AR AND MA REPRESENTATION OF PARTIAL AUTOCORRELATION FUNCTIONS, WITH APPLICATIONS

AKIHIKO INOUE

Abstract. We prove a representation of the partial autocorrelation function (PACF), or the Verblunsky coefficients, of a stationary process in terms of the AR and MA coefficients. We apply it to show the asymptotic behaviour of the PACF. We also propose a new definition of short and long memory in terms of the PACF.

1. Introduction

Let $\{X_n : n \in \mathbf{Z}\}$ be a real, zero-mean, weakly stationary process, defined on a probability space $(\Omega, \mathcal{F}, P)$ with spectral measure not of finite support, which we shall simply call a *stationary process*. We write $\{\gamma_n : n \in \mathbf{Z}\}$ for the autocovariance function of $\{X_n\}$: $\gamma_n := E[X_n X_0]$ for $n \in \mathbf{Z}$. For $\{X_n\}$, we have another sequence $\{\alpha_n\}_{n=0}^{\infty}$ called the *partial autocorrelation function* (PACF), where $\alpha_0 := \gamma_0$, $\alpha_1 := \gamma_1/\gamma_0$, and for $n \geq 2$, α_n is the correlation coefficient of the two residuals obtained from X_0 and X_n by regressing on the intermediate values $X_1, \dots X_{n-1}$ (see §2 below).

The autocovariance function $\{\gamma_n\}$ is positive definite, and the inequalities that this positive definiteness imposes may be inconvenient in some contexts. By contrast, the PACF $\{\alpha_n\}_{n=0}^{\infty}$ gives an *unrestricted parametrization*, in that the only inequalities restricting the α_n are the obvious ones implied by their being correlation coefficients, i.e., $\alpha_n \in [-1, 1]$, or $(-1, 1)$ in the non-degenerate case relevant here. This result is due to Barndorff–Nielsen and Schou [BS], Ramsey [Ra] in the time-series context. See also Dégerine [De], and for extensions to the non-stationary case, Dégerine and Lambert–Lacroix [DL]. However, in the context of mathematical analysis — specifically, the theory of orthogonal polynomials on the unit circle (OPUC) — the result dates back to 1935-6 to work of Verblunsky [V1, V2], where the PACF appears as the sequence of *Verblunsky coefficients*. For a survey of OPUC, see Simon [Si2], and for a textbook treatment, [Si3] (analytic theory), [Si4] (spectral theory). One of our main purposes here is to emphasize the importance of the PACF: "This knows everything", a remark we owe to Yukio Kasahara.

The question thus arises of a 'dictionary', allowing one to pass between statements on the covariance $\{\gamma_n : n \in \mathbf{Z}\}$, or the spectral measure μ defined by $\gamma_n = \int_{-\pi}^{\pi} e^{ik\theta}\mu(d\theta)$, and the PACF $\{\alpha_n\}_{n=0}^{\infty}$; see the last paragraph of [Si3], page 3, where this is stated as perhaps "the most central" question in OPUC. A prototype of such a result is *Baxter's theorem* (see [Ba1, Ba2]; see also Theorem 4.1 below

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and [Si3, Chapter 5]). Such a link exists, in the shape of the Levinson (or Szegö–Levinson–Durbin) algorithm, due to Szegö [Sz] in 1939 (Szegö recursion), Levinson [L] in 1947, Durbin [Du] in 1960; for a textbook account, see Pourahmadi [P], §7.2. For the history of Szegö recursion (Theorem 1.5.2 of [Si3]), see [Si3], page 69. But while very useful numerically, the Levinson algorithm is less suitable for theoretical studies, such as one of the questions that motivates us here — the behaviour of the PACF for large lags (that is, large n).

In this paper, to fill the gap above at least partially, we introduce a representation of the PACF which is given only in terms of another sequence $\{\beta_n\}_{n=0}^{\infty}$ defined by

$$(*) \qquad \beta_n, \text{ or } \beta(n), := \sum_{v=0}^{\infty} c_v a_{v+n} \qquad (n = 0, 1, \dots)$$

(see Inoue and Kasahara [IK1, page 8], [IK2, (2.23)]), where the two sequences $\{c_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ have statistical interpretations as the coefficients of the MA(∞) and AR(∞) representations of the process, respectively (see (MA) and (AR) below). They also have analytic interpretations as the coefficients in the Maclaurin expansions of the *Szegö function* $D(z)$ occurring in the theory of OPUC and its associate $-1/D(z)$, both of which are *outer functions* in Beurling's sense (see §2 below). For background, see [Si3], or — from a statistical point of view — Grenander and Szegö [GS], Rozanov [Ro], Ibragimov and Rozanov [IR].

We are particularly interested in the asymptotics of α_n as $n \to \infty$, and the representation of α_n is useful in investigating them since the representation enables us to study α_n directly via β_n or a_n and c_n. In a number of the specific examples of processes with *long memory* we treat, here and in [I2], [I3], [IK1], we observe behaviour of the form

$$(d/n) \qquad \alpha_n \sim d/n \qquad (n \to \infty)$$

(by the representation of α_n, we are able to improve the estimate of this type in [I2, I3, IK1], where only $|\alpha_n|$ was considered — we were unable to determine its sign). In (d/n), and throughout the paper, $a_n \sim b_n$ as $n \to \infty$ means $\lim_{n\to\infty} a_n/b_n = 1$. On the one hand, (d/n) seems very special behaviour, if we begin by specifying our model via the PACF, since by Verblunsky's theorem any value in $(-1, 1)$ can be taken by any α_n. On the other hand, it is more usual in practice to specify our model with long memory by other means, such as the AR and MA coefficients, and here such behaviour seems to be typical of the (quite broad) classes of example where one can carry out the computations and obtain an explicit asymptotic expression for α_n, such as the fractional ARIMA (FARIMA) models studied in [I3, IK1] and Theorem 2.5 below. In this connection, we note that in recent work of Simon [Si1], the case (d/n) is described as 'a prototypical example'. See also [Si3], [Si4] where [Si1] is developed further, and the papers by Golinskii and Ibragimov [GI], Damanik and Killip [DK] that inspired it. We note also that a generalization of (d/n), in which 'asymptotic to' is replaced by 'of the same order of magnitude as', is obtained as the conclusion in work of Ibragimov and Solev [IS], under conditions on the spectral density.

We denote by H the real Hilbert space spanned by $\{X_k : k \in \mathbf{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$. The norm of H is given by $\|Y\| := E[Y^2]^{1/2}$. For $n, m \in \mathbf{N}$ with $n \le m$, we denote by $H_{[n,m]}$, $H_{(-\infty,m]}$ and $H_{[m,\infty)}$ the subspaces of H spanned by $\{X_n, \dots, X_m\}$, $\{X_k : k \le m\}$, and $\{X_k : k \ge n\}$, respectively. The proof of the representation of the PACF is based on the approach introduced in [I2] which combines von Neumann's alternating projection theorem (see [P, Theorem 9.20]) and the following *intersection of past and future* property of $\{X_n\}$:

(IPF) $$H_{(-\infty,n]} \cap H_{[1,\infty)} = H_{[1,n]} \qquad (n = 1, 2, \dots).$$

This approach is also useful in continuous-time models; see Inoue and Nakano [IN] and the references therein. A useful sufficient condition for (IPF) is that $\{X_n\}$ is purely non-deterministic (PND) (see §2 below) and has spectral density $\Delta(\cdot)$ such that $\int_{-\pi}^{\pi} d\theta/\Delta(\theta) < \infty$ (see [I2], Theorem 3.1), which itself is a discrete-time version of the Seghier–Dym theorem [S], [Dy]. This theorem itself originates in work of Levinson and McKean [LM]. Naturally, (IPF) is closely related to the property

(CND) $$H_{(-\infty,0]} \cap H_{[1,\infty)} = \{0\}$$

called *complete non-determinism*; see Bloomfield et al. [BJH]. In fact, a stationary process is completely non-deterministic if and only both (PND) and (IPF) are satisfied (see [IK2], Theorem 2.3). In particular, if $\{X_n\}$ is PND and has spectral density $\Delta(\cdot)$ such that $\int_{-\pi}^{\pi} d\theta/\Delta(\theta) < \infty$, then it is CND.

In §2, we state the main results, including the representation of the PACF and its asymptotic behaviour of the type (d/n). Sections 3–5 are devoted to their proofs. In §6, we give a further application of the representation, that is, the asymptotics for the PACF of processes with regularly varying covariance functions. In §7, we close the paper with results for ARMA processes.

2. Main Results

As stated in §1, let H be the real Hilbert space spanned by $\{X_k : k \in \mathbf{Z}\}$ in $L^2(\Omega, \mathcal{F}, P)$, which has inner product $(Y_1, Y_2) := E[Y_1 Y_2]$ and norm $\|Y\| := (Y, Y)^{1/2}$. Also, for an interval $I \subset \mathbf{Z}$, we write H_I for the closed subspace of H spanned by $\{X_k : k \in I\}$ and $H_I^\perp$ for the orthogonal complement of H_I in H. Let P_I and $P_I^\perp$ be the orthogonal projection operators of H onto H_I and $H_I^\perp$, respectively. Thus $P_I^\perp Y = Y - P^\perp Y$ for $Y \in H$. The projection $P_I Y$ stands for the best linear predictor of Y based on the observations $\{X_k : i \in I\}$, and $P_I^\perp Y$ for its prediction error.

The partial autocorrelation function (PACF) $\{\alpha_n\}_{n=0}^\infty$ of $\{X_n\}$ is defined by

(PACF) $$\alpha_0 := \gamma_0, \qquad \alpha_n := U_n/V_n \qquad (n = 1, 2, \dots),$$

where $U_1 := (X_1, X_0)$, $V_1 := \|X_1\|^2$ and

$$U_n : = (P_{[1,n-1]}^\perp X_n, P_{[1,n-1]}^\perp X_0) \qquad (n = 2, 3, \dots),$$
$$V_n : = \|P_{[1,n-1]}^\perp X_n\|^2 \qquad (n = 2, 3, \dots)$$

(cf. Brockwell and Davis [BD, §3.4 and §5.2]). We have $V_n > 0$ for $n \geq 1$ since we have assumed that the spectral measure μ of $\{X_n\}$ has infinite support; $X_1, X_2, X_3, \dots$ are linearly independent since no nonzero trigonometric polynomial vanishes μ a.e. Also, since $\|X_0\| = \|X_1\|$ and $\|P_{[1,n-1]}^\perp X_0\| = \|P_{[1,n-1]}^\perp X_n\|$ for $n \geq 2$, α_n is actually the correlation coefficient between the residuals $P_{[1,n-1]}^\perp X_0$ and $P_{[1,n-1]}^\perp X_n$ (resp. X_0 and X_n) for $n \geq 2$ (resp. $n = 1$).

A stationary process $\{X_n\}$ is said to be *purely nondeterministic* (PND) if

(PND) $$\bigcap\nolimits_{n=-\infty}^{\infty} H_{(-\infty,n]} = \{0\}$$

or, equivalently, there exists a positive even and integrable function $\Delta(\cdot)$ on $(-\pi, \pi)$ such that

$$\gamma_n = \int_{-\pi}^{\pi} e^{in\theta}\Delta(\theta)d\theta \quad (n \in \mathbf{Z}), \qquad \int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta < \infty$$

(see [BD, §5.7], [Ro, Chapter II] and [GS, Chapter 10]). We call $\Delta(\cdot)$ the *spectral density* of $\{X_n\}$. Using $\Delta(\cdot)$, we define the *Szegö function* $D(\cdot)$ by

$$D(z) := \sqrt{2\pi} \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log \Delta(\theta) d\theta\right\} \qquad (z \in \mathbf{C},\ |z| < 1).$$

The function $D(z)$ is an outer function in the Hardy space H^2 of class 2 over the unit disk $|z| < 1$. Using $D(\cdot)$, we define the MA coefficients c_n by

$$D(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (|z| < 1),$$

and the AR coefficients a_n by

$$-\frac{1}{D(z)} = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1)$$

(see [I2, §4] and [IK2, §2.2] for background). Both $\{c_n\}$ and $\{a_n\}$ are real sequences, and $\{c_n\}$ is in l^2. The coefficients c_n and a_n are actually those that appear in the following MA(∞) and AR(∞) representations, respectively, of $\{X_n\}$ (under suitable condition such as $\{a_n\} \in l^1$ for the latter):

(MA) $$X_n = \sum_{j=-\infty}^{n} c_{n-j}\xi_j \qquad (n \in \mathbf{Z}),$$

(AR) $$\sum_{j=-\infty}^{n} a_{n-j}X_j + \xi_n = 0 \qquad (n \in \mathbf{Z}),$$

where $\{\xi_k\}$ is the innovation process defined by $\xi_k = \epsilon_k/\|\epsilon_k\|$ with ϵ_k the prediction error when we predict X_k from the whole past $\{X_m : m \leq k-1\}$, i.e., $\epsilon_k = P^{\perp}_{(-\infty,k-1]}X_k$ (cf. [IK2, §2]). From (MA), we have the following equality:

(2.1) $$\gamma_n = \sum_{v=0}^{\infty} c_v c_{|n|+v} \qquad (n \in \mathbf{Z}).$$

We wish to derive a representation of α_n which is given only in terms of $\beta(\cdot)$ defined by $(*)$. For this purpose, we consider the following two conditions (BC) and (O($1/n$)):

(BC) The process $\{X_n\}$ has summable autocovariance function $\{\gamma_n\}$, i.e., $\sum_{-\infty}^{\infty} |\gamma_n| < \infty$, and *positive* spectral density $\Delta(\cdot)$, i.e., $\min_{\theta\in[-\pi,\pi]} \Delta(\theta) > 0$.

(O($1/n$)) $\{X_n\}$ is PND, and satisfies both $\sum_0^{\infty} |a_n| < \infty$ and

(2.2) $$\sum_{v=0}^{\infty} |c_v a_{n+v}| = O(1/n) \qquad (n \to \infty).$$

Notice that if $\{\gamma_n\} \in l^1$, then $\{X_n\}$ has continuous spectral density $\Delta(\theta) = (2\pi)^{-1}\sum_{-\infty}^{\infty} \gamma_k e^{-ik\theta}$. We will see (Theorem 4.1) that (BC) holds if and only if

$\{X_n\}$ is PND, $\{a_n\} \in l^1$ and $\{c_n\} \in l^1$. We also see that (O($1/n$)) holds for many interesting processes including the FARIMA(p, d, q) processes with $0 < d < 1/2$ which we consider below. The condition (L(d, ℓ)) below imples (O($1/n$)) (see Proposition 5.1 below). ARMA processes satisfy both (BC) and (O($1/n$)).

Remarks. 1. Condition (BC) requires that the process have *summable autocovariance* and *positive spectral density*. It could thus be denoted (SP). We call it (BC) instead to emphasize its role as *Baxter's condition. Baxter's theorem* ([Ba1, Ba2]; see also Theorem 4.1 below) gives the equivalence of (BC) with summability of the PACF (i.e., $\{\alpha_n\} \in l^1$) subject only to the very weak condition that the spectral measure μ has infinite support. See Chapter 5 of [Si3], where Baxter's theorem is discussed in detail and proved.
2. Several different definitions of *short memory* (or its negation, *long memory*) are in current use. See for example Section 2 of the survey paper Baillie [Bai] for details and references. The most standard definition is that $\{X_n\}$ has long memory (resp. short memory) if $\sum_{k=-\infty}^{\infty} |\gamma_k| = \infty$ (resp. $< \infty$); see Beran [Be, page 6], [BD, §13.2]. The fact that Baxter's theorem is so powerful and useful suggests the possibility of using Baxter's condition to define short memory in a new sense: call the process *short memory* if Baxter's condition holds, *long memory* otherwise.
3. The difference between these approaches to short and long memory is well illustrated by the fractional ARIMA (or FARIMA) processes (see below for definitions), studied in e.g. [I3, IK1] and Theorems 2.4 and 2.5 below. The two main cases $d \in (-1/2, 0)$ and $d \in (0, 1/2)$ behave in the *same* way from the point of view of asymptotics of PACF (d/n for each; see Theorem 2.5 below) and prediction error (d^2/n for each; see (2.24) below) — but in *different* ways from the point of view of summability of the autocovariance function. Our contention is that the PACF $\{\alpha_n\}$, and/or the prediction error $\{\delta(n)\}$, are more informative about the essence of long-range dependence — the rate at which the information in the remote past decays with time — than the autocovariance function $\{\gamma_n\}$ which is usually used here.
4. Wu [W], §3 considers stationary processes which satisfy $\{\gamma_n\} \notin l^1$ (long-range dependence) but have $\lim_{K\to\infty} \sum_{k=-K}^{K} \gamma(k)$. He uses the Zygmund class of slowly varying functions; see Bingham et al. [BGT, §1.5.3], Zygmund [Z, V.2].

Under (BC) or (O($1/n$)), we define, for $n = 0, 1, \ldots$,

$$d_1(n) = \beta(n), \tag{2.3}$$

$$d_2(n) = \sum_{m_1=0}^{\infty} \beta(m_1 + n)\beta(m_1 + n), \tag{2.4}$$

and, for $k = 3, 4, \ldots$,

$$\begin{aligned} d_k(n) = &\sum_{m_{k-1}=0}^{\infty} \beta(m_{k-1} + n) \sum_{m_{k-2}=0}^{\infty} \beta(m_{k-1} + m_{k-2} + n) \\ &\cdots \sum_{m_2=0}^{\infty} \beta(m_3 + m_2 + n) \sum_{m_1=0}^{\infty} \beta(m_2 + m_1 + n)\beta(m_1 + n), \end{aligned} \tag{2.5}$$

the sums converging absolutely (see Proposition 4.3 below).

Here is the representation of the PACF $\{\alpha_n\}$.

Theorem 2.1. *We assume either* (BC) *or* (O(1/n)). *Then, for* $n = 1, 2, \dots,$

$$U_n = (c_0)^2 \sum\nolimits_{k=1}^{\infty} d_{2k-1}(n), \tag{2.6}$$

$$V_n = (c_0)^2 \left\{1 + \sum\nolimits_{k=1}^{\infty} d_{2k}(n)\right\}, \tag{2.7}$$

$$\alpha_n = \frac{\sum_{k=1}^{\infty} d_{2k-1}(n)}{1 + \sum_{k=1}^{\infty} d_{2k}(n)}, \tag{2.8}$$

all the sums converging absolutely.

If $\{X_n\}$ is PND, then $V_n \downarrow \|P^{\perp}_{(-\infty,-1]}X_0\|^2 = (c_0)^2$, whence $\alpha_n = U_n/V_n \sim (c_0)^{-2}U_n$, as $n \to \infty$ (see [I2, §2]). Thus an immediate consequence of Theorem 2.1 is the next corollary.

Corollary 2.2. *We assume either* (BC) *or* (O(1/n)). *Then,*

$$\alpha_n \sim \sum_{k=1}^{\infty} d_{2k-1}(n) \qquad (n \to \infty). \tag{2.9}$$

We turn to the results on the asymptotic behaviour of α_n as $n \to \infty$. We write $\mathcal{R}_0$ for the class of slowly varying functions at infinity: the class of positive, measurable ℓ, defined on some neighborhood $[A, \infty)$ of infinity, such that

$$\lim_{x\to\infty} \ell(\lambda x)/\ell(x) = 1 \quad \text{for all } \lambda > 0 \tag{2.10}$$

(see [BGT, Chapter 1]). For $\ell \in \mathcal{R}_0$ and $d \in (0, 1/2)$, we consider the following condition as a standard one for processes with *long memory* (see [IK2, (A2)]):

(L(d, ℓ)) $\{X_n\}$ is PND and $\{c_n\}$ and $\{a_n\}$ satisfy, respectively,

$$c_n \sim n^{-(1-d)}\ell(n) \qquad (n \to \infty), \tag{2.11}$$

$$a_n \sim n^{-(1+d)} \frac{1}{\ell(n)} \frac{d \sin(\pi d)}{\pi} \qquad (n \to \infty). \tag{2.12}$$

The condition (L(d, ℓ)) implies

$$\gamma_n \sim n^{-(1-2d)}\ell(n)^2 B(d, 1-2d) \qquad (n \to \infty) \tag{2.13}$$

(see [IK2, (2.22)]), whence $\{\gamma_n\} \notin l^1$, where $B(\cdot, \cdot)$ denotes the beta integral, i.e.,

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (p, q > 0).$$

Here is a result of the type (d/n) for processes with long memory.

Theorem 2.3. *For* $\ell \in \mathcal{R}_0$ *and* $d \in (0, 1/2)$, *(L(d, ℓ)) implies the asymptotic behaviour* (d/n) *of the PACF.*

For example, the FARIMA(p, d, q) model with $0 < d < 1/2$, which is regarded as a standard parametric model with long memory and which we consider below, satisfies (L(d, ℓ)) for some constant function ℓ (see Kokoszka and Taqqu [KT, Corollary 3.1]), whence, by this theorem, its PACF has the asymptotic behaviour (d/n). We see another class satisfying (L(d, ℓ)) in §6. We can write the property (d/n) as $d = \lim_{n\to\infty} n\alpha_n$, suggesting how to estimate the parameter d, which is important in a process with long memory. See [IK1, §5] for numerical calculation.

We consider the FARIMA. For $d \in (-1/2, 1/2)$ and $p, q \in \mathbf{N} \cup \{0\}$, a stationary process $\{X_n\}$ is said to be a (causal and invertible) fractional ARIMA(p, d, q) (or FARIMA(p, d, q)) process if it has a spectral density $\Delta(\cdot)$ of the form

$$\Delta(\theta) = \frac{1}{2\pi} \frac{|\Theta(e^{i\theta})|^2}{|\Phi(e^{i\theta})|^2} |1 - e^{i\theta}|^{-2d} \qquad (-\pi < \theta < \pi), \tag{2.14}$$

where $\Phi(z)$ and $\Theta(z)$ are polynomials with real coefficients of degrees p, q, respectively, satisfying the following condition:

$$\begin{aligned} &\Phi(z) \text{ and } \Theta(z) \text{ have no common zeros, and have no zeros} \\ &\text{in the closed unit disk } \{z \in \mathbf{C} : |z| \leq 1\}. \end{aligned} \tag{2.15}$$

The fractional ARIMA model was introduced independently by Granger and Joyeux [GJ] and Hosking [Ho]. See [Be, §2.5] and [BD, §13.2] for textbook treatments and [KT] for a formulation in terms of the backward shift operator B. If $d \in (-1/2, 1/2) \setminus \{0\}$, then the MA coefficient c_n, AR coefficients a_n and autocovariance function $\{\gamma_n\}$ of the FARIMA(p, d, q) process $\{X_n\}$ satisfy

$$c_n \sim n^{-(1-d)} \frac{K_1}{\Gamma(d)} \qquad (n \to \infty), \tag{2.16}$$

$$a_n \sim n^{-(1+d)} \frac{\Gamma(d)}{K_1} \cdot \frac{d \sin(\pi d)}{\pi} \qquad (n \to \infty), \tag{2.17}$$

$$\gamma_n \sim n^{-(1-2d)} \frac{(K_1)^2 \Gamma(1-2d) \sin(\pi d)}{\pi} \qquad (n \to \infty), \tag{2.18}$$

where

$$K_1 := \Theta(1)/\Phi(1) \tag{2.19}$$

(see [KT, Corollary 3.1] and [I3, §3]). In particular, if $0 < d < 1/2$, then $\{X_n\}$ has long memory. On the other hand, if $d = 0$, then $\{X_n\}$ reduces to the ordinary ARMA(p, q) process, and each of c_n, a_n and γ_n decays exponentially fast as $n \to \infty$ (see [BD, Chapter 3] and §7 below).

Theorems 2.1 and 2.3 cover FARIMA with $d \in (0, 1/2)$ but not the case $d \in (-1/2, 0)$. However, the FARIMA$(0, d, 0)$ with $d \in (-1/2, 1/2)$ has the PACF given by

$$\alpha_n = \frac{d}{n-d} \qquad (n = 1, 2, \dots) \tag{2.20}$$

(see [Ho, Theorem 1(f)] as well as [BD, (13.2.10)]), which suggests that (d/n) may also hold even if $d \in (-1/2, 0)$, and this is actually the case as we show below.

For FARIMA with $-1/2 < d < 0$, we define

$$\phi_n := \begin{cases} -a_0 & (n = 0), \\ a_{n-1} - a_n & (n = 1, 2, \dots). \end{cases}$$

and

$$\psi_n := -\sum_{k=n+1}^{\infty} c_k \qquad (n = 0, 1, \dots).$$

Notice that ϕ_n here corresponds to $-\phi_n$ in [IK1]. We define $q \in (1/2, 1)$ by

$$q := 1 + d.$$

Then, by [IK1, Lemma 4.1] and [KT, Corollary 3.1] (see also [I3, Lemma 2.1]), we have

$$\psi_n \sim n^{-(1-q)} \frac{K_1}{\Gamma(q)} \qquad (n \to \infty), \tag{2.21}$$

$$\phi_n \sim -n^{-(1+q)} \frac{\Gamma(q)}{K_1} \cdot \frac{q \sin(\pi q)}{\pi} \qquad (n \to \infty), \tag{2.22}$$

where K_1 is as in (2.19). We define

$$\beta_-(n) := \sum_{v=0}^{\infty} \psi_v \phi_{v+n+1} \qquad (n = 0, 1, \dots).$$

Notice that $\beta_-(n)$ corresponds to $-\beta(n)$ in [IK1]. We define, for $n = 0, 1, \dots,$

$$d_1(n) = \beta_-(n),$$

$$d_2(n) = \sum_{m_1=0}^{\infty} \beta_-(m_1 + n)\beta_-(m_1 + n),$$

and, for $k = 3, 4, \dots,$

$$d_k(n) = \sum_{m_{k-1}=0}^{\infty} \beta_-(m_{k-1} + n) \sum_{m_{k-2}=0}^{\infty} \beta_-(m_{k-1} + m_{k-2} + n)$$
$$\cdots \sum_{m_2=0}^{\infty} \beta_-(m_3 + m_2 + n) \sum_{m_1=0}^{\infty} \beta_-(m_2 + m_1 + n)\beta_-(m_1 + n),$$

the sums converging absolutely (see [IK2, Theorem 3.3]).

The representation of the PACF of FARIMA with $d \in (-1/2, 0)$ is given by the next theorem.

Theorem 2.4. *Let $p, q \in \mathbf{N} \cup \{0\}$ and $d \in (-1/2, 0)$, and let $\{X_n\}$ be a fractional ARIMA (p, d, q) process. Then, for $n = 1, 2, \dots,$ the representations* (2.6)–(2.8) *of U_n, V_n and α_n hold with all the sums converging absolutely.*

Here is the result of the type (d/n) for FARIMA.

Theorem 2.5. *Let $p, q \in \mathbf{N} \cup \{0\}$ and $d \in (-1/2, 1/2) \setminus \{0\}$, and let $\{X_n\}$ be a fractional ARIMA (p, d, q) process. Then the PACF has the asymptotics (d/n).*

This last theorem as well as Theorem 2.3 is an improvement of earlier work [I2, I3, IK1] for long-memory or FARIMA processes, asserting that

$$|\alpha_n| \sim |d|/n \qquad (n \to \infty). \tag{2.23}$$

Notice that while the earlier result cannot distinguish between the d and $-d$ cases (that is, between positive and negative differencing), Theorems 2.3 and 2.5 can. In [I2, I3, IK1], the asymptotic behaviour of mean-squared prediction error of the type

$$\delta(n) \sim d^2/n \qquad (n \to \infty) \tag{2.24}$$

was first derived and then used in Tauberian arguments to prove (2.23), where

$$\delta(n) := \frac{\|P^{\perp}_{[-n,0]} X_1\|^2 - \|P^{\perp}_{(-\infty,0]} X_1\|^2}{\|P^{\perp}_{(-\infty,0]} X_1\|^2} \qquad (n = 1, 2, \dots).$$

The proofs of Theorems 2.3 and 2.5, which are based on the representation of the PACF, are more direct and much simpler than those of the earlier result.

3. Inner products of prediction errors

In this section, we derive some expansions of V_n and U_n that we need to prove the representation of the PACF. The key is to extend [I2, Theorem 4.1] properly.

For $n, k \in \mathbf{N}$, we define the orthogonal projection operator P_n^k by

$$P_n^k := \begin{cases} P_{(-\infty,n-1]} & (k = 1, 3, 5, \dots), \\ P_{[1,\infty)} & (k = 2, 4, 6, \dots). \end{cases} \tag{3.1}$$

It should be noticed that $\{P_n^k : k = 1, 2, \dots\}$ is merely an alternating sequence of projection operators, first to the subspace $H_{(-\infty,n-1]}$, then to $H_{[1,\infty)}$, and so on.

Theorem 3.1. *Let $Y_1, Y_2 \in H$. We assume that $\{X_n\}$ is CND.*

(1) *We have*

$$\begin{aligned}(Y_1, Y_2) &= \big((P_1^1)^\perp Y_1, (P_1^1)^\perp Y_2\big) \\ &\qquad + \sum_{k=1}^{\infty} \big((P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_1, (P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_2\big).\end{aligned} \tag{3.2}$$

(2) *We have, for $n = 2, 3, \dots$,*

$$\begin{aligned}\left(P_{[1,n-1]}^\perp Y_1, P_{[1,n-1]}^\perp Y_2\right) &= \big((P_n^1)^\perp Y_1, (P_n^1)^\perp Y_2\big) \\ &\qquad + \sum_{k=1}^{\infty} \big((P_n^{k+1})^\perp P_n^k \cdots P_n^1 Y_1, (P_n^{k+1})^\perp P_n^k \cdots P_n^1 Y_2\big).\end{aligned} \tag{3.3}$$

If we put $Y_1 = Y_2$ in Theorem 3.1(2), then it reduces to [I2, Theorem 4.1].

Proof. (1) The orthogonal decompositions

$$H = H_{(-\infty,0]}^\perp \oplus H_{(-\infty,0]},$$
$$H = H_{[1,\infty)}^\perp \oplus H_{[1,\infty)}$$

of H imply the orthogonal decompositions

$$I_H = P_{(-\infty,0]}^\perp \oplus P_{(-\infty,0]}, \tag{3.4}$$

$$I_H = P_{[1,\infty)}^\perp \oplus P_{[1,\infty)} \tag{3.5}$$

of the identity map I_H, respectively. Repeated use of (3.4) and (3.5) yields, for $m = 2, 3, \dots$,

$$\begin{aligned}(Y_1, Y_2) &= \big((P_1^1)^\perp Y_1, (P_1^1)^\perp Y_2\big) \\ &\quad + \sum_{k=1}^{m-1} \big((P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_1, (P_1^{k+1})^\perp P_1^k \cdots P_1^1 Y_2\big) + R_1^m,\end{aligned}$$

where $R_1^m := \big(P_1^m \cdots P_1^1 Y_1, P_1^m \cdots P_1^1 Y_2\big)$. Since (CND) and von Neumann's alternating projection theorem (see [P, Theorem 9.20]) imply

$$\operatorname*{s-lim}_{m\to\infty} P_1^m \cdots P_1^1 = 0,$$

we have $\lim_{m\to\infty} R_1^m = 0$, whence (3.2).

(2) For $n = 2, 3, \ldots$, we have the orthogonal decompositions

$$H^{\perp}_{[1,n-1]} = H^{\perp}_{(-\infty,n-1]} \oplus \left(H^{\perp}_{[1,n-1]} \cap H_{(-\infty,n-1]}\right),$$
$$H^{\perp}_{[1,n-1]} = H^{\perp}_{[1,\infty)} \oplus \left(H^{\perp}_{[1,n-1]} \cap H_{[1,\infty)}\right)$$

of $H^{\perp}_{[1,n-1]}$, which in turn imply the orthogonal decompositions

$$P^{\perp}_{[1,n-1]} = P^{\perp}_{(-\infty,n-1]} \oplus P^{\perp}_{[1,n-1]} P_{(-\infty,n-1]}, \tag{3.6}$$
$$P^{\perp}_{[1,n-1]} = P^{\perp}_{[1,\infty)} \oplus P^{\perp}_{[1,n-1]} P_{[1,\infty)} \tag{3.7}$$

of $P^{\perp}_{[1,n-1]}$, respectively. Using (3.6) and (3.7) repeatedly, we find that, for $m = 2, 3, \ldots$,

$$\begin{aligned}\left(P^{\perp}_{[1,n-1]} Y_1, P^{\perp}_{[1,n-1]} Y_2\right) &= \left((P^1_n)^{\perp} Y_1, (P^1_n)^{\perp} Y_2\right) \\ &\quad + \sum_{k=1}^{m-1} \left((P^{k+1}_n)^{\perp} P^k_n \cdots P^1_n Y_1, (P^{k+1}_n)^{\perp} P^k_n \cdots P^1_n Y_2\right) + R^m_n,\end{aligned}$$

where $R^m_n := (P^{\perp}_{[1,n-1]} P^m_n \cdots P^1_n Y_1, P^{\perp}_{[1,n-1]} P^m_n \cdots P^1_n Y_2)$. From (IPF) implied by (CND) (see [IK2, Theorem 2.3]) and the alternating projection theorem, we get

$$\operatorname*{s-lim}_{m\to\infty} P^m_n \cdots P^1_n = P_{[1,n-1]},$$

whence

$$\lim_{m\to\infty} \|P^{\perp}_{[1,n-1]} P^m_n \cdots P^1_n Y_i\| = \|P^{\perp}_{[1,n-1]} P_{[1,n-1]} Y_i\| = 0 \qquad (i = 1, 2).$$

Thus $\lim_{m\to\infty} R^m_n = 0$, so that (3.3) follows. □

Remark. If $\{X_n\}$ is CND, then by the same arguments as above we see that

$$P^{\perp}_{[1,n-1]} = (P^1_n)^{\perp} + (P^2_n)^{\perp} P^1_n + (P^3_n)^{\perp} P^2_n P^1_n + \cdots.$$

Assuming (PND), we define

$$b^m_j := \sum_{k=0}^{m} c_k a_{j+m-k} \qquad (m, j \in \mathbf{N} \cup \{0\}).$$

Notice that b^m_j here is equal to that in [IK1] but it corresponds to b^{m+1}_{j-1} in [I2, I3].

Recall U_n and V_n from §2. Here are their representations in terms of the AR and MA coefficients.

Theorem 3.2. *We assume* (PND) *and* $\sum_{n=0}^{\infty} |a_n| < \infty$. *Then, for* $n = 1, 2, \ldots$,

$$U_n = (c_0)^2 \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} d_k(n,p) d_{k-1}(n,p), \tag{3.8}$$

$$V_n = (c_0)^2 \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} d_k(n,p)^2, \tag{3.9}$$

where, for $n \in \mathbf{N}$ *and* $p \in \mathbf{N} \cup \{0\}$, $d_0(n,p) := \delta_{p0}$,

$$d_1(n,p) := \sum_{m_1=0}^{\infty} a_{n+m_1+p} c_{m_1}, \tag{3.10}$$

and, for $k = 2, 3, \dots,$

$$d_k(n,p) := \sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{v=0}^{\infty} b_{n+p+v}^{m_1} c_v, \tag{3.11}$$

all the sums converging absolutely.

Proof. (Compare the proof of [I2, Theorem 4.5].) Notice that $\{X_n\}$ satisfies (IPF) (see [I2], Proposition 4.2 and Theorem 3.1), whence (CND) (see [IK2], Theorem 2.3). Hence, it follows from Theorem 3.1 that, for $n = 1, 2, \dots,$

$$\begin{aligned} U_n &= \left((P_n^2)^\perp P_n^1 X_n, (P_n^2)^\perp X_0\right) \\ &\quad + \sum_{k=2}^{\infty} \left((P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n, (P_n^{k+1})^\perp P_n^k \cdots P_n^2 X_0\right), \end{aligned} \tag{3.12}$$

$$V_n = \|(P_n^1)^\perp X_n\|^2 + \sum_{k=1}^{\infty} \|(P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n\|^2. \tag{3.13}$$

Let $n \in \mathbf{N}$. Suppose that k is even and ≥ 2. By [I2, Theorem 4.4], we have, for $n = 1, 2, \dots$ and $m = 0, 1, \dots,$

$$P_{(-\infty,n-1]} X_{m+n} = \sum_{j=0}^{\infty} b_{n+j}^m X_{-j} \qquad (\text{mod } H_{[1,n-1]} \text{ if } n \geq 2),$$

$$P_{[1,\infty)} X_{-m} = \sum_{j=0}^{\infty} b_{n+j}^m X_{j+n} \qquad (\text{mod } H_{[1,n-1]} \text{ if } n \geq 2),$$

whence

$$\begin{aligned} P_n^k \cdots P_n^1 X_n = c_0 &\sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \\ &\cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{m_0=0}^{\infty} b_{n+m_0}^{m_1} X_{m_0+n} \quad (\text{mod } H_{[1,n-1]} \text{ if } n \geq 2). \end{aligned}$$

Since we restrict to (PND), $\{X_n\}$ has no deterministic component in the Wold decomposition and it permits the moving-average representation (MA), where the orthonormal system $\{\xi_j : j \in \mathbf{Z}\}$ of H satisfies

$$H_{(-\infty,m]} = H_{(-\infty,m]}(\xi) \qquad (m \in \mathbf{Z})$$

with $H_{(-\infty,m]}(\xi)$ being the closed subspace of H spanned by $\{\xi_j : -\infty < j \leq m\}$ (see [Ro, Chapter II], [BD, §5.7]). Since

$$P_{(-\infty,n-1]}^\perp X_{m+n} = \sum_{j=0}^{m} c_{m-j} \xi_{j+n} \qquad (m = 0, 1, \dots),$$

we have

$$\begin{aligned} (P_n^{k+1})^\perp P_n^k \cdots P_n^1 X_n = c_0 &\sum_{m_{k-1}=0}^{\infty} a_{n+m_{k-1}} \sum_{m_{k-2}=0}^{\infty} b_{n+m_{k-2}}^{m_{k-1}} \\ &\cdots \sum_{m_1=0}^{\infty} b_{n+m_1}^{m_2} \sum_{m_0=0}^{\infty} b_{n+m_0}^{m_1} \sum_{j=0}^{m_0} c_{m_0-j} \xi_{j+n}, \end{aligned}$$

so that

$$\big((P_n^{k+1})^{\perp} P_n^k \cdots P_n^1 X_n, \xi_{p+n}\big) = \begin{cases} c_0 d_k(n,p) & (p = 0, 1, \dots), \\ 0 & (p = -1, -2, \dots). \end{cases}$$

Arguing similarly,

$$\big(\xi_{p+n}, (P_n^{k+1})^{\perp} P_n^k \cdots P_n^2 X_0\big) = \begin{cases} c_0 d_{k-1}(n,p) & (p = 0, 1, \dots), \\ 0 & (p = -1, -2, \dots). \end{cases}$$

Thus, from the Parseval equality, we get

$$\begin{aligned} &\big((P_n^{k+1})^{\perp} P_n^k \cdots P_n^1 X_n, (P_n^{k+1})^{\perp} P_n^k \cdots P_n^2 X_0\big) \\ &\qquad\qquad = (c_0)^2 \sum_{p=0}^{\infty} d_k(n,p) d_{k-1}(n,p), \end{aligned} \tag{3.14}$$

$$\|(P_n^{k+1})^{\perp} P_n^k \cdots P_n^1 X_n\|^2 = (c_0)^2 \sum_{p=0}^{\infty} d_k(n,p)^2. \tag{3.15}$$

Similarly, we have (3.14) and (3.15) for k odd, and also

$$\big((P_n^2)^{\perp} P_n^1 X_n, (P_n^2)^{\perp} X_0\big) = (c_0)^2 d_1(n,0) = (c_0)^2 \sum_{p=0}^{\infty} d_1(n,p) d_0(n,p), \tag{3.16}$$

$$\|(P_n^1)^{\perp} X_n\|^2 = (c_0)^2 = (c_0)^2 \sum_{p=0}^{\infty} d_0(n,p)^2. \tag{3.17}$$

The assertions (3.8) and (3.9) now follow if we substitute (3.14) and (3.16) into (3.12), and (3.15) and (3.17) into (3.13). □

We write $\sum^{\infty-}$ to indicate that the sum does not necessarily converge absolutely, i.e., $\sum_{k=m}^{\infty-} := \lim_{M\to\infty} \sum_{k=m}^{M}$. We need the next variant of Theorem 3.2 when we consider the fractional ARIMA(p, d, q) processes with $-1/2 < d < 0$.

Theorem 3.3. *We assume* (PND). *Then the representations* (3.8) *and* (3.9) *still hold if all the summations* $\sum^{\infty}$ *in* (3.10) *and* (3.11) *are replaced by* $\sum^{\infty-}$ *and if* $\sum_0^{\infty} |a_n| < \infty$ *is replaced by the two conditions* $\sum_0^{\infty} |c_n| < \infty$ *and* $\sum_0^{\infty} |a_n|^2 < \infty$.

Proof. By [I2, Proposition 4.2 and Theorem 3.1] and [IK2, Theorem 2.3], the conditions (PND) and $\{a_n\} \in l^2$ imply (IPF), whence (CND). Moreover, by [IK1, Proposition 2.1], we have, for $n \in \mathbf{N}$ and $m \in \mathbf{N} \cup \{0\}$,

$$P_{(-\infty,n-1]} X_{m+n} = \sum_{j=0}^{\infty-} b_{n+j}^m X_{-j} \qquad (\text{mod } H_{[1,n-1]} \text{ if } n \geq 2),$$

$$P_{[1,\infty)} X_{-m} = \sum_{j=0}^{\infty-} b_{n+j}^m X_{j+n} \qquad (\text{mod } H_{[1,n-1]} \text{ if } n \geq 2).$$

Using these equalities, we can prove the theorem as in the proof of Theorem 3.2. We omit the details. □

4. Proof of Theorem 2.1

First we give some necessary and sufficient conditions for (BC). Recall that the spectral measure of a stationary process is assumed to have infinite support.

Theorem 4.1. *For a stationary process $\{X_n\}$, the following conditions are equivalent:*

(1) *$\{X_n\}$ is PND and satisfies both $\sum_0^\infty |a_n| < \infty$ and $\sum_0^\infty |c_n| < \infty$;*
(2) *$\{X_n\}$ has a positive continuous spectral density and satisfies $\sum_0^\infty |c_n| < \infty$;*
(3) *$\{X_n\}$ has a positive continuous spectral density and satisfies $\sum_0^\infty |a_n| < \infty$;*
(4) *$\{X_n\}$ satisfies* (BC)*;*
(5) *$\{X_n\}$ has summable PACF: $\sum_0^\infty |\alpha_n| < \infty$.*

Proof. Notice that if $\{X_n\}$ has a positive continuous spectral density $\Delta(\cdot)$ on $[-\pi, \pi]$, then it is PND since $\int_{-\pi}^{\pi} |\log \Delta(\theta)| d\theta < \infty$ holds.

Suppose (1). We write $D(e^{i\theta})$ for the nontangential limit of $D(z)$, i.e.,

$$D(e^{i\theta}) = \lim_{r \to 1-0} D(re^{i\theta}) = \sum\nolimits_{n=0}^{\infty} c_n e^{in\theta} \qquad (-\pi \le \theta \le \pi).$$

Since $\{c_n\} \in l^1$ implies the continuity of $D(e^{i\theta})$, the spectral density $\Delta(\theta)$ is also continuous by the equality $\Delta(\theta) = 2\pi |D(e^{i\theta})|^2$. Letting $r \to 1-0$ in

$$\left(\sum\nolimits_{n=0}^{\infty} c_n r^n e^{in\theta}\right)\left(\sum\nolimits_{n=0}^{\infty} a_n r^n e^{in\theta}\right) = -1 \qquad (-\pi \le \theta \le \pi),$$

we obtain

$$\left(\sum\nolimits_{n=0}^{\infty} c_n e^{in\theta}\right)\left(\sum\nolimits_{n=0}^{\infty} a_n e^{in\theta}\right) = -1 \qquad (-\pi \le \theta \le \pi).$$

This implies that $D(e^{i\theta})$, whence $\Delta(\theta)$, has no zeros on $[-\pi, \pi]$. Thus $\Delta(\cdot)$ is positive, whence (2) and (3) follow.

Suppose (3). In the same way as above, we have

$$\frac{1}{\Delta(\theta)} = \frac{1}{2\pi} \left|\sum\nolimits_{n=0}^{\infty} a_n e^{in\theta}\right|^2 \qquad (-\pi \le \theta \le \pi).$$

This implies $\sum_{n=0}^{\infty} a_n e^{in\theta} \ne 0$ for every $\theta \in [-\pi, \pi]$. By Wiener's theorem for absolutely convergent Fourier series (cf. Lemma 11.6 in [Ru]), we obtain $\{c_n\} \in l^1$ (cf. Berk [Berk], page 493). Thus (1) follows. The proof of the implication (2) $\Rightarrow$ (1) is similar.

By (2.1), (2) implies (4). Conversely, we assume (4). Then we have $\{a_n\} \in l^1$, whence (3), by the arguments in Baxter [Ba2], pp. 139–140, which involve the Wiener–Lévy theorem.

The equivalence between (4) and (5) is Baxter's theorem ([Ba1, Ba2]; see also [Si3, Chapter 5]). This completes the proof. □

We put

$$B(n) := \sum_{v=0}^{\infty} |c_v a_{n+v}| \qquad (n \in \mathbf{N} \cup \{0\}).$$

For $n, k, u, v \in \mathbf{N} \cup \{0\}$, we define $D_k(n, u, v)$ recursively by

$$\begin{cases} D_0(n, u, v) := \delta_{uv}, \\ D_{k+1}(n, u, v) := \sum\nolimits_{w=0}^{\infty} B(n + v + w) D_k(n, u, w) \end{cases}$$

(see [IK2, §2.3]). We have, for example,

$$D_3(n,u,v) = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} B(n+v+v_1)B(n+v_1+v_2)B(n+v_2+u).$$

Proposition 4.2. *We assume either* (BC) *or* (O(1/n))*. Then, for* $k, n, v \in \mathbf{N} \cup \{0\}$,

$$\sum_{u=0}^{\infty} D_k(n,u,v) < \infty \quad \text{and} \quad \sum_{u=0}^{\infty} D_k(n,u,v)^2 < \infty,$$

respectively. In particular, we have $D_k(n,u,v) < \infty$ *for* $k, n, u, v \in \mathbf{N} \cup \{0\}$.

In view of Theorem 4.1, we can prove Proposition 4.2 in the same way as that of [IK2, Lemma 2.7], whence we omit it.

Recall $\beta(n)$ from $(*)$ and $d_k(n,p)$ from Theorem 3.2.

Proposition 4.3. *We assume either* (BC) *or* (O(1/n))*. Then we have, for* $n \in \mathbf{N}$ *and* $p \in \mathbf{N} \cup \{0\}$,

$$d_1(n,p) = \beta(n+p), \tag{4.1}$$

$$d_2(n,p) = \sum_{m_1=0}^{\infty} \beta(m_1+n)\beta(m_1+n+p), \tag{4.2}$$

and, for $k = 3, 4, \dots,$

$$\begin{aligned} d_k(n,p) = \sum_{m_{k-1}=0}^{\infty} \beta(m_{k-1}+n) \sum_{m_{k-2}=0}^{\infty} \beta(m_{k-1}+m_{k-2}+n) \\ \cdots \sum_{m_2=0}^{\infty} \beta(m_3+m_2+n) \sum_{m_1=0}^{\infty} \beta(m_2+m_1+n)\beta(m_1+n+p), \end{aligned} \tag{4.3}$$

the sums converging absolutely.

Proof. By Proposition 4.2, we can use the Fubini–Tonelli theorem to exchange the order of sums in (3.11), and we get (4.2) and (4.3) as in the proof of [I2, Theorem 4.6]. □

Proposition 4.4. *We assume either* (BC) *or* (O(1/n))*. Then, for* $i, j \in \mathbf{N}$,

$$\sum_{p=0}^{\infty} d_i(n,p)d_j(n,p) = d_{i+j}(n,0) \qquad (n = 1, 2, \dots). \tag{4.4}$$

Proof. For simplicity, we give details for the case $i = j = 4$ only. The general case can be treated in the same way. From Proposition 4.3, we have, for $n = 1, 2, \dots$ and $p = 0, 1, \dots,$

$$\begin{aligned} d_4(n,p) = \textstyle\sum_{m_1=0}^{\infty} \beta(m_1+n) \displaystyle\sum_{m_2=0}^{\infty} \beta(m_1+m_2+n) \\ \textstyle\sum_{m_3=0}^{\infty} \beta(m_2+m_3+n)\beta(m_3+p+n). \end{aligned} \tag{4.5}$$

By Proposition 4.2 and the Fubini theorem,

$$\begin{aligned} d_4(n,p) = \textstyle\sum_{m_3=0}^{\infty} \beta(m_3+p+n) \sum_{m_2=0}^{\infty} \beta(m_2+m_3+n) \\ \textstyle\sum_{m_1=0}^{\infty} \beta(m_1+m_2+n)\beta(m_1+n). \end{aligned} \tag{4.6}$$

Writing (m_7, m_6, m_5) for (m_1, m_2, m_3) in (4.5), we get

$$
\begin{aligned}
d_4(n,p) = {} & \sum\nolimits_{m_7=0}^{\infty} \beta(m_7+n) \sum\nolimits_{m_6=0}^{\infty} \beta(m_6+m_7+n) \\
& \sum\nolimits_{m_5=0}^{\infty} \beta(m_5+m_6+n)\beta(p+m_5+n).
\end{aligned} \tag{4.7}
$$

From (4.6), (4.7) and the Fubini theorem,

$$
\begin{aligned}
& \sum\nolimits_{p=0}^{\infty} d_4(n,p)d_4(n,p) = \sum\nolimits_{m_4=0}^{\infty} d_4(n,m_4)d_4(n,m_4) \\
& = \sum\nolimits_{m_4=0}^{\infty} \Big\{ \sum\nolimits_{m_7=0}^{\infty} \beta(m_7+n) \sum\nolimits_{m_6=0}^{\infty} \beta(m_6+m_7+n) \\
& \qquad\qquad \sum\nolimits_{m_5=0}^{\infty} \beta(m_5+m_6+n)\beta(m_4+m_5+n) \Big\} \\
& \quad \times \Big\{ \sum\nolimits_{m_3=0}^{\infty} \beta(m_3+m_4+n) \sum\nolimits_{m_2=0}^{\infty} \beta(m_2+m_3+n) \\
& \qquad\qquad \sum\nolimits_{m_1=0}^{\infty} \beta(m_1+m_2+n)\beta(m_1+n) \Big\},
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& \sum\nolimits_{m_7=0}^{\infty} \beta(m_7+n) \sum\nolimits_{m_6=0}^{\infty} \beta(m_6+m_7+n) \sum_{m_5=0}^{\infty} \beta(m_5+m_6+n) \\
& \quad \sum\nolimits_{m_4=0}^{\infty} \beta(m_4+m_5+n) \sum\nolimits_{m_3=0}^{\infty} \beta(m_3+m_4+n) \sum\nolimits_{m_2=0}^{\infty} \beta(m_2+m_3+n) \\
& \qquad \sum\nolimits_{m_1=0}^{\infty} \beta(m_1+m_2+n)\beta(m_1+n) \\
& = d_8(n,0).
\end{aligned}
$$

Thus the desired result for $i = j = 4$ follows. □

Proof of Theorem 2.1. By Proposition 4.3, we see that

$$d_k(n) = d_k(n,0) \qquad (k, n \in \mathbf{N}).$$

From this, Proposition 4.4 and Theorem 3.2, the theorem follows. □

5. Proofs of Theorems 2.3–2.5

Proposition 5.1. *For $d \in (0, 1/2)$ and $\ell \in \mathcal{R}_0$, we assume $(L(d,\ell))$.*

(1) *It holds that*

$$\beta_n \sim \frac{\sin(\pi d)}{\pi} n^{-1} \qquad (n \to \infty).$$

(2) *The condition* $(\mathrm{O}(1/n))$ *holds. More precisely, we have*

$$\sum_{v=0}^{\infty} |c_v a_{n+v}| \sim \frac{\sin(\pi d)}{\pi} n^{-1} \qquad (n \to \infty).$$

(3) *For $s \geq 0$ and $u \geq 0$, it holds that*

$$\beta([ns]+[nu]+n) \sim \frac{\sin(\pi d)}{\pi(s+u+1)} n^{-1} \qquad (n \to \infty).$$

(4) *For $r \in (1, \infty)$, there exists $N_1 \in \mathbf{N}$ such that*

$$|\beta([ns]+[nu]+n)| \leq \frac{r\sin(\pi d)}{\pi(s+u+1)} n^{-1} \quad (s \geq 0,\ u \geq 0,\ n \geq N_1). \tag{5.1}$$

Proof. The assertions (1) and (2) follow from [I1, Proposition 4.3]. Since we have $[ns] + [nu] + n \sim n(s + u + 1)$ as $n \to \infty$, (3) follows from (1). Let $r \in (1, \infty)$. Then $n/([ns] + [nu] + n) \to 1/(s + u + 1)$ as $n \to \infty$, uniformly in $s \geq 0$ and $u \geq 0$ (cf. [BGT, Theorem 1.5.2]), so that there exists $N_2 \in \mathbf{N}$ such that

$$\frac{1}{([ns] + [nu] + n)} \leq \frac{r^{1/2}}{n(s + u + 1)} \qquad (s \geq 0,\ u \geq 0,\ n \geq N_2),$$

while, from (1), there exists $N_3 \in \mathbf{N}$ such that

$$|\beta_n| \leq \frac{r^{1/2} \sin(\pi d)}{\pi n} \qquad (n \geq N_3).$$

If we put $N_1 := \max(N_2, N_3)$, then, for $s \geq 0$, $u \geq 0$, $n \geq N_1$, we have

$$|\beta([ns] + [nu] + n)| \leq \frac{r^{1/2} \sin(\pi d)}{\pi([ns] + [nu] + n)} \leq \frac{r \sin(\pi d)}{\pi(s + u + 1)} n^{-1}.$$

Thus (4) follows. □

Recall $d_k(n)$ from (2.3)–(2.5). For $k = 1, 2, \ldots$, we define the constant τ_k, which is equal to $f_k(0)$ in [IK2], by

$$\tau_1 = \frac{1}{\pi}, \quad \tau_2 = \frac{1}{\pi^2} \int_0^\infty \frac{ds_1}{(s_1 + 1)(s_1 + 1)} = \frac{1}{\pi^2},$$

and, for $k = 3, 4, \ldots$,

$$\tau_k = \frac{1}{\pi^k} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \frac{1}{(s_{k-1} + 1)} \left\{ \prod_{m=1}^{k-2} \frac{1}{(s_{m+1} + s_m + 1)} \right\} \frac{1}{(s_1 + 1)}.$$

Proposition 5.2. *For $d \in (0, 1/2)$ and $\ell \in \mathcal{R}_0$, we assume $(L(d, \ell))$.*

(1) *For $r \in (1, \infty)$ and $N_1 \in \mathbf{N}$ satisfying* (5.1),

(5.2) $$|d_k(n)| \leq n^{-1} \{r \sin(\pi d)\}^k \tau_k \qquad (u \geq 0,\ k \in \mathbf{N},\ n \geq N_1).$$

(2) *For $k \in \mathbf{N}$ and $u \geq 0$,*

(5.3) $$d_k(n) \sim n^{-1} \{\sin(\pi d)\}^k \tau_k \qquad (n \to \infty).$$

Proof. Let $k \geq 3$ and write

$$\begin{aligned} d_k(n) &= \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \beta([s_{k-1}] + n) \\ &\quad \times \left\{ \prod_{m=1}^{k-2} \beta([s_{m+1}] + [s_m] + n) \right\} \times \beta([s_1] + n) \\ &= n^{k-1} \int_0^\infty ds_{k-1} \cdots \int_0^\infty ds_1 \beta([ns_{k-1}] + n) \\ &\quad \times \left\{ \prod_{m=1}^{k-2} \beta([ns_{m+1}] + [ns_m] + n) \right\} \times \beta([ns_1] + n). \end{aligned}$$

Applying Proposition 5.1 and the dominated convergence theorem to this, we obtain (5.2) and (5.3). The cases $k = 1, 2$ can be treated in a similar fashion. □

Proposition 5.3. *For $k = 1, 2, \ldots$, we have $\tau_k \leq \pi^{-2}$.*

Proof. Let T be the linear bounded operator on $L^2((0,\infty),du)$ defined by

$$Tg(u) := \int_0^\infty \frac{1}{u+v} g(v)dv.$$

Then, by Hilbert's theorem (cf. [HLP, Theorems 316 and 317]), the operator norm $\|T\|$ is equal to π. Hence, for the inner product $(\cdot,\cdot)$ of $L^2((0,\infty),du)$ and $f(x) := 1/(1+x)$, we have $\tau_k \le \pi^{-k}(f, T^{k-2}f) \le \pi^{-2}$, yielding the proposition. □

Proof of Theorem 2.3. To apply the dominated convergence theorem, we choose $r > 1$ so that $0 < r\sin(\pi d) < 1$. Then, by Proposition 5.3, $\sum_{k=1}^\infty \tau_{2k-1}\{r\sin(\pi d)\}^{2k-1} < \infty$. Hence Proposition 5.2 and the dominated convergence theorem yield

$$\lim_{n\to\infty} n\sum_{k=1}^\infty d_{2k-1}(n) = \sum_{k=1}^\infty \tau_{2k-1}\sin^{2k-1}(\pi d). \tag{5.4}$$

By Lemma 5.4 below, the right-hand side is equal to d. Thus, by Corollary 2.2, (d/n) follows. □

Lemma 5.4. *For $|x| < 1$, we have $\sum_{k=1}^\infty \tau_{2k-1}x^{2k-1} = \pi^{-1}\arcsin x$.*

Proof. (Compare the proof of [I2, Lemma 6.5].) For $0 < d < 1/2$, let $\{Y_n : n \in \mathbf{Z}\}$ be a fractional ARIMA$(0,d,0)$ process such that $E[(Y_0)^2] = \Gamma(1-2d)/\Gamma^2(1-d)$. We denote by c'_n, a'_n, and α'_n the MA and AR coefficients, and PACF of $\{Y_n\}$, respectively. Then we have, for $n = 0, 1, \dots,$

$$c'_n = \frac{\Gamma(n+d)}{\Gamma(n+1)\Gamma(d)}, \qquad a'_n = \frac{\Gamma(n-d)d}{\Gamma(n+1)\Gamma(1-d)}$$

(see, e.g., [BD, §13.2]). We define $d'_k(n)$ similarly. Then since $\{Y_n\}$ satisfies $(\mathrm{L}(d,\ell'))$ with $\ell' \equiv 1/\Gamma(d)$, it follows from (5.4) that

$$\lim_{n\to\infty} n\sum_{k=1}^\infty d'_{2k-1}(n) = \sum_{k=1}^\infty \tau_{2k-1}\{\sin(\pi d)\}^{2k-1}.$$

However, since $\alpha'_n = d/(n-d)$, Corollary 2.2 gives

$$\lim_{n\to\infty} n\sum_{k=1}^\infty d'_{2k-1}(n) = \lim_{n\to\infty} n\alpha'_n = d.$$

Combining, we obtain $\sum_1^\infty \tau_{2k-1}\sin^{2k-1}(\pi d) = d$. The lemma follows if we substitute $\pi^{-1}\arcsin x$ with $0 < x < 1$ for d and use analytic continuation. □

Remark. From Lemma 5.4, it follows that

$$\tau_{2k-1} = \frac{1}{\pi}\cdot\frac{(2k-2)!}{2^{2k-2}((k-1)!)^2(2k-1)} \qquad (k = 1, 2, \dots).$$

Proof of Theorem 2.4. Let $\{X_n\}$ be a fractional ARIMA(p,d,q) process with spectral density (2.14) with (2.15). We assume that $-1/2 < d < 0$. Then (2.16) and (2.17) imply $\{c_n\} \in l^1$ and $\{a_n\} \in l^2$, respectively, so that we can use Theorem 3.3. Let $d_k(n,p)$ be as in Theorem 3.3. Recall ϕ_n, ψ_n and $\beta_-(n)$ from §2. By [IK1, Theorem 3.3], we have the same conclusions as those in Proposition 4.3 with $\beta(n)$ replaced by $\beta_-(n)$. Notice that, in [IK1, Theorem 3.3], the results are stated for $n \ge 2$ but we can prove the case $n = 1$ in the same way. The equality (4.4) holds in the same way as the proof of Proposition 4.4. Hence the theorem follows from Theorem 3.3. □

Proof of Theorem 2.5. If $0 < d < 1/2$, then (d/n) follows immediately from Theorem 2.3. We assume that $-1/2 < d < 0$. Let $q := d + 1 \in (1/2, 1)$ as in §2. Then, from (2.21), (2.22) and [I1, Proposition 4.3], it follows that

$$\beta_-(n) \sim -\frac{\sin(\pi q)}{\pi} n^{-1} \qquad (n \to \infty).$$

Running through the same arguments as those in the proof of Theorem 2.3, we see that

$$\lim_{n\to\infty} n\alpha_n = \lim_{n\to\infty} n \sum_{k=1}^{\infty} d_{2k-1}(n) = -\sum_{k=1}^{\infty} \tau_{2k-1} \sin^{2k-1}(\pi q) = d.$$

Thus, again, (d/n) holds. □

6. Model with regularly varying autocovariance function

In this section, we apply the representation of PACF to a stationary process $\{X_n\}$ which has regularly varying autocovariance function. We will also assume that $\{X_n\}$ is PND and satisfies the following conditions (cf. [I2, §2]):

(C1) $\qquad c_n \ge 0$ *for all* $n \ge 0$;

(C2) $\qquad \{c_n\}$ *is eventually decreasing to zero;*

(A1) $\qquad \{a_n\}$ *is eventually decreasing to zero.*

Notice that (C1) and (A1) imply $\{a_n\} \in l^1$ (see [I2, Proposition 4.3]). In [I2], the extra condition

(A2) $\qquad \{a_n - a_{n+1}\}$ *is eventually decreasing to zero*

is also required but we do not need it here. By [I2, Theorem 7.3], $\{X_n\}$ satisfies (C1)–(A1) (and also (A2)) if

(RP) $\qquad$ there exists a finite Borel measure σ on $[0, 1)$ such that $\gamma_n = \int_0^1 t^{|n|}\sigma(dt) \quad (n \in \mathbf{Z})$.

This property is called *reflection positivity* or *T-positivity*, which originates in quantum field theory; see, e.g., Osterwalder and Schrader [OS], Hegerfeldt [He] and Okabe [O]. A prototype of such a process is $\{X_n\}$ with $\gamma_n = (1 + |n|)^{-(1-2d)}$, $-\infty < d < 1/2$, which we consider in the Example below.

Let $\ell \in \mathcal{R}_0$, and choose a positive constant B so large that $\ell(\cdot)$ is locally bounded on $[B, \infty)$ (see [BGT, Corollary 1.4.2]). When we say $\int^\infty \ell(s)ds/s = \infty$, it means that $\int_B^\infty \ell(s)ds/s = \infty$. If so, then we define another slowly varying function $\tilde{\ell}$ by

$$\tilde{\ell}(x) := \int_B^x \frac{\ell(s)}{s} ds \qquad (x \ge B) \tag{6.1}$$

(see [BGT, §1.5.6]). The asymptotic behaviour of $\tilde{\ell}(x)$ as $x \to \infty$ does not depend on the choice of B since we have assumed that $\int^\infty \ell(s)ds/s = \infty$.

Here is the result on the asymptotic behaviour of α_n.

Theorem 6.1. *Let $-\infty < d < 1/2$ and $\ell \in \mathcal{R}_0$. We assume* (PND), (C1), (C2), (A1), *and*

$$\gamma_n \sim n^{2d-1}\ell(n) \qquad (n \to \infty). \tag{6.2}$$

(1) *If $0 < d < 1/2$, then (d/n) holds;*
(2) *if $d = 0$ and $\int^\infty \ell(s)ds/s = \infty$, then*

$$\alpha_n \sim n^{-1}\frac{\ell(n)}{2\tilde{\ell}(n)} \qquad (n \to \infty); \tag{6.3}$$

(3) *if $d = 0$ with $\int^\infty \ell(s)ds/s < \infty$ or $-\infty < d < 0$, then*

$$\alpha_n \sim \frac{n^{2d-1}\ell(n)}{\sum_{-\infty}^{\infty}\gamma_k} \qquad (n \to \infty). \tag{6.4}$$

This theorem is an improvement of [I2, Theorem 2.1] where only $|\alpha_n|$ is considered with additional assumption (A2).

We need some propositions to prove the theorem above.

Proposition 6.2. *Let $\ell \in \mathcal{R}_0$. If $\int^\infty \ell(s)ds/s = \infty$, then $\ell(n)/\tilde{\ell}(n)$ tends to 0 as $n \to \infty$. If $\int^\infty \ell(s)ds/s < \infty$, then $\ell(n)$ tends to 0 as $n \to \infty$.*

Proposition 6.2 follows immediately from [BGT, Proposition 1.5.9a].

Proposition 6.3. *Let $\ell \in \mathcal{R}_0$ and $-\infty < d \le 0$. We assume* (PND), (C1), (C2), (A1), *and* (6.2).

(1) *If $d = 0$ and $\int^\infty \ell(s)ds/s = \infty$, then*

$$c_n \sim n^{-1}\ell(n)\{2\tilde{\ell}(n)\}^{-1/2} \qquad (n \to \infty), \tag{6.5}$$

$$a_n \sim n^{-1}\ell(n)\{2\tilde{\ell}(n)\}^{-3/2} \qquad (n \to \infty), \tag{6.6}$$

$$\beta_n \sim n^{-1}\ell(n)\{2\tilde{\ell}(n)\}^{-1} \qquad (n \to \infty). \tag{6.7}$$

(2) *If $d = 0$ with $\int^\infty \ell(s)ds/s < \infty$ or $-\infty < d < 0$, then*

$$c_n \sim n^{2d-1}\ell(n)\left\{\sum_{-\infty}^{\infty}\gamma_k\right\}^{-1/2} \qquad (n \to \infty), \tag{6.8}$$

$$a_n \sim n^{2d-1}\ell(n)\left\{\sum_{-\infty}^{\infty}\gamma_k\right\}^{-3/2} \qquad (n \to \infty), \tag{6.9}$$

$$\beta_n \sim n^{2d-1}\ell(n)\left\{\sum_{-\infty}^{\infty}\gamma_k\right\}^{-1} \qquad (n \to \infty). \tag{6.10}$$

Proof. The assertions (6.5) and (6.6) follow from [I2, Theorem 5.2]. Using them, we obtain (6.7) (see [I2, (6.19)]). The assertions (6.8) and (6.9) follow from [I2, Theorem 5.3]. From them, we get (6.10) (see the proof of [I2, Theorem 6.7]). □

Proposition 6.4. *Let $\ell \in \mathcal{R}_0$ and $-\infty < d \le 0$. We assume* (PND), (C1), (C2), (A1), *and* (6.2).

(1) *For every $R \in (1, \infty)$, there exists $N \in \mathbf{N}$ such that*

$$\left|\frac{\beta([ns]+[nu]+n)}{\beta(n)}\right| \le \frac{R}{(s+u+1)} \qquad (s \ge 0,\ u \ge 0,\ n \ge N). \tag{6.11}$$

(2) *For every $r \in (0, 1)$, there exists $N \in \mathbf{N}$ such that*

$$|\beta([ns]+[nu]+n)| \le \frac{r}{\pi(s+u+1)}n^{-1} \qquad (s \ge 0,\ u \ge 0,\ n \ge N). \tag{6.12}$$

Proof. By Proposition 6.3 and [BGT, Theorem 1.5.2], we have

$$\beta([ns]+[nu]+n)/\beta(n) \to (s+u+1)^{2d-1} \qquad (n\to\infty)$$

uniformly in $s\geq 0$ and $u\geq 0$. Since

$$(s+u+1)^{2d-1}\leq (s+u+1)^{-1},$$

(1) follows. By Propositions 6.2 and 6.3, we have

$$\lim_{n\to\infty} n\beta_n = 0. \tag{6.13}$$

This and (1) show (2). □

Recall $d_k(n)$ from (2.3)–(2.5) and τ_k from §5.

Proposition 6.5. *Let $\ell\in\mathcal{R}_0$ and $-\infty<d\leq 0$. We assume* (PND), (C1), (C2), (A1), *and* (6.2).

1. *Let $r\in(0,1)$ and $R\in(1,\infty)$. Choose $N\in\mathbf{N}$ so that both* (6.11) *and* (6.12) *hold. Then $|d_k(n)/\beta_n|\leq \pi R r^{k-1}\tau_k$ for $k\in\mathbf{N},\ n\geq N$.*
2. *For $k\geq 2$, $\lim_{n\to\infty} d_k(n)/\beta_n = 0$.*

Proof. In the same way as the proof of (6.13), we see that (O$(1/n)$) holds. We assume $k\geq 3$; the cases $k=1,2$ can be treated in a similar way. We write

$$\begin{aligned}\frac{d_k(n)}{\beta(n)} &= \int_0^\infty ds_{k-1}\cdots\int_0^\infty ds_1 \frac{\beta([ns_{k-1}]+n)}{\beta(n)}\\ &\quad\times\left\{\prod_{m=1}^{k-2} n\beta([ns_{m+1}]+[ns_m]+n)\right\}\times n\beta([ns_1]+n).\end{aligned}$$

From this as well as (6.11) and (6.12), we get (1). The assertion (2) follows from (6.11)–(6.13) and the dominated convergence theorem. □

Proof of Theorem 6.1. Assume that $0<d<1/2$. Then, from [I2, Theorem 5.1], (L(d,ℓ')) holds with $\ell'(n):=[\ell(n)/B(d,1-2d)]^{1/2}$. Hence (d/n) follows immediately from Theorem 2.3.

Next, we assume $-\infty<d\leq 0$. By (A1) and (C1), $\sum_{v=0}^\infty |c_v a_{v+n}| = \beta_n$ if n is large enough. Hence from (6.7), (6.10) and Proposition 6.2, (O$(1/n)$) holds. By Proposition 6.5 and the dominated convergence theorem, we have

$$\lim_{n\to\infty}\frac{\sum_{k=1}^\infty d_{2k-1}(n)}{\beta_n} = 1+\lim_{n\to\infty}\sum_{k=2}^\infty \frac{d_{2k-1}(n)}{\beta_n} = 1.$$

Therefore, by Corollary 2.2, we see (2.10). Thus (2) and (3) follow from (6.7) and (6.10), respectively. □

Example. Let $-\infty<d<1/2$, and let $\{X_n\}$ be a stationary process with autocovariance function of the form $\gamma_n=(1+|n|)^{-(1-2d)}$. Then $\{X_n\}$ satisfies (RP) (cf. [I2, Example in §7]). Let $\{\alpha_n\}$ be the PACF of $\{X_n\}$. Applying Theorem 6.1 to $\{X_n\}$, we get the following result:

1. if $0<d<1/2$, then we have (d/n).
2. if $d=0$, then $\alpha_n\sim 1/(2n\log n)$ as $n\to\infty$.
3. if $-\infty<d<0$, then $\alpha_n\sim n^{2d-1}\cdot[2\zeta(1-2d)-1]^{-1}$ as $n\to\infty$.

Here $\zeta(s)$ is the Riemann zeta function.

Remarks. 1. Recall by (2.3) that β_n is the first term on the right of (2.9). By the arguments above, we see that, for the processes treated in Theorem 6.1,

$$\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}=\begin{cases}\frac{\pi d}{\sin(\pi d)} & (0<d<1/2),\\ 1 & (-\infty<d\le 0).\end{cases} \tag{6.14}$$

We raise, and leave open, the question of how generally this happens.
2. We note that in Theorems 2.3, 2.5 and 6.1 we have also

$$\alpha_n\sim\frac{\gamma_n}{\sum_{k=-n}^{n}\gamma_k}\qquad(n\to\infty)$$

(see [I2, §2 and §6] and [I3, §5] for the proofs). It would be interesting to know how generally this relation holds.

7. ARMA processes

In this section, we consider the fractional ARIMA$(p,0,q)$ processes, that is, the ARMA(p,q) processes. Let $p,q\in\mathbf{N}\cup\{0\}$, and let $\Phi(z)$ and $\Theta(z)$ be polynomials with real coefficients of degrees p, q, respectively, satisfying (2.15). Let $\{X_n\}$ be an ARMA(p,q) process with spectral density

$$\Delta(\theta)=\frac{1}{2\pi}\frac{|\Theta(e^{i\theta})|^2}{|\Phi(e^{i\theta})|^2}\qquad(-\pi<\theta<\pi).$$

We put $R:=0$ if $q=0$ and

$$R:=\max\left(1/|u_1|,\dots,1/|u_q|\right)\quad\text{if } q\ge 1,$$

where $u_1,\dots,u_q$ are the (complex) zeros of $\Theta(z)$:

$$\Theta(z)=\text{const.}\times(z-u_1)\cdots(z-u_q).$$

From the assumption (2.15), we see that $|u_k|>1$ for $k=1,\dots,q$, whence $R\in[0,1)$.

Let $\{\alpha_n\}$ be the PACF of the ARMA(p,q) process $\{X_n\}$. The next theorem implies that α_n decays exponentially as $n\to\infty$.

Theorem 7.1. *For every $r>R$, we have*

$$\alpha_n=O(r^n)\qquad(n\to\infty). \tag{7.1}$$

In particular, α_n decays exponentially fast as $n\to\infty$.

Proof. The Szegö function $D(z)$ of $\{X_n\}$ is given by $D(z)=\Theta(z)/\Phi(z)$ for $|z|<1$. Hence $-\Phi(z)/\Theta(z)=\sum_0^\infty a_n z^n$, so that, for every $r>R$,

$$a_n=O(r^n)\qquad(n\to\infty).$$

Treating $D(z)=\Theta(z)/\Phi(z)$ similarly, we see that c_n also decays exponentially as $n\to\infty$, in particular, $\{c_n\}\in l^1$. Therefore, we see that, for every $r>R$,

$$\beta_n=O(r^n)\qquad(n\to\infty).$$

The assertion (7.1) follows easily from this and Theorem 2.1. Since r can be chosen so that $R<r<1$, (7.1) implies exponential decay of α_n. □

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Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: `inoue@math.sci.hokudai.ac.jp`