A PREDICTION PROBLEM IN $L^2(w)$

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Abstract. For a nonnegative integrable weight function w on the unit circle T, we provide an expression for $p = 2$, in terms of the series coefficients of the outer function of w, for the weighted L^p distance $\inf_f \int_T |1 - f|^p w d\mu$, where μ is the normalized Lebesgue measure and f ranges over trigonometric polynomials with frequencies in $[\{\dots, -3, -2, -1\} \setminus \{-n\}] \cup \{m\}$, $m \geq 0$, $n \geq 2$. The problem is open for $p \neq 2$.

1. Introduction

Many prediction problems of stationary stochastic processes (cf. [2, 7, 10, 14]) are equivalent to finding the distance from the constant function 1 to a subspace $\mathcal{M}(S) = \overline{sp}\{e_k : k \in S\}$ in $L^p(w)$, where S is a subset of the integers $\mathbb{Z}$, $e_k = e^{-ik\lambda}$, w is a nonnegative integrable function on the unit circle T, $0 < p < \infty$, and $L^p(w)$ is the weighted L^p space on T with norm $\|f\|_p = \{\int_T |f|^p w d\mu\}^{1/p}$. Here μ is the Lebesgue measure on T, so normalized that $\mu(T) = 1$. Write

$$\sigma_p(w, S) = \inf_{f \in \mathcal{M}(S)} \|1 - f\|_p$$

for the distance. For example, $\mathcal{M}(S)$ is populated by polynomials $f = a_1 z + a_2 z^2 + \cdots + a_n z^n$, $z = e^{i\lambda}$, and their limits in $L^p(w)$ when the index set S is the halfline S_0, i.e.,

$$S_0 = \{\dots, -3, -2, -1\}.$$

In this case, the well-known Szegö theorem asserts that, for $p > 0$,

$$\sigma_p(w, S_0) = \exp\left\{\frac{1}{p}\int_T \log w d\mu\right\} \tag{1.1}$$

if $\log w \in L^1$, otherwise $\sigma_p(w, S_0) = 0$ (see, e.g., Gamelin [5, p. 156]). The work in Nakazi [10] for the index set $S_1 = S_0 \cup \{1, 2, \dots, n\}, n \geq 1$, has generated considerable interest in computing $\sigma_p(w, S)$ when the index set S is S_0 with finitely many points of $\mathbb{Z}$ added or deleted. To name some related contributions, let us mention here Cheng et al. [2], Frank and Klotz [4], Klotz and Riedel [6], Kolmogorov [7], Miamee and Pourahmadi [9], Pourahmadi [13, 14], and Urbanik [15]. At present, the best known general result is Theorem 2 of Cheng et al. [2] which states that, for such an $S, \sigma_p(w, S)$ is positive if and only if $\log w \in L^1(d\mu)$. However, the problem of computing $\sigma_p(w, S)$ and the function f_0 in $\mathcal{M}(S)$ attaining it has remained largely elusive, even for $p = 2$, except in a few special cases enumerated in Section 2. In this paper we solve the problem for a reasonably general index set S that could shed

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light on some difficulties commonly encountered in this area of research. Section 3 presents the results for $p = 2$ and contains some open problems for the general p. It seems that a successful solution of prediction problems for the $p = 2$ case can be traced to striking the right balance between duality and orthogonalization. Unfortunately, the collapse of this balance does occur often in the $p \neq 2$ case, since the notion of orthogonality is not well developed here.

2. Duality and orthogonalization

Throughout the paper we assume $\log w \in L^1(d\mu)$, so that $w(e^{i\lambda}) = |\phi(e^{i\lambda})|^2$ for some outer function ϕ in the Hardy class H^2. Let b_k's and a_k's be the coefficients in the following series expansions:

$$\phi(z) = \sum_{k=0}^{\infty} b_k z^k, \quad \frac{1}{\phi(z)} = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 1.$$

Note that $|b_0|^2 = \exp\{\int_T \log w d\mu\} = |a_0|^{-2}$ and that

$$b_0 a_0 = 1, \qquad \sum_{k=0}^{l} b_k a_{l-k} = 0, \quad l = 1, 2, 3, \ldots. \tag{2.1}$$

Explicit expressions for the b_k's and a_k's in terms of the Fourier coefficients of $\log w$ can be found in Nakazi and Takahashi [11] and Pourahmadi [12].

For the index set $S_0 - n = \{\ldots, -n-3, -n-2, -n-1\}, n \geq 0$, which corresponds to removing the first n frequencies from S_0, it is known that

$$\sigma_2^2(w, S_0 - n) = \sum_{k=0}^{n} |b_k|^2 \tag{2.2}$$

(see [7, 11, 2]). This is the so-called $(n+1)$-step prediction variance. For the index set $S_1 = S_0 \cup \{1, 2, \cdots, n\}$, which corresponds to adding the next n frequencies to S_0, it is shown in Nakazi [10] that

$$\sigma_2^2(w, S_1) = \left(\sum_{k=0}^{n} |a_k|^2\right)^{-1} \tag{2.3}$$

if $w^{-1} \in L^1(d\mu)$. The rather curious "inverse" relationship between the distances in (2.2) and (2.3), and also the need for the unnatural condition $w^{-1} \in L^1(d\mu)$ were explained by establishing a *duality* between $L^2(w)$ and $L^2(w^{-1})$ as Banach spaces (see [9, 2]) and noting that the complement $S_1^c = \mathbb{Z}_0 \setminus S_1$ of S_1 in $\mathbb{Z}_0$ is equivalent to the halfline $S_0 - n$, where $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. Consequently, a general and more challenging prediction problem based on S_1 in $L^2(w)$ was reduced to an ordinary prediction problem in $L^2(w^{-1})$. More generally, for any index set $S \subset \mathbb{Z}_0$ with finitely many points of $\mathbb{Z}$ added or deleted, let $S^c = \mathbb{Z}_0 \setminus S$ be the complement of S in $\mathbb{Z}_0$, and for a fixed $p \in (1, \infty)$, define q and r by $(1/q) + (1/p) = 1$ and $r = 1/(1-p)$, respectively. Then the same duality argument shows that

$$\sigma_p(w, S) = \sigma_q(w^r, S^c)^{-1} \tag{2.4}$$

if $w^r \in L^1(d\mu)$. Though the latter unnatural restriction can be weakened [2] to $\log w \in L^1(d\mu)$, the quantity $\sigma_q(w^r, S^c)$ might not be well-defined. Fortunately, for the index set S_1, this difficulty was resolved in [2, Theorem 3] using another dual extremal problem in [3] related to the projection of L^p onto the Hardy space H^p. However, for the general S, defining the right hand side of (2.4) remains an open problem. Ideally one would like to apply (2.4) when one problem is simpler than the other, but (2.4) is of no use when the prediction problems corresponding to S and S^c are equally difficult or even identical. In the former situation, a suitable *orthogonalization* coupled with (2.4) seems to provide a good recipe for solving some prediction problems. For example, for $n \geq 2$, the complement of $S_2 = S_0 \setminus \{-n\}$ in $\mathbb{Z}_0$ is equivalent to $S_3 = S_0 \cup \{n\}$, corresponding to deleting and adding a single observation to S_0, respectively. Neither problem is particularly simple but the latter seems simpler. In [2, Theorems 5, 6], an orthogonalization method is used to compute $\sigma_2(w, S_3)$, then the duality relation (2.4) to give $\sigma_2(w, S_2)$, yielding

$$\sigma_2^2(w, S_3) = |b_0|^2 \frac{\sum_{k=0}^{n-1} |b_k|^2}{\sum_{k=0}^{n} |b_k|^2}, \quad \sigma_2^2(w, S_2) = |a_0|^{-2} \frac{\sum_{k=0}^{n} |a_k|^2}{\sum_{k=0}^{n-1} |a_k|^2}. \tag{2.5}$$

In this paper, we compute $\sigma_2(w, S_4)$ for the more general index set $S_4 = S_2 \cup \{m\}$ with $n \geq 2$ and $m \geq 0$, i.e.,

$$S_4 = \{\dots, -n-3, -n-2, -n-1\} \cup \{-n+1, \dots, -1\} \cup \{m\}.$$

This index set has features of both S_2 and S_3. In fact, it reduces to S_2 when $m = 0$, while its complement S_4^c in $\mathbb{Z}_0$ has the same form as S_4, so that the duality relation (2.4) is of no use. Here, too, we show that an orthogonalization technique, the key step of which is to compute the projection $P_{\mathcal{M}}^{e_m}$ of e_m onto the subspace $\mathcal{M} = \mathcal{M}(S_2)$, can be used to solve the problem. To set the notation, let $\hat{e}_k$ stand for the orthogonal projection of e_k onto the subspace $\mathcal{M}_1 = \mathcal{M}(S_0 - n)$. Since $e_k - \hat{e}_k$, $k = -n+1, \dots, -1$, are orthogonal to $\mathcal{M}_1$, the subspaces $\mathcal{M}$ and $\mathcal{M}(S_4)$ can be written as the following orthogonal sums:

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 \oplus \overline{sp}\{e_k - \hat{e}_k : k = -n+1, \dots, -1\}, \\ \mathcal{M}(S_4) &= \mathcal{M} \oplus \overline{sp}\{e_m - P_{\mathcal{M}}^{e_m}\}. \end{aligned} \tag{2.6}$$

Thus, computing $P_{\mathcal{M}}^{e_m}$, its coprojection and norm are the first priority. The following identity which is a generalization of [2, Theorem 6] is of independent interest and curious so far as its relation with $\sigma_2^2(w, S_0 - m)$ and $\sigma_2^2(w, \tilde{S}_1)$, where $\tilde{S}_1 = S_0 \cup \{1, \dots, n-1\}$ (which is S_1 with $n-1$ instead of n), is concerned:

$$\begin{aligned} \|e_m - P_{\mathcal{M}}^{e_m}\|^2 = \sigma_2^2(w, S_2 - m) &= Q^{-1}|c_{m,n}|^2 + \sum_{j=0}^{m} |b_j|^2 \\ &= |c_{m,n}|^2 \sigma_2^2(w, \tilde{S}_1) + \sigma_2^2(w, S_0 - m), \end{aligned} \tag{2.7}$$

where $\|\cdot\| = \|\cdot\|_2$ and

$$Q = \sum_{i=0}^{n-1} |a_i|^2, \quad c_{m,n} = -\sum_{k=0}^{m} b_{m-k} a_{n+k}. \tag{2.8}$$

The constant $c_{m,n}$ is indeed the coefficient of e_{-n} in the formal series expansion of the $(m+1)$-step predictor $P_{\mathcal{M}(S_0)}^{e_m}$ (see [16]). Finally, the desired distance is

$$\sigma_2^2(w, S_4) = \sigma_2^2(w, S_2) - |b_0|^2 \frac{|\bar{b}_m - \bar{\alpha}_m a_n|^2}{\|e_m - P_{\mathcal{M}}^{e_m}\|^2}, \tag{2.9}$$

where

$$\alpha_m = Q^{-1} c_{m,n}. \tag{2.10}$$

In contrast to (2.2), (2.3) and (2.5), where the distances depend either on $\{b_k\}$ or $\{a_k\}$ alone, those in (2.7) and (2.9) do depend on both. Explicit forms of these distances provide useful tools for assessing the impacts of adding (deleting) a vector to decreasing (increasing) such distances. In particular, it follows from (2.7) that removing e_{-n} from S_0 will not increase the distance of e_m from $\mathcal{M}$ if $c_{m,n}$ is zero. Similarly, from (2.9), adding e_m to S_2 will not decrease $\sigma_2^2(w, S_2)$ if $\bar{b}_m = \bar{\alpha}_m a_n$. These phenomena are bound to have interesting prediction-theoretic interpretations and statistical consequences (cf. [16, 14]). It would be useful and instructive to have a few concrete examples of weight functions w or stationary processes displaying these phenomena.

3. The results and proofs for $p = 2$

Throughout this section, for a complex matrix $A = (a_{ij})$, we write $\bar{A}$, A' and A^* for the matrices $(\bar{a}_{ij})$, (a_{ji}) and $(\bar{a}_{ji})$, respectively. Using the outer function $\phi \in H^2$, we define $\xi_k = e^{-ik\lambda}/\phi(e^{i\lambda})$ and note that $\{\xi_k : k \in \mathbb{Z}\}$ is a complete orthonormal basis for $L^2(w)$ such that $\overline{sp}\{e_k : k \le n\} = \overline{sp}\{\xi_k : k \le n\}$, $n \in \mathbb{Z}$, and that $e_n = \sum_{j=0}^{\infty} b_j \xi_{n-j}$, $n \in \mathbb{Z}$. We express various (co)projections in terms of ξ_k's.

Theorem 3.1. *Suppose w is a nonnegative integrable function with* $\log w \in L^1(d\mu)$. *Then we have the following:*

(1) $P_{\mathcal{M}}^{e_m} = \hat{e}_m + \sum_{k=1}^{n-1} \beta_{k,m}(e_{-k} - \hat{e}_{-k})$, *where* $\beta_m = (\beta_{n-1,m}, \dots, \beta_{1,m})'$ *satisfies* (3.3) *below.*
(2) $e_m - P_{\mathcal{M}}^{e_m} = \alpha_m \sum_{i=0}^{n-1} \bar{a}_i \xi_{i-n} + \sum_{j=0}^{m} b_j \xi_{m-j}$, *where α_m is as in* (2.10).
(3) $\|e_m - P_{\mathcal{M}}^{e_m}\|^2 = Q^{-1}|c_{m,n}|^2 + \sum_{j=0}^{m} |b_j|^2$, *where Q and $c_{m,n}$ are as in* (2.8).

For $m = 0$, Theorem 3.1 gives the explicit form of $P_{\mathcal{M}}^{e_0}$, which is needed for projecting e_0 on $\mathcal{M}(S_4)$. In view of (2.6), we also need to project e_0 on the one-dimensional subspace $\overline{sp}\{e_m - P_{\mathcal{M}}^{e_m}\}$ or determine the coefficient

$$\gamma = \frac{(e_0, e_m - P_{\mathcal{M}}^{e_m})}{\|e_m - P_{\mathcal{M}}^{e_m}\|^2}, \tag{3.1}$$

where $(\cdot, \cdot)$ is the inner product of $L^2(w)$, i.e., $(f, g) = \int_T f \bar{g} w d\mu$. The relevant results are summarized in the next theorem.

Theorem 3.2. *Suppose w is a nonnegative integrable function with* $\log w \in L^1(d\mu)$. *Then the following hold:*

(1) $\gamma = b_0(\bar{b}_m - \bar{\alpha}_m a_n)\|e_m - P_{\mathcal{M}}^{e_m}\|^{-2}$.
(2) $P_{\mathcal{M}(S_4)}^{e_0} = \hat{e}_0 + \sum_{k=1}^{n-1} \beta_{k,0}(e_{-k} - \hat{e}_{-k}) + \gamma(e_m - P_{\mathcal{M}}^{e_m})$, *where $\beta_{k,0}$ is as in* (3.3) *but with $m = 0$.*
(3) $e_0 - P_{\mathcal{M}(S_4)}^{e_0} = (\alpha_0 - \gamma\alpha_m)\sum_{i=0}^{n-1} \bar{a}_i \xi_{i-n} + (b_0 - \gamma b_m)\xi_0 - \gamma \sum_{j=0}^{m-1} b_j \xi_{m-j}$.
(4) $\|e_0 - P_{\mathcal{M}(S_4)}^{e_0}\|^2$ *is as in* (2.9).

Let $e = (e_{-(n-1)}, \dots, e_{-1})'$ and $\hat{e} = (\hat{e}_{-(n-1)}, \dots, \hat{e}_{-1})'$. For computing the projection of e_m onto the $(n-1)$-dimensional span of the entries of $e - \hat{e}$, the $(n-1) \times (n-1)$ matrix $A = (a_{ij})$ and $(n-1)$-vector $c = (c_1, \dots, c_{n-1})'$ with the following components are needed:

$$\begin{aligned} a_{ij} &= \left(e_{-(n-i)} - \hat{e}_{-(n-i)}, e_{-(n-j)} - \hat{e}_{-(n-j)}\right), \quad i, j = 1, 2, \dots, n-1, \\ c_i &= \left(e_{-(n-i)} - \hat{e}_{-(n-i)}, e_m\right), \quad i = 1, 2, \dots, n-1. \end{aligned}$$

We define the $(n-1)$-vector b by $b = (b_1, b_2, \ldots, b_{n-1})'$ and the $(n-1) \times (n-1)$ lower triangle matrix T by

$$T = \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-2} & b_{n-3} & \cdots & b_0 \end{bmatrix}.$$

Then, since $e_{-k} - \hat{e}_{-k} = \sum_{j=0}^{n-k} b_j \xi_{-k-j}$, the following representation of $e - \hat{e}$ is immediate:

$$e - \hat{e} = \xi_{-n} b + T\xi, \tag{3.2}$$

where $\xi = (\xi_{-(n-1)}, \ldots, \xi_{-1})'$. From this, we obtain

$$A = TT^* + bb^*, \quad c = \bar{b}_{m+n} b + T\bar{b}_{r,m},$$

where $b_{r,m} = (b_{m+n-1}, \ldots, b_{m+1})'$ is a reversed and shifted version of the vector b above. With these notations, the normal equation for β_m in Theorem 3.1 (1) is

$$A\bar{\beta}_m = c. \tag{3.3}$$

Further, we define $a = (a_1, \ldots, a_{n-1})'$. Then, by (2.1), (2.8) and (2.10),

$$c_{m,n} = b_{m+n} a_0 + b'_{r,m} a, \qquad \alpha_m = Q^{-1}(b_{m+n} a_0 + b'_{r,m} a).$$

Also, $Q = |b_0|^{-2}(1 + |b_0|^2 a^* a)$, in view of $a^* a = \sum_{i=1}^{n-1} |a_i|^2$.

Since the matrix A is a rank-one perturbation of $G = TT^*$, it can be inverted easily using the inverse of G and the relationship between a_k's and b_k's described in (2.1). The inverse of A and other relevant results are summarized in the next lemma.

Lemma 3.3. (1) *We have* $b = -b_0 T a$ *and* $b^* G^{-1} b = |b_0|^2 a^* a$.
(2) $A^{-1} = G^{-1} - (1 + |b_0|^2 a^* a)^{-1} G^{-1} bb^* G^{-1} = (T^{-1})^* [I - Q^{-1} aa^*] T^{-1}$.
(3) $\bar{\beta}_m = A^{-1} c = (T^{-1})^* [I - Q^{-1} aa^*](\bar{b}_{r,m} - b_0 \bar{b}_{m+n} a)$.
(4) $\beta'_m b = Q^{-1}(b_{m+n} a^* a - \bar{a}_0 b'_{r,m} a)$.
(5) $b_{m+n} - \beta'_m b = Q^{-1}(b_{m+n} a_0 + b'_{r,m} a)\bar{a}_0 = \alpha_m \bar{a}_0$.
(6) $b'_{r,m} - \beta'_m T = Q^{-1}(b_{m+n} a_0 + b'_{r,m} a) a^* = \alpha_m a^*$.

The proofs of the assertions in Lemma 3.3 are straightforward; so we omit them.

Proof of Theorem 3.1. The derivation of (3.3) above already proves (1). Using the representation in (3.2) and the definition of $e_m - \hat{e}_m$, we have

$$\begin{aligned} e_m - P_{\mathcal{M}}^{e_m} &= e_m - \hat{e}_m - \beta'_m (e - \hat{e}) = \sum_{k=0}^{m+n} b_k \xi_{m-k} - \beta'_m (b\xi_{-n} + T\xi) \\ &= (b_{m+n} - \beta'_m b)\, \xi_{-n} + \left(b'_{r,m} - \beta'_m T\right) \xi + \sum_{k=0}^{m} b_k \xi_{m-k}. \end{aligned}$$

The assertion (2) follows from this and Lemma 3.3 (5), (6). Finally, we obtain (3) from (2). □

Proof of Theorem 3.2. Using Theorem 3.1 (2) and the latter identity in (2.1), we get

$$(e_0, e_m - P_{\mathcal{M}}^{e_m}) = b_0 \bar{b}_m + \bar{\alpha}_m \sum_{k=1}^{n} b_k a_{n-k} = b_0(\bar{b}_m - \bar{\alpha}_m a_n),$$

whence (1). By (2.6) and (3.1), $P^{e_0}_{\mathcal{M}(S_4)} = P^{e_0}_{\mathcal{M}} + \gamma(e_m - P^{e_m}_{\mathcal{M}})$. So (2) follows from Theorem 3.1 (1), and (3) is obtained by applying Theorem 3.1 (2) to

$$e_0 - P^{e_0}_{\mathcal{M}(S_4)} = (e_0 - P^{e_0}_{\mathcal{M}}) - \gamma(e_m - P^{e_m}_{\mathcal{M}}).$$

This identity is also needed for the proof of (4). Since $P^{e_0}_{\mathcal{M}} \perp e_m - P^{e_m}_{\mathcal{M}}$,

$$(e_0, e_m - P^{e_m}_{\mathcal{M}}) = (e_0 - P^{e_0}_{\mathcal{M}}, e_m - P^{e_m}_{\mathcal{M}}),$$

which, in view of (3.1), gives

$$\gamma(e_m - P^{e_m}_{\mathcal{M}}, e_0 - P^{e_0}_{\mathcal{M}}) = \bar{\gamma}(e_0 - P^{e_0}_{\mathcal{M}}, e_m - P^{e_m}_{\mathcal{M}}) = |\gamma|^2 \|e_m - P^{e_m}_{\mathcal{M}}\|^2.$$

Thus,

$$\begin{aligned}\|e_0 - P^{e_0}_{\mathcal{M}(S_4)}\|^2 &= \|(e_0 - P^{e_0}_{\mathcal{M}}) - \gamma(e_m - P^{e_m}_{\mathcal{M}})\|^2 \\ &= \|e_0 - P^{e_0}_{\mathcal{M}}\|^2 - |\gamma|^2 \|e_m - P^{e_m}_{\mathcal{M}}\|^2.\end{aligned}$$

Now, $\|e_0 - P^{e_0}_{\mathcal{M}}\|^2 = \sigma_2^2(w, S_2)$ because $\mathcal{M} = \mathcal{M}(S_2)$. On the other hand, from (1), we have

$$|\gamma|^2 \|e_m - P^{e_m}_{\mathcal{M}}\|^2 = |b_0|^2 |\bar{b}_m - \bar{\alpha}_m a_n|^2 \|e_m - P^{e_m}_{\mathcal{M}}\|^{-2}.$$

Therefore, we obtain (4) establishing the desired distance formula (2.9). □

Of course, it is of great interest to compute $\sigma_p(w, S_i)$, $i = 0, 1, 2, 3, 4$, for $p \neq 2$. For $i = 0$, the $(n+1)$-step prediction problem has been solved [1, 10] under the additional assumption that $P_n(z) = \sum_{k=0}^n c_k z^k \neq 0$, for all $|z| < 1$, where c_k's are defined by

$$\phi^{p/2}(z) = \left(\sum\nolimits_{k=0}^{\infty} b_k z^k\right)^{p/2} = \sum_{k=0}^{\infty} c_k z^k.$$

Using this result and the duality relation (2.4), $\sigma_p(w, S_1)$ is found in [2]. It seems quite likely that the one-dimensional orthogonalization technique used in [2, Theorem 5] can be extended to the $L^p(w)$ setting, and then using the duality relation (2.4), one can also compute $\sigma_p(w, S_2)$. Along this line the extension to S_4 may require assumptions on the location of zeros of $P_n(z)$ for several n, which raises the question of existence of nontrivial weight functions w satisfying such conditions.

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