A review of chiral phase transition in the 1+1 dimensional Gross-Neveu model

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1. Introduction

Quantum Chromodynamics (QCD) is a $SU(3)$ gauge theory that explains the strong interaction. The phase diagram supplies us with important clues for studying the birth of the universe and hadronic physics. When drawing the diagram, numerical methods by Lattice QCD are applicable at low densities. On the other hand, toy models and perturbation theory are effective for higher density regions. I mainly review the analysis by J. Lenz *et al.* [1], aiming to draw a phase diagram based on the Gross-Neveu (GN) model, which is one of the famous toy models for QCD.

2. Phase transitions in elementary particle physics

There are two properties of QCD which are the keys when studying the phase diagram. Other than the $SU(3)$ gauge symmetry, QCD Lagrangian given by eq.(1) has a $U(N_f)_L \times U(N_f)_R$ chiral symmetry in the massless limit, $M \to 0$. When the symmetry is spontaneously broken, quarks become massive.

$$\mathcal{L}_{QCD(Minkowski)} = \bar{\psi}\left[i\gamma^\mu D_\mu - M\right]\psi - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} \tag{1}$$

covariant derivative
$$D_\mu = \partial_\mu - igA^a_\mu T^a \tag{2}$$

gluon field strength tensor
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - igA^b_\mu A^c_\nu [T^b, T^c] \tag{3}$$

Dirac / fermion field
$$\psi = (q_1,\ q_2,\ \dots,\ q_{N_f})^T \tag{4}$$

Another important property is asymptotic freedom. The interaction term of the Dirac fields and the gauge fields is proportional to a coupling constant g. This is an indicator of the strength of the interaction. It becomes smaller at higher energies, which means the interaction gets weaker. At that time quarks move without being affected by the gluons. One of the difficulties in investigating QCD is that perturbation theory is not applicable to lower energy studies where the coupling constant is large. The region can be numerically reached by Lattice QCD, while it suffers from a notorious *sign problem* for higher densities. Hence to study such regions, toy models and perturbative methods have been used. One of the famous models is the Gross-Neveu model [2].

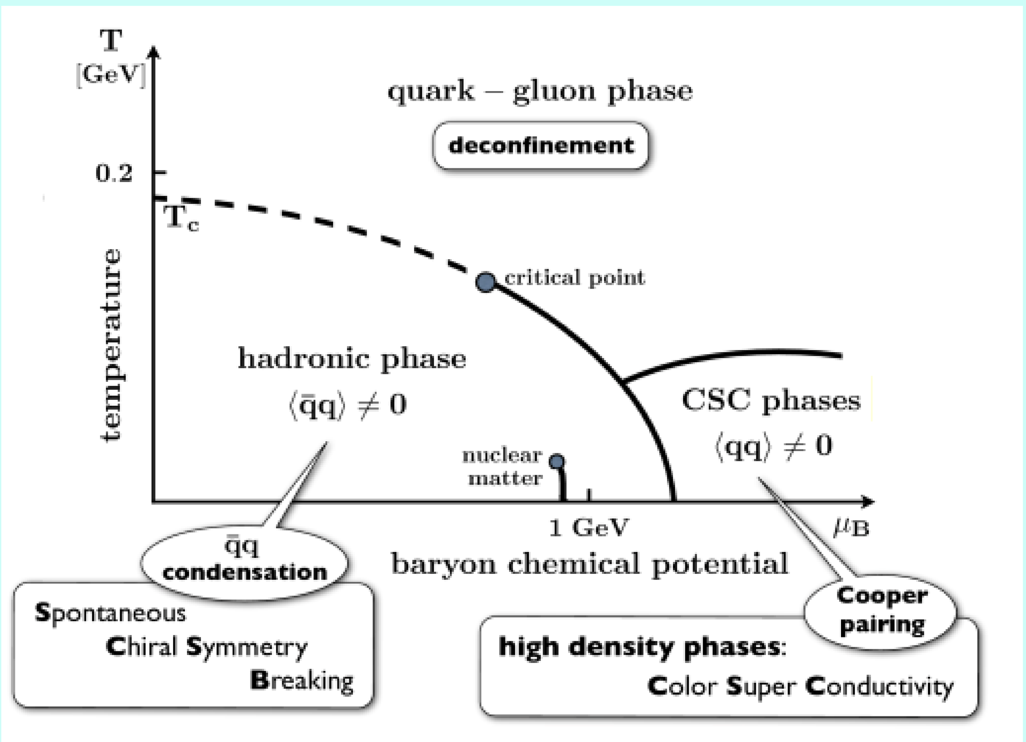

Fig. 1 The QCD phase diagram in recent studies [3].

the quark-gluon phase
The system is disordered at high temperatures. The chiral symmetry is restored.

the hadronic phase
Quarks and gluons are condensed below T_C and μ_C. The order parameter $\langle \bar{q}q\rangle$ is finite, indicating SSB of the chiral symmetry.

the CSC phases
Quarks condense pairwise. This resembles the superconducting phase in electromagnetics.

3. Analysis in the Gross-Neveu model

The GN model [2] was advocated by D. J. Gross and A. Neveu in 1974 as a toy model for chiral symmetry breaking studies. The Lagrangian in a 1+1 dimensional spacetime eq.(5) assumes N_f flavors of quarks.

$$\mathcal{L}_{GN(Euclidean)} = \bar{\psi}i\gamma_\mu\partial_\mu\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2 \tag{5}$$

As is the case with QCD, the GN Lagrangian contains an interaction term, the four-fermi term, which is proportional to the positive power of the coupling constant. This realizes asymptotic freedom. The theory also shares the chiral Z_2 invariance with QCD, where the Dirac fields are transformed as $\psi \to i\gamma_5\psi$. Although the GN theory itself is Z_2 symmetric, symmetry broken states are preferred under certain conditions. This phenomenon is called *spontaneous symmetry breaking* (SSB).

We investigate SSB of the chiral Z_2 symmetry predicted by the GN model in 1+1 dimensions based on the methods from statistical mechanics. The order parameter is the expectation value of $\bar{\psi}\psi$, given by eq.(6).

$$\langle\bar{\psi}\psi\rangle = \frac{1}{Z}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-S_\psi}\ \bar{\psi}\psi \tag{6}$$

$$S_\psi \equiv \int d^2x\, \mathcal{L}_{GN} \tag{7}$$
$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ e^{-S_\psi} \tag{8}$$

The path integral over the Dirac fields in eq.(6) is not analytically evaluable due to the four-fermi term. Then we prescribe an auxiliary scalar field σ and rewrite the Lagrangian into a bilinear form. The Ward-Takahashi Identity formula states that $\langle\sigma\rangle$ is proportional to $\langle\bar{\psi}\psi\rangle$, and therefore we can regard the former as the order parameter. Besides, we introduce β and μ, which are an inverse temperature and a chemical potential, respectively. Consistency with statistical mechanics implies β corresponds to imaginary temporal periodicity $-it_{period}$. μ is explicitly put in the Dirac operator as in eq.(11).

$$\langle\sigma\rangle = \frac{1}{Z}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \int \mathcal{D}\sigma\ e^{-S_\sigma}\ \sigma \tag{9}$$

$$S_\sigma = \int d^2x\left(\bar{\psi}iD\psi + \frac{N_f}{2g^2}\sigma^2\right) \tag{10}$$
$$D = \gamma_\mu\partial_\mu + \sigma + \mu\gamma_0 \tag{11}$$

Now we can evaluate eq.(9) with respect to $\bar{\psi}$ and ψ following the integration rule in complex Grassman Algebra. This rule was introduced to the Dirac fields to express the Pauli exclusion principle of fermions. We eventually obtain an effective action eq.(13) in the Boltzmann factor.

$$\langle\sigma\rangle = \frac{1}{Z}\int \mathcal{D}\sigma\ e^{-N_f S_{eff}}\ \sigma \tag{12}$$
$$S_{eff} = \frac{1}{2g^2}\int d^2x\,\sigma^2 - \log\det D \tag{13}$$

In the large limit, $N_f \to \infty$, the saddle point approximation becomes exact. This means we can identify $\langle\sigma\rangle$ by minimizing the effective action.

4. Numerical calculations

To draw the GN phase diagram, we need to find a set of minima of the effective action. As S_{eff} can be decomposed as $\beta L \times U_{eff}$, we find the minima of renormalized effective potential eq.(14).

$$\begin{aligned}\hat{U}_{eff} &= \hat{U}_0 + \Delta\hat{U} \\ &= \frac{\hat{\sigma}^2}{4\pi}(\log\hat{\sigma}^2 - 1) - \frac{1}{\pi}\int_0^\infty d\hat{k}_1\,\frac{\hat{k}_1^2}{\hat{\epsilon}_{k_1}}\left(\frac{1}{1+e^{\hat{\beta}(\hat{\epsilon}_{k_1}-\hat{\mu})}} + \frac{1}{1+e^{\hat{\beta}(\hat{\epsilon}_{k_1}+\hat{\mu})}}\right)\end{aligned} \tag{14}$$

The term $\hat{U}_0$ is the effective potential at vanishing T and μ, while $\Delta\hat{U}$ is a correction by the finite T and μ effects. We define dimensionless values with *hats*; $\hat{O} \equiv O/\sigma_0$ where σ_0 is the non-trivial minimum of $\hat{U}_0$. By numerically seeking the minima of $\hat{U}_{eff}$ with respect to $\hat{\sigma}$, the GN phase diagram is obtained.

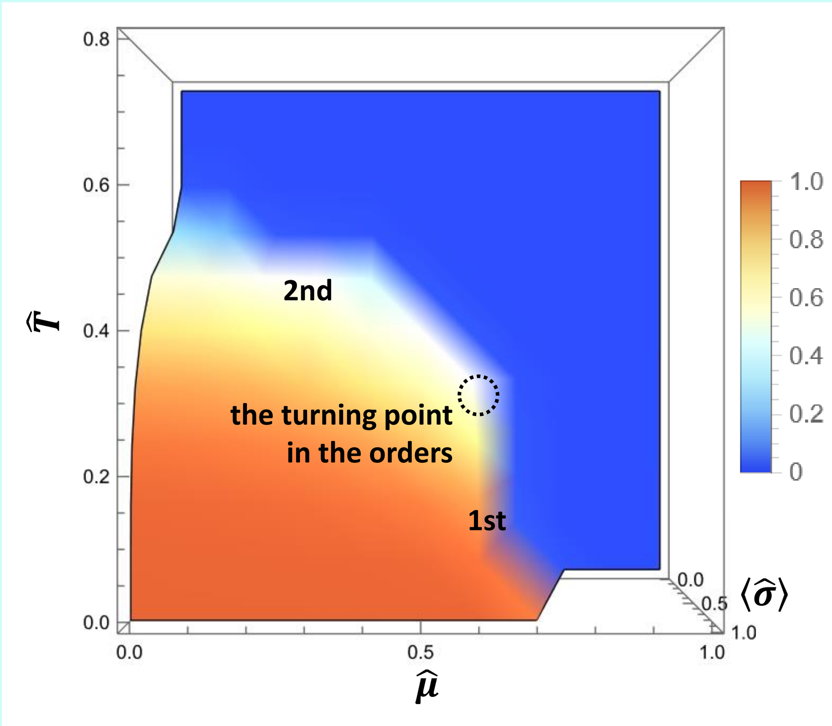

Fig. 2 The two-phase diagram by the GN phase model [drawn with Mathematica].

the blue colored region: chiral Z_2 symmetric

the red colored region: spontaneously Z_2 broken

The order parameter jumps at $\hat{\mu} \approx 0.7$, whereas it smoothly grows around $\hat{T} \approx 0.5$. These are the reflection of the phase transition orders. First order phase transition occurs at vanishing temperatures, while second order one at very low densities. The critical temperature $\hat{T}_C$ and chemical potential $\hat{\mu}_C$ well accord with the following estimations by $\hat{U}_{eff}$.

$\hat{\mu}_C = 1/\sqrt{2} \approx 0.707$ (analytically derived)

$\hat{T}_C = e^\gamma/\pi$ [1] ≈ 0.567. (semi-analytically derived)

In this section, we have assumed that the order parameter σ does not depend on the spatial coordinate, x. The GN model in 1+1 dimensions, however, predicts another phase where σ is spatially varying at high densities.

5. Advocated another phase in the Gross-Neveu model

If we take the dependence of σ on x into consideration, the story becomes more complex. We newly construct a Hamiltonian h_σ eq.(15) and read the Dirac operator D in terms of the Hamiltonian.

$$h_\sigma = \gamma_0\gamma_1\partial_1 + \gamma_0\sigma(x) \tag{15}$$
$$D = \gamma_0(\partial_0 + \mu + h_\sigma) \tag{16}$$

The order parameters should be a family of solutions of a self-consistency equation: $\delta S_{eff}/\delta\sigma(x) = 0$. As it is badly difficult to directly solve it, M. Thies and K. Urlichs (2003) [4] focused on energy. At higher densities where asymptotic freedom becomes predominant, we can apply perturbation theory to eq.(15). The renormalized energy density proved to be minimized by different forms of $\sigma(x)$, depending on the density circumstances as described in eqs.(17): at a high μ, (18): at a low μ.

$$\sigma_{high}(x) \propto \sin\left(2\pi\frac{\rho}{N_f}x\right) \tag{17}$$
$$\sigma_{low}(x) \propto \begin{cases}\tanh x & (0 < x < \Delta) \\ 1 & (\Delta < x < a/4)\end{cases} \tag{18}$$

The structure of kinks and anti-kinks expected in low densities is approximated to eq.(18). The period Δ, lattice spacing $a = L/N$, and the coefficients for both types of order parameters monotonously decrease as the baryon density ρ grows. The inhomogeneous phase proved to be energetically favored above $\hat{\mu}_C$. Hence the 1+1 dimensional Gross-Neveu phase diagram is revised as Fig. 3 [4].

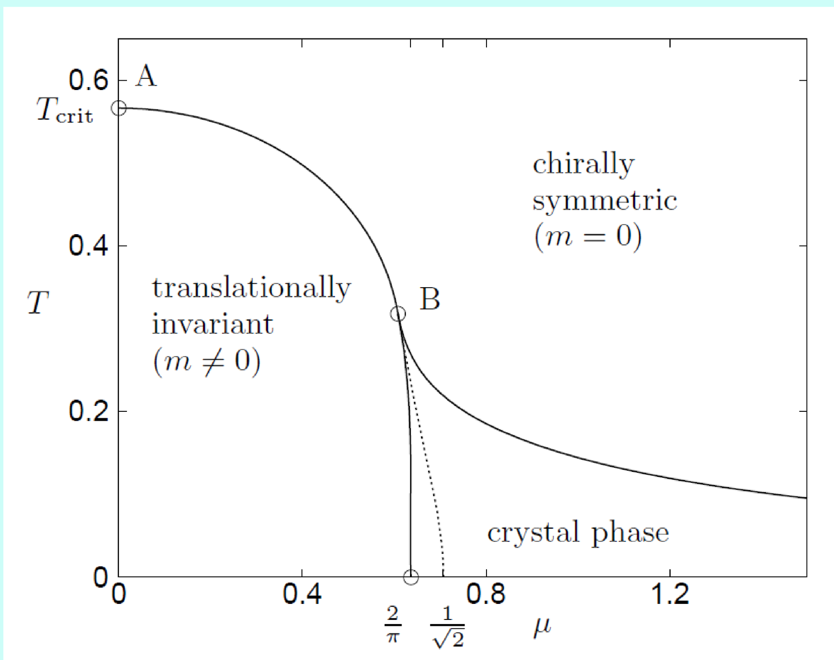

Fig. 3 The revised GN phase diagram [4].

the dashed line:
first order phase transition with the constant σ

the two solid lines above $\hat{\mu}_C$:
second order phase transition associated with $\sigma(x)$

The critical chemical potential for hom.-inhom. phase transition is estimated to be lower than the previous one ($2/\pi < 1/\sqrt{2}$). The point $B \approx (0.608, 0.318)$ is a triple point called the *Lifshitz point* in condensed matter physics [5]. The curve AB is common for the two-phase and three-phase diagrams.

6. Summary

I have reviewed the GN phase diagram mainly based on the work by J. Lenz *et al.* [1]. The GN model predicts another phase than the Z_2 symmetric phase and the homogeneous Z_2 broken phase. By considering the spatially varying order parameter $\sigma(x)$, we see the inhomogeneous phase at high densities. The orders of phase transition are different between two-phase and three-phase. The *raison d'être* of the inhomogeneous phase is still under discussion. It might be a secret phase in QCD, or, can be just a feature of the toy model. In any case, it is worth considering.

7. Acknowledgements

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8. References

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