On the asymptotic behavior of the prediction error of a stationary process

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Abstract. We give an example of a long-memory stationary process for which we can calculate explicitly the prediction error from a finite part of the past. The long-time behavior of the prediction error is discussed.

1. Introduction and results

Let $X = (X(t), t \in \mathbb{R})$ be a real, centered, weakly stationary process defined on a probability space $(\Omega, \mathcal{F}, P)$. For $T \geq 0$, we denote by $P_{[-T,0]}$ the orthogonal projection operator of $L^2(\Omega, \mathcal{F}, P)$ onto the subspace spanned by $\{X(u) : -T \leq u \leq 0\}$. Similarly, we write $P_{(-\infty,0]}$ for the orthogonal projection operator onto the subspace spanned by $\{X(u) : -\infty < u \leq 0\}$. For $T > 0$ and $t > 0$, we define $Q(T,t)$ and $Q(\infty,t)$ by

$$Q(T,t) := E[\{X(t) - P_{[-2T,0]}X(t)\}^2],$$
$$Q(\infty,t) := E[\{X(t) - P_{(-\infty,0]}X(t)\}^2].$$

We adopted $2T$ rather than T in the above to follow the notation of Dym and McKean [3].

We are concerned with the asymptotic behavior of $Q(T,t) - Q(\infty,t)$ as $T \to \infty$. We are especially interested in the case where the stationary process X is a *long-memory process* (see Beran [1]). The main difficulty of this problem comes from that of the calculation of $Q(T,t)$ itself. In this paper, we give an example of a long-memory process for which we can calculate $Q(T,t)$ explicitly. The authors know no other example of such a long-memory process.

We write R for the autocovariance function of X: $R(t) = E[X(t)X(0)]$ for $t \in \mathbb{R}$. Let μ be the spectral measure of X: $R(t) = \int_{-\infty}^{\infty} e^{i|t|\gamma}\mu(d\gamma)$. For $\frac{1}{2} < \alpha < 1$, we define the constants $c = c(\alpha)$ and $d = d(\alpha)$ by

$$c := \left\{\pi\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right\}^{\frac{1}{2\alpha}}, \qquad d := \frac{2^{3-2\alpha}\sin(\alpha\pi)}{\Gamma(\alpha-\frac{1}{2})^2}. \tag{1.1}$$

As usual, we write K_ν for the modified Bessel function (cf. Watson [6, 3.7]).

Theorem 1. *Let $T > 0$, $t > 0$ and $\frac{1}{2} < \alpha < 1$. Set $T_1 := T + t$. Let X be a real, centered, weakly stationary process with spectral measure μ on $\mathbb{R}$ of the form*

$$\mu(d\gamma) = \frac{\sin(\alpha\pi)}{\pi} \cdot \frac{|\gamma|^{1-2\alpha}}{1+\gamma^2} d\gamma. \tag{1.2}$$

Then $Q(T,t) = d\{Q_1(T,t) + Q_2(T,t)\}$ with

$$Q_1(T,t) := \int_T^{T_1} \left\{ \int_s^{T_1} u^{1-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}} K_{-\alpha}(u) du \right\}^2 \frac{T_1^2}{sK_{-\alpha}(s)^2} ds,$$
$$Q_2(T,t) := \int_T^{T_1} \left\{ \int_s^{T_1} u^{2-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}} K_{1-\alpha}(u) du \right\}^2 \frac{ds}{sK_{1-\alpha}(s)^2}.$$

For the stationary process X in the theorem above, we can show that

$$R(t) \sim \frac{1}{\Gamma(2\alpha - 1)} t^{-(2-2\alpha)} \quad (t \to \infty) \tag{1.3}$$

(see §3). Therefore X is a long-memory process.

Theorem 2. *Let α, t and X be as in Theorem 1. Then*

$$Q(T,t) - Q(\infty,t) \sim \sin(\alpha\pi) \left\{ \frac{1}{\Gamma(\alpha - \frac{1}{2})} \int_0^t s^{\alpha-\frac{1}{2}} e^{s-t} ds \right\}^2 \frac{1}{T} \quad (T \to \infty).$$

Let X be as in Theorem 1 and let f be the spectral density of X:

$$f(\gamma) = \frac{\sin(\alpha\pi)}{\pi} \cdot \frac{|\gamma|^{1-2\alpha}}{1+\gamma^2} \qquad (\gamma \in \mathbb{R}). \tag{1.4}$$

We write h for the outer function of X:

$$h(\zeta) := \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1+\gamma\zeta}{\gamma - \zeta} \cdot \frac{\log f(\gamma)}{1+\gamma^2} d\gamma \right\} \qquad (\Im\zeta > 0). \tag{1.5}$$

The *canonical representation kernel* F of X is defined by $F := (2\pi)^{-1/2}\hat{h}$, where $\hat{h}$ is the Fourier transform of $h(\cdot) := \text{l.i.m.}_{\eta\downarrow 0} h(\cdot + i\eta) \in L^2(\mathbb{R})$. We have the following relation between F and h:

$$h(\zeta) = (2\pi)^{-\frac{1}{2}} \int_0^{\infty} e^{i\zeta t} F(t) dt \qquad (\Im\zeta > 0). \tag{1.6}$$

Corollary. *Let $\frac{1}{2} < \alpha < 1$ and $t > 0$. Let X be as in Theorem 1 with autocovariance function R and canonical representation kernel F. Then*

$$Q(T,t) - Q(\infty,t) \sim \left(\int_0^t F(s) ds \right)^2 \times \int_{2T}^{\infty} \left\{ \frac{R(u)}{\int_{-u}^{u} R(s) ds} \right\}^2 du \quad (T \to \infty). \tag{1.7}$$

Recently, the authors obtained similar results for the stationary processes with autocovariance function R of the form

$$R(t) = \int_0^\infty e^{-|t|\lambda}\sigma(d\lambda) \qquad (t \in \mathbb{R}),$$

where σ is a finite Borel measure on $(0,\infty)$. There, we assume that

$$R(t) \sim t^{-p}\ell(t) \qquad (t \to \infty)$$

with $0 < p < \infty$ and ℓ slowly varying. It is perhaps surprising that the asymptotic relation (1.7) still holds as it is even for the case $p \geq 1$ there. The proofs are quite different from that of the present paper. The detail will appear elsewhere. The first author also obtained relevant results for the partial autocorrelation functions of stationary time series ([4]).

2. Proof of Theorem 1

In the proof below, we apply the theory of *strings*, due to M. G. Krein, as described in [3]. The key is to use Rule 6.9.4 in [3, p. 268].

Step 1. Recall $\frac{1}{2} < \alpha < 1$ and c from (1.1). We write m for the function

$$m(x) := \frac{c^2}{\alpha(1-\alpha)} x^{(1-\alpha)/\alpha} \qquad (0 \leq x < \infty).$$

The purpose of this step is to obtain the functions A, B, C, D, K and the measure $d\Delta$ associated with the string specified by m; see [3, Ch. 5] for background.

For $z \in \mathbb{C}$, we consider the following differential equation

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} A(x,z) = -z^2 A(x,z) m'(x) \qquad (0 < x < \infty), \\ A(0+,z) = 1, \quad \dfrac{\partial}{\partial x} A(0+,z) = 0. \end{cases}$$

The solution to the above is given by

$$A(x,z) := \pi \left\{ \frac{\alpha}{\sin(\alpha\pi)} \right\}^{\frac{1}{2}} z^\alpha x^{\frac{1}{2}} J_{-\alpha}(2czx^{\frac{1}{2\alpha}}) \qquad (0 < x < \infty),$$

where J_ν is the Bessel function of the first kind ([6, 3.1]).

We set, as in [3, p. 172],

$$C(x,z) := A(x,z) \int_0^x \{A(y,z)\}^{-2} dy \qquad (0 < x < \infty,\ \Im z \neq 0).$$

Since

$$\frac{\pi}{2\sin(\alpha\pi)} \cdot \frac{d}{dx} \left\{ \frac{J_\alpha(x)}{J_{-\alpha}(x)} \right\} = \frac{1}{x J_{-\alpha}(x)^2}$$

(cf. [6, 5.11(1)]), we have

$$\frac{\alpha\pi}{\sin(\alpha\pi)} \cdot \frac{d}{dx}\left\{\frac{J_\alpha(2czx^{\frac{1}{2\alpha}})}{J_{-\alpha}(2czx^{\frac{1}{2\alpha}})}\right\} = \frac{1}{xJ_{-\alpha}(2czx^{\frac{1}{2\alpha}})^2}.$$

This and the asymptotic representation

$$J_\nu(x) \sim \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \qquad (x \to 0+) \tag{2.1}$$

(see [6, 3.12]) give

$$C(x,z) = \left\{\frac{\alpha}{\sin(\alpha\pi)}\right\}^{\frac{1}{2}} z^{-\alpha} x^{\frac{1}{2}} J_\alpha(2czx^{\frac{1}{2\alpha}}) \qquad (0 < x < \infty,\ \Im z \neq 0).$$

As usual, we write $H_\nu^{(1)}$ for the Bessel function of the third kind ([6, 3.6]). Let the function D be as in [3, §5.4]. By [3, p. 175], [6, 3.6(2)], and the asymptotic representations

$$J_\nu(z) = (\pi z/2)^{-\frac{1}{2}}\left[\cos\left\{z - \tfrac{1}{4}\pi(1+2\nu)\right\} + O(z^{-1})\right] \quad (|z| \to \infty), \tag{2.2}$$

$$H_\nu^{(1)}(z) \sim (\pi z/2)^{-\frac{1}{2}} \exp[i\{z - \tfrac{1}{4}\pi(1+2\nu)\}] \qquad (|z| \to \infty) \tag{2.3}$$

(see [6, 7.2]), we have, for $\Im z \neq 0$,

$$\begin{aligned} D(0,z) &= \lim_{x\to\infty} \frac{C(x,z)}{A(x,z)} = \pi^{-1} z^{-2\alpha} \lim_{x\to\infty} \frac{J_\alpha(2czx^{\frac{1}{2\alpha}})}{J_{-\alpha}(2czx^{\frac{1}{2\alpha}})} \\ &= \pi^{-1} z^{-2\alpha} \lim_{x\to\infty}\left\{e^{i\alpha\pi} - i\sin(\alpha\pi)\frac{H_{-\alpha}^{(1)}(2czx^{\frac{1}{2\alpha}})}{J_{-\alpha}(2czx^{\frac{1}{2\alpha}})}\right\} \\ &= \pi^{-1} z^{-2\alpha} \exp\{i\alpha\pi \cdot \operatorname{sgn}(\Im z)\}. \end{aligned}$$

Therefore, by [3, p. 175] and [6, 3.7(2)], we obtain, for $0 < x < \infty$ and $\Im z \neq 0$,

$$\begin{aligned} D(x,z) &= D(0,z)A(x,z) - C(x,z) \\ &= \left\{\frac{\alpha}{\sin(\alpha\pi)}\right\}^{\frac{1}{2}} z^{-\alpha} x^{\frac{1}{2}} \left[\exp\{i\alpha\pi \cdot \operatorname{sgn}(\Im z)\} J_{-\alpha}(2czx^{\frac{1}{2\alpha}}) - J_\alpha(2czx^{\frac{1}{2\alpha}})\right] \end{aligned}$$

or, by [6, 3.6(2)],

$$D(x,z) = \begin{cases} i\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}} z^{-\alpha} x^{\frac{1}{2}} H_{-\alpha}^{(1)}(2czx^{\frac{1}{2\alpha}}) & (\Im z > 0), \\ -i\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}} z^{-\alpha} x^{\frac{1}{2}} H_{-\alpha}^{(2)}(2czx^{\frac{1}{2\alpha}}) & (\Im z < 0). \end{cases}$$

In particular, by [6, 3.7(8)],

$$D(x,i) = \frac{2}{\pi}\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}} x^{\frac{1}{2}} K_{-\alpha}(2cx^{\frac{1}{2\alpha}}) \qquad (0 < x < \infty). \tag{2.4}$$

Recall μ from (1.2). We write $d\Delta$ for the measure $(1+\gamma^2)d\mu(\gamma)$ on $\mathbb{R}$:

$$d\Delta(\gamma) := \frac{\sin(\alpha\pi)}{\pi}|\gamma|^{1-2\alpha}d\gamma.$$

By simple calculation, we obtain

$$\int_0^\infty \frac{\gamma^{1-2\alpha}}{\gamma^2 - z^2}d\gamma = \frac{\pi}{2\sin(\alpha\pi)}z^{-2\alpha}\exp\{i\alpha\pi\cdot\mathrm{sgn}(\Im z)\} \qquad (\Im z \neq 0).$$

Therefore

$$D(0,z) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{d\Delta(\gamma)}{\gamma^2 - z^2} \qquad (\Im z \neq 0).$$

See [3, §5.5] for the implication of this equality.

As in Rule 6.9.4 in [3, p. 268], we set

$$K(x) := -\frac{D(x,i)}{(\partial D/\partial x)(x,i)} \qquad (x > 0).$$

By (2.4) and [6, 3.71(6)], we have

$$\frac{\partial D}{\partial x}(x,i) = -\frac{2c}{\alpha\pi}\{\alpha\sin(\alpha\pi)\}^{\frac{1}{2}}x^{\frac{1-\alpha}{2\alpha}}K_{1-\alpha}(2cx^{\frac{1}{2\alpha}})$$

and hence

$$K(x) = c^{-1}\alpha x^{1-\frac{1}{2\alpha}}\frac{K_{-\alpha}(2cx^{\frac{1}{2\alpha}})}{K_{1-\alpha}(2cx^{\frac{1}{2\alpha}})} \qquad (0 < x < \infty).$$

Let B as in [3, §5.7]. Then it follows from [6, 3.2] that, for $0 < x < \infty$ and $\gamma \in \mathbb{R}$,

$$B(x,\gamma) = -\frac{1}{\gamma}\frac{\partial}{\partial x}A(x,\gamma) = c\pi\{\alpha\sin(\alpha\pi)\}^{-\frac{1}{2}}\gamma^\alpha x^{\frac{1-\alpha}{2\alpha}}J_{1-\alpha}(2c\gamma x^{\frac{1}{2\alpha}}).$$

Step 2. Recall $T > 0$, $t > 0$ and $T_1 = T + t$. By [3, §6.10] and Rule 6.9.4 in [3, p. 268] applied to the string specified by m, we obtain

$$Q(T,t) = Q_{\mathrm{even}}(T,t) + Q_{\mathrm{odd}}(T,t),$$

where

$$Q_{\mathrm{even}}(T,t) = \pi\int_x^\infty \left[\frac{2}{\pi}\int_0^\infty \cos(\gamma T_1)\left\{A(y,\gamma) - \gamma K(y)B(y,\gamma)\right\}\frac{d\Delta(\gamma)}{1+\gamma^2}\right]^2 \frac{dy}{K(y)^2},$$

$$Q_{\mathrm{odd}}(T,t) = \pi\int_x^\infty \left[\frac{2}{\pi}\int_0^\infty \sin(\gamma T_1)\left\{\gamma A(y,\gamma) + K(y)B(y,\gamma)\right\}\frac{d\Delta(\gamma)}{1+\gamma^2}\right]^2 dm(y)$$

with $T=\int_0^x\{m'(y)\}^{1/2}dy$, or $x=\{T/(2c)\}^{2\alpha}$. By change of variables $s=2cy^{\frac{1}{2\alpha}}$, we have

$$Q_{\text{even}}(T,t)=\frac{2\sin(\alpha\pi)}{\pi}\int_T^\infty\{I_1(s)-I_2(s)\}^2\frac{ds}{sK_{-\alpha}(s)^2},$$
$$Q_{\text{odd}}(T,t)=\frac{2\sin(\alpha\pi)}{\pi}\int_T^\infty\{I_3(s)-I_4(s)\}^2\frac{ds}{sK_{1-\alpha}(s)^2},$$

where

$$I_1(s):=sK_{1-\alpha}(s)\int_0^\infty\cos(\gamma T_1)J_{-\alpha}(s\gamma)\frac{\gamma^{1-\alpha}}{1+\gamma^2}d\gamma,$$
$$I_2(s):=sK_{-\alpha}(s)\int_0^\infty\cos(\gamma T_1)J_{1-\alpha}(s\gamma)\frac{\gamma^{2-\alpha}}{1+\gamma^2}d\gamma,$$
$$I_3(s):=sK_{1-\alpha}(s)\int_0^\infty\sin(\gamma T_1)J_{-\alpha}(s\gamma)\frac{\gamma^{2-\alpha}}{1+\gamma^2}d\gamma,$$
$$I_4(s):=sK_{-\alpha}(s)\int_0^\infty\sin(\gamma T_1)J_{1-\alpha}(s\gamma)\frac{\gamma^{1-\alpha}}{1+\gamma^2}d\gamma.$$

Step 3. Recall the constant d from (1.1). In this step, we show that $Q_{\text{even}}(T,t)=d\cdot Q_1(T,t)$.

We first note that I_1-I_2 is continuous on $(0,\infty)$ by the asymptotic representations (2.1) and (2.2). By [5, p. 68], (13.19), we have

$$\int_0^\infty\frac{x^{\nu+1}}{1+x^2}J_\nu(ax)\cos(xy)d\gamma=\cosh(y)K_\nu(a)\qquad(y<a,\ -1<\nu<\tfrac{3}{2}),$$

so that $I_1(s)-I_2(s)=0$ for $s>T_1$ hence for $s\geq T_1$ by continuity.

The calculation of $I_1(s)-I_2(s)$ for $T<s<T_1$ is more tricky. By

$$\frac{d}{dx}\{x^\alpha J_{-\alpha}(x)\}=-x^\alpha J_{1-\alpha}(x),\qquad\frac{d}{dx}\{x^{1-\alpha}J_{1-\alpha}(x)\}=x^{1-\alpha}J_{-\alpha}(x),$$
$$\frac{d}{dx}\{x^\alpha K_{-\alpha}(x)\}=-x^\alpha K_{1-\alpha}(x),\qquad\frac{d}{dx}\{x^{1-\alpha}K_{1-\alpha}(x)\}=-x^{1-\alpha}K_{-\alpha}(x)$$

(see [6, 3.2 and 3.71]), we have

$$\frac{\partial}{\partial s}\{I_1(s)-I_2(s)\}=-sK_{-\alpha}(s)\int_0^{\infty-}\cos(\gamma T_1)\gamma^{1-\alpha}J_{-\alpha}(s\gamma)d\gamma\qquad(T<s<T_1).$$

In fact, by (2.2) and the second integral mean-value theorem ([7, §4.14]), the improper integral on the right-hand side converges uniformly in s on each compact subset of (T,T_1), whence we may interchange the derivative and the integral. Now the above equality and (13.13) in [5, p. 67] give

$$\frac{\partial}{\partial s}\{I_1(s)-I_2(s)\}=-\frac{2^{1-\alpha}\pi^{\frac{1}{2}}T_1}{\Gamma(\alpha-\frac{1}{2})}s^{1-\alpha}(T_1^2-s^2)^{\alpha-\frac{3}{2}}K_{-\alpha}(s)$$

for $T < s < T_1$, and so we have

$$I_1(s) - I_2(s) = \frac{2^{1-\alpha}\pi^{\frac{1}{2}}T_1}{\Gamma(\alpha - \frac{1}{2})}\int_s^{T_1} u^{1-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}}K_{-\alpha}(u)du \quad (T < s < T_1).$$

Thus $Q_{\mathrm{even}}(T,t) = d \cdot Q_1(T,t)$.

Step 4. We show that $Q_{\mathrm{odd}}(T,t) = d \cdot Q_2(T,t)$. The proof is quite analogous to that for Q_{even} in Step 3. First, $I_3 - I_4$ is continuous on $(0,\infty)$. Next, since

$$\int_0^\infty \frac{x^\nu}{1+x^2}J_\nu(ax)\sin(xy)d\gamma = \sinh(y)K_\nu(a) \qquad (y < a,\ -1 < \nu < \tfrac{5}{2})$$

(see [5, p. 166], (13.20)), $I_3 - I_4$ vanishes on $[T_1,\infty)$. Finally, it follows from [5, p. 164], (13.9) that, for $T < s < T_1$,

$$\begin{aligned}\frac{\partial}{\partial s}\{I_3(s) - I_4(s)\} &= -sK_{1-\alpha}(s)\int_0^{\infty-} \sin(\gamma T_1)\gamma^{1-\alpha}J_{1-\alpha}(s\gamma)d\gamma \\ &= -\frac{2^{1-\alpha}\pi^{\frac{1}{2}}}{\Gamma(\alpha - \frac{1}{2})}s^{2-\alpha}(T_1^2 - s^2)^{\alpha-\frac{3}{2}}K_{1-\alpha}(s),\end{aligned}$$

and so

$$I_3(s) - I_4(s) = \frac{2^{1-\alpha}\pi^{\frac{1}{2}}}{\Gamma(\alpha - \frac{1}{2})}\int_s^{T_1} u^{2-\alpha}(T_1^2 - u^2)^{\alpha-\frac{3}{2}}K_{1-\alpha}(u)du \quad (T < s < T_1).$$

Thus $Q_{\mathrm{odd}}(T,t) = d \cdot Q_2(T,t)$. □

3. Proof of (1.3)

Let α and X be as in Theorem 1, and let R be the autocovariance function of X. We set

$$\ell(x) := \frac{x^2}{1+x^2}, \qquad k(x) := x^{2\alpha-3}\cos(1/x) \qquad (0 < x < \infty).$$

Then, for $t > 0$,

$$R(t) = \frac{2\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{\gamma^{1-2\alpha}\cos(\gamma t)}{1+\gamma^2}d\gamma = \frac{2\sin(\alpha\pi)t^{2\alpha-2}}{\pi}\int_0^\infty k(x)\ell(xt)dx.$$

Choose $\delta > 0$ such that $\delta < \min(2 - 2\alpha, 2\alpha - 1)$. Then the improper integrals

$$\int_{0+}^1 x^{-\delta}k(x)dx, \qquad \int_1^{\infty-} x^\delta k(x)dx$$

exist. Therefore, by the Bojanic–Karamata theorem (cf. Bingham et al. [2, Th. 4.1.5]),

$$\int_0^\infty k(x)\ell(xt)dx \to \int_{0+}^{\infty-} k(x)dx \qquad (t \to \infty).$$

Since

$$\int_{0+}^{\infty-} k(x)dx = \int_0^{\infty-} x^{1-2\alpha}\cos x dx = \frac{\pi}{2\sin(\alpha\pi)\Gamma(2\alpha-1)},$$

(1.3) follows. □

4. Proof of Theorem 2

Step 1. First we consider $Q_1(T,t)$. By change of variables $s' = s - T$, $u' = u - T$, we obtain

$$Q_1(T,t) = 2^{2\alpha-3}\int_0^t ds \frac{(T+t)^2}{\left\{(T+s)^{\frac{1}{2}}K_{-\alpha}(T+s)\right\}^2}$$
$$\times\left[\int_s^t (T+u)^{\frac{1}{2}}K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2.$$

By the asymptotic representation

$$x^{\frac{1}{2}}K_\nu(x) = (\pi/2)^{\frac{1}{2}}e^{-x}\left\{1+\tfrac{1}{8}(4\nu^2-1)x^{-1}+O(x^{-2})\right\} \qquad (x\to\infty) \tag{4.1}$$

(see [6, 7.23]), we have, as $T\to\infty$,

$$(T+u)^{\frac{1}{2}}K_{-\alpha}(T+u) = (\pi/2)^{\frac{1}{2}}e^{-T-u}\left\{1+\tfrac{1}{8}(4\alpha^2-1)T^{-1}+O(T^{-2})\right\}.$$

This, together with

$$(1+x)^p = 1+px+O(x^2) \qquad (x\to 0+),$$

gives

$$\begin{aligned}
&(T+u)^{\frac{1}{2}}K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}\\
&\quad= (\pi/2)^{\frac{1}{2}}e^{-T-u}T^{-1}\\
&\qquad\times\left[1+\left\{\tfrac{1}{8}(4\alpha^2-1)+\left(\tfrac{1}{2}-\alpha\right)u+\tfrac{1}{2}(\alpha-\tfrac{3}{2})(t+u)\right\}T^{-1}+O(T^{-2})\right]\\
&\quad= (\pi/2)^{\frac{1}{2}}e^{-T-u}T^{-1}\\
&\qquad\times\left[1+\tfrac{1}{2}\left\{(\alpha^2-\tfrac{1}{4})-2t+(\alpha+\tfrac{1}{2})(t-u)\right\}T^{-1}+O(T^{-2})\right].
\end{aligned}$$

Hence, as $T\to\infty$,

$$\begin{aligned}
&\left[\int_s^t (T+u)^{\frac{1}{2}}K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha}\{T+\tfrac{1}{2}(t+u)\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2\\
&= \frac{\pi e^{-2(t+T)}}{2T^2}\left[\left(\int_0^{t-s} u^{\alpha-\frac{3}{2}}e^u du\right)^2 + \left(\int_0^{t-s} u^{\alpha-\frac{3}{2}}e^u du\right)\right.\\
&\quad\left.\times\left(\int_0^{t-s}\left\{(\alpha^2-\tfrac{1}{4})-2t+(\alpha+\tfrac{1}{2})u\right\}u^{\alpha-\frac{3}{2}}e^u du\right)T^{-1}+O(T^{-2})\right].
\end{aligned}$$

Similarly, we have

$$\frac{(T+t)^2}{(T+s)K_{-\alpha}(T+s)^2} = \frac{2T^2e^{2(T+s)}}{\pi}\left[1+\left\{2t-(\alpha^2-\tfrac{1}{4})\right\}T^{-1}+O(T^{-2})\right].$$

Combining, we obtain

$$\frac{(T+t)^2}{(T+s)K_{-\alpha}(T+s)^2}$$
$$\times\left[\int_s^t (T+u)^{\frac{1}{2}}K_{-\alpha}(T+u)(T+u)^{\frac{1}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2$$
$$= e^{-2(t-s)}\left(\int_0^{t-s} u^{\alpha-\frac{3}{2}}e^u du\right)^2$$
$$+ (\alpha+\tfrac{1}{2})e^{-2(t-s)}\left(\int_0^{t-s} u^{\alpha-\frac{3}{2}}e^u du\right)\left(\int_0^{t-s} u^{\alpha-\frac{1}{2}}e^u du\right)T^{-1}+O(T^{-2}).$$

Thus

$$Q_1(T,t) = 2^{2\alpha-3}J_0(t) + 2^{2\alpha-3}(\alpha+\tfrac{1}{2})J_1(t)T^{-1} + O(T^{-2}) \qquad (T\to\infty) \tag{4.2}$$

with

$$J_0(t) := \int_0^t e^{-2s}\left(\int_0^s u^{\alpha-\frac{3}{2}}e^u du\right)^2 ds,$$
$$J_1(t) := \int_0^t e^{-2s}\left(\int_0^s u^{\alpha-\frac{3}{2}}e^u du\right)\left(\int_0^s u^{\alpha-\frac{1}{2}}e^u du\right) ds.$$

Step 2. Next we consider $Q_2(T,t)$. By change of variables $s' = s-T$, $u' = u-T$, we obtain

$$Q_2(T,t) = 2^{2\alpha-3}\int_0^t ds \frac{1}{\left\{(T+s)^{\frac{1}{2}}K_{1-\alpha}(T+s)\right\}^2}$$
$$\times\left[\int_s^t (T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u)(T+u)^{\frac{3}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}(t-u)^{\alpha-\frac{3}{2}}du\right]^2.$$

By (4.1), we have, as $T\to\infty$,

$$(T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u) = (\pi/2)^{\frac{1}{2}}e^{-T-u}\left[1+\tfrac{1}{8}\{4(1-\alpha)^2-1\}T^{-1}+O(T^{-2})\right].$$

Therefore,

$$(T+u)^{\frac{1}{2}}K_{1-\alpha}(T+u)(T+u)^{\frac{3}{2}-\alpha}\left\{T+\tfrac{1}{2}(t+u)\right\}^{\alpha-\frac{3}{2}}$$
$$= (\pi/2)^{\frac{1}{2}}e^{-T-u}$$
$$\times\left[1+\tfrac{1}{2}\left\{(1-\alpha)^2-\tfrac{1}{4}+(\alpha-\tfrac{3}{2})(t-u)\right\}T^{-1}+O(T^{-2})\right].$$

Hence, as $T \to \infty$,

$$\left[\int_s^t (T+u)^{\frac12} K_{1-\alpha}(T+u)(T+u)^{\frac32-\alpha}\{T+\tfrac12(t+u)\}^{\alpha-\frac32}(t-u)^{\alpha-\frac32}du\right]^2$$
$$= \frac{\pi e^{-2(t+T)}}{2}\left[\left(\int_0^{t-s} u^{\alpha-\frac32}e^u du\right)^2 + \left(\int_0^{t-s} u^{\alpha-\frac32}e^u du\right)\right.$$
$$\left.\times \left(\int_0^{t-s} \left\{(1-\alpha)^2 - \tfrac14 + (\alpha-\tfrac32)u\right\} u^{\alpha-\frac32}e^u du\right) T^{-1} + O(T^{-2})\right].$$

Similarly, we have

$$\frac{1}{(T+s)K_{1-\alpha}(T+s)^2} = \frac{2e^{2(T+s)}}{\pi}\left[1 - \left\{(1-\alpha)^2 - \tfrac14)\right\} T^{-1} + O(T^{-2})\right].$$

Combining, we obtain

$$\frac{1}{(T+s)K_{1-\alpha}(T+s)^2}$$
$$\times \left[\int_s^t (T+u)^{\frac12} K_{1-\alpha}(T+u)(T+u)^{\frac32-\alpha}\left\{T+\tfrac12(t+u)\right\}^{\alpha-\frac32}(t-u)^{\alpha-\frac32}du\right]^2$$
$$= e^{-2(t-s)}\left(\int_0^{t-s} u^{\alpha-\frac32}e^u du\right)^2$$
$$+ (\alpha-\tfrac32)e^{-2(t-s)}\left(\int_0^{t-s} u^{\alpha-\frac32}e^u du\right)\left(\int_0^{t-s} u^{\alpha-\frac12}e^u du\right) T^{-1} + O(T^{-2}).$$

Thus

$$Q_2(T,t) = 2^{2\alpha-3}J_0(t) + 2^{2\alpha-3}(\alpha-\tfrac32)J_1(t)T^{-1} + O(T^{-2}) \qquad (T\to\infty). \tag{4.3}$$

Step 3. Since $Q(T,t) \downarrow Q(\infty,t)$ as $T \to \infty$, it follows from (4.2) and (4.3) that

$$Q(\infty,t) = \frac{2\sin(\alpha\pi)}{\Gamma(\alpha-\frac12)^2}\int_0^t e^{-2s}\left(\int_0^s u^{\alpha-\frac32}e^u du\right)^2 ds \tag{4.4}$$

and that

$$Q(T,t) - Q(\infty,t) = \frac{(2\alpha-1)\sin(\alpha\pi)}{\Gamma(\alpha-\frac12)^2}J_1(t)T^{-1} + O(T^{-2}). \qquad (T\to\infty).$$

It remains to show that

$$(2\alpha-1)J_1(t) = \left(\int_0^t s^{\alpha-\frac12}e^{s-t}ds\right)^2. \tag{4.5}$$

Integrating by parts, we have

$$(\alpha-\tfrac12)\int_0^s u^{\alpha-\frac32}e^u du = s^{\alpha-\frac12}e^s - \int_0^s u^{\alpha-\frac12}e^u du.$$

Hence the left-hand side of (4.5) is equal to $2\{J_2(t) - J_3(t)\}$, where

$$J_2(t) := \int_0^t s^{\alpha-\frac{1}{2}} e^s \left(\int_0^s u^{\alpha-\frac{1}{2}} e^u du \right) e^{-2s} ds,$$

$$J_3(t) := \int_0^t e^{-2s} \left(\int_0^s u^{\alpha-\frac{1}{2}} e^u du \right)^2 ds.$$

However

$$J_2(t) = \int_0^t \frac{d}{ds} \left[\frac{1}{2} \left(\int_0^s u^{\alpha-\frac{1}{2}} e^u du \right)^2 \right] e^{-2s} ds = \frac{1}{2} \left(\int_0^t s^{\alpha-\frac{1}{2}} e^{s-t} ds \right)^2 + J_3(t),$$

and so (4.5) follows. □

5. Proof of Corollary

¿From (1.3), we see that

$$\frac{R(u)}{\int_{-u}^u R(s)ds} \sim (\alpha - \tfrac{1}{2}) u^{-1} \qquad (u \to \infty).$$

So

$$\int_{2T}^\infty \left\{ \frac{R(u)}{\int_{-u}^u R(s)ds} \right\}^2 du \sim \frac{1}{2} (\alpha - \tfrac{1}{2})^2 T^{-1} \qquad (T \to \infty). \tag{5.1}$$

Recall f and h from (1.4) and (1.5). By applying Exercises 2.3.4 and 2.7.2 of [3] to the rational functions $1/(1 - i\zeta)$ and $-i\zeta/(1 - i\zeta)^2$, we obtain

$$\frac{1}{1 - i\zeta} = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{1 + \gamma\zeta}{\gamma - \zeta} \cdot \frac{\log(1 + \gamma^2)^{-1}}{1 + \gamma^2} d\gamma \right\} \qquad (\Im\zeta > 0),$$

$$-i\zeta = \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{1 + \gamma\zeta}{\gamma - \zeta} \cdot \frac{\log(\gamma^2)}{1 + \gamma^2} d\gamma \right\} \qquad (\Im\zeta > 0)$$

(note that both $1/(1 - i\zeta)$ and $-i\zeta$ are positive on the upper imaginary axis). Therefore

$$h(\zeta) = \left\{ \frac{\sin(\alpha\pi)}{\pi} \right\}^{\frac{1}{2}} \frac{(-i\zeta)^{\frac{1}{2}-\alpha}}{1 - i\zeta} \qquad (\Im\zeta > 0). \tag{5.2}$$

We set

$$G(t) := \frac{\{2\sin(\alpha\pi)\}^{\frac{1}{2}}}{\Gamma(\alpha - \frac{1}{2})} \int_0^t e^{s-t} s^{\alpha-\frac{3}{2}} ds \qquad (0 < t < \infty).$$

Then, by (5.2) and (1.6), we have

$$(2\pi)^{-\frac{1}{2}} \int_0^\infty e^{i\zeta t} G(t) dt = \left\{ \frac{\sin(\alpha\pi)}{\pi} \right\}^{\frac{1}{2}} \frac{(-i\zeta)^{\frac{1}{2}-\alpha}}{1 - i\zeta}$$

$$= h(\zeta) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{i\zeta t} F(t) dt.$$

Hence $F = G$, and so

$$\int_0^t F(s)ds = \frac{\{2\sin(\alpha\pi)\}^{\frac{1}{2}}}{(\alpha - \frac{1}{2})\Gamma(\alpha - \frac{1}{2})} \int_0^t e^{s-t} s^{\alpha - \frac{1}{2}} ds. \tag{5.3}$$

This, together with Theorem 2 and (5.1), gives the corollary. □

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