KYUSHU UNIVERSITY 100th 2011
知の新世紀を拓く

常微分方程式で表す摩擦と接触

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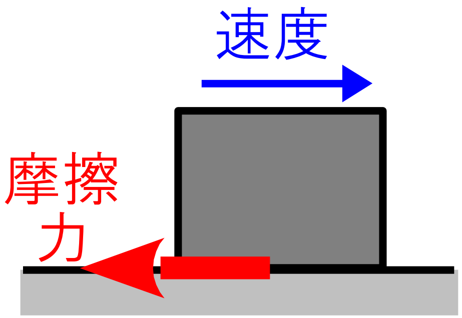
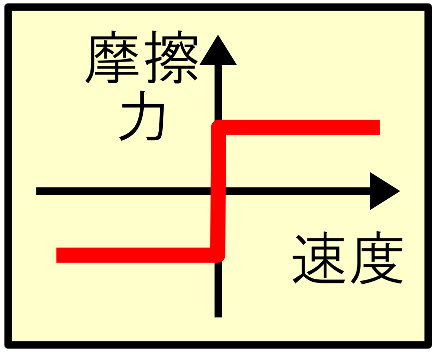
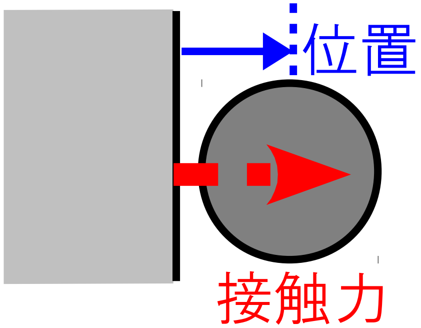
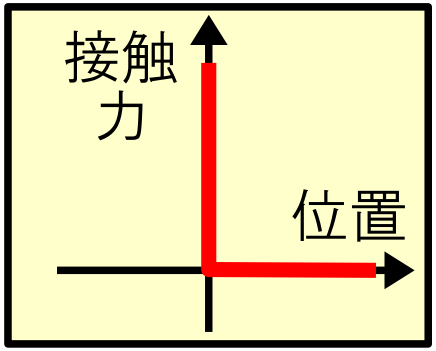
摩擦と接触：力と運動の関係が不連続

- 摩擦（クーロン摩擦）
- 接触（片側剛体拘束）





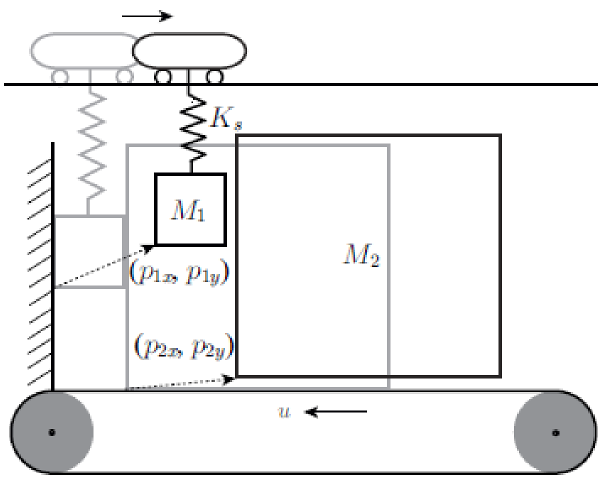




- ルンゲ・クッタ法などの数値積分法が使えない.
- 滑らかに近似しても挙動が不自然.
  - 摩擦
    ⇒ チャタリング or 静止摩擦力以下でもすべる.
  - 接触（バネダンパで近似すると）
    ⇒ 粘性項によって吸着力が発生.

本発表で提案するもの

- 摩擦・接触を含む系を，不連続性のエッセンスを維持したまま，常微分方程式に近似する手順



$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} \in \begin{bmatrix} \dot{p} \\ (\mathrm{dio}(p_{1x}) - \mathrm{dio}(p_{2x} - p_{1x}))/M_1 \\ (-\Omega_1 - \Omega_2 - K_s p_{1y})/M_1 \\ (-\Omega_3 + \mathrm{dio}(p_{2x} - p_{1x}))/M_2 \\ (\Omega_2 + \mathrm{dio}(p_{2y}))/M_2 - g \end{bmatrix}$$

【正しいが計算できない式】

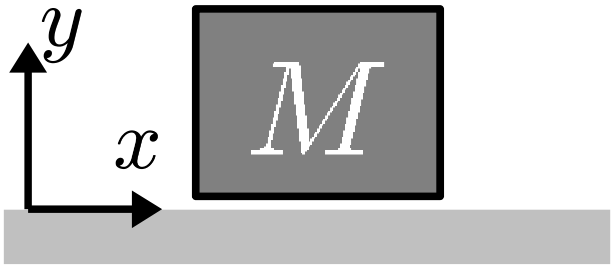
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ \dot{p} \\ e_1 \\ e_2 \\ e_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ (\psi_1 - \psi_2)/M_1 \\ (-\theta_1 - \theta_2 - K_s p_{1y})/M_1 \\ (-\theta_3 + \psi_2)/M_2 \\ (\theta_2 + \psi_3)/M_2 - g \\ \max(-e_1/\beta_1, -(p_{1x} + e_1)/\alpha_1) \\ \max(-e_2/\beta_2, -(p_{2x} - p_{1x} + e_2)/\alpha_2) \\ \max(-e_3/\beta_3, -(p_{2y} + e_3)/\alpha_3) \\ (\theta_1/K_4 - a_1)/\beta_4 \\ (\theta_2/K_5 - a_2)/\beta_5 \\ (\theta_3/K_6 - a_3)/\beta_6 \end{bmatrix}$$

【計算できる常微分方程式】

- 2つの不連続関数を用いて状態方程式を記述．
- ある規則に沿って，不連続なパーツを連続なパーツで置換．
- 新しい状態変数を追加．

摩擦・接触を含むシステムの数式表現

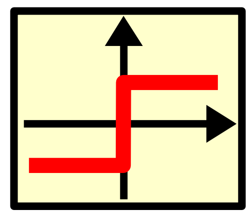
- 例：摩擦のある床と質点の運動方程式

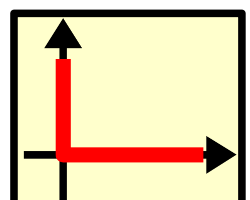


$$M\ddot{p}_x \in \mu\,\mathrm{dio}(p_y)\,\mathrm{sgn}(\dot{p}_x)$$

$$M\ddot{p}_y \in \mathrm{dio}(p_y) - mg$$

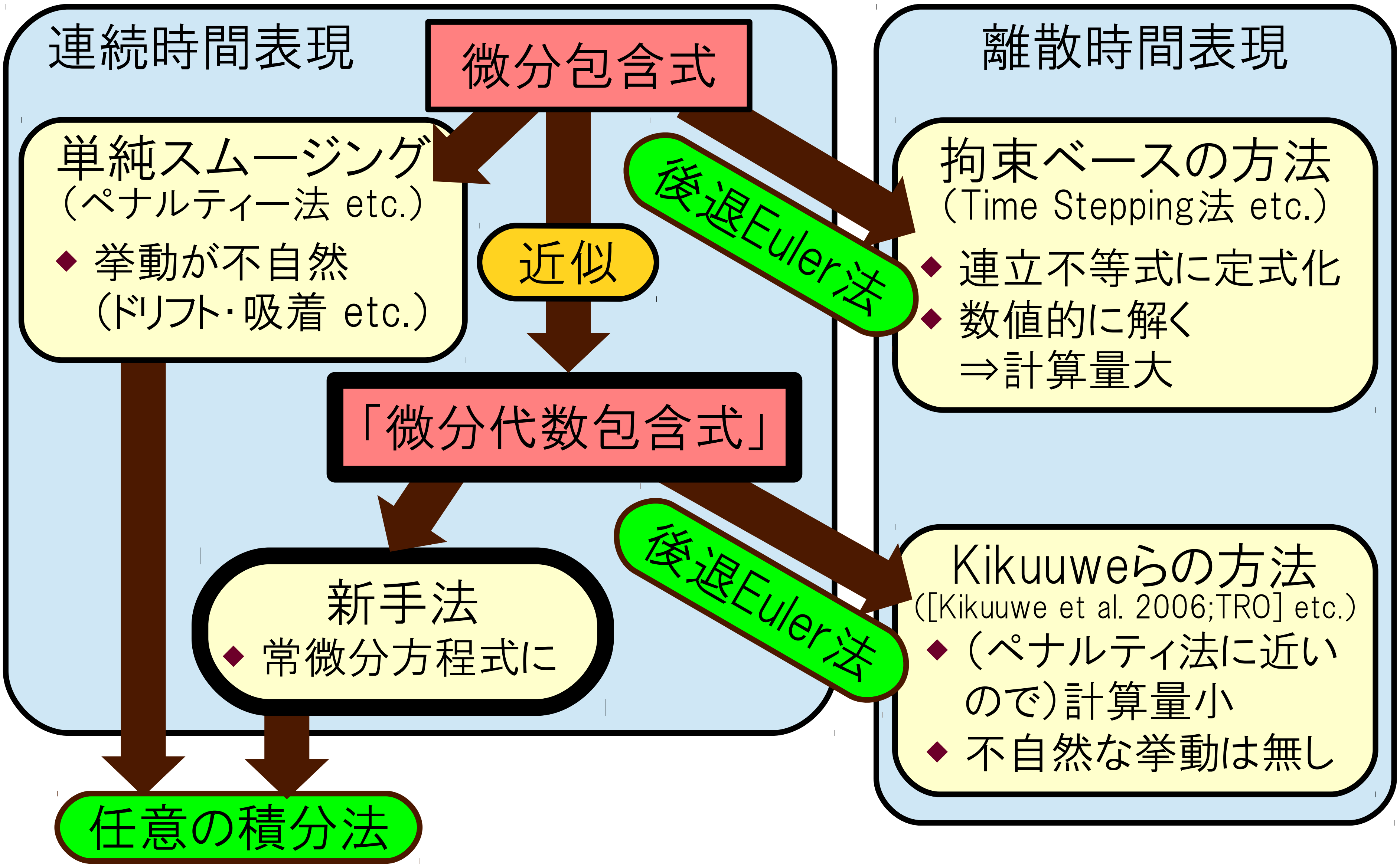
（微分包含式; differential inclusions）

- 符号関数：$\mathrm{sgn}(x) \triangleq$  $= \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$

- ダイオード関数：$\mathrm{dio}(x) \triangleq$  $= \begin{cases} [0, \infty) & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

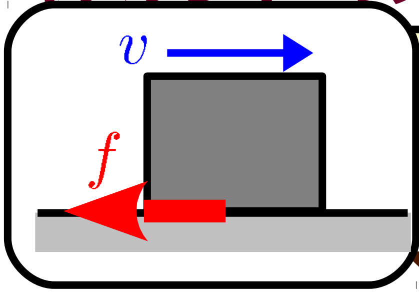
- 微分包含式表現：正しいが計算できない．

不連続なシステムのシミュレーション



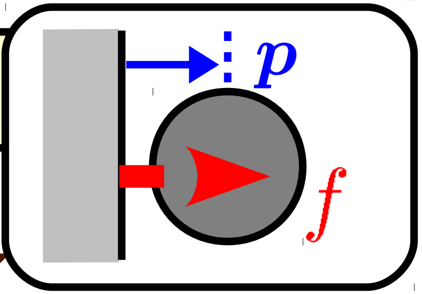
微分代数近似+後退オイラー法

[Kikuuweら, 2006 TRO]
[菊植・藤本, 2007 JRSJ]



$f \in -F\mathrm{sgn}(v)$

$f \in \mathrm{dio}(p)$



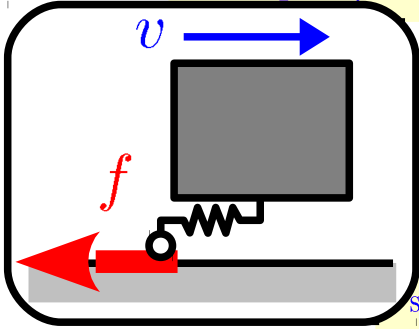
微分代数包含式で近似

$f \in -F\mathrm{sgn}(v - \dot{a})$

$F\mathrm{sgn}(v - \dot{a}) \ni K(a + \beta\dot{a})$

$f \in \mathrm{dio}(p + e)$

$\mathrm{dio}(p + e) \ni K(e + \beta\dot{e})$



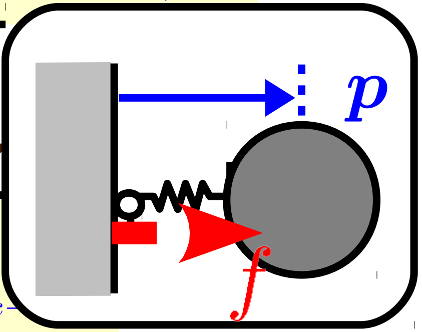
後退オイラー法

$= -K\left(a_k + \beta\frac{a_k - a_{k-1}}{T}\right)$

$\mathrm{sgn}\left(v_k - \frac{a_k - a_{k-1}}{T}\right) \ni K\left(a_k + \beta\frac{a_k - a_{k-1}}{T}\right)$

$f_k = K\left(e_k + \beta(e_k - e_{k-1})/T\right)$

$\mathrm{dio}(p_k + e_k) \ni K\left(e_k + \beta(e_k - e_{k-}\right.$



計算できる式に等価変換

$$a_k = \frac{\beta a_{k-1} + T\mathrm{sat}(F/K, a_{k-1} + (\beta + T)v_k)}{\beta + T}$$

$$f_k = K\left(a_k + \beta(a_k - a_{k-1})/T\right)$$

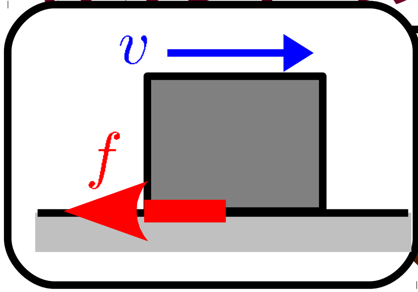
$$e_k = \max\left(\beta e_{k-1}/(T + \beta), -p_k\right)$$

$$f_k = K\left(e_k + \beta(e_k - e_{k-1})/T\right)$$

- 静止摩擦で滑らない.
- 吸着しない

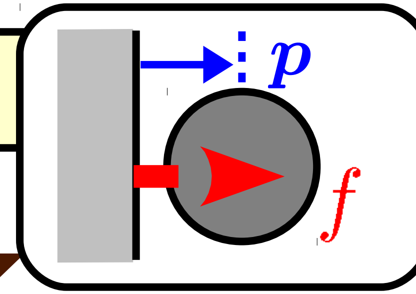
微分代数近似+後退オイラー法

[Kikuuweら, 2006 TRO]
[菊植・藤本, 2007 JRSJ]



$$f \in -F\mathrm{sgn}(v)$$

$$f \in \mathrm{dio}(p)$$



微分代数包含式で近似

$$f \in -F\mathrm{sgn}(v - \dot{a})$$

$$F\mathrm{sgn}(v - \dot{a}) \ni K(a + \beta$$

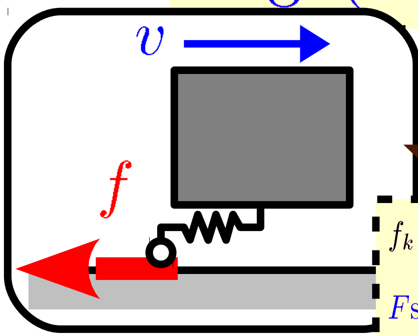
$$f \in \mathrm{dio}(p + e)$$

$$x \in Z\mathrm{sgn}(a - x) \iff x = \mathrm{sat}(Z, a)$$

$$x \in \mathrm{dio}(a + x) \iff x = \max(0, -a)$$

[Kikuuweら, 2006 TRO, 菊植・藤本, 2007 JRSJ]

後退

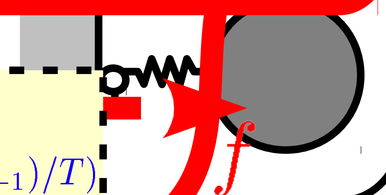


$$f_k = -K\left(a_k + \beta\frac{a_k - a_{k-1}}{T}\right)$$

$$F\mathrm{sgn}\left(v_k - \frac{a_k - a_{k-1}}{T}\right) \ni K\left(a_k + \beta\frac{a_k - a_{k-1}}{T}\right)$$

$$f_k = K\left(e_k + \beta(e_k - e_{k-1})/T\right)$$

$$\mathrm{dio}(p_k + e_k) \ni K\left(e_k + \beta(e_k - e_{k-1})/T\right)$$



計算できる式に等価変換

$$a_k = \frac{\beta a_{k-1} + T\mathrm{sat}(F/K, a_{k-1} + (\beta + T)v_k)}{\beta + T}$$

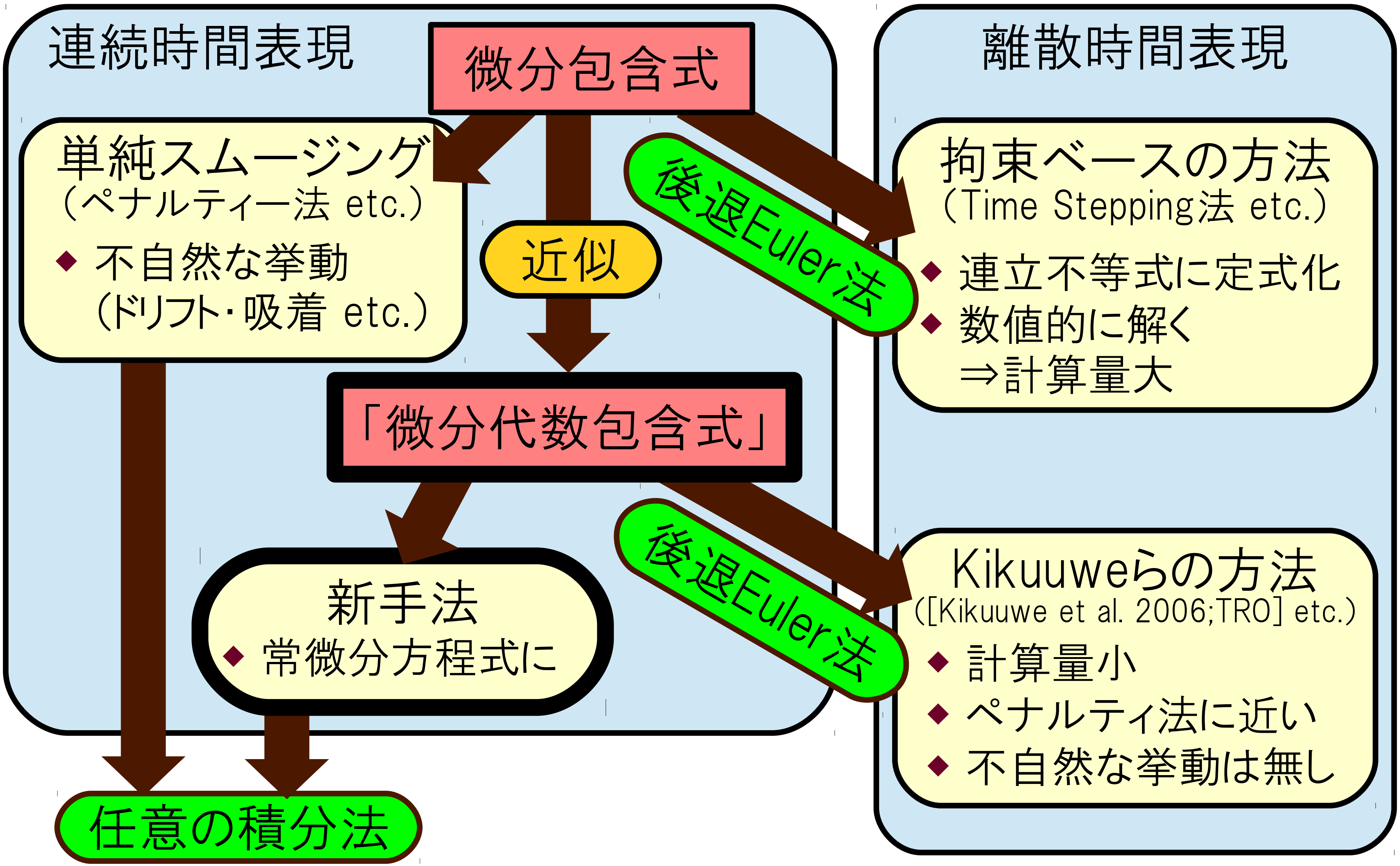
$$f_k = K\left(a_k + \beta(a_k - a_{k-1})/T\right)$$

$$e_k = \max\left(\beta e_{k-1}/(T + \beta), -p_k\right)$$

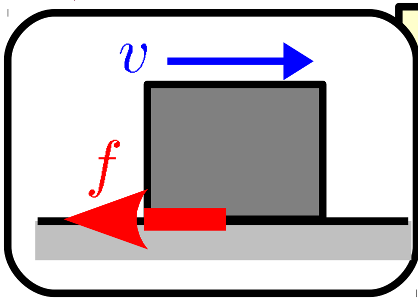
$$f_k = K\left(e_k + \beta(e_k - e_{k-1})/T\right)$$

- 静止摩擦で滑らない.
- 吸着しない

新手法：任意の積分法が利用可能

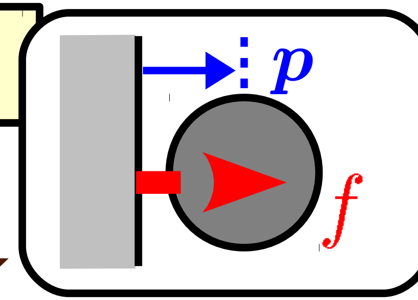


新手法：常微分方程式へ



$$f \in -F\mathrm{sgn}(v)$$

$$f \in \mathrm{dio}(p)$$



微分代数包含式で近似

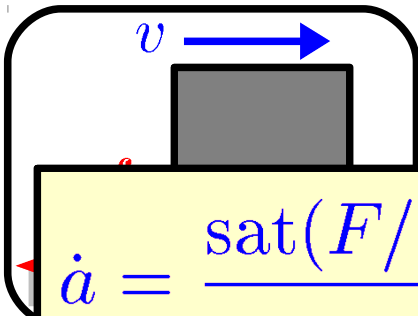
$$f \in -F\mathrm{sgn}(v - \dot{a})$$

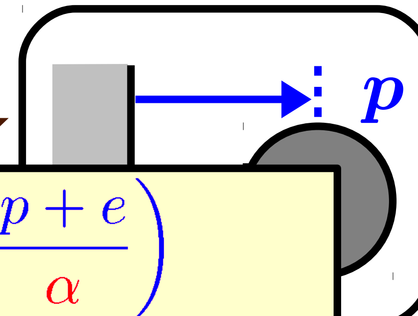
$$F\mathrm{sgn}(v - \dot{a}) \ni K(a + \beta\dot{a})$$

$$f \in \mathrm{dio}(p + e + \alpha\,\dot{e})$$

$$\mathrm{dio}(p + e + \alpha\dot{e}) \ni K(e + \beta\dot{e})$$

常微分方程式に等価変換





$$\dot{a} = \frac{\mathrm{sat}(F/K, a + \beta v) - a}{\beta}$$

$$f = -\mathrm{sat}(F, K(a + \beta v))$$

$$\dot{e} = \max\left(-\frac{e}{\beta}, -\frac{p+e}{\alpha}\right)$$

$$f = K\max\left(0, e - \frac{\beta(p+e)}{\alpha}\right)$$

任意の積分法

新手法：常微分方程式へ

$f \in -F\mathrm{sgn}(v)$

$f \in \mathrm{dio}(p)$

- $\triangle\mathrm{sgn}(\square)$ を $\mathrm{sat}(\triangle, K(a+\beta\square))$で置換.
- $\dot{a} = \dfrac{\mathrm{sat}(\triangle/K, a+\beta\square) - a}{\beta}$を追加.

- $\mathrm{dio}(\bigcirc)$ を $\max\left(0, Ke - \dfrac{B(\bigcirc + e)}{\alpha}\right)$で置換.
- $\dot{e} = \max\left(-\dfrac{e}{\beta}, -\dfrac{\bigcirc + e}{\alpha}\right)$ を追加.

$$\dot{a} = \frac{\mathrm{sat}(F/K, a+\beta v) - a}{\beta}$$

$$f = -\mathrm{sat}(F, K(a+\beta v))$$

$$\dot{e} = \max\left(-\frac{e}{\beta}, -\frac{p+e}{\alpha}\right)$$

$$f = K\max\left(0, e - \frac{\beta(p+e)}{\alpha}\right)$$

任意の積分法

結局・・・

[1] $\mathrm{dio}(\bigcirc)$ を $\max\left(0, Ke - \dfrac{B(\bigcirc + e)}{\alpha}\right)$ で置換

[2] $\triangle\mathrm{sgn}(\square)$を $\mathrm{sat}(\triangle, K(a + \beta\square))$ で置換

[3] 対応する $\left\{\begin{aligned} \dot{e} &= \max\left(-\dfrac{e}{\beta}, -\dfrac{\bigcirc + e}{\alpha}\right) \\ \dot{a} &= \dfrac{\mathrm{sat}(\triangle/K, a + \beta\square) - a}{\beta} \end{aligned}\right\}$ を追加

例:

$$f \in \mu\,\mathrm{dio}(p_y)\,\mathrm{sgn}(v_x)$$

[1] $$f \in \mu\max\left(0, Ke - \frac{B(p_y+e)}{\alpha}\right)\mathrm{sgn}(v_x)$$

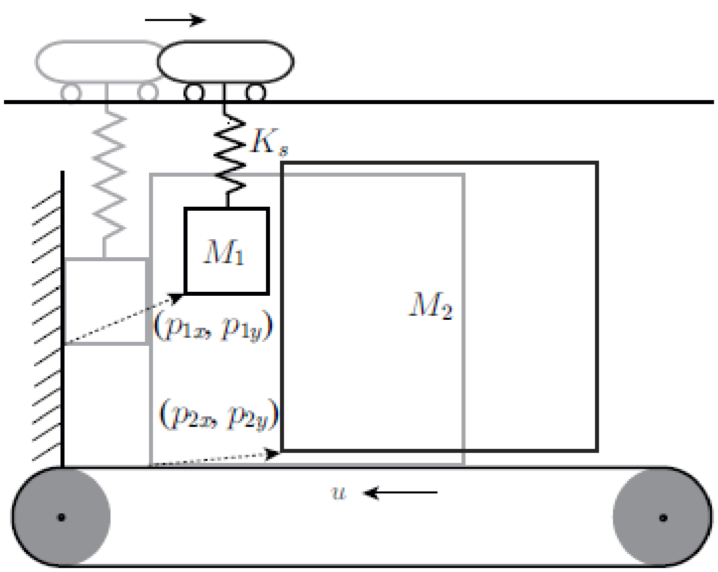
[2] $$f = \mathrm{sat}\left(\mu\max\left(0, Ke - \frac{B(p_y+e)}{\alpha}\right), K(a + \beta v_x)\right)$$

[3] $$\left\{\begin{aligned} \dot{e} &= \max\left(-\frac{e}{\beta}, -\frac{p_y+e}{\alpha}\right) \\ \dot{a} &= \frac{\mathrm{sat}\left(\mu\max\left(0,Ke-\frac{B(p_y+e)}{\alpha}\right)/K, a+\beta v_x\right)}{\beta} \end{aligned}\right\}$$

任意の積分法

例題：複数の摩擦接触を含むシステム

- まず，dioとsgnを用いた状態空間表現（＝微分包含式）で表す．



$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ \hdashline \dot{p} \end{bmatrix} \in \begin{bmatrix} \dot{p} \\ \hdashline (\mathrm{dio}(p_{1x}) - \mathrm{dio}(p_{2x} - p_{1x}))/M_1 \\ (-\Omega_1 - \Omega_2 - K_s p_{1y})/M_1 \\ (-\Omega_3 + \mathrm{dio}(p_{2x} - p_{1x}))/M_2 \\ (\Omega_2 + \mathrm{dio}(p_{2y}))/M_2 - g \end{bmatrix}$$

ただし

$$\Omega_1 \triangleq \mu_1 \mathrm{dio}(p_{1x})\mathrm{sgn}(\dot{p}_{1y})$$

$$\Omega_2 \triangleq \mu_2 \mathrm{dio}(p_{2x} - p_{1x})\mathrm{sgn}(\dot{p}_{1y} - \dot{p}_{2y})$$

$$\Omega_3 \triangleq \mu_3 \mathrm{dio}(p_{2y})\mathrm{sgn}(\dot{p}_{2x} + u)$$

- このままだと数値的に扱えないが･･･

例題）変換の手順

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} \in \begin{bmatrix} \dot{p} \\ (\mathrm{dio}(p_{1x}) - \mathrm{dio}(p_{2x}-p_{1x}))/M_1 \\ (-\Omega_1 - \Omega_2 - K_s p_{1y})/M_1 \\ (-\Omega_3 + \mathrm{dio}(p_{2x}-p_{1x}))/M_2 \\ (\Omega_2 + \mathrm{dio}(p_{2y}))/M_2 - g \end{bmatrix}$$

ただし

$$\Omega_1 \triangleq \mu_1 \mathrm{dio}(p_{1x})\mathrm{sgn}(\dot{p}_{1y})$$

$$\Omega_2 \triangleq \mu_2 \mathrm{dio}(p_{2x}-p_{1x})\mathrm{sgn}(\dot{p}_{1y}-\dot{p}_{2y})$$

$$\Omega_3 \triangleq \mu_3 \mathrm{dio}(p_{2y})\mathrm{sgn}(\dot{p}_{2x}+u)$$

- まずdioを置換
- 次にsgnを置換
- 変数を追加

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} p \\ \dot{p} \\ e_1 \\ e_2 \\ e_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ (\psi_1 - \psi_2)/M_1 \\ (-\theta_1 - \theta_2 - K_s p_{1y})/M_1 \\ (-\theta_3 + \psi_2)/M_2 \\ (\theta_2 + \psi_3)/M_2 - g \\ \max(-e_1/\beta_1, -(p_{1x}+e_1)/\alpha_1) \\ \max(-e_2/\beta_2, -(p_{2x}-p_{1x}+e_2)/\alpha_2) \\ \max(-e_3/\beta_3, -(p_{2y}+e_3)/\alpha_3) \\ (\theta_1/K_4 - a_1)/\beta_4 \\ (\theta_2/K_5 - a_2)/\beta_5 \\ (\theta_3/K_6 - a_3)/\beta_6 \end{bmatrix}$$

ただし

$$\psi_1 \triangleq K_1 \max(0, e_1 - \beta_1(p_{1x}+e_1)/\alpha_1)$$

$$\psi_2 \triangleq K_2 \max(0, e_2 - \beta_2(p_{2x}-p_{1x}+e_2)/\alpha_2)$$

$$\psi_3 \triangleq K_3 \max(0, e_3 - \beta_3(p_{2y}+e_3)/\alpha_3)$$

$$\theta_1 \triangleq K_4 \mathrm{sat}(\mu_1\psi_1/K_4, a_1 + \beta_4\dot{p}_{1y})$$

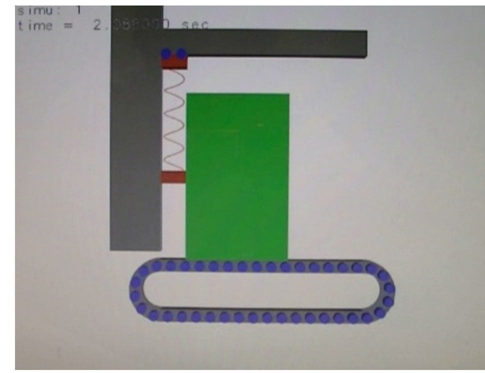
$$\theta_2 \triangleq K_5 \mathrm{sat}(\mu_2\psi_2/K_5, a_2 + \beta_5(\dot{p}_{1y}-\dot{p}_{2y}))$$

$$\theta_3 \triangleq K_6 \mathrm{sat}(\mu_3\psi_3/K_6, a_3 + \beta_6(\dot{p}_{2x}+u))$$

- 任意の積分法で積分できる常微分方程式に！

まとめ

- 摩擦と接触を含む機械システムの運動方程式を常微分方程式で表す数式操作の方法を示した.
  - 運動方程式をdioとsgnを用いて表記し,
    [1] まずdioを置換
    [2] 次にsgnを置換
    [3] 状態変数を追加



$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ \dot{p} \end{bmatrix} \in \begin{bmatrix} \dot{p} \\ (\mathrm{dio}(p_{1x}) - \mathrm{dio}(p_{2x} - p_{1x}))/M_1 \\ (-\Omega_1 - \Omega_2 - K_s p_{1y})/M_1 \\ (-\Omega_3 + \mathrm{dio}(p_{2x} - p_{1x}))/M_2 \\ (\Omega_2 + \mathrm{dio}(p_{2y}))/M_2 - g \end{bmatrix}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ \dot{p} \\ e_1 \\ e_2 \\ e_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ (\psi_1 - \psi_2)/M_1 \\ (-\theta_1 - \theta_2 - K_s p_{1y})/M_1 \\ (-\theta_3 + \psi_2)/M_2 \\ (\theta_2 + \psi_3)/M_2 - g \\ \max(-e_1/\beta_1, -(p_{1x} + e_1)/\alpha_1) \\ \max(-e_2/\beta_2, -(p_{2x} - p_{1x} + e_2)/\alpha_2) \\ \max(-e_3/\beta_3, -(p_{2y} + e_3)/\alpha_3) \\ (\theta_1/K_4 - a_1)/\beta_4 \\ (\theta_2/K_5 - a_2)/\beta_5 \\ (\theta_3/K_6 - a_3)/\beta_6 \end{bmatrix}
$$

- 今後の課題
  - パラメータ選定指針の明確化