma')$ in $\mathbb{P}^2$. □

Therefore we can classify the curves $D(\Sigma)$ of type σ_k by their embedding topology by means of purely lattice-theoretic computation, namely, by calculating the orbit decomposition of

$$\binom{\{\text{the roots of the } E_8\text{-lattice}\}/\langle \pm 1 \rangle}{k}$$

under the action of $W(E_8)/\langle \pm 1 \rangle$, which is of order $348,364,800$.

Theorem

For each integer k satisfying $1 < k < 119$, there is a Zariski $N(k)$-tuple consisting of plane curves of degree $7 + 2k$, where

$$N(k) \geq \frac{1}{348364800} \binom{120}{k}.$$

The values of $N(k) = N(120 - k)$ for small k are as follows:

k	1	2	3	4	5	6	7	8	9
$N(k)$	1	2	5	15	48	212	1116	7388	56946 .

We have

$$N(60) > 2.77 \times 10^{26}.$$

A preprint is available from arXiv:2507.15210

Thank you very much for listening!