広島大学

Inclusive processes from lattice QCD

(or , how we go beyond quark-hadron duality)

Shoji Hashimoto（KEK, SOKENDAI）

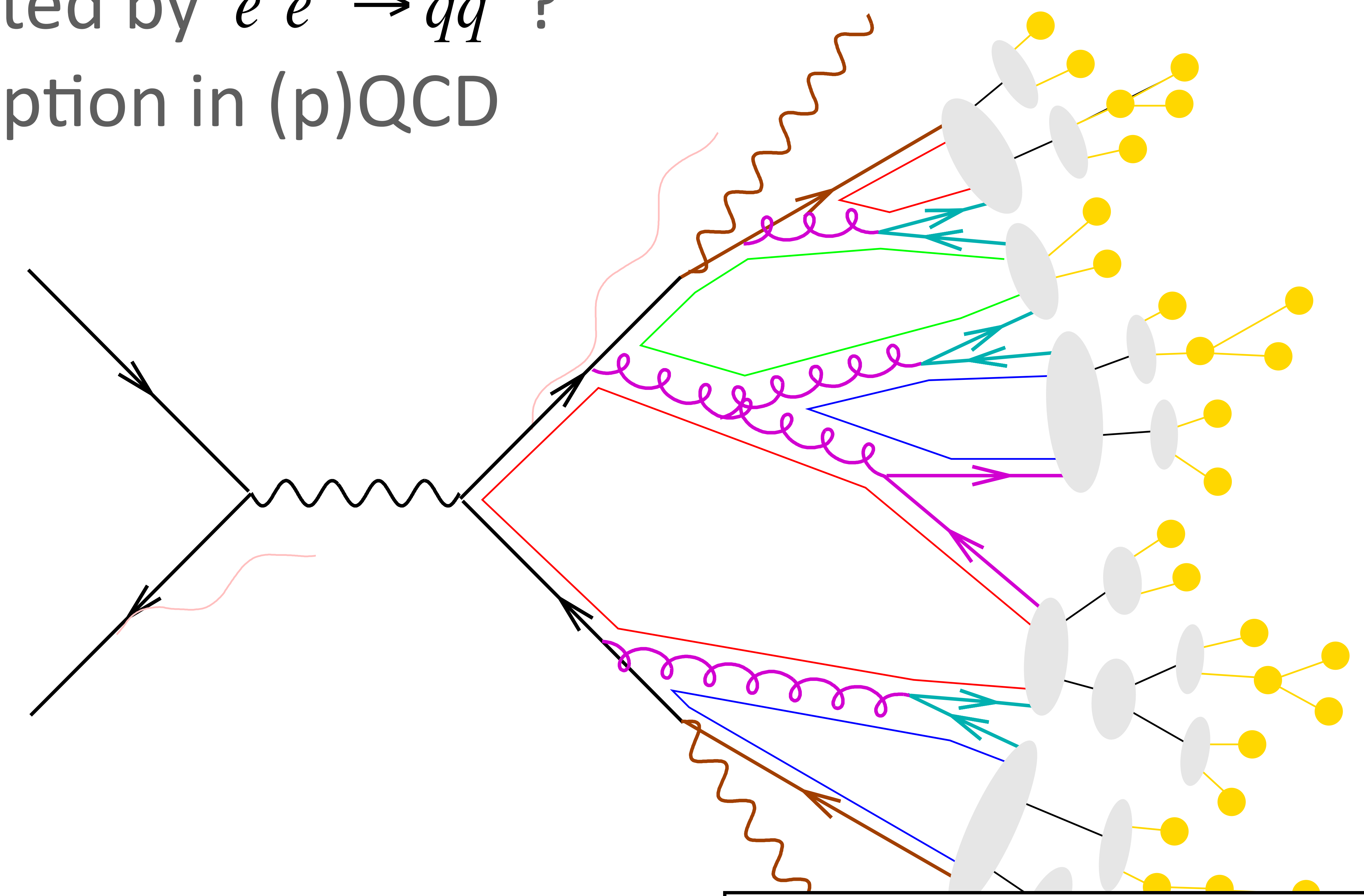
SOKENDAI

Feb 20, 2023

Quark-hadron duality ?

Well approximated by $e^+e^- \to q\bar{q}$?
= Basic assumption in (p)QCD

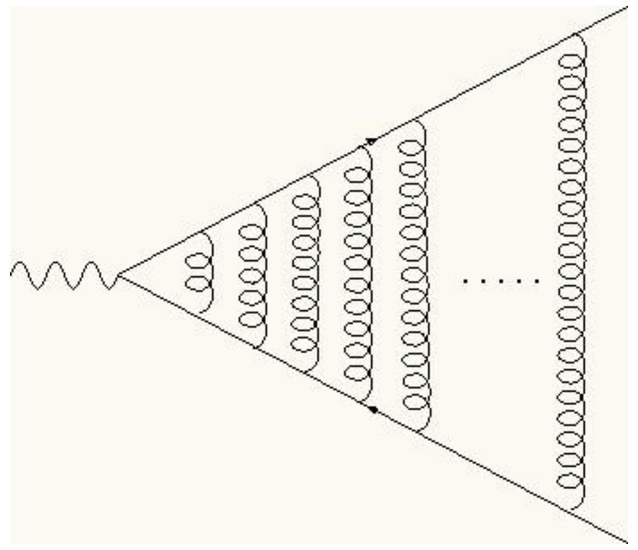
from Zeppenfeld's lecture



- What is the condition?
- How do you estimate the error?
- What can be done if not satisfied?

Duality badly violated…

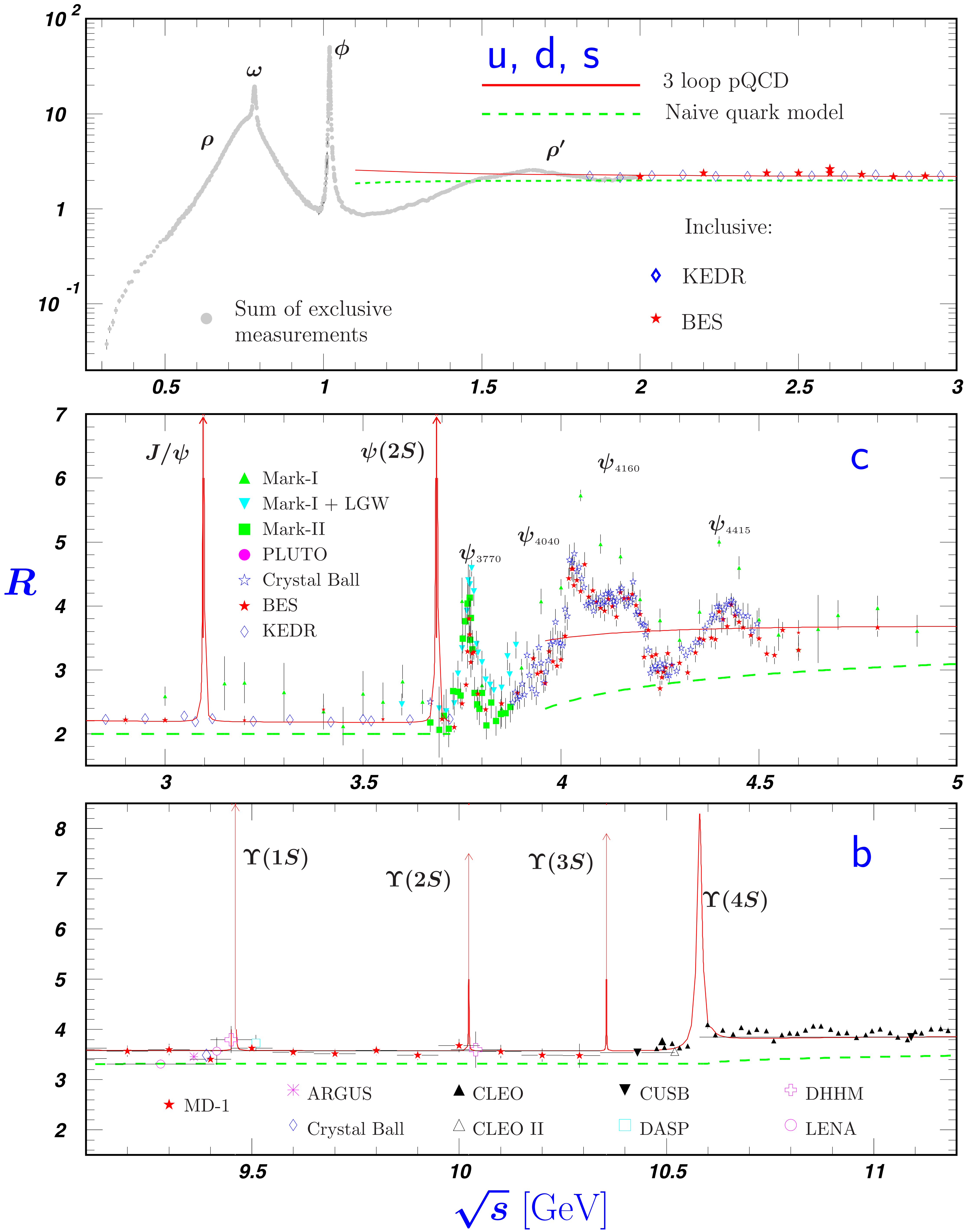
- A lot of resonances!
  - Highly non-perturbative even for quarkonium.



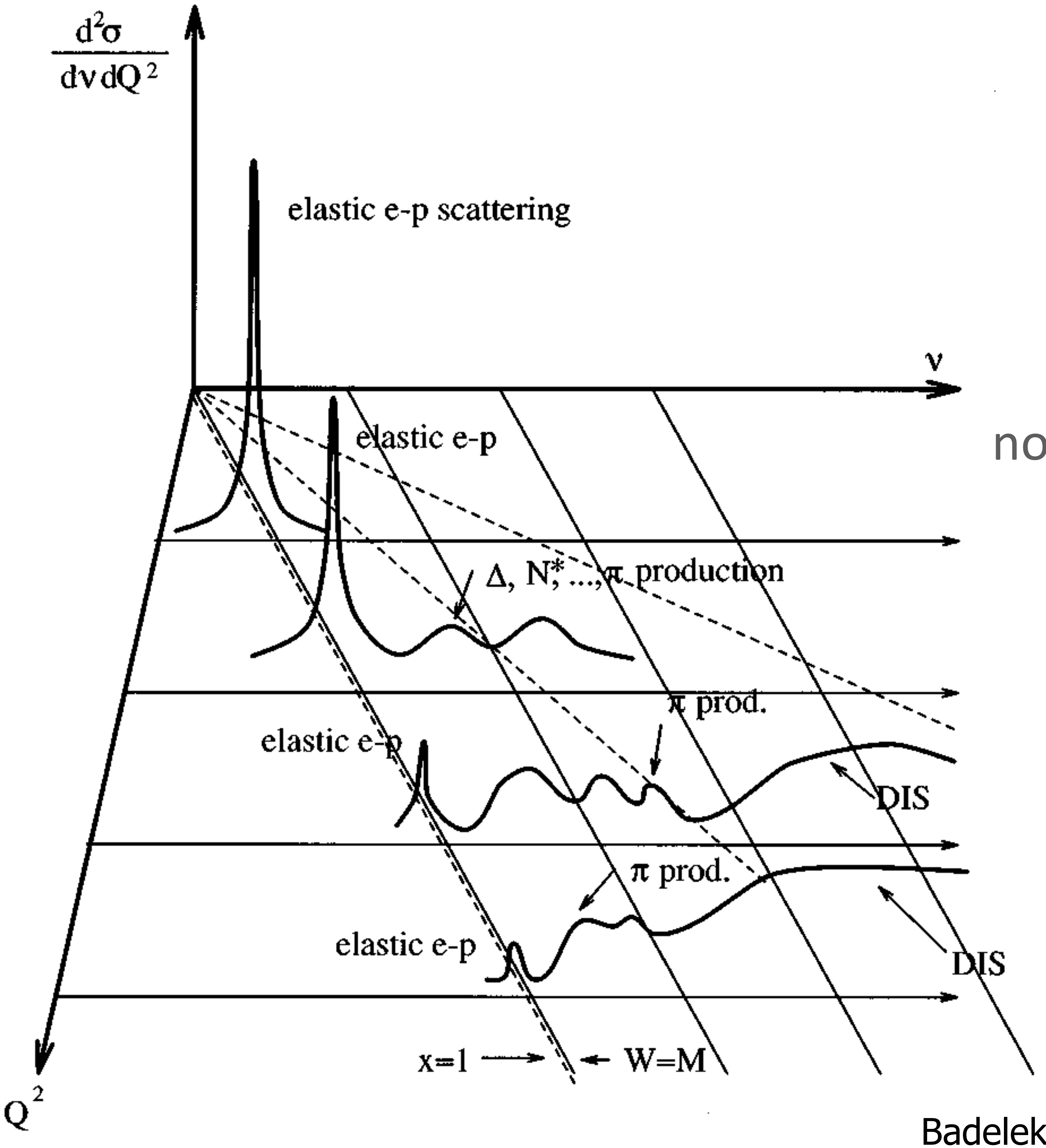
$$\sim \left(\frac{\alpha_s}{v}\right)^n$$

Need to resum, yet incomplete

  - Even harder for the light sector



e-p scattering:



non-perturbative

??

perturbative

Badelek, Kwiecinski, RMP 68, 445 (1996)

Duality works when…

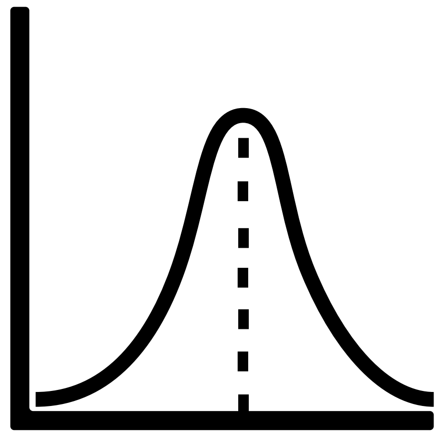
Poggio, Quinn, Weinberg, PRD13, 1958 (1976)

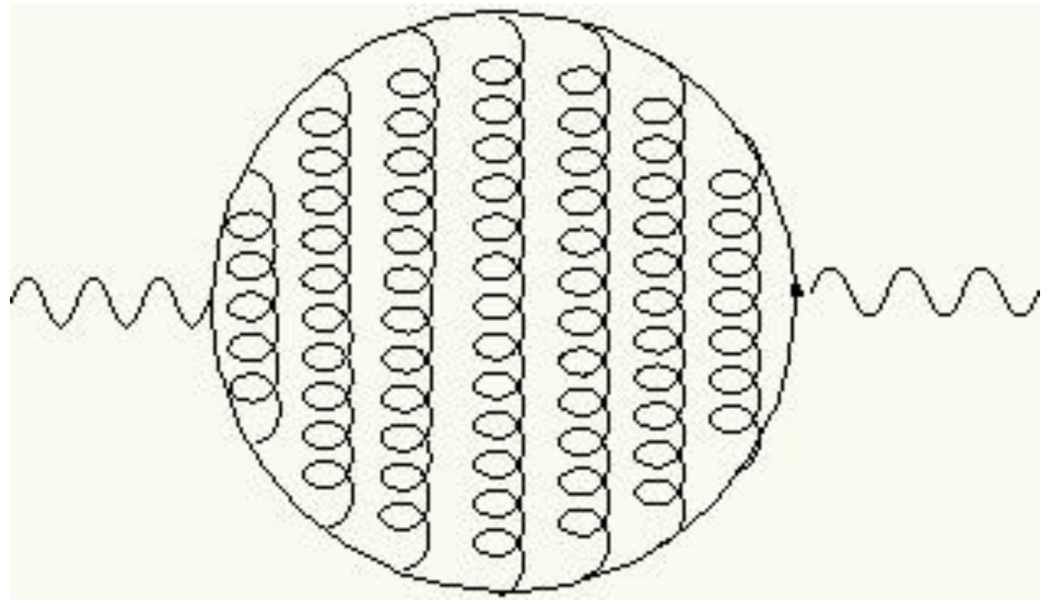
- Sufficiently smeared:

$$\mathrm{Im}\Pi(s) \propto R(s) = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

  - Consider a quantity **smeared** over some range.

$$\begin{aligned}\bar{R}(s,\Delta) &\equiv \frac{\Delta}{\pi}\int_0^\infty ds' \frac{R(s')}{(s-s')^2+\Delta^2} \\ &= \frac{1}{2\pi i}\int_0^\infty ds' R(s')\left(\frac{1}{s-s'+i\Delta} - \frac{1}{s-s'-i\Delta}\right) \\ &= \frac{1}{2i}\left[\Pi(s+i\Delta) - \Pi(s-i\Delta)\right]\end{aligned}$$





  - One can avoid the threshold singularity.
  - Δ must be larger than Λ_{QCD}^2 to avoid non-perturbative physics.

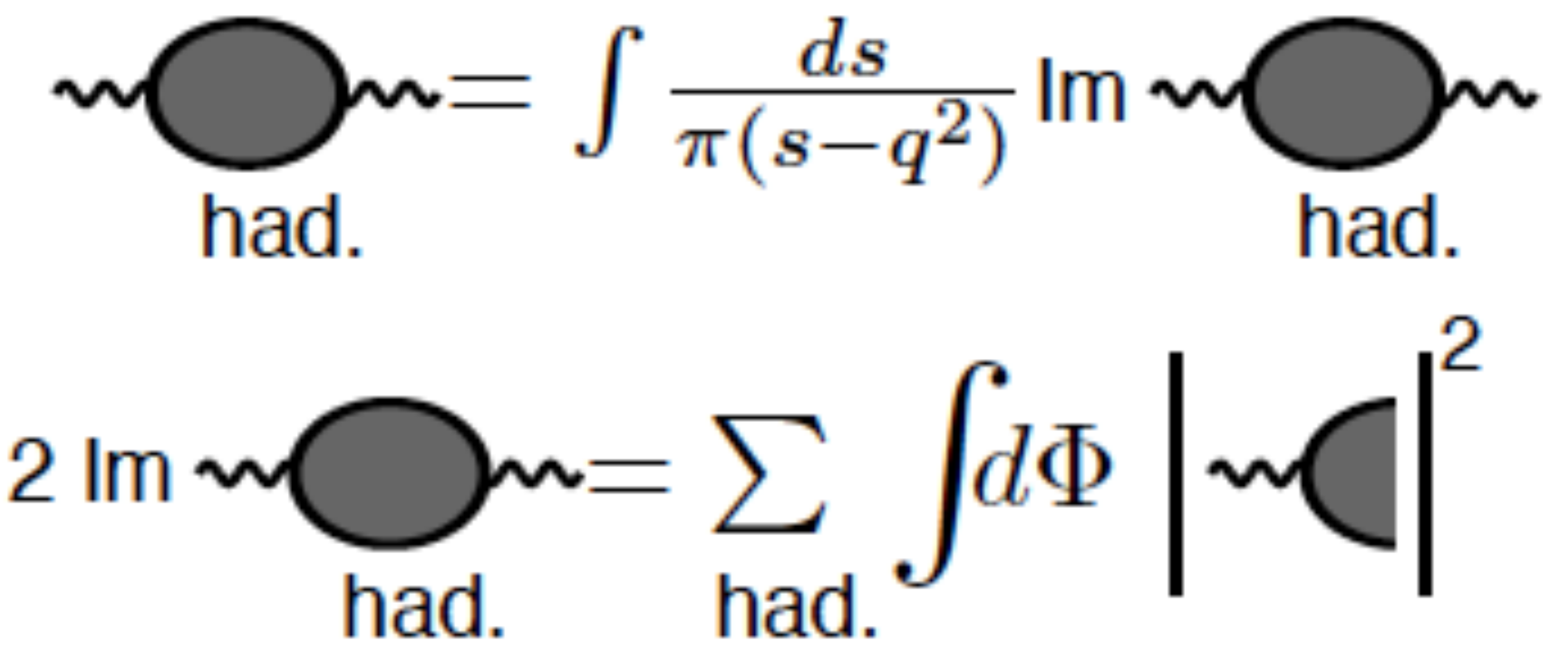
QCD sum rule

Shifman, Vainshtein, Zakharov, NPB147 385, 448 (1979)

$\Pi(Q^2)$: calculable by pQCD and OPE (+ Borel sum)

smearing over energy

$\Pi(Q^2)$



space-like region: $Q^2 = -q^2 > 0$

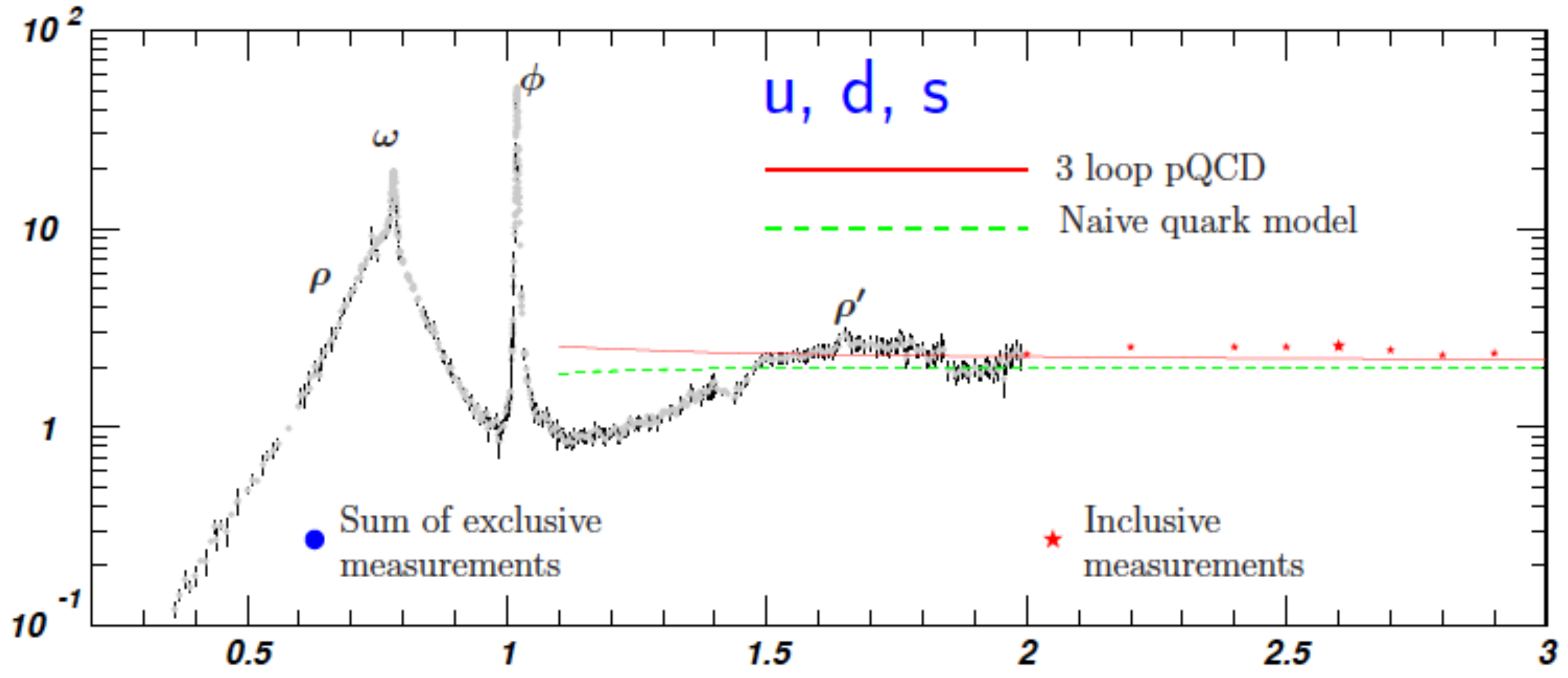
pQCD should work

time-like region: $s = q^2 > 4m_\pi^2$

exp available

smearing over phase-space

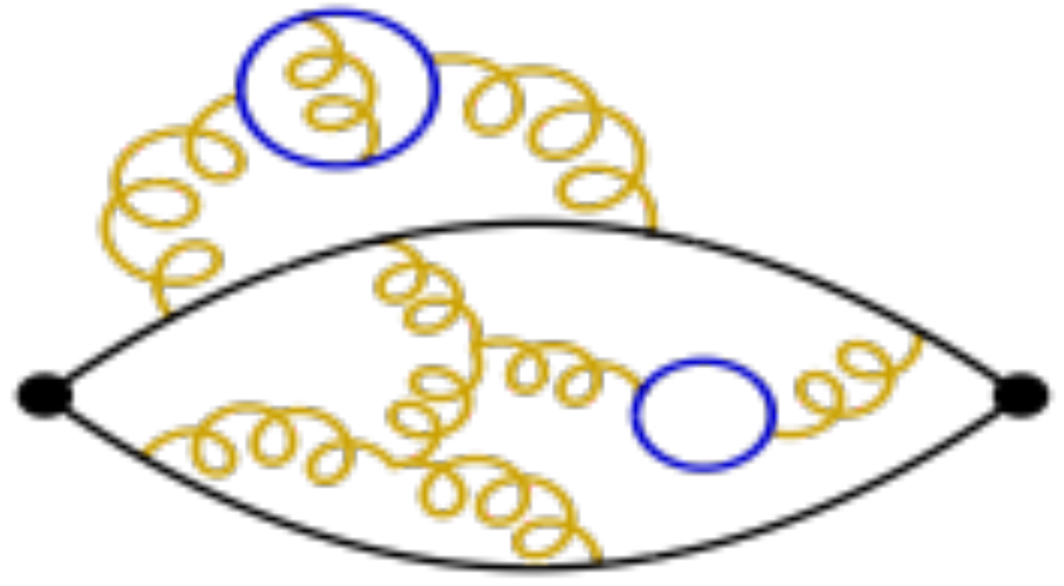
Yes, sufficiently smeared!



$\Pi(Q^2)$: why not lattice?

Well, it's surely possible!

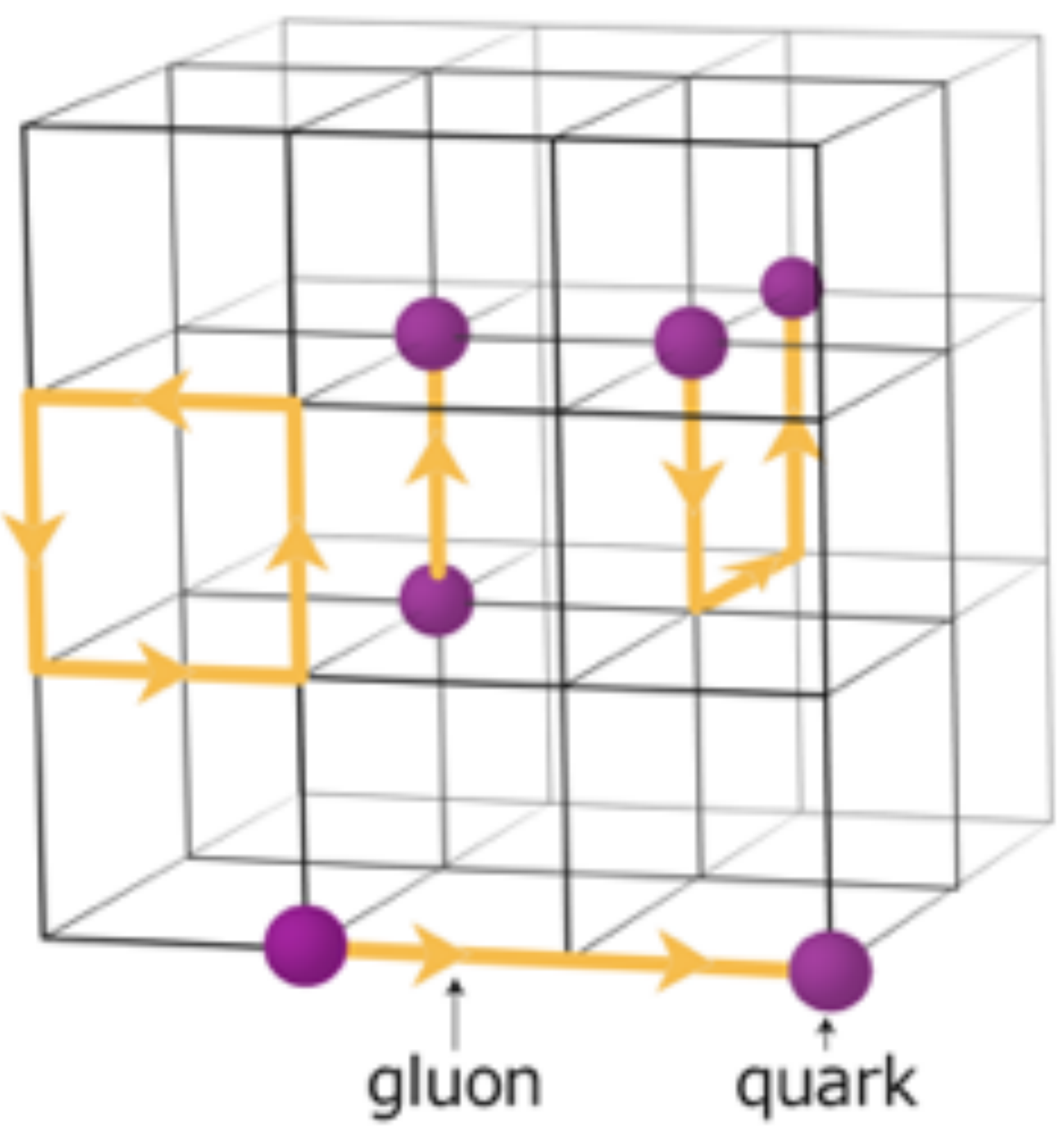
$$\Pi_{\mu\nu}(x) = \langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle$$



- Calculation on the Euclidean lattice naturally provide the space-like $\Pi(Q^2)$.
  - A bread-and-butter calculation, though need large resources to be realistic.
  - An input for hadronic-vacuum-polarization (HVP) contribution of muon g-2.
- Remember: the smearing is crucial to compare with exp.
  - But, then, no assumption is involved, plus fully non-perturbative.

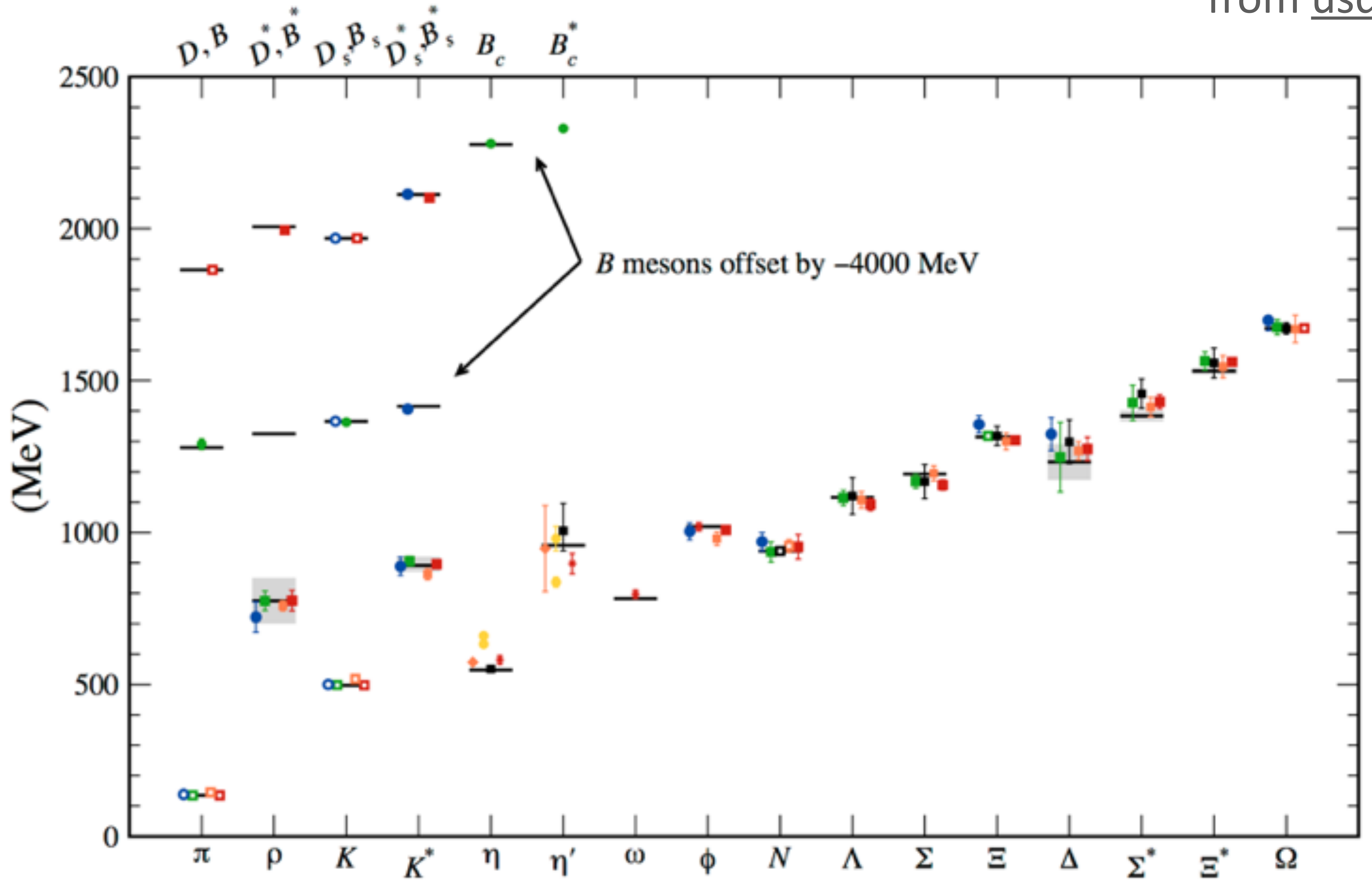
Euclidean lattice QCD

LQCD = ab initio calculation of QCD, but on the Euclidean space



- Define the quark and gluon fields on the **Euclidean** lattice.
- Perform the path integral numerically (Monte Carlo).

from usqcd.org



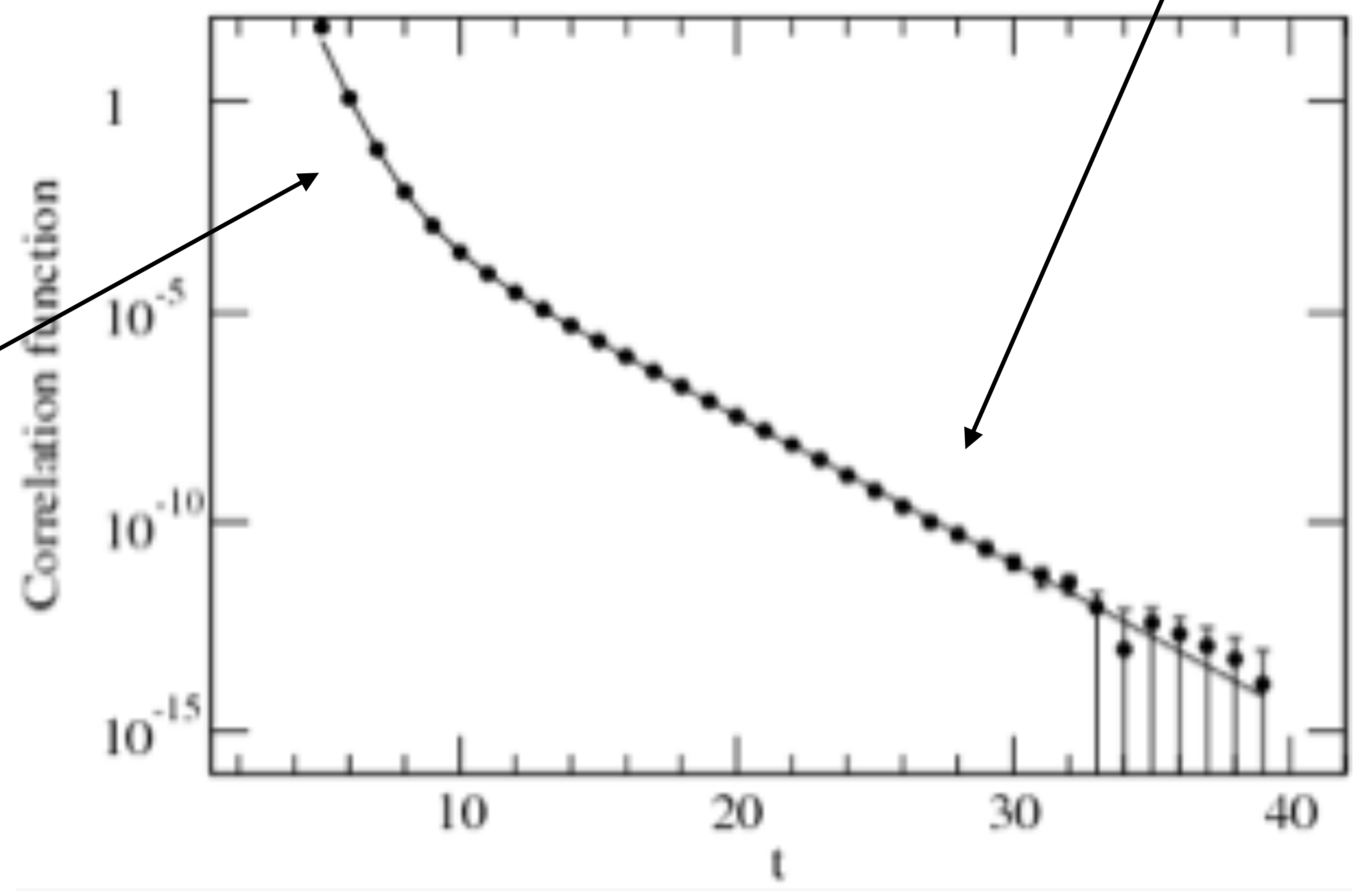
Case study: vacuum polarization

Euclidean correlator

- e^{-Et} instead of e^{-iEt}

read off the exponential slope at long distances
→ hadron energy (or mass)

$$\int d^3\mathbf{x}\langle\mathcal{O}(\mathbf{x},t)\mathcal{O}^\dagger(0)\rangle$$

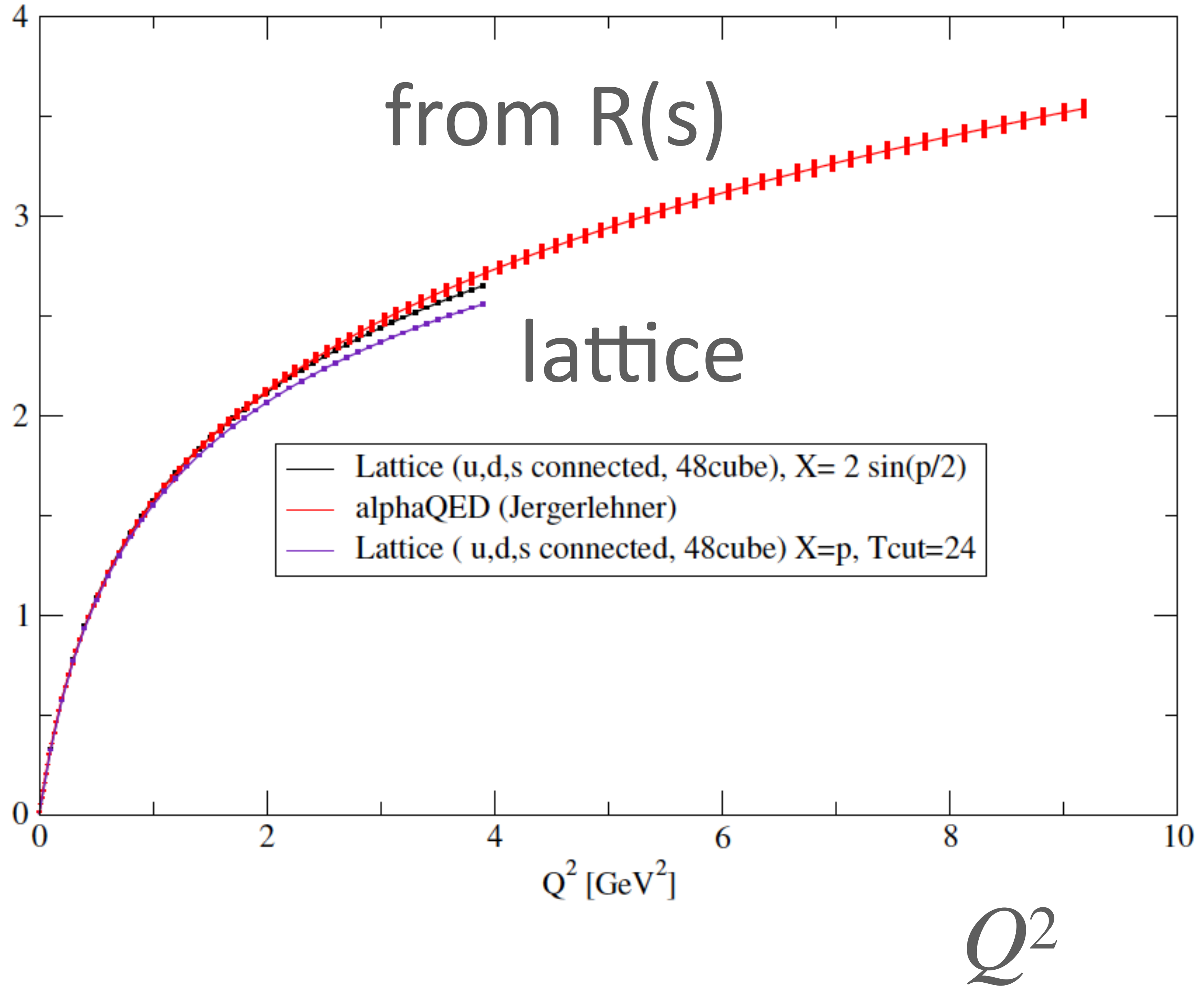


More physics info contained in the short-distance region

Go space-like

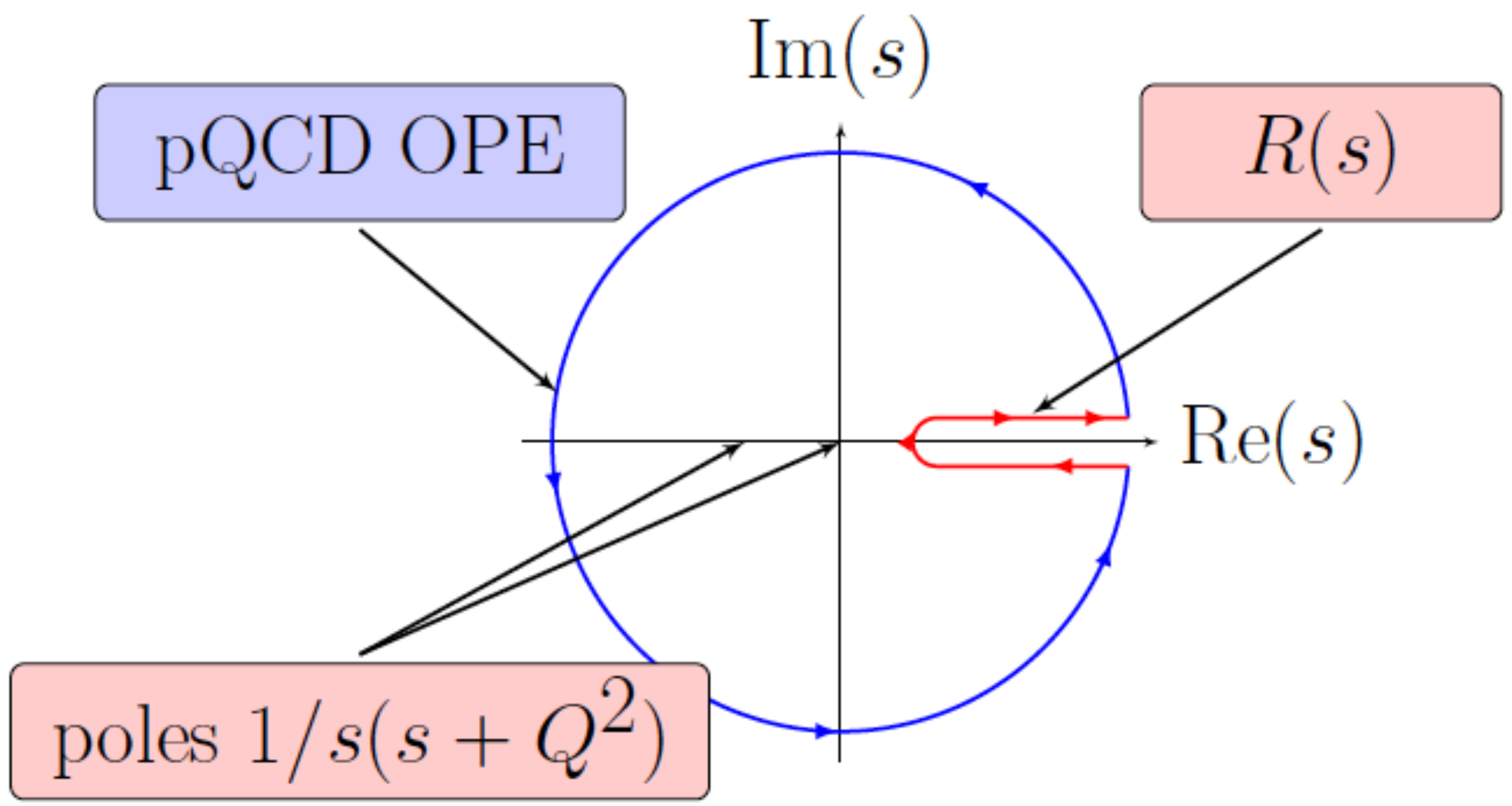
Fourier transform of lattice data
to produce the space-like $\Pi(Q^2)$

RBC/UKQCD:
Izubuchi@g-2 WS (2017)



smearing proveded by

$$\hat{\Pi}(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}$$



Smearing in general

Some (weighted) integrals:

- Space-like correlator: $$\Pi(-Q^2) = \frac{1}{\pi}\int_0^\infty ds\, \frac{\mathrm{Im}\Pi(s+i\epsilon)}{s+Q^2} = \int_0^\infty ds\, \frac{\rho(s)}{s+Q^2}$$
  - weighted integral over s (or ω)
  - can be written as a Fourier transform of the Euclidean lattice correlator
- HVP contribution to Muon g-2: $$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s}\, \frac{1}{\pi}\mathrm{Im}\Pi(s)\, K(s)$$
  - weighted integral over s (or ω)
  - can also be written as an integral (or a sum) of lattice correlator

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, C(t)\tilde{f}(t)$$

Bernecker-Meyer (2011)

Connection to the lattice correlator

all possible states contribute ↓

correlator: $$C(t) = \int_0^\infty d\omega\, \rho(\omega) e^{-\omega t} \qquad \sim \langle 0 | J\, e^{-\hat{H} t}\, J | 0 \rangle$$

sum over states:
(or smearing)
$$\Gamma = \int_0^\infty d\omega\, K(\omega) \rho(\omega) \qquad \sim \langle 0 | J K(\hat{H}) J | 0 \rangle$$

Approximation of the form

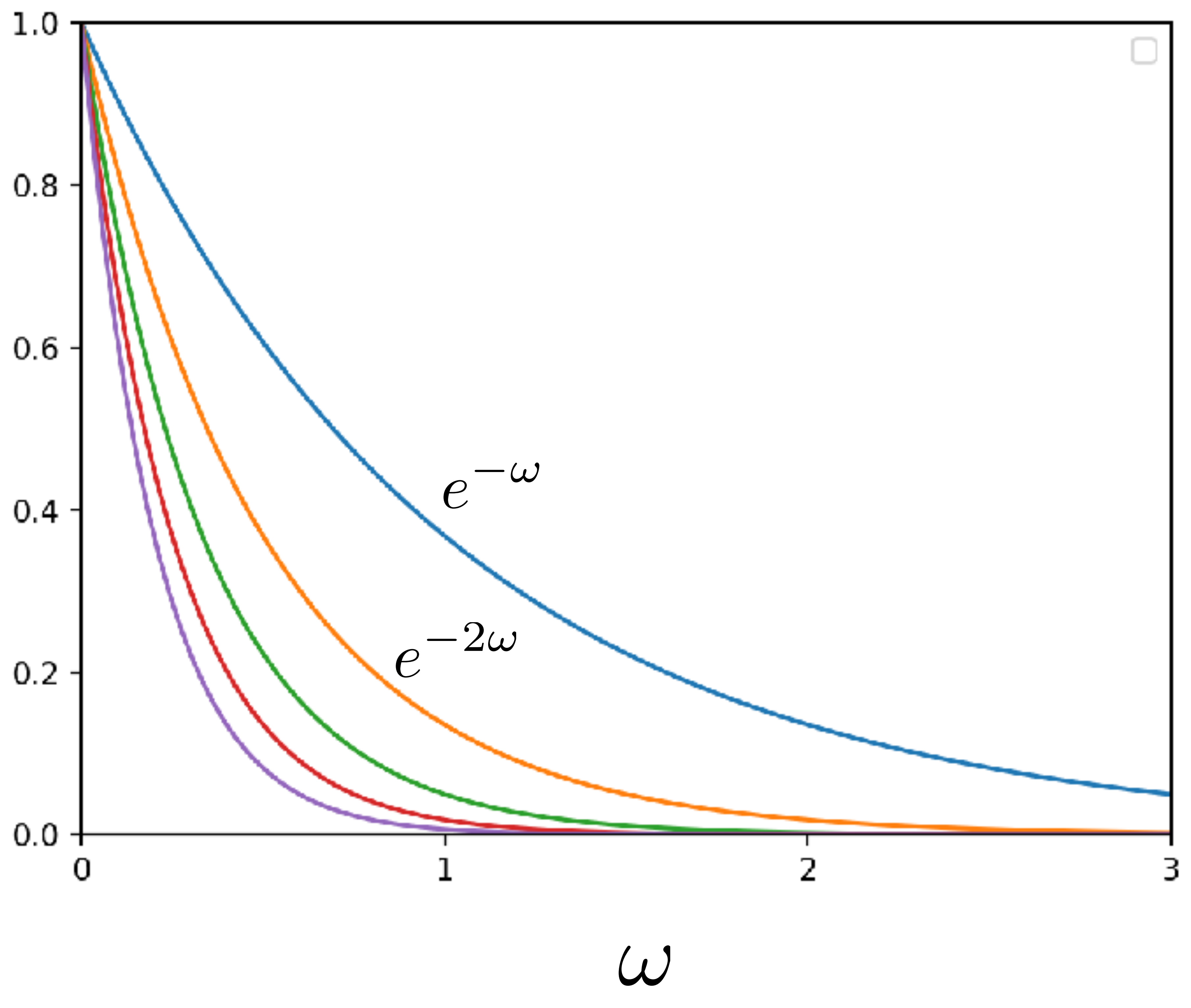
$$K(\hat{H}) = c_0 + c_1 e^{-\hat{H}} + c_2 e^{-2\hat{H}} + c_3 e^{-3\hat{H}} + \cdots$$

can relate Γ to the correlator.

c.f. spectral func: $$\rho(\omega) \propto \sum_X \delta(\omega - E_X) |\langle X | J | 0 \rangle|^2 \qquad \sim \langle 0 | J\, \delta(\omega - \hat{H})\, J | 0 \rangle$$

Approximation?

$$K(\hat{H}) \simeq k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-N\hat{H}}$$



- Not always possible; when the function varies rapidly, in particular.
- Some methods developed recently.

  - Modified Backus-Gilbert

    Hansen, Lupo, Tantalo, arXiv:1903.06476
  - Or, Chebyshev polynomial

    Bailas, Ishikawa, SH, arXiv:2001.11779

Chebyshev polynomials

Bailas, SH, Ishikawa (2000)

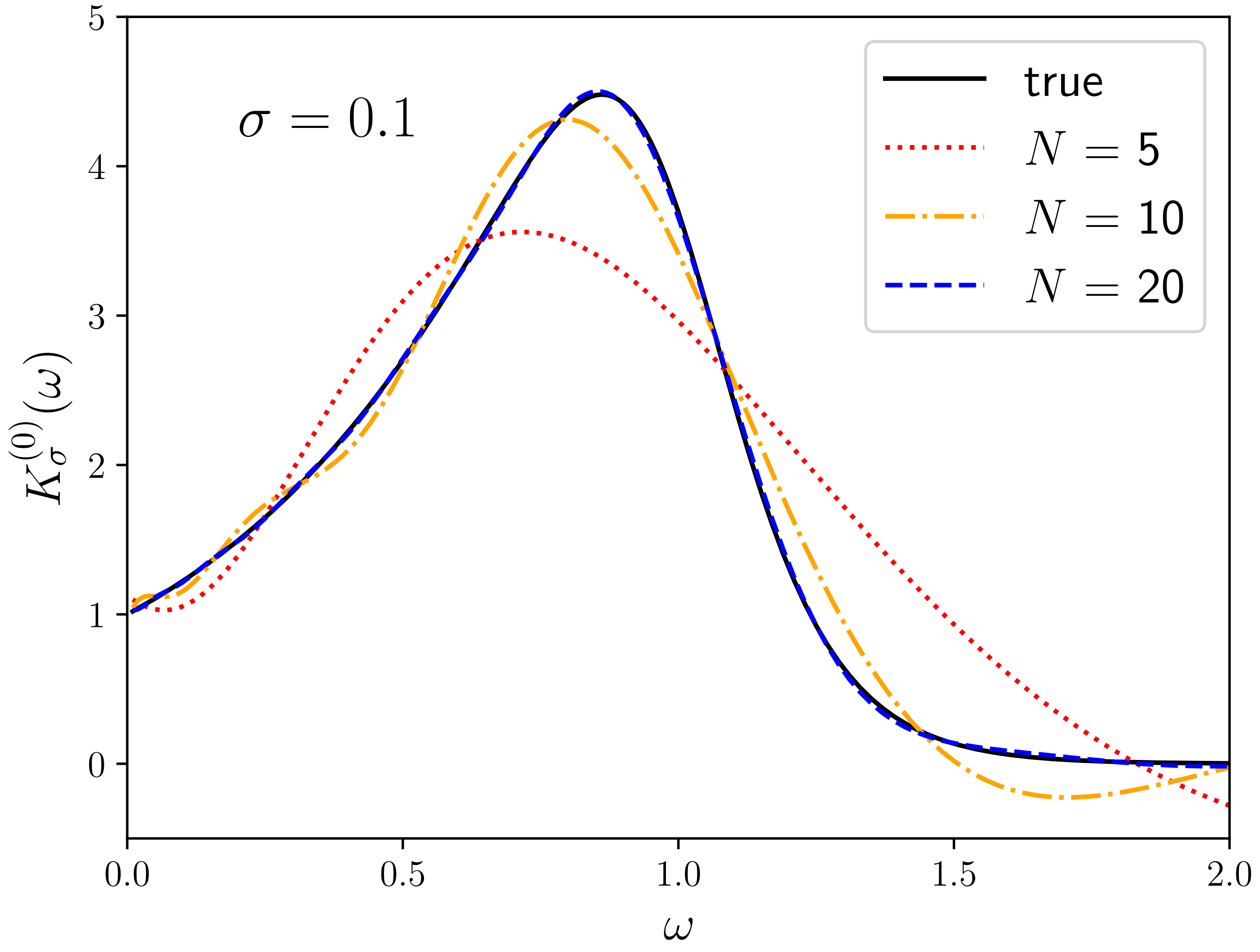
$$K(\hat{H}) \simeq \sum_{j=0}^{N} c_j T_j(e^{-\hat{H}})$$

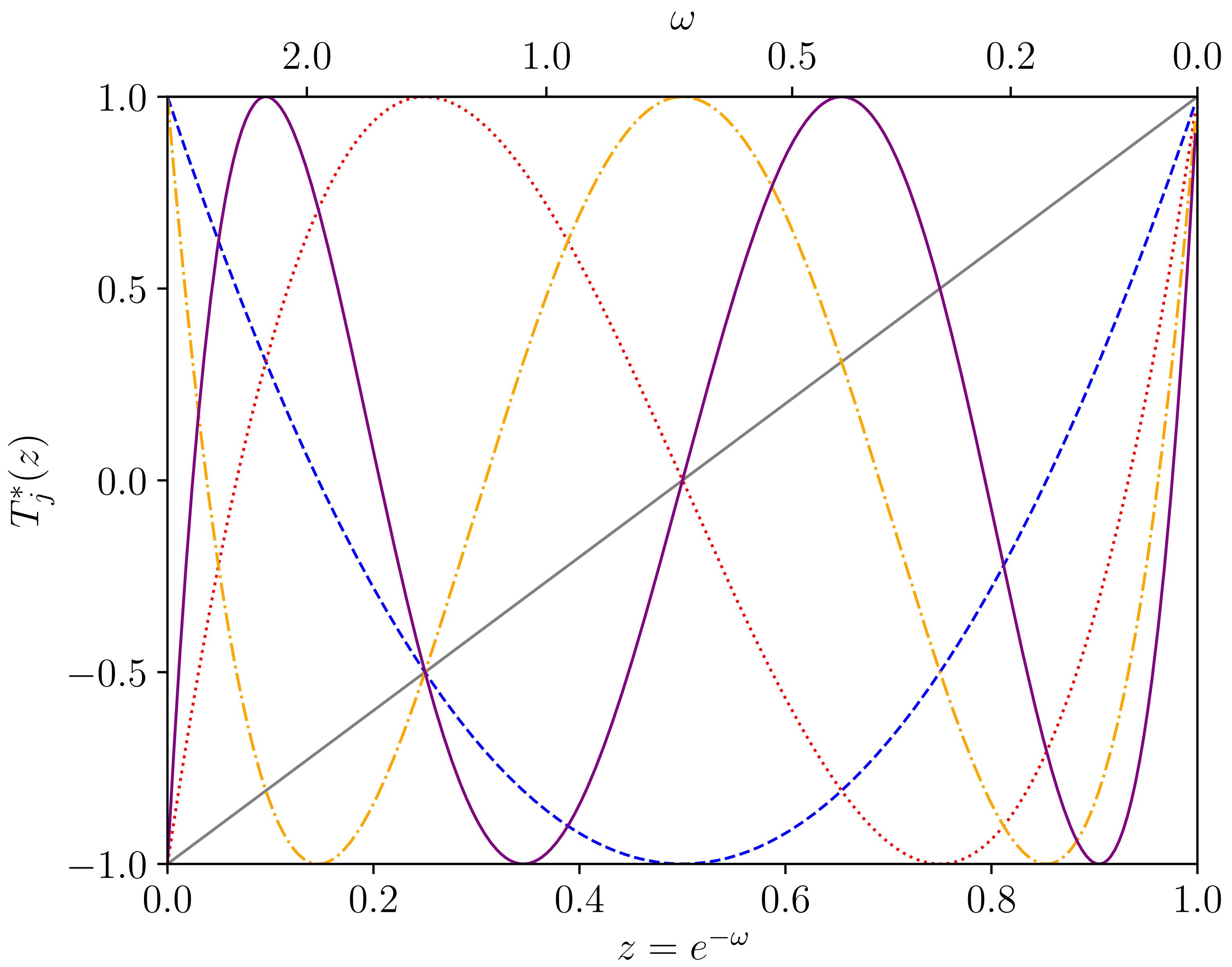
(shifted) Chebyshev polynomials

$$T_0^*(x) = 1$$
$$T_1^*(x) = 2x - 1$$
$$T_2^*(x) = 8x^2 - 8x + 1$$
$$\vdots$$
$$T_{j+1}^*(x) = 2(2x-1)T_j^*(x) - T_{j-1}^*(x)$$

- Coefficients can be easily calculated.
- The “best” approx (= maximal deviation is minimal)
- Only smooth functions can be approximated.
- (The constraint $|T_j(z)| < 1$ helps stabilize.)

example of the Chebyshev approx:



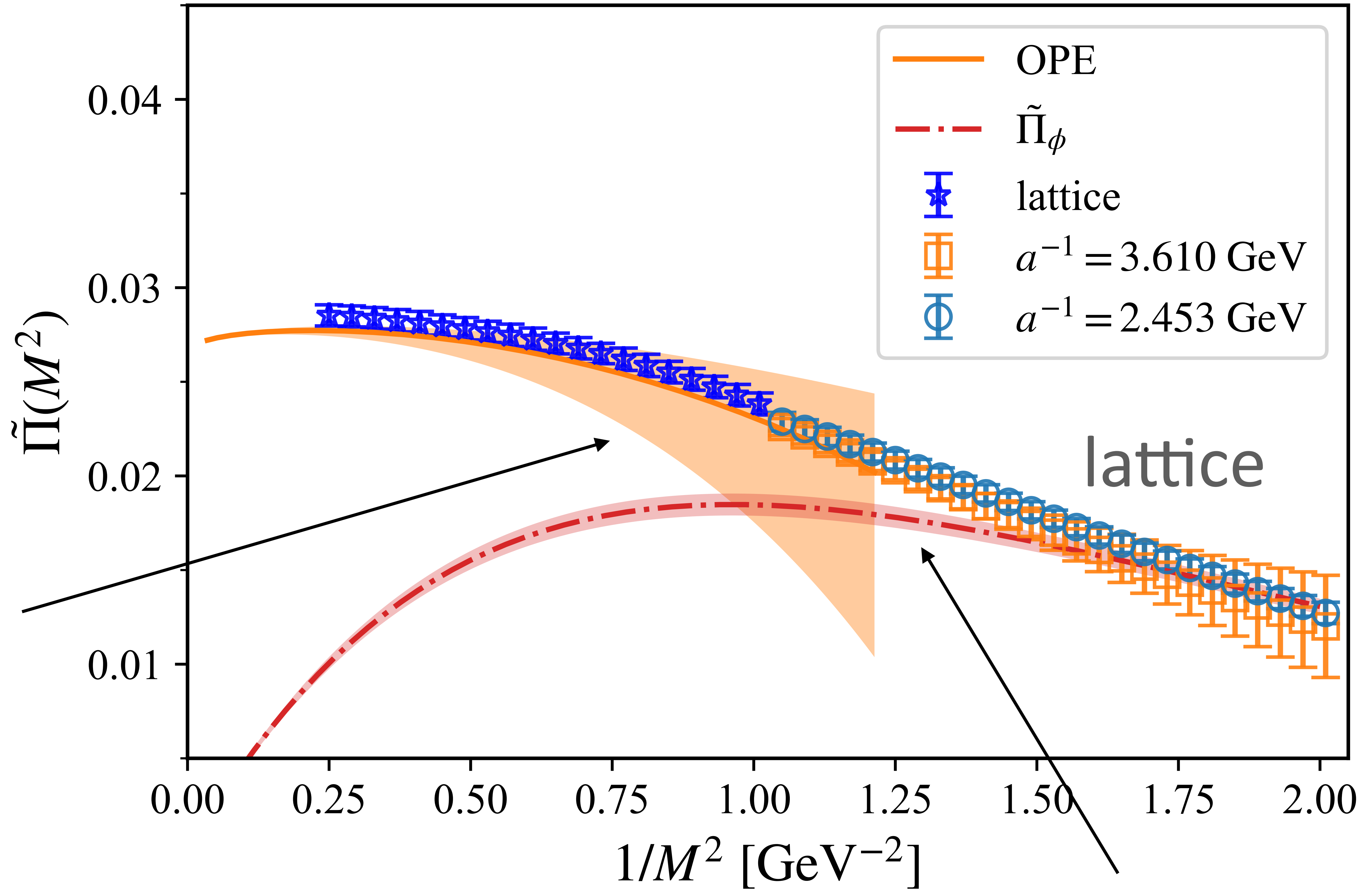


Borel sum (as in QCD sum rule)

Ishikawa, SH, Phys. Rev. D104, 074521 (2021)

$$\int ds\, e^{-s/M^2} \mathrm{Im}\Pi(s)$$

$s\bar{s}$ channel



lattice

pQCD + OPE

ϕ meson contrib

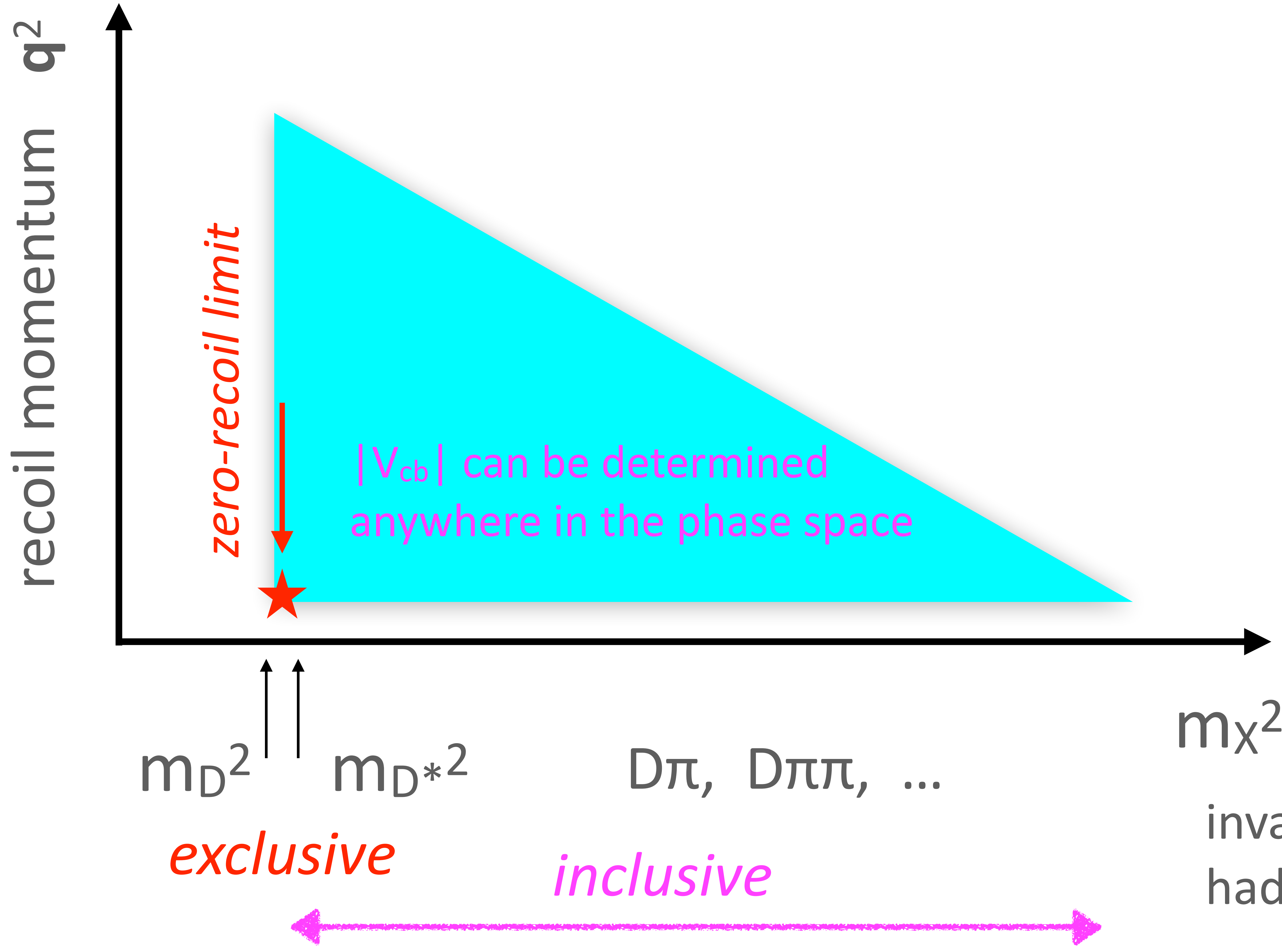
B meson semileptonic decays: total inclusive rate?

Based on the collaborations of

- Gambino, SH, Phys. Rev. Lett. 125 (2020) 032001; arXiv:2005.13730
- Gambino, SH, Machler. Panero, Sanfilippo, Simula, Smecca, Tantalo, JHEP 07 (2022) 083; arXiv:2203.11762
- Barone, Kellerman, SH, Juttner, Kaneko, arXiv:2211.15623, arXiv:2211.16830

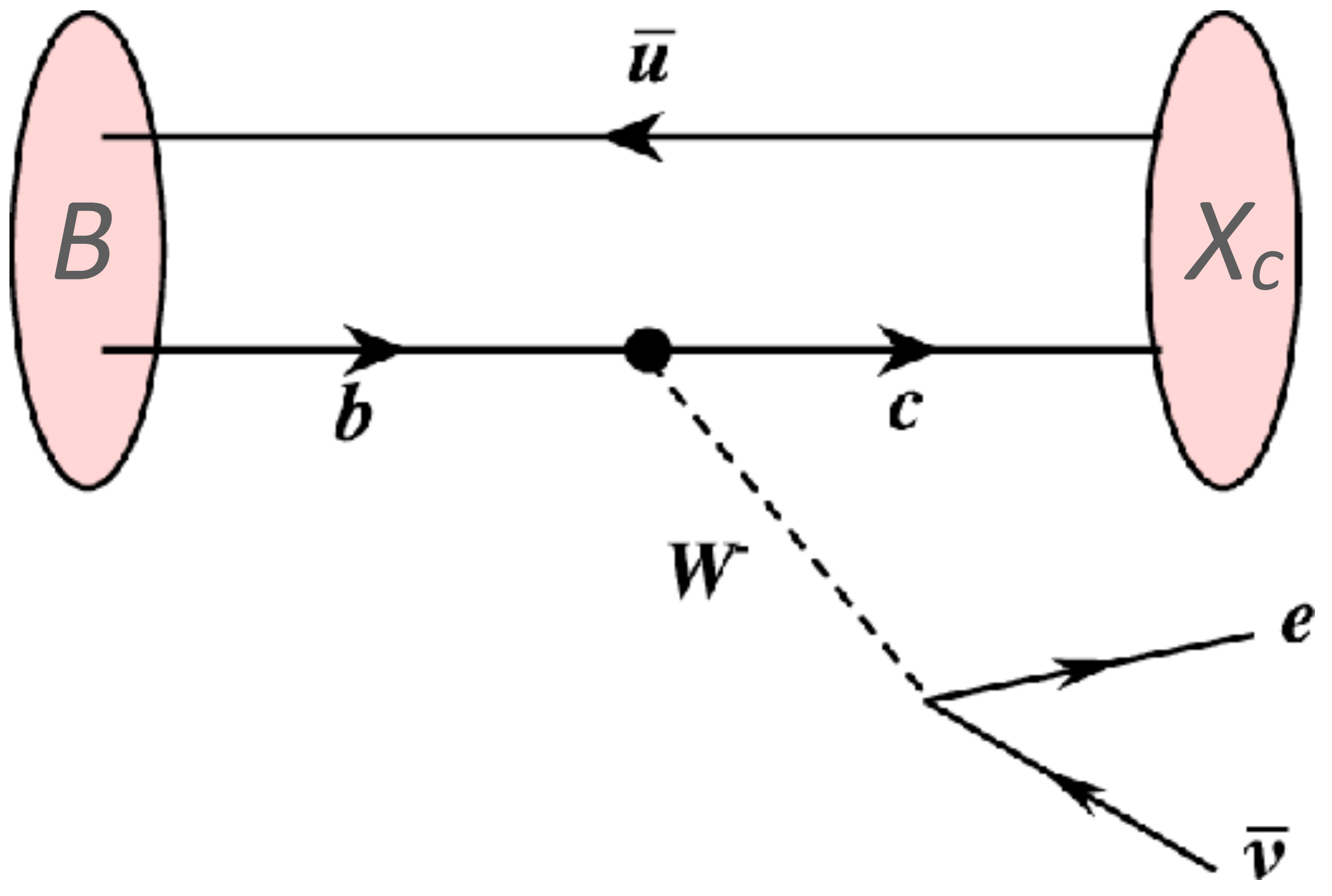
see also, Hansen, Meyer, Robaina, Phys. Rev. D96, 094513 (2017); arXiv:1704.08993

Inclusive and exclusive B semileptonic decays



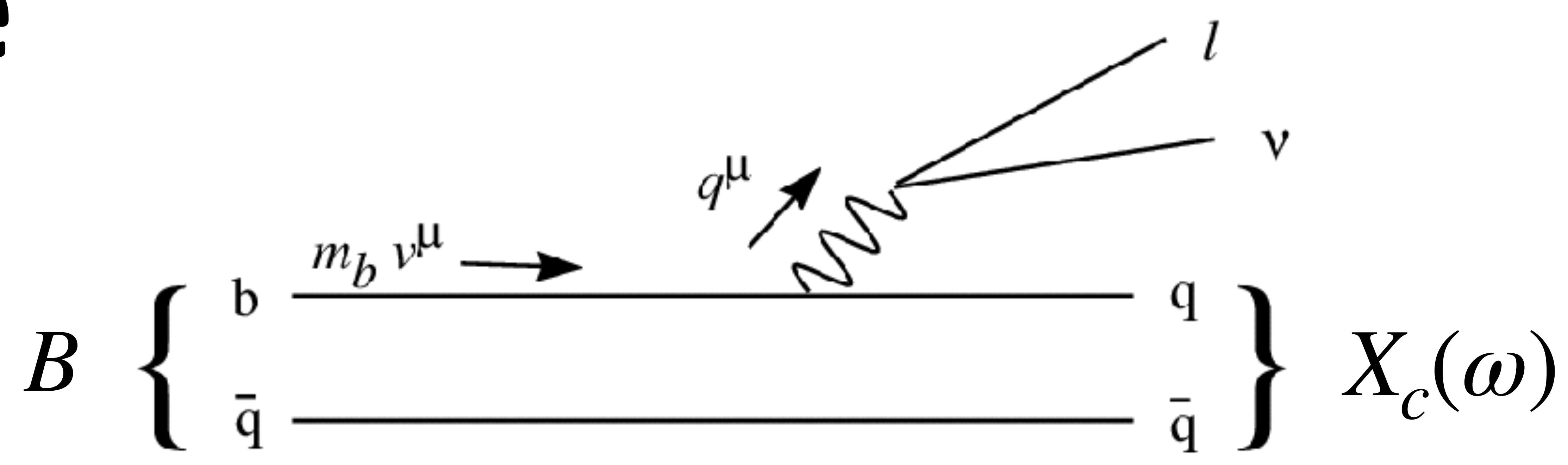
m_X^2

invariant mass of the hadronic system



exclusive particular final states (D, D*, …)
inclusive sum over final states

Inclusive semi-leptonic rate



Differential decay rate:

$$d\Gamma \sim |V_{cb}|^2 l^{\mu\nu} W_{\mu\nu}$$

Structure function (or hadronic tensor):

$$W_{\mu\nu} = \sum_X (2\pi)^2 \delta^4(p_B - q - p_X) \frac{1}{2M_B} \langle B(p_B)|J_\mu^\dagger(0)|X\rangle\langle X|J_\nu(0)|B(p_B)\rangle$$

$$\rightarrow \langle B(\mathbf{0})|\tilde{J}_\mu^\dagger(-\boldsymbol{q};t)\ \delta(\omega - \hat{H})\ \tilde{J}_\nu(\boldsymbol{q};0)|B(\mathbf{0})\rangle$$

Total decay rate:

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

kinematical (phase-space) factor

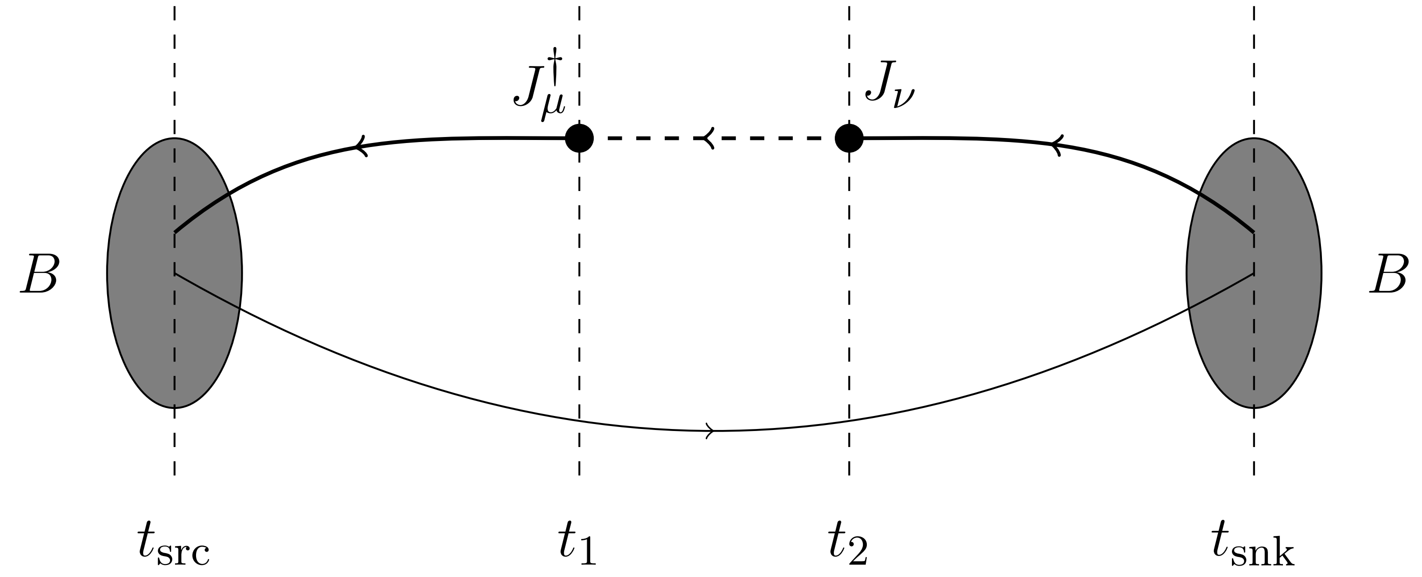
Energy integral to be evaluated:

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$

$$= \langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})K(\hat{H};\boldsymbol{q}^2)\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$

Compton amplitude obtained on the lattice:

$$\langle B(\boldsymbol{0})|\tilde{J}^\dagger_\mu(-\boldsymbol{q};t)\ \ \tilde{J}_\nu(\boldsymbol{q};0)|B(\boldsymbol{0})\rangle \longrightarrow \langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})e^{-\hat{H}t}\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$



Using :

$$K(\hat{H}) = k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-k_N\hat{H}}$$

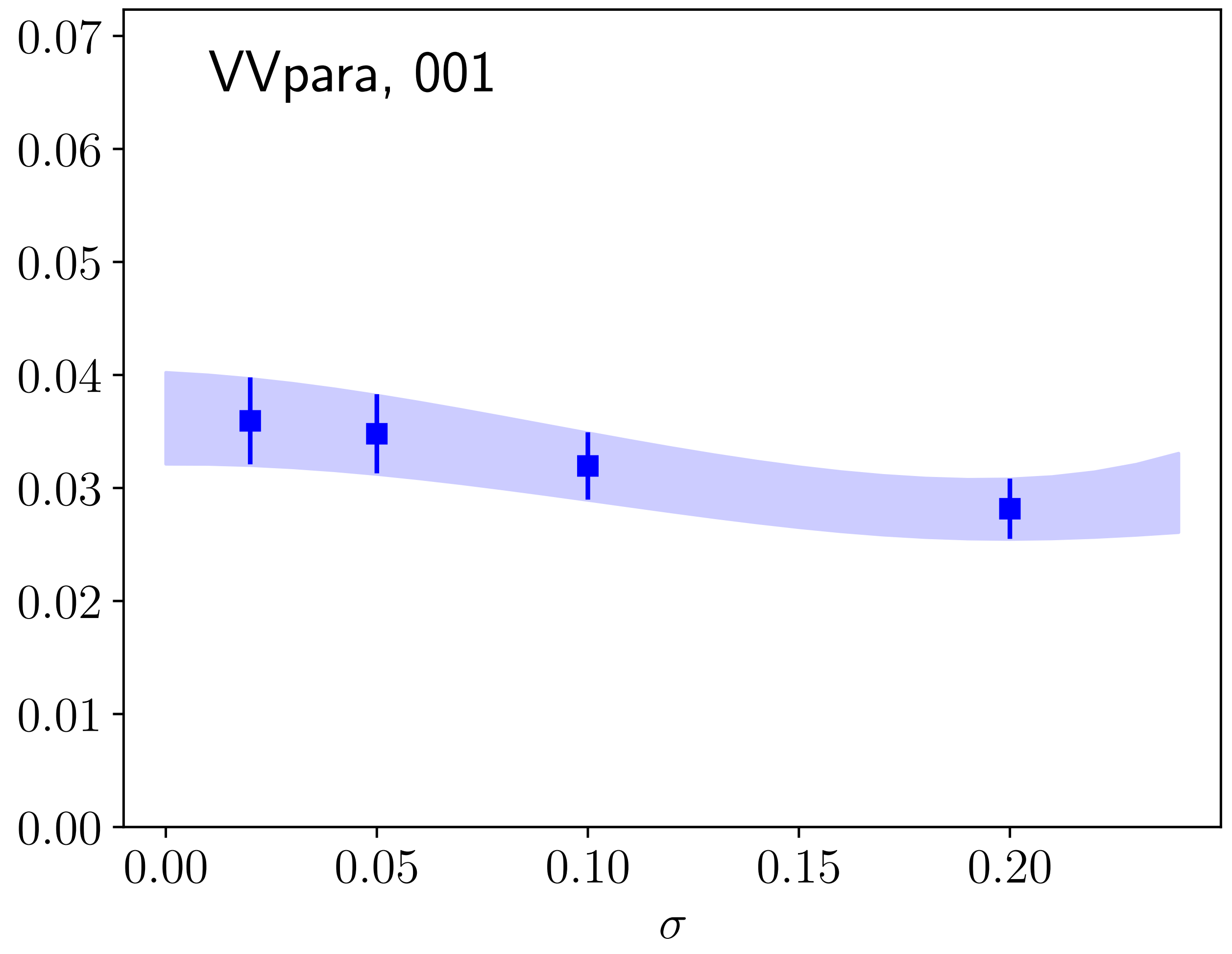
Phase-space factor as a kernel

upper limit

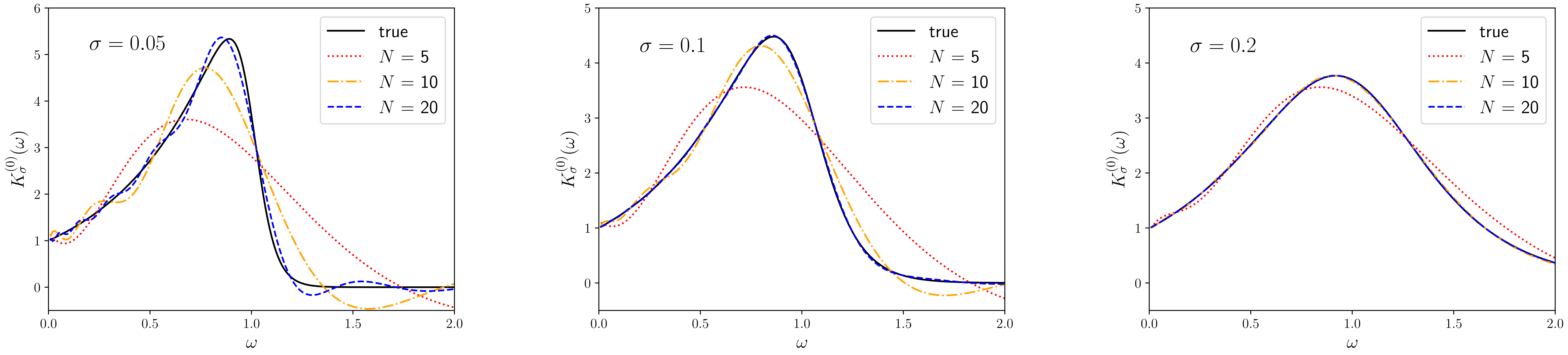
$$K(\omega) \sim e^{2\omega t_0} \underline{(m_B - \omega)^l} \boxed{\theta(m_B - |\mathbf{q}| - \omega)}$$

kinematical

smear by sigmoid with a width σ;
Need to take the σ→ 0 limit



narrower smearing wider

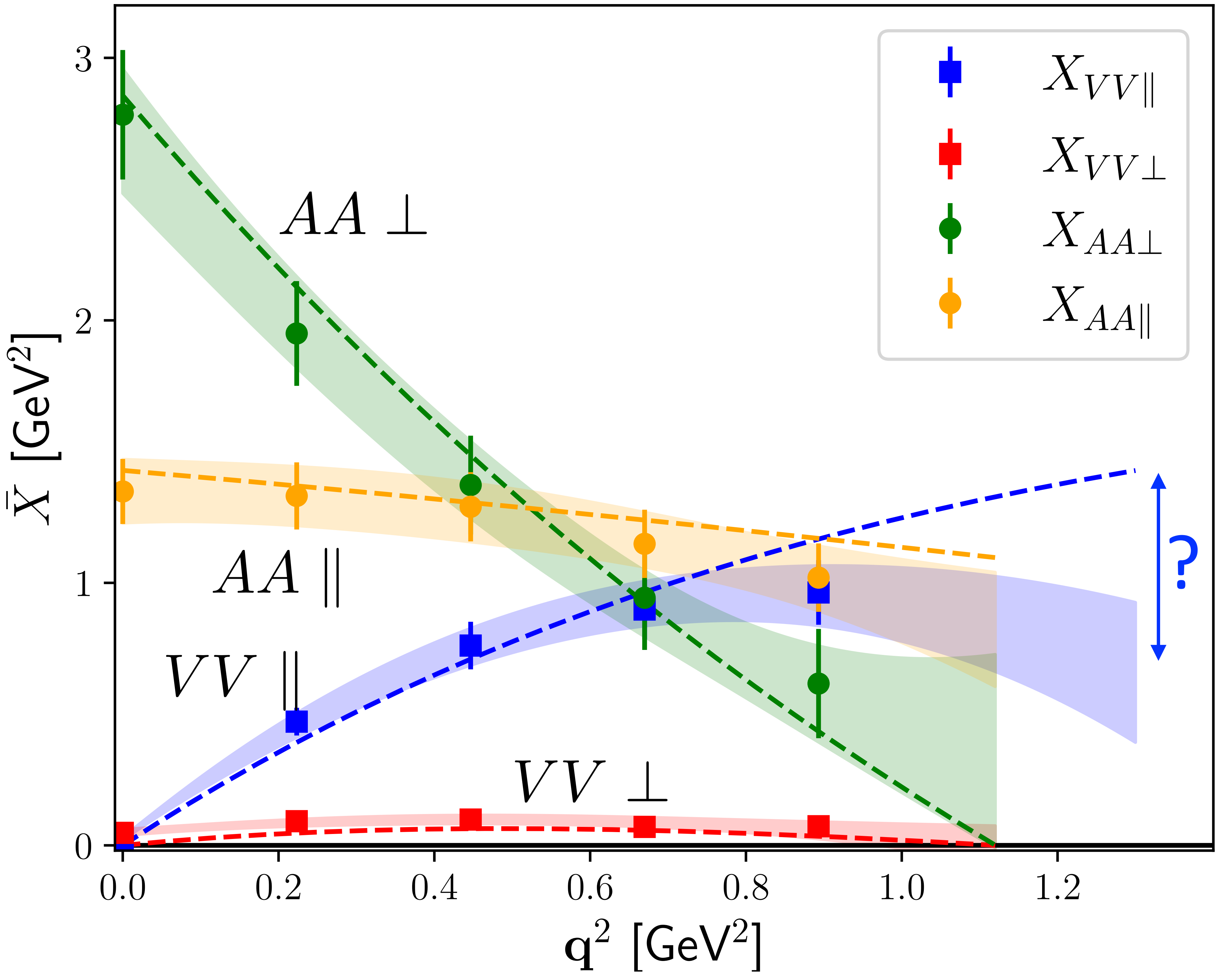


Inclusive decay rate

- Prototype lattice calculation
  - $B_s \to Xc$
  - the b quark is lighter than physical.

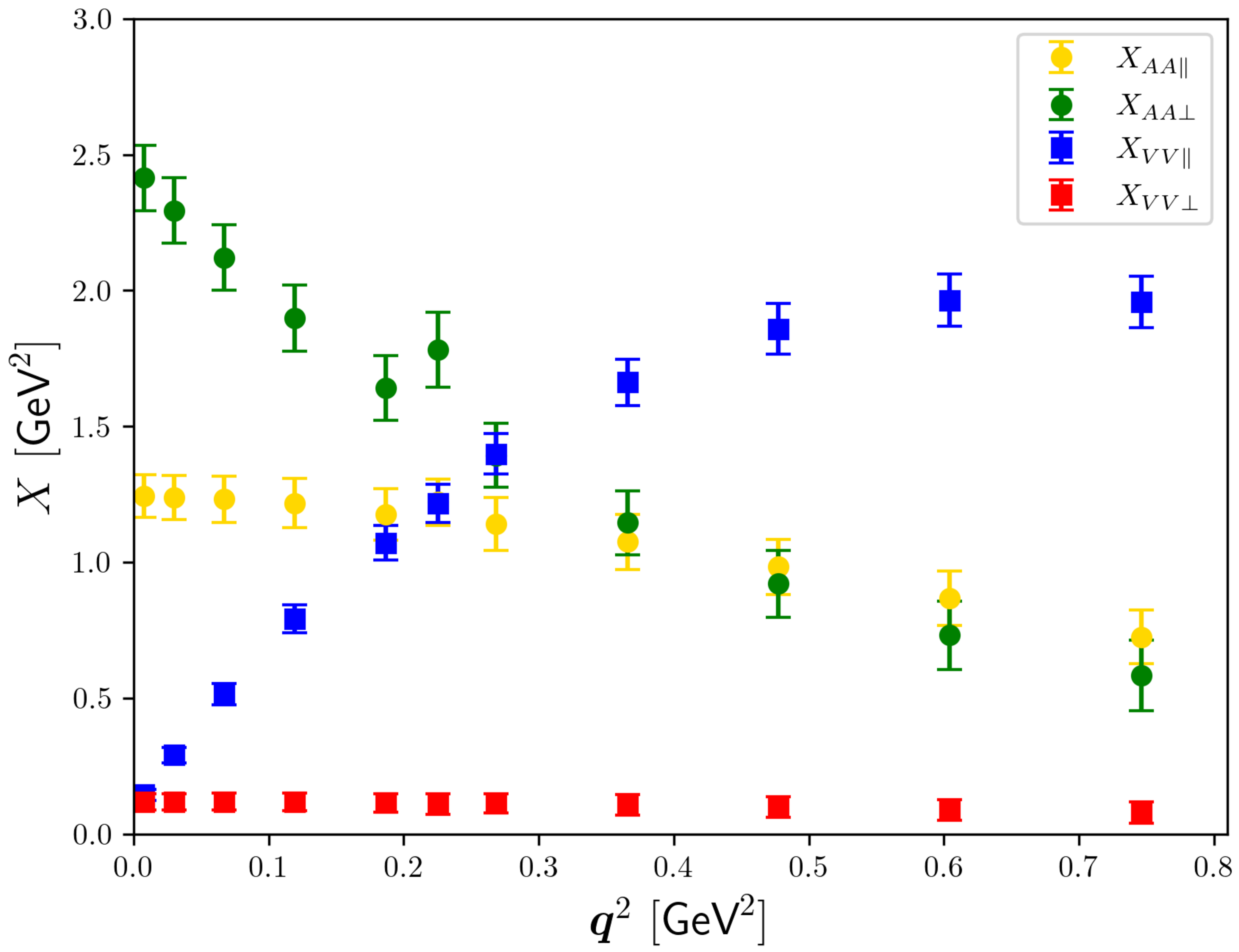
- Decay rate in each channel
  - VV and AA
  - parallel or perpendicular to the recoil momentum
  - compared to “exclusive” (dashed lines)
    - $VV_{||}$ is dominated by B→D
    - Others are by B→D*

differential rate / |**q**|



JLQCD data from
Gambino et al., 2203.11762

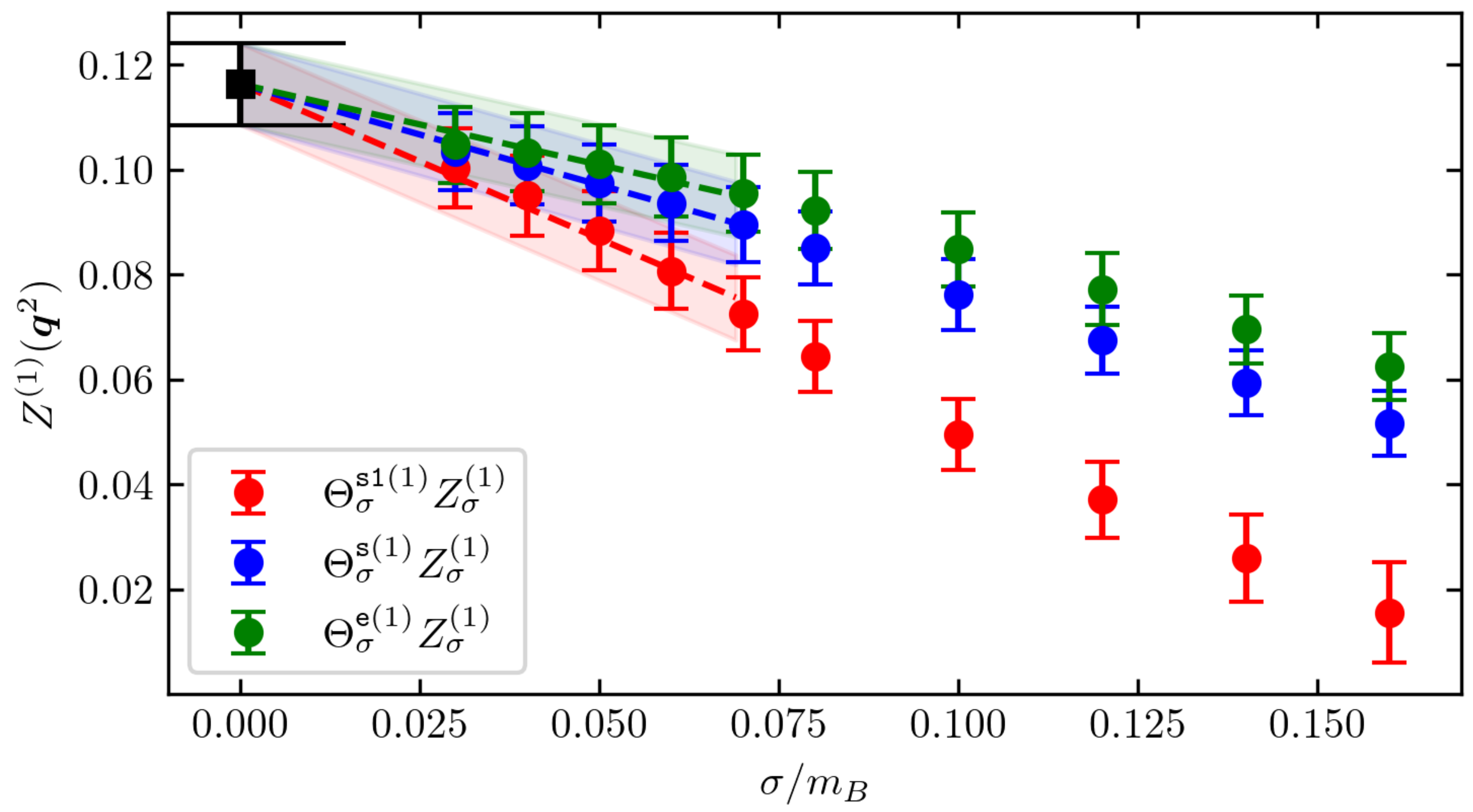
Inclusive decay rate



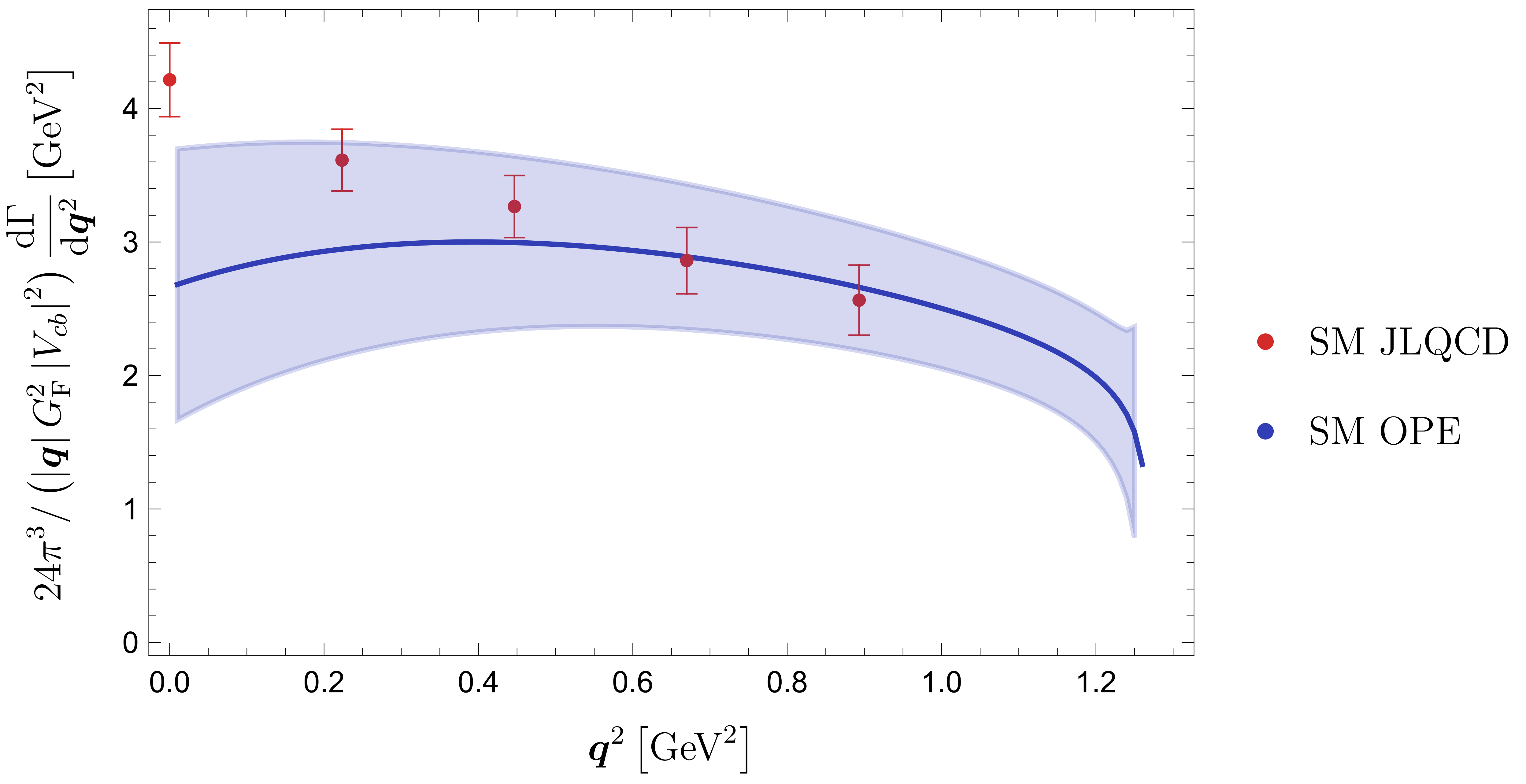
ETMC data from
Gambino et al., 2203.11762

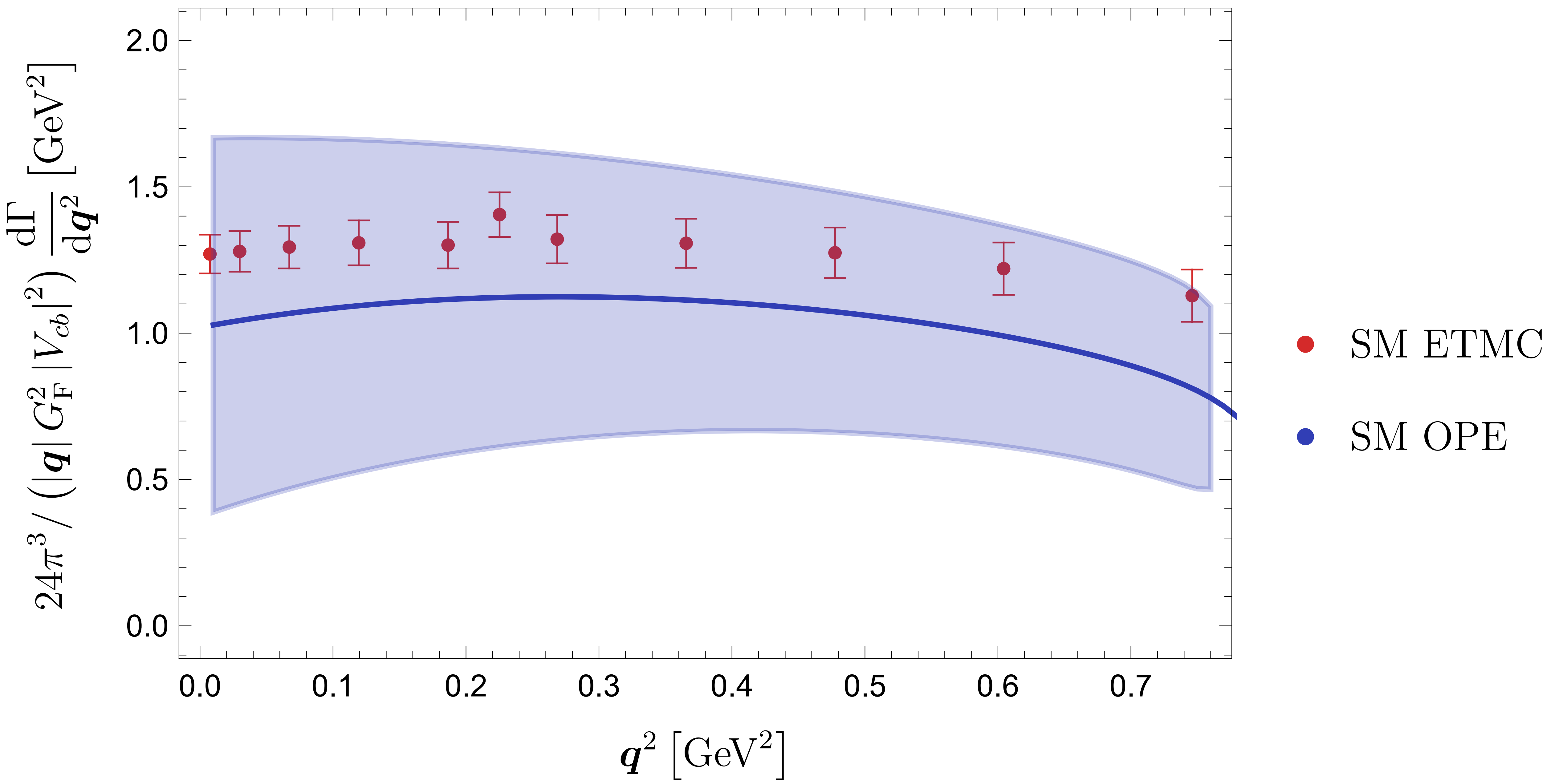
From 2203.11762
Analysis with Backus-Gilbert (by Smecca et al)

- Backus-Gilbert works equally well
- σ→0 limit is taken (with different smearings)



- calculated at many **q**^{2} points
- lighter b quark





From 2203.11762

OPE calculation by Gambino and Machler

- PT including $O(\alpha_s)$, OPE up to $O(1/m^3)$
- Hadronic parameters μ_π^2 etc are taken from the phono analysis.
- b quark mass is adjusted to match the lattice calculations.
- OPE breaks down near the $\mathbf{q}^2$ endpoint.

✓ Good agreement.

✓ Error of OPE is from the hadronic parameters. Large because of small m_b.

✓ Better for moments $\langle M_X^2 \rangle$, $\langle E_l \rangle$, …

Sum over states: dangerous game?

Sum over states with a kernel $K(s)$: $\int_0^\infty ds\, K(s)\rho(s)$

Crucially depends on our ability to approximate the energy integral.

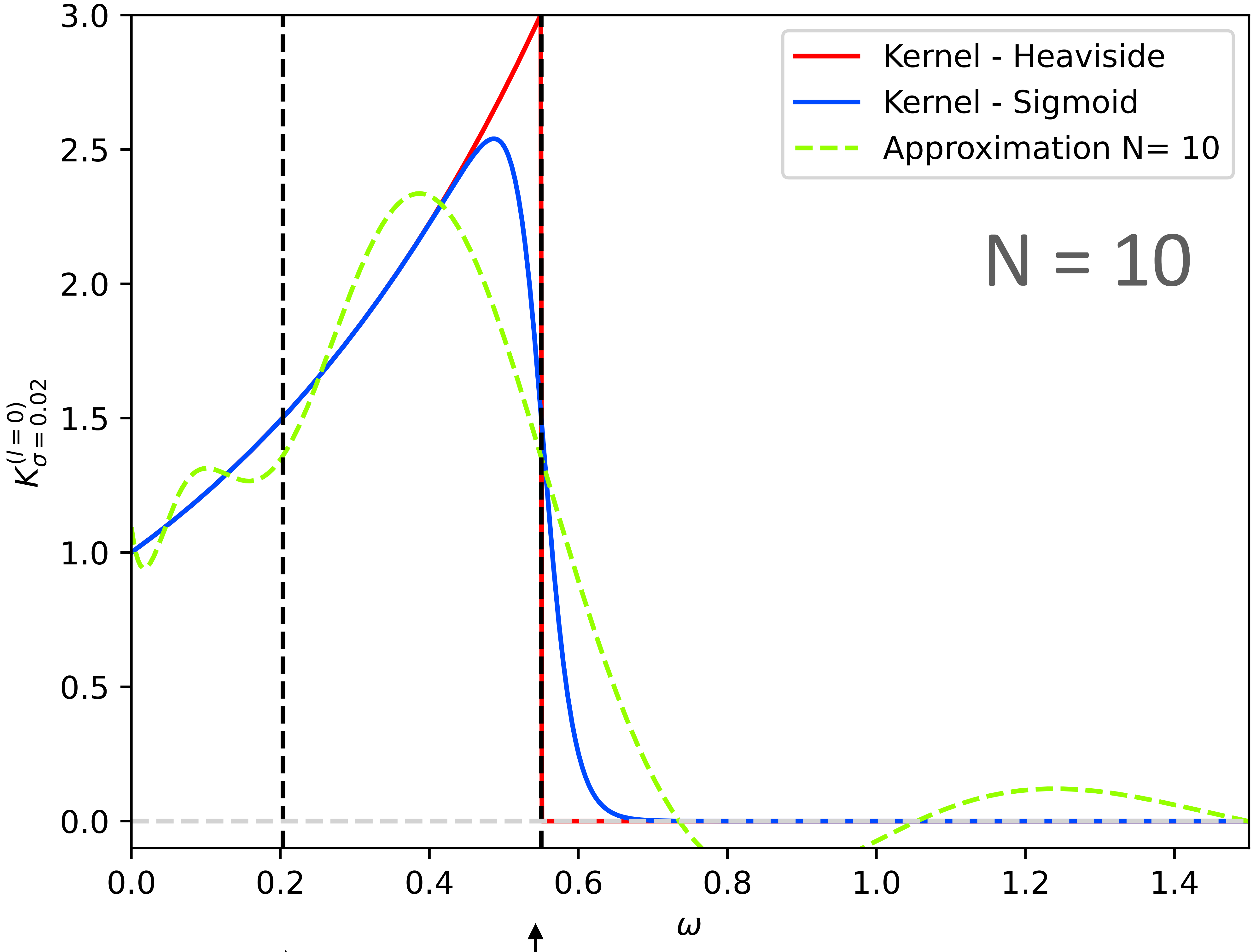
- Possible to treat any $K(s)$?
- No, because $K(s) = \delta(s)$ leads us back to the ill-posed problem (reconstruction of full spectral function from lattice data!)

- Then, what is the limitation or potential systematic effect?

Kernel approximation: an example

$$K(\omega) \sim e^{2\omega t_0}(m_B - \omega)^l \theta(m_B - |\mathbf{q}| - \omega)$$

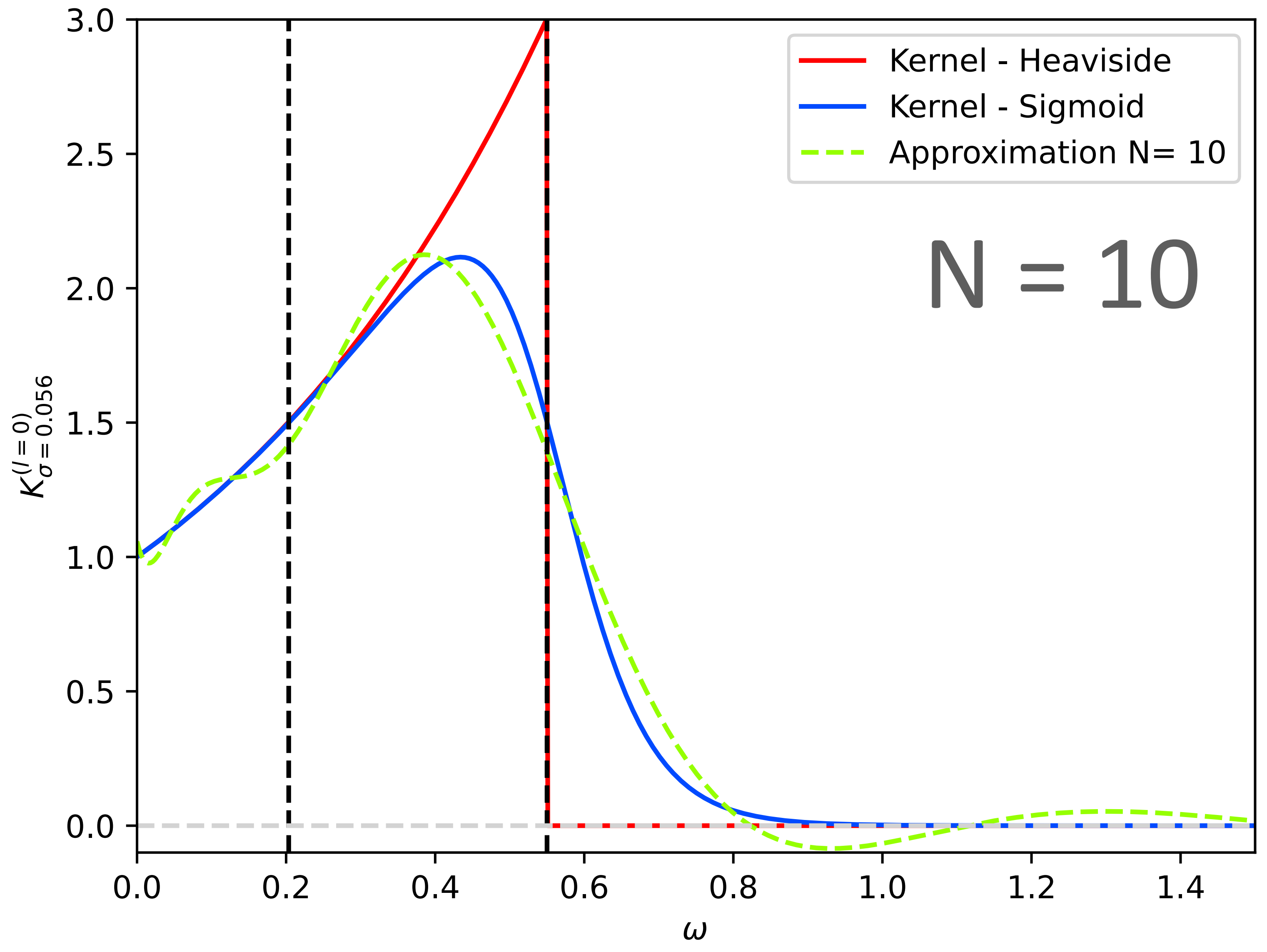
narrow smearing (σ = 0.02)



upper limit

lowest energy state

medium (σ = 0.056)



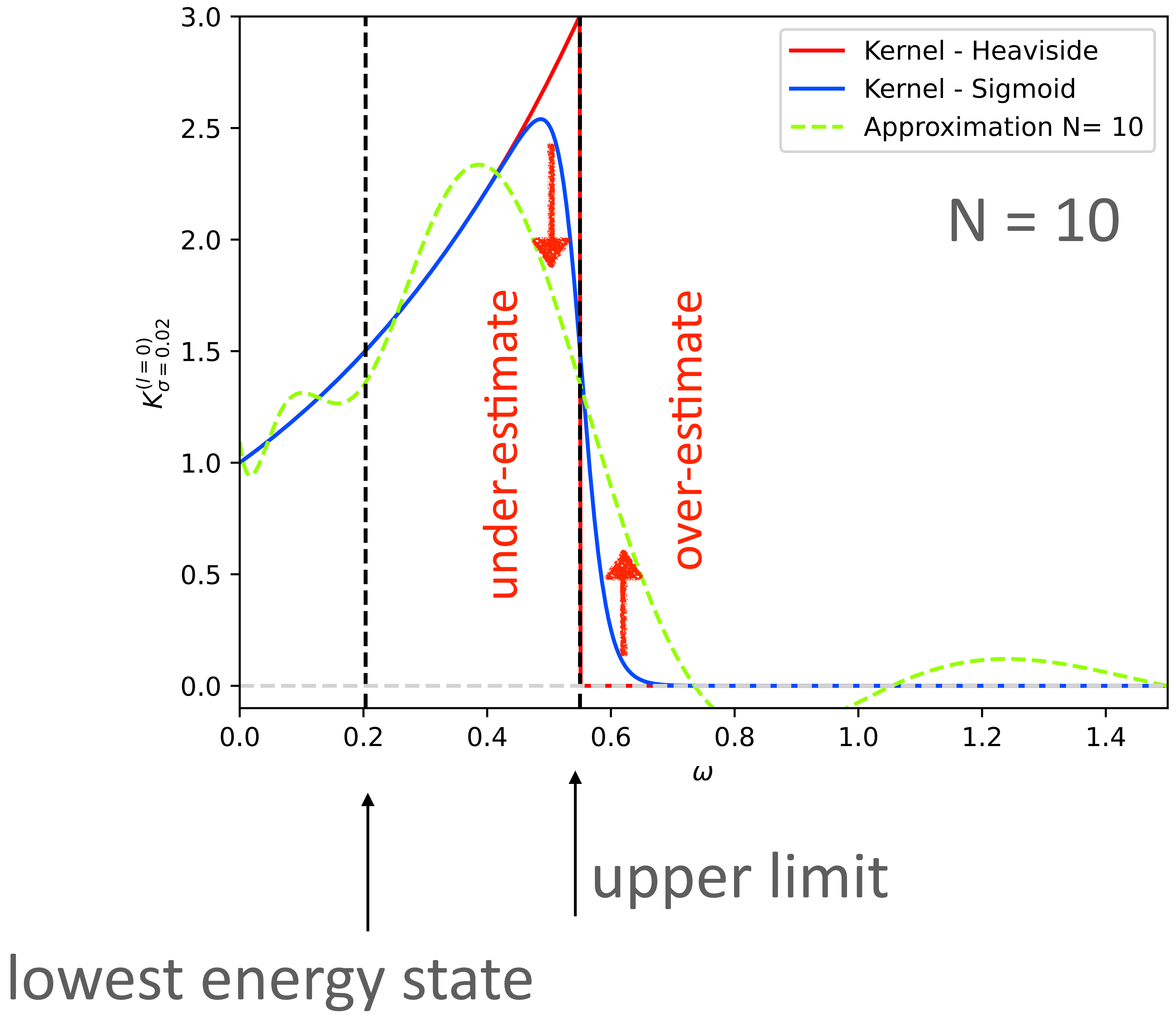
Smearing:

- Too wide = away from the true func
- Too narrow = bad approx

Kernel approximation: an example

$$K(\omega) \sim e^{2\omega t_0}(m_B - \omega)^l \theta(m_B - |\mathbf{q}| - \omega)$$

narrow smearing (σ = 0.02)



Good news:

- Error cancels due to the oscillating approximation (Chebyshev polynomial) when the states distribute evenly.

Bad news:

- Physical spectrum may be non-flat. (A large gap between ground and excited states, for instance. Finite volume spectrum is not even continuous.)
- The integral range gets narrower for larger $\mathbf{q}^2$. The problem gets harder. (But we can keep the ground state only, there.)

Prospects

"The devil is in the details."

- Still in the early stage. Concerning the errors, I am optimistic, but more studies are necessary for various kinematical setups.
- Real calculation of $B \to X_c$, X_u at physical masses still to be done (actually on-going).

- Many potential applications
  - D and B. Not just total rate, but moments, e.g. $\langle M_X^2 \rangle$, $\langle E_l \rangle$
  - Comparison with OPE, then to determine MEs (see 2203.11762)
  - Borel sum (as in the SVZ sum rule; see Ishikawa-SH, 2103.06539)
  - lepton-nucleon scattering, not-so-deep inelastic scattering (see Fukaya et al. 2010.01253, Yoo et al. 2111.15194)

So, what happened to duality?

Smearing is the key (as anticipated).

- Once sufficiently smeared (over energy spectrum), one can transform the problem to the one calculable in PT or on the lattice. Fully nonperturbative in the latter. Systematic errors can be addressed rigorously.

Jets, hadronization, … ?

- Large momentum is a stumbling block on the lattice, yet. Go quantum?