Effects of educational subsidies and pay-as-you-go pensions on long-run growth under an endogenous growth model*

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Abstract

This study considers a three-period overlapping generations model with an endogenous growth setting, in which an agent borrows in the first period and repays the loan in the second period under a perfect credit market. Two educational subsidy schemes are considered: one is provided when an agent borrows and the other is provided when the agent repays his or her loan. This study compares the growth rates under each educational subsidy scheme for a balanced growth path and provides the sufficient conditions under which the growth rate in one scheme is larger than that in the other. A key to determining the size relationship of growth rates is whether the production of goods and services is physical-capital-intensive. Namely, when the production is sufficiently physical-capital-intensive, the interest rate in the credit market tends to be high, which discourages an agent from borrowing for his or her human capital investment. Hence, in this situation, an educational subsidy should be provided when an agent borrows to achieve a higher growth rate. If the production is not so physical-capital-intensive, with some additional conditions, educational subsidies should be provided when an agent repays his or her loan to achieve a higher growth rate.

Keywords: Endogenous growth, educational subsidy, pay-as-you-go pension, balanced growth path, growth rate

JEL Classification: O40, I22, H52, H55

1 Introduction

Since the studies of Lucas (1988) and Azariadis and Drazen (1990), human capital externalities have become important for understanding the sources of income differences across countries. However, a laissez-faire equilibrium allocation may not be dynamically efficient under human capital externalities. To correct the inefficiencies caused by these externalities, some form of educational subsidy is needed.

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If individuals need to borrow to pursue education in a perfect credit market, the question is when educational subsidies should be provided: when individuals borrow or when they repay their loans? If the timing of providing educational subsidies differs, an individual’s decision-making regarding human capital investment and savings will also differ, and this difference will cause different growth rates for a balanced growth path. Therefore, this study considers a three-period overlapping generations model with physical and human capital, investigating which educational subsidy scheme will lead to a higher growth rate for a balanced growth path.

The model is the same as those of Docquier et al. (2007) and Del Rey and Lopez-Garcia (2013). Namely, it is a three-period overlapping generations model, in which an agent borrows for education in the first period and repays the loan in the second period. As in Docquier et al. (2007) and Del Rey and Lopez-Garcia (2013), a perfect credit market is assumed. Under this setting, two educational subsidy schemes are considered: one is when educational subsidies are provided in the first period, when an agent borrows, and the other is when subsidies are provided in the second period, when an agent repays his or her loan. In addition to educational subsidies, the government implements pay-as-you-go pensions. The pay-as-you-go pension and educational subsidy are financed by labor income tax and I assume the government budget is balanced in each period. Hence, if the cost of education is subsidized in the first period of an agent’s life, the educational subsidy is an inter-generational transfer scheme. If the student loans are subsidized in the second period of an agent’s life, the educational subsidy is an intra-generational transfer scheme. Under each educational subsidy scheme, a perfect foresight competitive equilibrium for a balanced growth path, called the balanced growth path (BGP) equilibrium, is characterized and the growth rates of the BGP equilibrium under both schemes are compared.

The motivation of this study is closely related to that of Eckwert and Zilcha (2014). Namely, Eckwert and Zilcha (2014) considers a two period model, where human capital is under-invested and shows that two subsidization schemes similar to those in this paper induce individuals to invest more in terms of human capital and lead to a more socially desirable income distribution. Additionally, Eckwert and Zilcha (2014) show that subsidizing the cost of education dominates subsidizing student loans in the generalized Lorenz sense, and compares the two educational subsidy schemes from the perspective of income distribution, whereas this study compares them from the growth rate perspective in the long run. In contrast to the main results of Eckwert and Zilcha (2014), this study implies there is no dominance relationship between the two schemes from the long-run growth rate perspective.

In an endogenous growth setting, some studies, such as Blankenau (2005) and Docquier et al. (2007), consider a model for subsidizing the cost of education, while other papers, such as Yakita (2004), Del Rey and Lopez-Garcia (2013), Del Rey and Lopez-Garcia (2015), and Del Rey and Lopez-Garcia (2019), consider a model with subsidizing loans. To the best of my knowledge, this study is the first to compare the effects of different educational subsidy schemes under an endogenous growth setting.

As previously mentioned, the model in this study is the same as those of Docquier et al. (2007) and Del Rey and Lopez-Garcia (2013), who show how to implement optimal allocations in a decentralized economy by using educational subsidies and pay-as-you-go pensions. Further, Docquier et al. (2007) and Del Rey and Lopez-Garcia (2013)}
uses educational subsidies for the cost of education and Del Rey and Lopez-Garcia (2013) uses those for loans. As per footnote 3 of Del Rey and Lopez-Garcia (2013), how to design educational subsidies does not matter for implementation, whereas this study shows that it matters for the long-run growth rate.

This study considers not only educational subsidies but also pay-as-you-go pensions and assumes that the government budget is balanced in each period; hence, the labor income tax revenue needs to be split between the educational subsidies and pay-as-you-go pensions. Kaganovich and Zilcha (1999) considers a similar setting to this study, except that parents pay the cost of education for their children, and investigates how the allocation of tax revenues between educational subsidies and pay-as-you-go social security affects growth and welfare. Since parents pay the cost of education, Kaganovich and Zilcha (1999) do not consider the educational subsidy that covers part of the student loan. Moreover, Fan et al. (2018) examines the optimal allocation of tax revenues between children’s public education and elderly’s public consumption in a different setting from this study. Although this study’s main focus is not on the optimal allocation of tax revenue, it shows that the allocation of tax revenue matters in determining whether educational subsidies accelerate economic growth.

The main contribution of this paper is providing the sufficient conditions under which one of the two growth rates under the two educational subsidy schemes is larger than the other. A key to determining the size relationship of the two growth rates is whether the output production is physical-capital-intensive. When the production function is sufficiently physical-capital-intensive, the interest rate tends to be high. If a young agent borrows the cost of education, he or she needs to repay it with interest. A higher interest rate increases the interest payment. If an educational subsidy is provided when a young agent borrows, the interest payment will become small, even if the agent borrows a large amount. This merit is large when the interest rate is high. Therefore, when the production function is sufficiently physical-capital-intensive, the growth rate in an economy subsidizing the cost of education is higher than that in an economy subsidizing student loans. Conversely, when the production function is sufficiently effective-labor-intensive, the opposite is likely to hold because the interest rate tends to be low.

The remainder of this paper is organized as follows. Section 2 characterizes the BGP equilibrium in the model subsidizing the cost of education, and Section 3 describes the BGP equilibrium in the model subsidizing loans. In Section 4, the growth rates under the previous BGP equilibria are compared, and Section 5 concludes the paper.

### 2 Model: Subsidizing the cost of education

Time is discrete and continues forever, namely $t = 1, 2, \ldots$. An agent lives for three periods: young, middle, and old. Therefore, in each period $t$, three generations coexist. When an agent is young, in period $t - 1$, he or she decides on the educational expenditure, $e_{t-1}$, to accumulate human capital. Assuming that a young agent has no wealth or income, he or she must borrow to manage the educational expenditure. Additionally, a young agent receives educational subsidies provided by the government. If a young agent wants to spend $e_{t-1}$ to accumulate human capital, then the agent receives an educational subsidy, denoted by $\sigma_{t-1} e_{t-1}$ ($\sigma_{t-1} \in [0, 1]$) and borrows $(1 - \sigma_{t-1}) e_{t-1}$ in the credit market. A young agent who borrows $(1 - \sigma_{t-1}) e_{t-1}$ in period $t - 1$ will repay the debt in the next period with interest rate $r_i$. Furthermore, I assume a perfect credit market. If a young agent spends $e_{t-1}$ for human capital accumulation in period $t - 1$, the agent’s human capital in period $t$ will be:

$$h_t = \theta e_{t-1}^\eta h_{t-1}^{1-\eta}.$$
where \( \theta > 0 \) and \( \eta \in (0, 1) \). Assume that \( e_0 > 0 \) and \( h_0 > 0 \).

A middle-aged agent in period \( t \) is endowed with one unit of time and supplies this unit for labor inelastically. A middle-aged agent with human capital \( h_t \) receives an effective labor income of \( w_t h_t \), consumes \( c_t \), repays debt \( (1 + r_t)(1 - \sigma_{t-1})e_{t-1} \), and saves \( s_t \), where \( w_t \) is the effective wage in period \( t \) and \( r_t \) is the interest rate in period \( t \). Additionally, a middle-aged agent will pay a labor income tax whose tax rate is \( \tau \in [0, 1) \). I assume that tax rate \( \tau \) is time-invariant and set by the government. Then, the budget constraint for the middle-aged agent is:

\[
c_t + s_t + (1 + r_t)(1 - \sigma_{t-1})e_{t-1} = (1 - \tau)w_t h_t. \tag{1}
\]

When an agent becomes old, the agent retires. An old agent in period \( t + 1 \) has interest income \( (1 + r_{t+1})s_t \), receives social security benefits \( P_{t+1} \), and consumes \( d_{t+1} \). Therefore, the budget constraint for an old agent is:

\[
d_{t+1} = (1 + r_{t+1})s_t + P_{t+1}. \tag{2}
\]

From Equations (1) and (2), the lifetime budget constraint is:

\[
c_t + \frac{d_{t+1}}{1 + r_{t+1}} + (1 + r_t)(1 - \sigma_{t-1})e_{t-1} = (1 - \tau)w_t h_t + \frac{P_{t+1}}{1 + r_{t+1}}. \tag{3}
\]

An agent’s lifetime utility is expressed as:

\[
U(c_t, d_{t+1}) := \ln(c_t) + \beta \ln(d_{t+1}),
\]

where \( \beta > 0 \).

Let \( N_t \) be the population of middle-aged agents in period \( t \). Population growth rate \( n > -1 \) is given exogenously. Therefore, the population of middle-aged agents in period \( t + 1 \) is:

\[
N_{t+1} = (1 + n)N_t.
\]

Assume that \( N_0 > 0 \).

There exists a representative firm in the economy whose production function is expressed as:

\[
Y_t = AF(K_t, H_t) := AK_t^\alpha H_t^{1-\alpha},
\]

where \( A > 0 \) represents the productivity, \( \alpha \in (0, 1) \), and \( K_t \) and \( H_t \) are the aggregate capital stock and aggregate effective labor in the economy, respectively. Given effective wage \( w_t \) and real rental rate of capital \( r_t \), the firm’s profit in period \( t \) is:

\[
AK_t^\alpha H_t^{1-\alpha} - (1 + r_t)K_t - w_t H_t.
\]

I assume that physical capital is fully depreciated after production. Let \( k_t := K_t/H_t \) denote the physical capital per effective unit of labor in period \( t \).

The government provides educational subsidies for young agents and social security benefits for old ones by taxing middle-aged agents. I assume that the government uses the \( \lambda \in [0, 1] \) ratio of tax revenue.
for educational subsidies and the $1 - \lambda$ ratio for social security. Additionally, the government balances the budget for every period. Therefore, the government’s budget constraints in period $t$ are:

$$\lambda N_t \tau w_t h_t = N_{t+1} \sigma_t e_t, \quad (4)$$

$$\lambda N_t \tau w_t h_t = N_{t-1} P_t, \quad (5)$$

An equilibrium concept is a standard perfect foresight competitive equilibrium. The formal definition is as follows:

**Definition 2.1.** Given $N_0 > 0$, $h_0 > 0$, $e_0 > 0$, $s_0 > 0$, $\sigma_0 > 0$, $\tau \in [0, 1)$, and $\lambda \in [0, 1]$, an equilibrium consists of a consumption sequence $(d_t^*, (c_t^*, d_{t+1}^*), \tau)_{t=1}^\infty$; a sequence of educational expenditure $(e_t^*)_{t=1}^\infty$; a sequence of savings $(s_t^*)_{t=1}^\infty$; a sequence of human capital $(h_t^*)_{t=1}^\infty$; a sequence of production inputs $(K_t^*, H_t^*)_{t=1}^\infty$; a sequence of prices $(w_t^*, r_t^*)_{t=1}^\infty$, and a sequence of educational subsidy and social security benefits $(\sigma_t^*, P_t^*)_{t=1}^\infty$ so that:

1. For all $t \geq 2$, given $(r_t^*, w_t^*, s_t^*)$, $h_{t-1}^*$, $\sigma_{t-1}^*$, and $P_{t-1}^*$, $(e_t^*, c_t^*, d_t^*)$ is a solution to:

$$\max_{c_t, d_t} \ln(c_t) + \beta \ln(d_t)$$

s.t.

$$c_t + \frac{d_{t+1}}{1 + r_{t+1}} + (1 + r_t^*)(1 - \sigma_{t-1}^*)c_{t-1} = (1 - \tau)w_t^*h_t + \frac{P_{t+1}^*}{1 + r_{t+1}^*},$$

$$h_t = \theta e_{t-1}^\eta (h_{t-1}^*)^{1-\eta}.$$

2. Given $(r_t^*, w_t^*, r_1^*)$, $e_0 > 0$, $h_0 > 0$, $\sigma_0$, and $P_2^*$, $(c_1^*, d_2^*)$ is a solution to:

$$\max_{c_1, d_2} \ln(c_1) + \beta \ln(d_2)$$

s.t.

$$c_1 + \frac{d_2}{1 + r_2^*} + (1 + r_1^*)(1 - \sigma_0)e_0 = (1 - \tau)w_1^*h_1 + \frac{P_2^*}{1 + r_2^*},$$

$$h_1 = \theta e_0^\eta (h_0^*)^{1-\eta}.$$

3. For the initial old agent, $d_{1}^* = (1 + r_1^*)s_0 + P_1^*$.

4. $(w_t^*, r_t^*)_{t=1}^\infty$ satisfies:

$$w_t^* = (1 - \alpha)A(k_t^*)^\alpha, \quad 1 + r_t^* = \alpha A(k_t^*)^{\alpha - 1}.$$ 

5. Given $\tau$ and $\lambda$, $(\sigma_t^*, P_t^*)_{t=1}^\infty$ satisfies:

$$\lambda N_t \tau w_t^* h_t^* = N_{t+1} \sigma_t^* e_t^*,$$

$$\lambda N_t \tau w_t^* h_t^* = N_{t-1} P_t^*.$$ 

6. The capital and a labor market clear in each period. For all $t$, $K_t^* + N_t(1 - \sigma_{t-1}^*)e_{t-1}^* = N_{t-1} s_{t-1}^*$ and $H_t^* = N_t h_t^*$.

A BGP equilibrium is an equilibrium where all per capita variables grow at a constant rate.
2.1 Characterizing equilibrium

The first-order conditions for a middle-aged agent’s problem in period \( t \) induce:

\[
d_{t+1} = \beta (1+r_{t+1})c_t, \quad (6)
\]

\[
w_t(1-\eta)\theta \eta e^{\alpha-1}h^{1-\eta}_{t-1} = (1-\sigma_t)(1+r_t). \quad (7)
\]

Multiplying both sides of Equation (6) by \( e_{t-1} \), I obtain:

\[
w_t(1-\eta)\eta h_t = (1+r_t)(1-\sigma_t)e_{t-1}. \quad (8)
\]

Substituting Equations (5) and (6) into Equation (3), I have:

\[
c_t = \frac{1}{1+\beta} \left[ (1-\eta)(1-\tau)w_t h_t + \frac{P_{t+1}}{1+r_{t+1}} \right].
\]

Using Equation (9), a middle-age agent saves:

\[
s_t = \frac{\beta}{1+\beta} (1-\eta)(1-\tau)w_t h_t - \frac{1}{1+\beta}\frac{P_{t+1}}{1+r_{t+1}}. \quad (9)
\]

From Equation (6), a middle-aged agent in period \( t+1 \) spends:

\[
e_t = \frac{w_{t+1}(1-\tau)\eta h_{t+1}}{(1+r_{t+1})(1-\sigma_t)}. \quad (10)
\]

for human capital accumulation. Note that, without considering the general equilibrium effect, a young agent invests more as the interest rate and tax rate become lower and the subsidy rate becomes larger.

The government budget constraint for educational subsidies (Equation (5)) induces:

\[
\lambda \tau w_t h_t = (1+n)\sigma_t \frac{w_{t+1}(1-\tau)\eta h_{t+1}}{(1+r_{t+1})(1-\sigma_t)}. \quad (11)
\]

Rearranging this equation, \( \sigma_t \) in equilibrium satisfies:

\[
\sigma_t = \frac{(1+r_{t+1})\lambda \tau w_t h_t}{(1+n)w_{t+1}(1-\tau)\eta h_{t+1} + (1+r_{t+1})\lambda \tau w_t h_t} = \frac{\lambda \tau}{1+n \frac{w_{t+1}}{w_t} (1-\tau) \frac{h_{t+1}}{h_t} + \lambda \tau}. \quad (11)
\]

Since the educational subsidy is in this case financed \textit{inter}-generationally, the variables in period \( t+1 \) such as \( r_{t+1}, w_{t+1}, \) and \( h_{t+1} \) appear in \( \sigma_t \) in Equation (11) because of the general equilibrium effect.

Dividing the government’s budget constraint for social security benefits by the population of middle-aged agents in period \( t+1 \), that is, \( N_t \), I obtain:

\[
(1-\lambda)\tau w_{t+1} h_{t+1} = \frac{P_{t+1}}{1+n} \Rightarrow (1+n)(1-\lambda)\tau w_{t+1} h_{t+1} = P_{t+1}. \quad (12)
\]

From the capital market and labor market clearing conditions,

\[
k_{t+1} = \frac{K_{t+1}}{H_{t+1}} = \frac{N_t s_t - N_t(1-\sigma_t)e_t}{N_{t+1} h_{t+1}} = \frac{1}{1+n \frac{s_t}{h_{t+1}}} \frac{s_t}{h_{t+1}} - \frac{(1-\sigma_t) e_t}{h_{t+1}}
\]
Therefore, the equilibrium sequence of $k_t$ holds in equilibrium. Inserting Equation (13) into this equation yields:

$$k_{t+1} = \frac{1}{1+n} \left[ \frac{\beta}{1+\beta} (1-\eta)(1-\tau)w_t \frac{h_t}{h_{t+1}} - \frac{1}{1+\beta 1+n \tau_r + h_{t+1}} \right] - \frac{(1-\alpha)e_t}{h_{t+1}}. \tag{13}$$

Because $w_t = (1-\alpha)Ak_t^\alpha$, $1 + r_t = \alpha Ak_t^{\alpha-1}$, and $\frac{(1-\alpha)e_t}{h_{t+1}} = \frac{w_t(1-\tau)\eta}{h_{t+1}}$ at equilibrium, the following equation is derived from Equation (13):

$$k_{t+1} = \frac{\alpha \beta (1-\eta)(1-\alpha)(1-\tau)A}{(1+\beta)\alpha + \eta(1-\tau)(1-\alpha)(1+\beta) + (1-\lambda)\tau(1-\alpha)} \frac{1}{1+n} k_t^\alpha \frac{h_t}{h_{t+1}}. \tag{14}$$

From Equation (14), the dynamics of the economy at equilibrium are characterized by Equations (15) and (16). Equation (15) implies that, at equilibrium, $k_{t+1}/k_t$ is constant, that is:

$$k_{t+1}/k_t = \tilde{Z} := \frac{\alpha \beta (1-\eta)(1-\alpha)(1-\tau)A}{(1+\beta)\alpha + \eta(1-\tau)(1-\alpha)(1+\beta) + (1-\lambda)\tau(1-\alpha)} \frac{1}{1+n}. \tag{17}$$

Combining this equation with Equation (16), I obtain:

$$\frac{h_{t+1}}{h_t} = \theta \frac{1}{\gamma} \left[ (1-\tau)\eta \frac{1-\alpha}{\alpha} \right]^{\eta_{(1-\gamma)}} \left[ 1 + \frac{\alpha \lambda \tau A}{(1+n)(1-\tau)\eta \tilde{Z}} k_t^\alpha \frac{h_t}{h_{t+1}} \right]^{\eta_{(1-\gamma)}} k_t^\alpha \frac{h_t}{h_{t+1}}. \tag{18}$$

Inserting Equation (18) into Equation (13), the dynamics of $k_t$ in equilibrium are:

$$k_{t+1} = \frac{\tilde{Z}}{\theta (1-\alpha) \eta \left[ (1-\tau)\eta \tilde{Z} + \alpha \lambda \tau A \frac{1}{1+n} \right]} k_t^\alpha (1-\eta).$$

Therefore, the equilibrium sequence of $k_t$ converges to:

$$\tilde{k}^* := \left\{ \frac{\tilde{Z}}{\theta (1-\alpha) \eta \left[ (1-\tau)\eta \tilde{Z} + \alpha \lambda \tau A \frac{1}{1+n} \right]} \right\}^{\eta_{(1-\gamma)}}.$$

From Equation (17), the growth factor under the BGP equilibrium is:

$$\frac{h_{t+1}}{h_t} = 1 + \tilde{g}^* := \theta \frac{1}{\gamma} \left[ \frac{1-\alpha}{\alpha} \right]^{\eta_{(1-\gamma)}} \left\{ \tilde{Z}^{\alpha} \left[ (1-\tau)\eta \tilde{Z} + \alpha \lambda \tau A \frac{1}{1+n} \right]^{1-\alpha} \right\}^{\eta_{(1-\gamma)}}.$$
At the BGP equilibrium:
\[
\frac{\bar{c}_{t+1}}{c_t} = \frac{\bar{d}_{t+1}}{d_t} = \frac{\bar{e}_{t+1}}{e_t} = \frac{\bar{P}_{t+1}}{P_t} = 1 + \bar{g}^* 
\]
and
\[
\frac{\bar{\sigma}_{t+1}}{\sigma_t} = 1.
\]

The proposition below is a summary of the above results.

**Proposition 2.1.** For any \(h_0 > 0, s_0 > 0, \tau \in [0, 1], \) and \(\lambda \in [0, 1],\) there is a unique equilibrium under which \(\frac{\bar{c}_{t+1}}{c_t} = \frac{\bar{d}_{t+1}}{d_t} = \frac{\bar{e}_{t+1}}{e_t} = \frac{\bar{P}_{t+1}}{P_t} = 1 + \bar{g}^*\) and \(\frac{\bar{\sigma}_{t+1}}{\sigma_t} = 1\). Furthermore, the BGP equilibrium is globally stable.

Below, I investigate how changes in \(\tau\) and \(\lambda\) affect the growth rate, \(\bar{g}^*\).

**Proposition 2.2.**
1. Let
\[
\bar{\lambda}^* := \eta \frac{(1+\beta)[\alpha + (1-\alpha)\eta] + \frac{1+\alpha\beta}{1-\alpha} \frac{1}{\beta(1-\eta)(1-\alpha)}}{(1+\beta)[\alpha + (1-\alpha)\eta]^2 + \eta} > 0.
\]
If \(\lambda > (\lambda^*)\), then \(\frac{\partial \bar{g}^*}{\partial \tau} > (\lambda^*)0\) for a sufficiently small \(\tau\).

2. For all \(\lambda \in (0, 1],\) \(\frac{\partial \bar{g}^*}{\partial \tau} < 0\) for a sufficiently high \(\tau\).

3. For all \(\tau \in (0, 1),\) \(\frac{\partial \bar{g}^*}{\partial \lambda} > 0\) for all \(\lambda \in [0, 1].\) At \(\tau = 0,\) \(\frac{\partial \bar{g}^*}{\partial \lambda} = 0\) for all \(\lambda \in [0, 1].\)

**Proof.** See the Appendix. \(Q.E.D.\)

This proposition implies that the amount of tax revenue used for educational subsidies is important regarding whether an increase in the tax rate accelerates growth under the BGP equilibrium. An increase in \(\tau\) decreases the disposable labor income, which in turn lowers the return from education. Then, an increase in \(\tau\) discourages a young agent from investing in his or her human capital, which causes a decreases in the growth rate. The role of educational subsidies mitigates this negative effect on growth. However, if \(\lambda\) is very small and because most of the tax revenue is used for transfer from middle-age agents to old agents, educational subsidies are not as effective in mitigating this negative effect on growth. Therefore, an increase in \(\tau\) negatively affects the growth rate. Similarly, if \(\tau\) is too high, the return from education is too small. Hence, a young agent does not invest in his or her human capital, and this behavior depresses growth.
3 Model: Subsidizing student loans

Here, I examine another way to provide educational subsidies. In the previous section, in period $t$, a young agent receives educational subsidy $\sigma_{t-1} e_{t-1}$. Therefore, a young agent borrows $(1 - \sigma_{t-1}) e_{t-1}$ in period $t - 1$ and repays debt $(1 + r_t)(1 - \sigma_{t-1}) e_{t-1}$ in period $t$. Another way to provide educational subsidies is to provide subsidies when a middle-aged agent repays his or her debt.

All but a middle-aged agent’s budget constraint and the government’s budget constraint for educational subsidies are the same as before. The middle-aged agent’s budget constraint changes from Equation (11) to:

$$ c_t + s_t + (1 - \sigma_t)(1 + r_t) e_{t-1} = (1 - \tau)w_t h_t. $$ (19)

Because an old agent’s budget constraint is the same as in Equation (12), an agent’s lifetime budget constraint is:

$$ c_t + \frac{d_{t+1}}{1 + r_{t+1}} + (1 - \sigma_t)(1 + r_t) e_{t-1} = (1 - \tau)w_t h_t + \frac{P_{t+1}}{1 + r_{t+1}}. $$

The government’s budget constraint for educational subsidies changes from Equation (13) to:

$$ \lambda N_t \tau w_t h_t = N_t \sigma_t (1 + r_t) e_{t-1}. $$

The equilibrium concept is the same as that described in the previous section except for the capital market clearing condition. The capital market clearing condition in the equilibrium definition is:

$$ K^*_t + N_t e^*_t = N_{t-1} s^*_{t-1}. $$

for all $t$.

3.1 Characterizing equilibrium

The first-order conditions for a young agent’s problem induce Equation (13) and:

$$ w_t (1 - \tau) \eta e^{-\eta}_{t-1} h_{t-1}^{1-\eta} = (1 - \sigma_t)(1 + r_t). $$ (20)

Multiplying both sides of Equation (13) by $e_{t-1}$, I obtain:

$$ w_t (1 - \tau) \eta h_t = (1 - \sigma_t)(1 + r_t) e_{t-1}. $$ (21)

Substituting Equations (13) and (21) into an agent’s lifetime budget constraint, I have:

$$ c_t = \frac{1}{1 + \beta} \left[ (1 - \eta)(1 - \tau)w_t h_t + \frac{P_{t+1}}{1 + r_{t+1}} \right]. $$

Using Equation (19), a middle-aged agent saves:

$$ s_t = \frac{\beta}{1 + \beta} (1 - \eta)(1 - \tau)w_t h_t - \frac{1}{1 + \beta} \frac{P_{t+1}}{1 + r_{t+1}}. $$ (22)
From Equation (21), a middle-aged agent in period $t$ spends:

$$e_{t-1} = \frac{w_t (1 - \tau) \eta h_t}{(1 + r_t)(1 - \sigma_t)}$$

(23)

for human capital accumulation. Since, the government’s budget constraint for educational subsidies must be satisfied at equilibrium,

$$\lambda \tau w_t h_t = \sigma_t \frac{w_t (1 - \tau) \eta h_t}{1 - \sigma_t}$$

holds, where I use the government’s budget constraint for educational subsidies per middle-aged agent. At equilibrium, $\sigma_t$ is set to satisfy this equation. By rearranging the equation, $\sigma_t$ must satisfy the following:

$$\sigma_t = \frac{\lambda \tau}{(1 - \tau) \eta + \lambda \tau}.$$  

(24)

In contrast to Equation (11) in the previous section, since the educational subsidy is in this case financed intra-generationally, the variables in period $t + 1$ do not appear in $\sigma_t$ in Equation (24) and $\sigma_t$ is constant over time.

From the capital market and labor market clearing conditions,

$$k_{t+1} = \frac{K_{t+1}}{H_{t+1}} = \frac{N_t s_t - N_{t+1} e_t}{N_{t+1} h_{t+1}} = \frac{1}{1 + \frac{\sigma_t}{n_{t+1}}} \frac{s_t}{h_{t+1}} - \frac{e_t}{h_{t+1}}.$$  

holds at equilibrium. Substituting Equation (22) into this equation yields:

$$k_{t+1} = \frac{1}{1 + \frac{\lambda}{\eta} (1 - \tau) \eta} \frac{h_t}{h_{t+1}} - \frac{\frac{P_{t+1}}{1 + \frac{\lambda}{\eta} (1 - \tau) \eta} - e_t}{h_{t+1}}.$$  

(25)

Since $w_t = (1 - \alpha) A k_t^\alpha$, $1 + r_t = \alpha A k_t^{\alpha - 1}$, and $\frac{\alpha}{h_{t+1}} = \frac{w_{t+1} (1 - \tau) \eta}{(1 - \sigma_{t+1}) (1 + r_{t+1})}$ at equilibrium and $P_{t+1}$ satisfies Equation (22), from Equation (25), I obtain:

$$k_{t+1} = \hat{Z} k_t^\alpha \frac{h_t}{h_{t+1}}.$$  

(26)

where

$$\hat{Z} := \frac{\alpha \beta (1 - \eta)(1 - \alpha)(1 - \tau) A}{(1 + \beta) \alpha + (1 + \beta) (1 - \alpha)(1 - \tau) \eta + \lambda \tau + (1 - \lambda) \tau (1 - \alpha)} \frac{1}{1 + \frac{\lambda}{\eta} (1 - \tau) \eta}.$$

(27)

From Equation (23):

$$h_{t+1} = \frac{\alpha}{h_t} \left[ \frac{(1 - \tau) \eta}{1 - \sigma_{t+1}} \frac{1}{\alpha} k_{t+1} \right]^{\eta n}.$$  

(28)

Using Equation (24), Equation (28) can be rewritten as:

$$h_{t+1} = \theta^{\frac{\eta}{n}} \left( \frac{1 - \alpha}{\alpha} \right)^{\frac{n}{\eta}} [(1 - \tau) \eta + \lambda \tau]^{\frac{n}{\eta}} k_{t+1}^{\frac{n}{\eta}}.$$  

(29)
Combining Equations (26) and (29) yields:

\[ k_{t+1} = \frac{\hat{Z}^{1-\eta}}{\theta \left( \frac{1-\alpha}{\alpha} \right)^{\eta} [1 - \tau]^\eta} \frac{k_t^{(1-\eta)}}{e_t}. \]

Therefore, the equilibrium sequence of \( k_t \) converges to:

\[ \hat{k}^* := \frac{\hat{Z}^{1-\eta}}{\theta \left( \frac{1-\alpha}{\alpha} \right)^{\eta} [1 - \tau]^\eta} \frac{Z^{1-\eta}}{\theta \left( \frac{1-\alpha}{\alpha} \right)^{\eta} [1 - \tau]^\eta}. \]

From Equation (27), the growth factor at BGP equilibrium is:

\[ \frac{h_{t+1}}{h_t} = 1 + \hat{g}^* := \frac{\hat{Z}}{(k^*)^{1-\alpha}} = \theta \left( \frac{1-\alpha}{\alpha} \right)^{\eta} \left( \frac{1 - \alpha}{\alpha} \right)^{\eta} [1 - \tau]^\eta + \lambda \tau \right)^{\eta} \frac{Z^{1-\eta}}{\theta \left( \frac{1-\alpha}{\alpha} \right)^{\eta} [1 - \tau]^\eta}. \]

At the BGP equilibrium:

\[ \frac{c_{t+1}}{c_t} = \frac{d_{t+1}}{d_t} = \frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\hat{d}_{t+1}}{\hat{d}_t} = \frac{\hat{P}_{t+1}}{\hat{P}_t} = 1 + \hat{g}^* \]

and

\[ \frac{\hat{\sigma}_{t+1}}{\hat{\sigma}_t} = 1. \]

Note that the ratio of educational subsidies, \( \sigma_t \), is constant at the BGP equilibrium to satisfy the government’s budget constraint. However, since educational expenditure \( e_t \) increases over time, the amount of educational subsidy, \( \sigma_t e_t \), also increases over time.

The following proposition summarizes the above results.

**Proposition 3.1.** For any \( h_0 > 0 \), \( s_0 > 0 \), \( \tau \in [0, 1] \), and \( \lambda \in [0, 1] \), there exists a unique equilibrium, in which \( \frac{c_{t+1}}{c_t} = \frac{d_{t+1}}{d_t} = \frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\hat{d}_{t+1}}{\hat{d}_t} = \frac{\hat{P}_{t+1}}{\hat{P}_t} = 1 + \hat{g}^* \) and \( \frac{\hat{\sigma}_{t+1}}{\hat{\sigma}_t} = 1 \). Furthermore, the BGP equilibrium is globally stable.

The proposition below explains how a change in \( \tau \) and \( \lambda \) affects \( \hat{g}^* \).

**Proposition 3.2.**

1. Let

\[ \hat{\lambda}^* := \eta \frac{(1 + \beta)[\alpha + (1 - \alpha)\eta] + \frac{1 + \alpha \beta}{1 - \alpha}}{\alpha[1 + \beta(1 - \eta)] + (1 - \alpha)\eta}. \]

If \( \lambda < \hat{\lambda}^* \), then \( \frac{\partial \hat{g}^*}{\partial \tau} < 0 \) for all \( \tau \in [0, 1] \).

2. If \( \lambda > \hat{\lambda}^* \), then there exists a unique \( \hat{\tau} \in (0, 1) \) so that \( \frac{\partial \hat{g}^*}{\partial \tau} > 0 \) for all \( \tau \in [0, \hat{\tau}) \) and \( \frac{\partial \hat{g}^*}{\partial \tau} < 0 \) for all \( \tau \in (\hat{\tau}, 1) \).

3. For all \( \tau \in (0, 1) \), \( \frac{\partial \hat{g}^*}{\partial \lambda} > 0 \) for all \( \lambda \in [0, 1] \). At \( \tau = 0 \), \( \frac{\partial \hat{g}^*}{\partial \lambda} = 0 \) for all \( \lambda \in [0, 1] \).
Proof. See the Appendix. \( Q.E.D. \)

The intuition behind this result is similar to that for Proposition 2.2: when \( \lambda \) is too small, a slight increase in \( \tau \) depresses the growth rate because most of the tax revenue is spent for transfers from middle-aged agents to old agents. When \( \lambda \) is sufficiently large and \( \tau \) is sufficiently small, a slight increase in \( \tau \) can boost the growth rate, because an excessively high tax rate decreases the disposable labor income and a young agent is discouraged from investing in his or her human capital.

4 Comparison of the two educational subsidy schemes

4.1 \( \tilde{g}^* \) versus \( \tilde{g}^* \)

This section analyzes which educational subsidy leads to a higher growth rate in the BGP equilibrium.

Proposition 4.1. 1. If \( \lambda \tau = 0 \), then \( \tilde{g}^* = \tilde{g}^* \).

2. Assume \( \lambda \tau > 0 \). Then, for a sufficiently large \( \alpha \in (0, 1) \), \( \tilde{g}^* > \tilde{g}^* \).

3. Assume \( \lambda \tau > 0 \). Further, assume that \( \alpha \in (0, 1) \) and \( \eta \in (0, 1) \) are sufficiently small. Then, \( \tilde{g}^* < \tilde{g}^* \) for all sufficiently small \( \tau \in (0, 1) \).

Proof. See the Appendix. \( Q.E.D. \)

When either \( \lambda \) or \( \tau \) is 0, there is no educational subsidy. Then, both models are exactly the same and there is no difference between the two growth rates.

When \( \lambda \tau > 0 \), the result depends on the parameter values. Specifically, the key parameter is \( \alpha \in (0, 1) \). When \( \alpha \) is large, production is physical-capital-intensive. Therefore, the interest rate tends to be high. If a young agent borrows a certain amount, he or she needs to repay it with interest. Higher interest rates increase the interest payments in the second period of the agent’s life. If an educational subsidy is provided when a young agent borrows, the interest payment will be small, even if the young agent borrows a lot. This effect is large when the interest rate is high. Therefore, for a sufficiently large \( \alpha \in (0, 1) \), the growth rate in the economy subsidizing the cost of education is higher than that in the economy subsidizing student loans. By contrast, for a sufficiently small \( \alpha \in (0, 1) \) and with additional parameters, the opposite is likely to hold because the interest rate tends to be low.

4.2 \( \tilde{\lambda}^* \) versus \( \tilde{\lambda}^* \)

Next, I compare \( \tilde{\lambda}^* \) with \( \tilde{\lambda}^* \). From Propositions 4.1 and 4.2, for a slight increase in \( \tau \) to accelerate the growth rate under the BGP equilibrium, \( \lambda \) must be higher than \( \lambda^* \). Which of the two systems requires a higher \( \lambda^* \) for an increase in the tax rate to accelerate the growth rate? As shown below, the result depends on \( \alpha \) and \( \eta \):

Proposition 4.2. Let

\[
\bar{\alpha} := \frac{\beta}{1 + 2\beta}.
\]
1. If $\alpha \geq \bar{\alpha}$, then $\hat{\lambda}^* < \hat{\lambda}^*$.

2. If $\alpha < \bar{\alpha}$, then there exists a unique $\eta \in (0, 1)$ so that $\hat{\lambda}^* > \hat{\lambda}^*$ for all $\eta \in (0, \bar{\eta})$ and $\hat{\lambda}^* < \hat{\lambda}^*$ for all $\eta \in (\bar{\eta}, 1)$.

**Proof.** See the Appendix. \[Q.E.D.\]

Assume that production is physical-capital-intensive. Then, when $\lambda \in (\hat{\lambda}^*, \hat{\lambda}^*)$, the introduction of an educational subsidy will accelerate the growth rate in the economy by subsidizing the cost of education, whereas it will slow down the growth rate by subsidizing student loans. For student loan subsidies to be effective, higher tax revenues need to be distributed to education. Analogous to Proposition 4.1, the opposite is likely to be true for an economy where production is labor-intensive.

## 5 Concluding remarks

This study considers a three-period OLG model with educational subsidies and pay-as-you-go pensions, in which a young agent borrows for his or her education in a perfect credit market and a middle-aged agent repays these loans. Two educational subsidy schemes are considered: one is to provide subsidies when a young agent borrows and the other is to provide subsidies when a middle-aged agent repays his or her loan. I also characterize a unique BGP equilibrium under each educational subsidy scheme and compare their growth rates, finding that the size relationship of growth rates depends on whether the production is sufficiently physical-capital-intensive. If production is highly physical-capital-intensive, the interest rate tends to be high in the credit market. This discourages a young agent from borrowing because the interest payments will become a burden for the agent. In this case, the existence of an educational subsidy is helpful for a young agent, allowing him or her to borrow and invest in human capital. Therefore, in this case, an educational subsidy provided when a young agent borrows leads to a higher growth rate than that the one provided when a middle-aged agent repays his or her loan. If the production is not so highly physical-capital-intensive, the opposite is likely to hold.

When educational subsidies are provided, several educational subsidy schemes can be considered. It might be true that the scheme does not matter for optimal allocations in a decentralized economy. However, this study points out that the scheme matters for the long-run growth rate and the key is how physical-capital-intensive production is. Therefore, if a policy maker aims to achieve a higher growth rate in the long run by providing educational subsidies, the policy maker should design an educational subsidy scheme depending on how physical-capital-intensive the production is.

### A Appendix

#### A.1 Proof of Proposition 2.2

**Proof.** Let:

$$
\bar{G}(\tau, \lambda) := \bar{Z}^\alpha \left[ (1 - \tau)\eta\bar{Z} + \frac{\alpha \lambda \tau A}{1 + n} \right]^{1 - \alpha}.
$$

The signs of $\frac{\partial \bar{G}}{\partial \tau}$ and $\frac{\partial \bar{G}}{\partial \lambda}$ are equivalent to those of $\frac{\partial \bar{C}}{\partial \tau}$ and $\frac{\partial \bar{C}}{\partial \lambda}$, respectively.
First, I show the proof for statements 1, 2, and 3. Taking the partial derivative of $\tilde{G}$ with respect to $\tau$, I obtain:

$$\frac{\partial \tilde{G}}{\partial \tau} = \hat{Z}^{\alpha-1} \left[ (1-\tau) \eta \tilde{Z} + \frac{\alpha \lambda \tau A}{1+n} \right] - \alpha \times \left\{ \alpha \left[ (1-\tau) \eta \tilde{Z} + \frac{\alpha \lambda \tau A}{1+n} \right] \frac{\partial \tilde{Z}}{\partial \tau} + \tilde{Z}(1-\alpha) \left[ -\eta \tilde{Z} + (1-\tau) \eta \frac{\partial \tilde{Z}}{\partial \tau} + \frac{\alpha \lambda A}{1+n} \right] \right\}.$$

Note that, for all $\tau \in [0, 1)$ and all $\lambda \in [0, 1]$, $Z > 0$. To investigate the sign of $\frac{\partial \tilde{G}}{\partial \tau}$, I examine the sign of:

$$\tilde{J}(\tau, \lambda) := \alpha \left[ (1-\tau) \eta \tilde{Z} + \frac{\alpha \lambda \tau A}{1+n} \right] \frac{\partial \tilde{Z}}{\partial \tau} + \tilde{Z}(1-\alpha) \left[ -\eta \tilde{Z} + (1-\tau) \eta \frac{\partial \tilde{Z}}{\partial \tau} + \frac{\alpha \lambda A}{1+n} \right].$$

Using the expression of $\tilde{Z}$ and:

$$\frac{\partial \tilde{Z}}{\partial \tau} = -\frac{1}{1+n} \frac{\alpha \beta (1-\eta)(1-\alpha) A \left[ (1+\beta) \alpha + (1-\lambda) (1-\alpha) \right]}{(1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha)} < 0,$$

$\tilde{J}(\tau, \lambda)$ can be rewritten as:

$$\tilde{J}(\tau, \lambda) = \frac{\alpha^2 A^2 \beta (1-\eta)(1-\alpha)}{(1+n)^2 \left[ (1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha) \right]} \tilde{j}(\tau, \lambda),$$

where

$$\tilde{j}(\tau, \lambda) := -(1-\tau) \eta \frac{\beta (1-\eta)(1-\alpha)(1-\tau)}{(1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha)} \left[ (1+\beta) \alpha + (1-\lambda) (1-\alpha) \right]$$

$$- \alpha \lambda \tau \left[ (1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha) \right]$$

$$+ (1-\alpha) \eta \frac{\beta (1-\eta)(1-\alpha)(1-\tau)}{(1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha)}.$$

Because\n
$$\frac{\alpha^2 A^2 \beta (1-\eta)(1-\alpha)}{(1+n)^2 \left[ (1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha) \right]} > 0,$$

the sign of $\tilde{j}$ is equivalent to that of $\tilde{J}$. From this, $\tilde{j}(\tau, \lambda) > 0$ if and only if $\tilde{j}_L(\tau, \lambda) > \tilde{j}_R(\tau, \lambda)$, where:

$$\tilde{j}_L(\tau, \lambda) := (1-\tau)(1-\alpha) \lambda - \alpha \lambda \tau \frac{(1+\beta) \alpha + (1-\lambda) (1-\alpha)}{(1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha)}$$

$$\tilde{j}_R(\tau, \lambda) := (1-\tau) \eta \frac{\beta (1-\eta)(1-\alpha)(1-\tau)}{(1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha)} \left[ (1+\beta) \alpha + (1-\lambda) (1-\alpha) \right]$$

$$+ (1-\alpha) \eta \frac{\beta (1-\eta)(1-\alpha)(1-\tau)}{(1+\beta) \alpha + \eta (1-\tau)(1-\alpha)(1+\beta) + (1-\lambda) \tau(1-\alpha)}.$$
Note that both \( \tilde{j}_L \) and \( \tilde{j}_R \) are decreasing and continuous in \( \tau \):

\[
\begin{align*}
\tilde{j}_L(0) &= (1 - \alpha) \lambda \geq 0, \\
\tilde{j}_L(1) &= -\alpha \lambda \frac{(1 + \beta) \alpha + (1 - \lambda) (1 - \alpha)}{(1 + \beta) \alpha + (1 - \lambda) (1 - \alpha)} < 0, \\
\tilde{j}_R(0) &= \eta (1 - \alpha) \beta (1 - \eta) \left[ (1 + \beta) \alpha + (1 - \lambda) (1 - \alpha) \right] > 0 \\
\end{align*}
\]

and

\[
\tilde{j}_R(1) = 0.
\]

Hence, for \( \lambda \in (0, 1) \), \( \tilde{j}_L(1) < \tilde{j}_R(1) \). This and the continuities of \( \tilde{j}_L \) and \( \tilde{j}_R \) imply that, for a sufficiently high \( \tau \in (0, 1) \), \( \tilde{j}(\tau, \lambda) < 0 \) for \( \lambda \in [0, 1) \). This implies that \( \frac{\partial \tilde{j}}{\partial \tau} < 0 \).

Now, consider the condition under which \( \tilde{j}_L(0) > \tilde{j}_R(0) \). If this holds, for a sufficiently small \( \tau \in [0, 1) \), \( \tilde{j}(\tau, \lambda) > 0 \). From the expressions of \( \tilde{j}_L(0) \) and \( \tilde{j}_R(0) \), \( \tilde{j}_L(0) > \tilde{j}_R(0) \) if and only if:

\[
\lambda > \tilde{\lambda}^* := \eta \frac{(1 + \beta) \alpha + (1 - \lambda) \eta}{(1 - \eta) \beta (1 - \alpha)} + \eta > 0.
\]

Therefore, if Equation (30) holds, for a sufficiently small \( \tau \in [0, 1) \), \( \tilde{j}(\tau, \lambda) > 0 \). This implies that \( \frac{\partial \tilde{j}}{\partial \tau} > 0 \).

If \( \lambda < \tilde{\lambda}^* \), then \( \tilde{j}_L(0, \lambda) < \tilde{j}_R(0, \lambda) \). This implies that for a sufficiently small \( \tau \in (0, 1) \), \( \tilde{j}(\tau, \lambda) < 0 \).

Second, I provide the proof for statement 4. Taking the partial derivative of \( \tilde{G} \) with respect to \( \lambda \), I obtain:

\[
\frac{\partial \tilde{G}}{\partial \lambda} = \alpha \tilde{Z}^{\alpha - 1} \frac{\partial \tilde{Z}}{\partial \lambda} \left[ (1 - \tau) \eta \tilde{Z} + \frac{\alpha \lambda \tau A}{1 + n} \right]^{1 - \alpha} + \tilde{Z}^{\alpha} (1 - \alpha) \left[ (1 - \tau) \eta \tilde{Z} + \frac{\alpha \lambda \tau A}{1 + n} \right]^{-\alpha} \left[ (1 - \tau) \eta \frac{\partial \tilde{Z}}{\partial \lambda} + \alpha \lambda \tau A \right].
\]

Since

\[
\frac{\partial \tilde{Z}}{\partial \lambda} = \frac{\alpha \beta (1 - \eta) (1 - \alpha)^2 (1 - \tau) \lambda \tau}{[(1 + \beta) \alpha + \eta (1 - \tau) (1 - \alpha) (1 + \beta) + (1 - \lambda) \tau (1 - \alpha)]^2} \frac{1}{1 + n} > 0
\]

for all \( \tau \in (0, 1) \) and \( \tilde{Z} > 0 \) for all \( \lambda \in [0, 1] \) and all \( \tau \in [0, 1) \), \( \frac{\partial \tilde{G}}{\partial \tau} > 0 \) for all \( \lambda \in [0, 1] \) and all \( \tau \in (0, 1) \).

When \( \tau = 0 \), \( \frac{\partial \tilde{Z}}{\partial \lambda} = 0 \) for all \( \lambda \in [0, 1] \). Therefore, \( \frac{\partial \tilde{G}}{\partial \tau} = 0 \) when \( \tau = 0 \). \( \boxed{Q.E.D.} \)

A.2 Proof of Proposition 3.2

Proof. Let:

\[
\hat{G}(\tau, \lambda) : = [(1 - \tau) \eta + \lambda \tau]^{\frac{(1 - \alpha) \eta}{(1 - \eta) \beta (1 - \alpha)}} Z^{\frac{\eta}{1 - \eta} - \frac{\eta}{(1 - \eta) \beta (1 - \alpha)}}.
\]

Note that the signs of \( \frac{\partial \hat{G}}{\partial \tau} \) and \( \frac{\partial \hat{G}}{\partial \lambda} \) are equivalent to those of \( \frac{\partial \tilde{G}}{\partial \tau} \) and \( \frac{\partial \tilde{G}}{\partial \lambda} \), respectively.
First, I prove statements 1 and 2.
Taking the partial derivative of \( \hat{G} \) with respect to \( \tau \), I obtain:
\[
\frac{\partial \hat{G}}{\partial \tau} = \frac{\eta}{1 - \alpha(1 - \eta)} [(1 - \tau)\eta + \lambda \tau] \frac{1 - \alpha(1 - \eta)}{1 - \alpha(1 - \eta)} \frac{1}{1 - \alpha(1 - \eta)} \frac{1}{1 - \alpha(1 - \eta)} \left\{(1 - \alpha)Z(\lambda) + [(1 - \tau)\eta + \lambda \tau] \frac{\partial \hat{Z}}{\partial \tau}\right\}.
\]
Because \( \frac{\eta}{1 - \alpha(1 - \eta)} [(1 - \tau)\eta + \lambda \tau] \frac{1 - \alpha(1 - \eta)}{1 - \alpha(1 - \eta)} \frac{1}{1 - \alpha(1 - \eta)} \frac{1}{1 - \alpha(1 - \eta)} > 0 \), the sign of \( \frac{\partial \hat{G}}{\partial \tau} \) is equivalent to that of \( (1 - \alpha)Z(\lambda) + [(1 - \tau)\eta + \lambda \tau] \frac{\partial \hat{Z}}{\partial \tau} \). Then:
\[
(1 - \alpha)Z(\lambda) + [(1 - \tau)\eta + \lambda \tau] \frac{\partial \hat{Z}}{\partial \tau} = \frac{\hat{Z}}{1 - \tau} \hat{J}(\tau, \lambda),
\]
where
\[
\hat{J}(\tau, \lambda) := (1 - \alpha)(1 - \tau)(\lambda - \eta) - [(1 - \tau)\eta + \lambda \tau] \frac{1 + \beta\alpha}{1 + \beta\lambda + (1 - \alpha)(1 - \alpha)} (1 - \tau)
\]
\[
\geq \hat{J}_R(\tau) := \frac{(1 - \alpha)(\lambda - \eta)}{(1 + \beta\alpha + (1 - \alpha)(1 - \alpha))[(1 - \tau)\eta + \lambda \tau] + (1 - \lambda)\tau(1 - \alpha)}.
\]
Note that \( \hat{J}_L(\tau) \) is linear in \( \tau \), strictly decreasing in \( \tau \),\( \hat{J}_L(0) = (1 - \alpha)(\lambda - \eta) + (1 - \alpha)(1 - \alpha)(1 - \alpha) > 0 \), and \( \hat{J}_L(1) = 0 \). However, taking the derivative of \( \hat{J}_R \) with respect to \( \tau \), I obtain:
\[
\frac{\partial \hat{J}_R}{\partial \tau} = \frac{\lambda[(1 + \beta)\eta + \eta(1 - \alpha)] - \eta[(1 + \beta)\alpha + 1 - \alpha]}{((1 + \beta)\alpha + (1 - \alpha)(1 - \alpha)(1 - \alpha))[(1 - \tau)\eta + \lambda \tau] + (1 - \lambda)\tau(1 - \alpha)^2}.
\]
If \( \lambda \leq \hat{\lambda} := \frac{\eta[(1 + \beta)\alpha + 1 - \alpha]}{(1 + \beta)\alpha + (1 - \alpha)(1 - \alpha)} \), \( \hat{J}_R \) is decreasing in \( \tau \). Because \( \hat{J}_R(1) = \frac{\lambda}{(1 + \beta)\alpha + (1 - \alpha)(1 - \alpha)} \) and \( \hat{J}_R(0) < \hat{J}_R(1) \), if \( \lambda \leq \hat{\lambda} \), for all \( \tau \in [0, 1] \), \( \hat{J}_R(\tau) < \hat{J}_R(\tau) \), this implies \( \hat{J}(\tau, \lambda) < 0 \).
Assume that \( \lambda > \hat{\lambda} \). Then, \( \hat{J}_R \) is strictly increasing in \( \tau \). In this case, if \( \hat{J}_L(0) < \hat{J}_R(0) \), then \( \hat{J}_L(\tau) < \hat{J}_R(\tau) \) for all \( \tau \). If \( \hat{J}_L(0) \geq \hat{J}_R(0) \), there exists a unique \( \tilde{\tau} \in [0, 1] \) so that \( \hat{J}_L(\tau) > \hat{J}_R(\tau) \) for all \( \tau \in [0, \tilde{\tau}] \) and \( \hat{J}_L(\tau) < \hat{J}_R(\tau) \) for all \( \tau \in (\tilde{\tau}, 1) \). Because \( \hat{J}_R(0) = \frac{\eta}{(1 + \beta)(1 - \alpha)(1 - \alpha)} \), \( \hat{J}_L(0) > \hat{J}_R(0) \) if and only if:
\[
\lambda > \hat{\lambda}^* := \eta \frac{(1 + \beta)(\alpha + (1 - \alpha)\eta) + 1 - \alpha}{(1 + \beta)(\alpha + (1 - \alpha)\eta) + 1 - \alpha}.
\]
Note that \( \hat{\lambda}^* > \hat{\lambda} > \eta \). Hence, if \( \lambda > \hat{\lambda}^* \), \( \hat{J}_L(0) > \hat{J}_R(0) \), \( \hat{J}_L(1) < \hat{J}_R(1) \), \( \hat{J}_L(\tau) \) is strictly decreasing in \( \tau \) and \( \hat{J}_R(\tau) \) is strictly increasing in \( \tau \). Moreover, both \( \hat{J}_L \) and \( \hat{J}_R \) are continuous in \( \tau \). This implies there
exists a unique $\hat{\tau} \in (0,1)$ so that $\hat{f}(\tau, \lambda) > 0$ for all $\tau < \hat{\tau}$ and $\hat{f}(\tau, \lambda) < 0$ for all $\tau > \hat{\tau}$. This completes the proofs of 1 and 2 in Proposition 5.3.

Second, I prove statement 3. Taking the partial derivative of $\hat{G}$ with respect to $\lambda$, I obtain:

$$\frac{\partial \hat{G}}{\partial \lambda} = \frac{\eta}{1-\alpha(1-\eta)}[(1-\tau)\eta + \lambda \tau]^{(1-\alpha)\eta} \hat{Z}^{1-\alpha} \left\{ (1-\alpha)\tau \hat{Z} + [(1-\tau)\eta + \lambda \tau] \frac{\partial \hat{Z}}{\partial \lambda} \right\}.$$ 

Then:

$$(1-\alpha)\tau \hat{Z} + [(1-\tau)\eta + \lambda \tau] \frac{\partial \hat{Z}}{\partial \lambda} = (1-\alpha)\tau \hat{Z},$$

$$\times \left\{ 1 - \frac{[(1-\tau)\eta + \lambda \tau]\beta}{(1+\beta)(1-\alpha)(1-\tau)\eta + \lambda \tau + (1-\lambda)\tau(1-\alpha)} \right\}.$$ 

In the above equation, the expression between the brackets is:

$$1 - \frac{[(1-\tau)\eta + \lambda \tau]\beta}{(1+\beta)(1-\alpha)(1-\tau)\eta + \lambda \tau + (1-\lambda)\tau(1-\alpha)} = \frac{[(1-\lambda)(1-\alpha) + (\lambda - \eta)(1-\alpha - \alpha \beta)]\tau + (1+\beta)\alpha(1-\eta) + \eta}{(1+\beta)(1-\alpha)(1-\tau)\eta + \lambda \tau + (1-\lambda)\tau(1-\alpha)}.$$ 

Note that the numerator of this equation is a linear equation of $\tau$. When $\tau = 0$, the value is $(1+\beta)\alpha(1-\eta) + \eta > 0$. If the value at $\tau = 1$ is strictly positive, the above value is greater than 0 for all $\tau \in (0,1)$. At $\tau = 1$, the value of the numerator is $1 + \alpha \beta (1-\lambda) > 0$. Therefore, for all $\tau \in (0,1)$:

$$1 - \frac{[(1-\tau)\eta + \lambda \tau]\beta}{(1+\beta)(1-\alpha)(1-\tau)\eta + \lambda \tau + (1-\lambda)\tau(1-\alpha)} > 0,$$

for all $\lambda \in [0,1]$. Hence, $\frac{\partial \hat{G}}{\partial \lambda} > 0$ for all $\tau \in (0,1)$ and all $\lambda \in [0,1]$.

Because $\frac{\partial \hat{Z}}{\partial \lambda} = \hat{Z} \frac{\beta(1-\alpha)\tau}{(1+\beta)(1-\alpha)(1-\tau)\eta + \lambda \tau + (1-\lambda)\tau(1-\alpha)}$, $\frac{\partial \hat{G}}{\partial \lambda} = 0$ when $\tau = 0$ for all $\lambda \in [0,1]$.  

Q.E.D.

A.3 Proof of Proposition 5.5

Proof. From Equations (19) and (30), $\hat{g}^x \geq \tilde{g}^x$ if and only if:

$$\hat{Z}^\alpha \left[ (1-\tau)\eta \hat{Z} + \frac{\alpha \lambda \tau A}{1+n} \right]^{1-\alpha} \geq \hat{Z}[(1-\tau)\eta + \lambda \tau]^{1-\alpha}$$

or

$$\left[ \frac{\beta(1-\eta)(1-\alpha)(1-\tau)}{(1+\beta)(1-\alpha + \eta(1-\tau)(1-\alpha)(1+\beta) + (1-\lambda)\tau(1-\alpha))} \right]^\alpha \times \left[ \frac{\beta(1-\eta)(1-\alpha)(1-\tau)}{(1+\beta)(1-\alpha)(1+\beta)(1-\alpha) + \lambda \tau} \right]^{1-\alpha} \geq \frac{\beta(1-\eta)(1-\alpha)(1-\tau)}{(1+\beta)(1-\alpha + \eta(1-\tau)(1-\alpha)(1+\beta) + (1-\lambda)\tau(1-\alpha))}.$$ 

(31)
Note that if $\lambda \tau = 0$, both sides of Equation (31) are the same, which completes the proof of statement 1. Assuming $\lambda \tau > 0$, equation (31) is equivalent to:

\[
\begin{aligned}
&\left\{ \frac{(1 + \beta)(1 + \alpha)(1 - \alpha)((1 - \tau)\eta + \lambda \tau) + (1 - \lambda)\tau(1 - \alpha)}{(1 + \beta)\alpha + \eta(1 - \tau)(1 - \alpha)(1 + \beta) + (1 - \lambda)\tau(1 - \alpha)} \right\}^\alpha \\
&\geq \left\{ \frac{(1 - \tau)\eta(1 + \alpha)(1 - \alpha)(1 - \tau)\eta + \lambda \tau + \lambda \tau(1 - \alpha)}{(1 + \beta)\alpha + \eta(1 - \tau)(1 - \alpha)(1 + \beta) + (1 - \lambda)\tau(1 - \alpha)} \right\}^{1 - \alpha}.
\end{aligned}
\]

As $\alpha$ approaches 1, the term between the brackets on the left-hand side of Equation (32) approaches 1 as well. Moreover, since:

\[
\frac{(1 + \beta)\alpha + (1 + \beta)(1 - \alpha)((1 - \tau)\eta + \lambda \tau) + (1 - \lambda)\tau(1 - \alpha)}{(1 + \beta)\alpha + \eta(1 - \tau)(1 - \alpha)(1 + \beta) + (1 - \lambda)\tau(1 - \alpha)} > 1
\]

for all $\alpha \in (0, 1)$. For a sufficiently large $\alpha \in (0, 1)$, the left-hand side of Equation (32) is strictly larger than 1. Analogously, the term between the bracket on the right-hand side of Equation (32) approaches 0, as $\alpha$ goes to 1. Therefore, for a sufficiently large $\alpha \in (0, 1)$, the right-hand side of Equation (32) is strictly smaller than 1. This completes the proof for statement 2.

As $\alpha$ and $\eta$ approach 0, the term between the brackets of the left-hand side of Equation (32) approaches:

\[
\frac{(1 + \beta)\lambda}{1 - \lambda} + 1 - \lambda.
\]

(33)

Analogously, if $\alpha$ and $\eta$ go to 0, the term between the brackets on the right-hand side of Equation (32) approaches:

\[
\frac{\beta(1 - \tau)}{(1 + \beta)\lambda \tau + (1 - \lambda)\tau(1 - \alpha)}.
\]

(34)

As $\tau$ approaches 0, Equation (32) approaches $+\infty$, whereas Equation (33) approaches $\frac{(1 + \beta)\lambda}{1 - \lambda} + 1 - \lambda < +\infty$. This implies that for a sufficiently small $\alpha$ and $\eta$:

\[
\left\{ \frac{(1 + \beta)\alpha + (1 + \beta)(1 - \alpha)((1 - \tau)\eta + \lambda \tau) + (1 - \lambda)\tau(1 - \alpha)}{(1 + \beta)\alpha + \eta(1 - \tau)(1 - \alpha)(1 + \beta) + (1 - \lambda)\tau(1 - \alpha)} \right\}^\alpha
\]

\[
\less
\left\{ \frac{(1 - \tau)\eta(1 + \alpha)(1 - \alpha)(1 - \tau)\eta + \lambda \tau + \lambda \tau(1 - \alpha)}{(1 + \beta)\alpha + \eta(1 - \tau)(1 - \alpha)(1 + \beta) + (1 - \lambda)\tau(1 - \alpha)} \right\}^{1 - \alpha}
\]

holds for a sufficiently small $\tau \in (0, 1)$, which completes the proof of statement 3. \textit{Q.E.D.}

A.4 Proof of Proposition 4.2

Proof. $\lambda^* < \lambda^*$ if and only if:

\[
\frac{(1 + \beta)^2\alpha + (1 - \alpha)\eta}{\beta(1 - \eta)(1 - \alpha)} > \frac{\alpha[1 + \beta - \eta]}{\alpha[1 + \beta - \eta] + (1 - \alpha)\eta}.
\]

18
This is equivalent to:

$$(1 + \beta)[\alpha + (1 - \alpha)\eta]^2 > \beta \alpha(1 - \alpha)(1 - \eta)^2.$$ 

Let $I_L(\eta) := (1 + \beta)[\alpha + (1 - \alpha)\eta]^2$ and $I_R(\eta) := \beta \alpha(1 - \alpha)(1 - \eta)^2$. Note that $I_L(\eta)$ is strictly increasing and is strictly convex in $\eta$ on $(0, 1)$, while $I_R(\eta)$ is strictly decreasing and is strictly concave in $\eta$ on $(0, 1)$. Also note that:

\[
\begin{align*}
I_L(0) &= (1 + \beta)\alpha^2, \\
I_L(1) &= 1 + \beta, \\
I_R(0) &= \beta \alpha(1 - \alpha)
\end{align*}
\]

and

$$I_R(1) = 0.$$ 

Therefore, if $I_L(0) \geq I_R(0)$, then $I_L(\eta) > I_R(\eta)$ for all $\eta \in (0, 1)$. If $I_L(0) < I_R(0)$, then there exists a unique $\eta \in (0, 1)$ such that $I_L(\eta) < I_R(\eta)$ for all $\eta \in (0, \eta)$ and $I_L(\eta) > I_R(\eta)$ for all $\eta \in (\eta, 1)$. Consequently, $I_L(0) < I_R(0)$ if and only if:

$$\alpha < \alpha := \frac{\beta}{1 + 2\beta}.$$ 

Hence, if $\alpha \geq \alpha$, then $\hat{\lambda}^* < \lambda^*$. If $\alpha < \alpha$, there exists a unique $\eta \in (0, 1)$ so that $\tilde{\lambda}^* > \hat{\lambda}^*$ for all $\eta \in (0, \eta)$ and $\tilde{\lambda}^* < \lambda^*$ for all $\eta \in (\eta, 1)$.

**Q.E.D.**

**Data Availability Statement**

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

**References**


19


