Basic Idea of Numerical Stochastic Perturbation Theory and application to the Gross-Witten-Wadia model

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  (What is the Motivation for NSPT.)

- HSPT (HMD-based NSPT)
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Introduction

Recent development of resurgence theory suggests that relation between divergence of perturbation series and non-perturbative effects.

Ex.) instanton, renormalon

This relation is called "Resurgence Structure".

- What is divergence of perturbative series?

  If we take infinite sums of asymptotic expansion, this series will diverge at finite order.

$$Z(g) = \sum_{k=0}^{Ntrunc} Z^{(k)} g^k$$

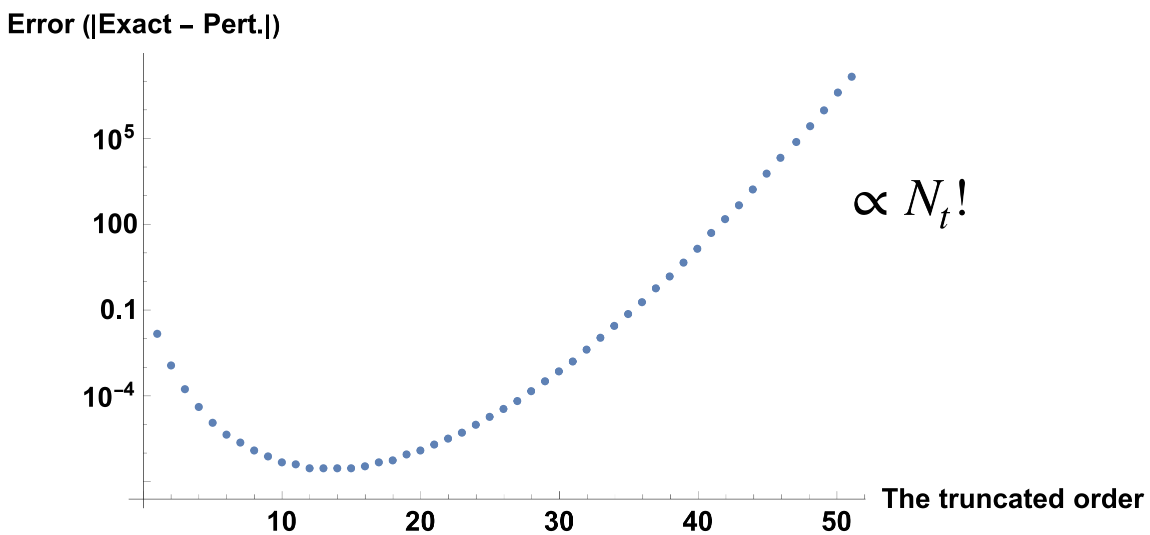


Fig.1 Difference between exact result and perturbative series in simple example (phi4).

- Resurgence Structure

By using analytic function (Borel summation), we can extract non-perturbative information from the behavior of divergence for the asymptotic series.

Ex.) $(S_{0+} - S_{0-}) Z_0(g) = e^{-S_{inst}} S\Phi_1(g)$

Borel summation

Perturbative series around vacuum

Non-perturbative information

Perturbative series around instanton

Introduction

The resurgence structure is expected to be a new non-perturbative analysis.

A few systems in QM/QFT show resurgence structure.

Cf.) Large-N GWW model A. Ahmed, G. V. Dunne (2017) E. Alfinito, M. Beccaria (2017) .

Large-N 2D O(N) sigma model H. Nishimura, et. al., J. High Energy. Phys. **2022**, 151 (2022).

CP(N-1) model G. V. Dunne, M. Unsal, J. High Energy. Phys. **2012**, 170 (2012) etc.

→ To extract non-perturbative information, we need higher order coefficients.

However it is very difficult to evaluate higher order coefficients exactly !

We need to compute $N!$ Feynman diagram.

Factorical divergence !

→ We need another approach to obtaining perturbative coefficients.

Numerical Stochastic Perturbation Theory (NSPT)

There is no need to evaluate Feynman diagram, instead we integrate hierarchy partial differential equation on computer.

NSPT

・Basic idea of NSPT

We expand field and action in terms of coupling constant and integrate hierarchy stochastic differential equation (Langevin eq. Molecular dynamics eq.) .

$$\phi = \sum g^k \phi^{(k)}$$
$$(\phi^{(0)}, \phi^{(1)}, \cdots, \phi^{(N_t)})$$

$\rightarrow$

$$< O[\phi] > = \sum g^k < O^{(k)}[\phi^{(0)}, \cdots, \phi^{(k)}] >$$
$$< O^{(k)}[\phi^{(0)}, \cdots, \phi^{(k)}] > = \frac{1}{N_s} \sum_{i=1}^{Ns} O^{(k)}[\phi_i^{(0)}, \cdots, \phi_i^{(k)}]$$

・Monte Carlo simulation Non-perturbative simulation

Evaluating the Path Integral

$$< O[\phi] > = \frac{1}{Z} \int \mathscr{D}\phi O[\phi] \mathrm{e}^{-S[\phi]}$$

Partition function : $Z = \int \mathscr{D}\phi \mathrm{e}^{-S[\phi]}$

Expectation value of observable $O[\phi]$

Probability Density $\frac{1}{Z}\mathrm{e}^{-S[\phi]}$

To evaluate P.I., we need to generate the configuration which follows probability density $P(\phi) = \frac{1}{Z}\mathrm{e}^{-S[\phi]}$.

→ Markov Chain Monte Carlo (MCMC) method

Ex.) Stochastic quantization (Langevin method), Molecular Dynamics, Metropolis, Hybrid Monte Carlo

NSPT

- Hybrid Monte Carlo Method (HMC), (Non-perturbative simulation)

  First to make hamiltonian formalism, we introduce canonical conjugate momenta P.

$$Z = \int d\phi \mathrm{e}^{-S[\phi]} \rightarrow \int d\phi dp \mathrm{e}^{-H[\phi,P]} \qquad H(\phi, P) = \frac{1}{2}P^2 + S[\phi]$$

  In this simulation, the field evolution is described by molecular dynamics equation,

$$\frac{d}{dt}\phi(x;t) = P(x;t)$$

$$\frac{d}{dt}P(x;t) = F(\phi(x:t)) = -\frac{\partial H}{\partial \phi}$$

t is simulation time

- Simulation Steps

1. Start from any initial state of ϕ
2. Generate momentum P from Gaussian distribution.
3. Update field ϕ and momentum P by MD equation.
4. (Metropolis test)
5. Calculate observable and refresh momentum.

To make NSPT formalism, we expand ϕ and P.

NSPT

- HMD-based NSPT (HSPT)

Expanding the MD equation.

$$\phi(x;t) = \sum_{k=0}^{N_t} g^k \phi^{(k)}(x;t)\ ,\ P(x;t) = \sum_{k=0}^{N_t} g^k P^{(k)}(x;t)$$

g : coupling constant,

N_t : truncated order

$$P(x;t) = P^{(0)}(x;t) + gP^{(1)}(x;t) + g^2 P^{(2)}(x;t) + \cdots$$

$$\frac{d}{dt}\phi(x;t) = P(x;t) \qquad \rightarrow \qquad \frac{d}{dt}\phi^{(k)}(x;t) = P(x;t)^{(k)}$$

$$\frac{d}{dt}P(x;t) = F(\phi(x:t)) \qquad \frac{d}{dt}P^{(k)}(x;t) = F^{(k)}[\phi^{(0)}(x;t), \cdots, \phi^{(k-1)}(x;t)]$$

$$< O[\phi] > = \sum_{k=0}^{N_t} g^k O[(\phi^{(0)}, \cdots, \phi^{(k-1)})]$$

- Simulation Steps
  1. Start from any initial state of ϕ
  2. Generate the lowest order momentum from Gaussian distribution.
     Other $P^{(k)}$ is zero.
  3. Update field ϕ and momentum P by integrating MD equation
     Order by order
     (Metropolis test)
  4. Calculate observables and refresh momentum.

NSPT for GWW model

To confirm validity of NSPT simulation, we calculate Wilson Loop in GWW model in which the Trans-series can be solved analytically.

The complete expansion formula which contain all non-perturbative effects.

- GWW (Gross-Witten-Wadia) model (equivalent to 2D $U(N_c)$ Yang-Mills theory)

$$S(U) = -\frac{1}{2g^2}\mathrm{Tr}(U + U^\dagger) \quad , \quad Z(g) = \int dU \exp[-S(U)]$$

D. Gross, E. Witten Phys. Rev. D **21**, 446 (1980).
S. R. Wadia, Phys. Lett., B93, **403** (1980).

U : $N_c \times N_c$ unitary matrix

This matrix integral can be evaluated as a Toeplitz determinant.

$$Z(g) = \int dU \exp[-S(U)] = \det[I_{j-k}(x)]_{j,k=1,\cdots,N_c}$$

j, k : matrix index

$(x \equiv 2/g^2)$

$I_\nu(x)$: modified Bessel function

A. Ahmed, G. V. Dunne (2017)
E. Alfinito, M. Beccaria (2017)

Wilson loop also can be written using Toeplitz matrix.

$$< \mathrm{Tr}U^k > = \mathrm{Tr}(M_0^{-1}M_k),\ \left(M_k\right)_{ij} = I_{k+i-j}(x)$$

We want to see the behavior of higher order coefficients for W.L. !

The Bessel function have trans-series formula at large x.

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\nu+n+\frac{1}{2})}{n!\Gamma(\nu-n+\frac{1}{2})}\frac{1}{(2x)^n} + \frac{e^{-x+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi x}}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n+\frac{1}{2})}{n!\Gamma(\nu-n+\frac{1}{2})}\frac{1}{(2x)^n}$$

at Large $|x|$.

$\rightarrow$ we can obtain trans-series of W.L in $\lambda = N_c g^2$.

NSPT for GWW model

To confirm validity of NSPT simulation, we calculate Wilson Loop in GWW model in which the Trans-series can be solved analytically.

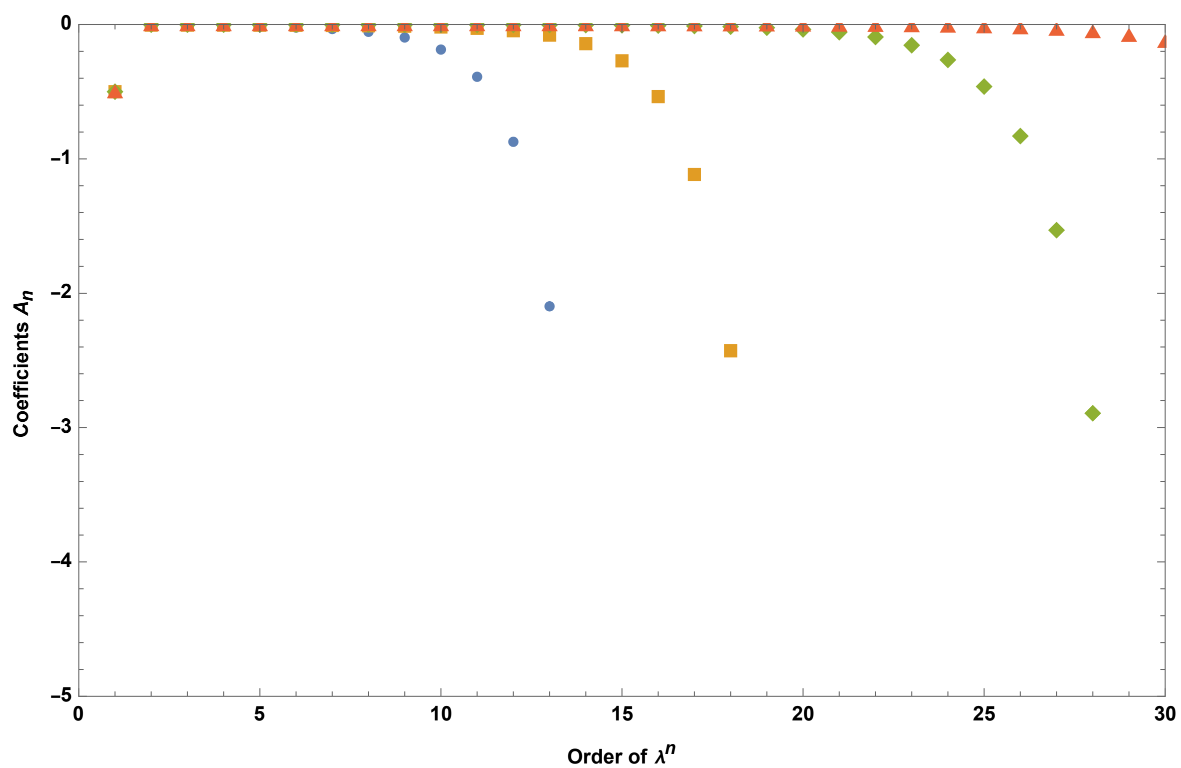
$$< \mathrm{Tr}U > = \frac{1}{N_c}\mathrm{Tr}(M_0^{-1}M_1), \qquad \left(M_k\right)_{ij} = I_{k+i-j}(x)$$

$$= \sum_{m=0}^{\infty} A_m \lambda^m + \mathscr{C} e^{-\frac{C}{\lambda}} \sum_{m=0}^{\infty} B_m \lambda^m$$

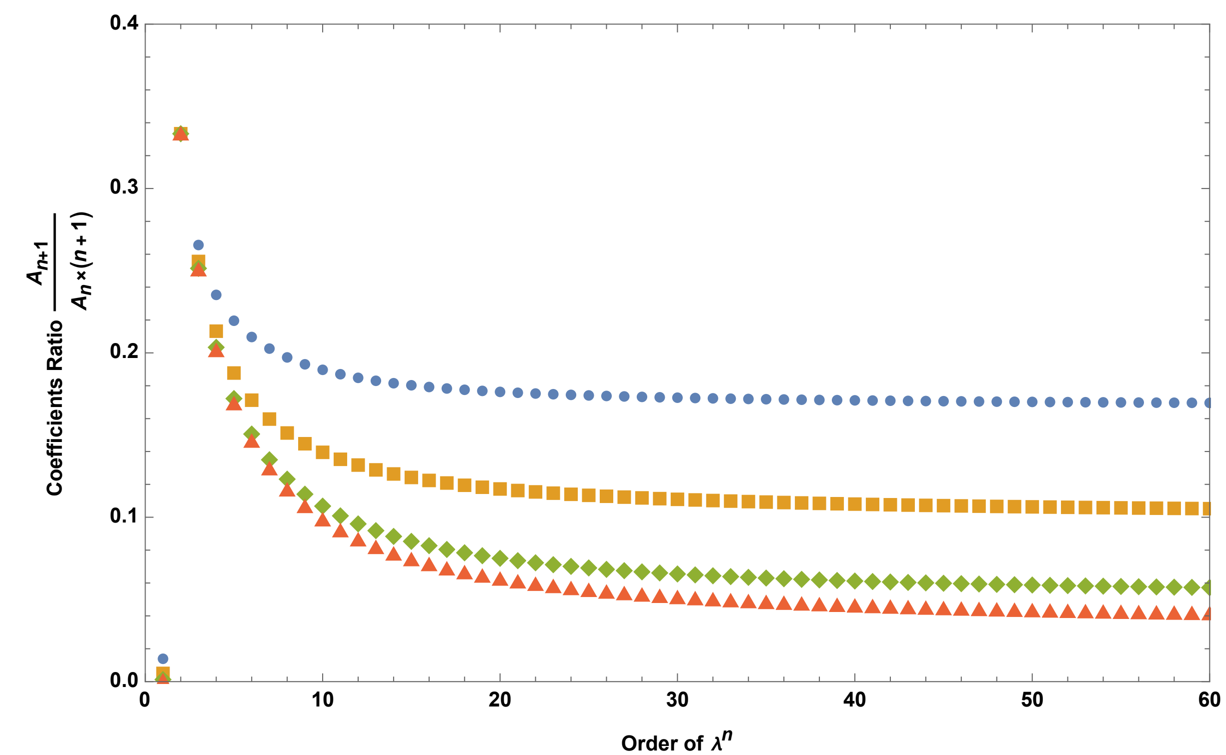
We want to see the behavior of higher order coefficients for W.L. !

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\nu+n+\frac{1}{2})}{n!\Gamma(\nu-n+\frac{1}{2})} \frac{1}{(2z)^n} + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n+\frac{1}{2})}{n!\Gamma(\nu-n+\frac{1}{2})} \frac{1}{(2z)^n}$$

$$\text{Ratio : } R_n = \frac{A_{n+1}}{A_n(n+1)} \quad \left(\propto \frac{(n+1)!}{n!(n+1)} = 1 \right)$$



Coefficients Nc=3 , Nc=5 , Nc=10 , Nc=15.



Ratio of Coefficients Nc=3 , Nc=5 , Nc=10 , Nc=15.

→ Our purpose is to reproduce this behavior by using NSPT.

NSPT for GWW model

・Setup for HSPT simulation

Cf.) Ishikawa JPS talk. 2022.

$$S(U) = -\frac{1}{2g^2}\mathrm{Tr}(U + U^\dagger) \quad , \quad Z(g) = \int dU \exp[-S(U)]$$

Introduce canonical momentum P which is canonical conjugate to U.

$$H(U,P) = \frac{1}{2}\mathrm{Tr}(P^2) - \frac{1}{2g^2}\mathrm{Tr}(U + U^\dagger)$$

and expand U, P in coupling $h = sg$.

$$U = \sum_{k=0}^{N_t} h^k U^{(k)} = U^{(0)}\left(= I_{N\times N}\right) + hU^{(1)} + \cdots + h^{N_t}U^{(N_t)}, \quad hP = \sum_{k=1}^{N_t} h^k P^{(k)} = hP^{(1)} + \cdots + h^{N_t}P^{(N_t)}$$

MD equation (Evolution eq.) is

$$\frac{d}{dt}U(t) = iP(t)U(t)$$
$$\frac{d}{dt}P(t) = \frac{i}{2g^2}(U - U^\dagger)$$

$\rightarrow$

$$U^{(k)}(t+dt) = \left(\mathrm{e}^{ihP(t)} * U(t)\right)^{(k)}$$
$$hP(t+dt)^{(k)} = hP(t)^{(k)} + \frac{is^2}{2}(U - U^\dagger)^{(k)}$$

$*$: convolution operator

$$A * B = \sum_{k=0}^{\infty} h^k A^{(k)} \sum_{k=0}^{\infty} h^k B^{(k)} = \sum_{k=0}^{\infty}\sum_{l=0}^{k} A^{(k-l)}B^{(l)}h^k$$
$$(A*B)^{(k)} = \sum_{l=0}^{k} A^{(k-l)}B^{(l)}$$

Compute the coefficients of W.L. up to $\mathcal{O}(\lambda^{15}(= h^{30}))$.

(Coefficient divergence start from $\mathcal{O}(\lambda^{15})$ in $N_c = 5$ W.L.)

Setup

Matrix size : $N_c = 5$, trajectory length : $\tau = 0.01$, # of MD steps : 5

Result for GWW model

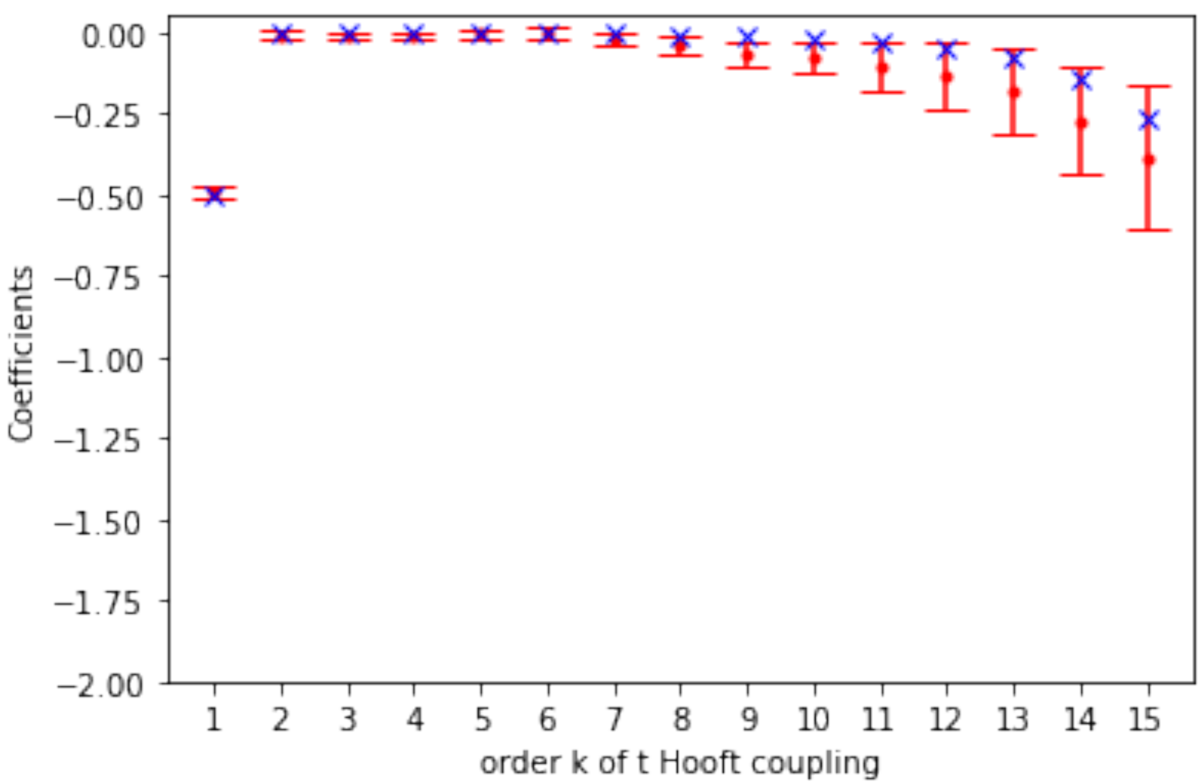
- **The numerical results of Wilson loop up to $\mathcal{O}(\lambda^{15}(=h^{30}))$.**

Coefficient divergence start from $\mathcal{O}(\lambda^{15})$ at $N_c = 5$.

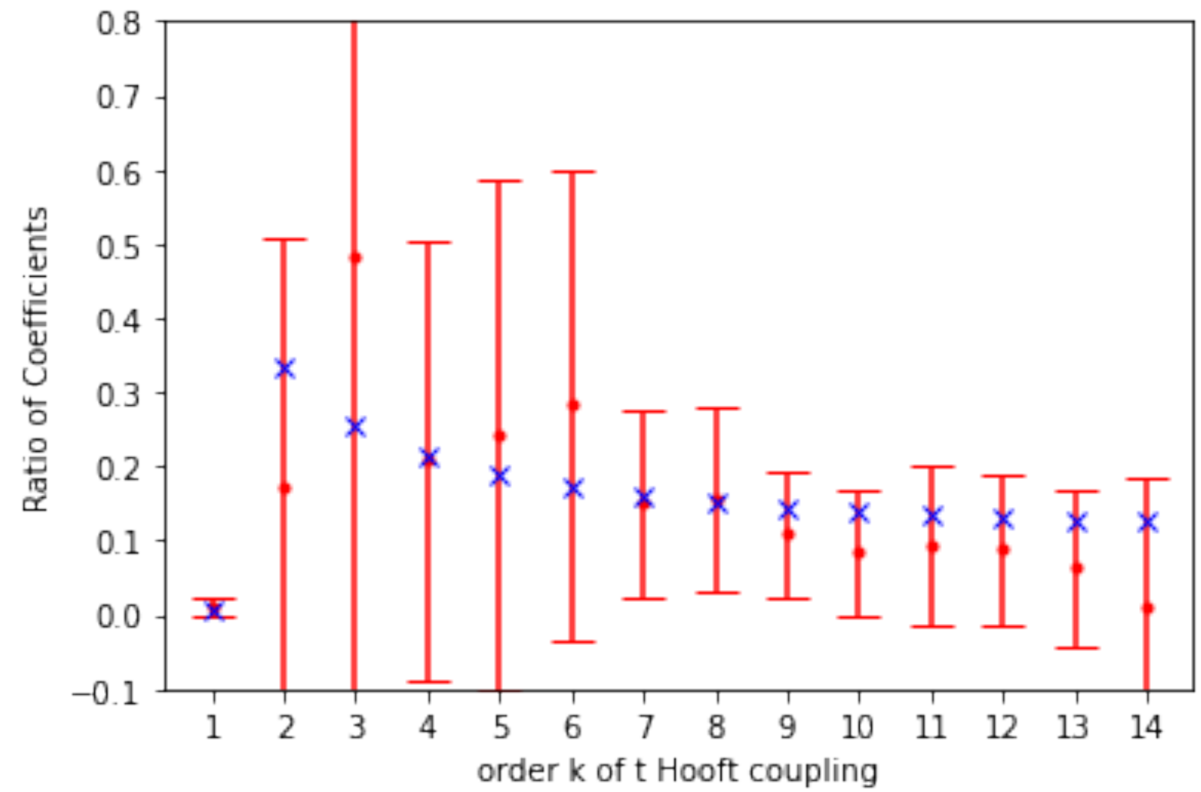
HMD Setup

Matrix size : $N_c = 5$, trajectory length : $\tau = 0.01$, # of MD steps : 5

$N_{sample} = 50$, (impose no unitary condition)



Order of ’t Hooft coupling and Coefficients of W.L. expanded in ’t Hooft coupling λ.



Variance increase with order.

← Pepe effect (appearance of rare effects)

Ratio is convergence at higher order.

We want to know higher order behavior

Result for GWW model

Because of unitarity breaks down a little bit in simulation, we need to maintain unitarity.

To preserve unitarity, we impose unitary condition at some MD steps.

・Unitary condition

$$UU^\dagger = I_{N_c\times N_c}\ ,\ U = \exp(iA)$$

$$\rightarrow \ln(U) = -\ln(U^\dagger)$$

$$A = A^\dagger \qquad \left(A_{ij} = \frac{1}{2}(A_{ij} + \bar{A}_{ji})\right)$$

In NSPT simulation, we should impose this condition order by order.

・Unitary condition in HSPT

$$(U * U^\dagger)^{(k)} = I_{N_c\times N_c}\delta_{0,k}\ ,\ U^{(k)} = \left(\exp(iA)\right)^{(k)}$$

$$\rightarrow \ (\ln(U))^{(k)} = -(\ln(U^\dagger))^{(k)}$$

$$A^{(k)} = (A^\dagger)^{(k)}$$

Expensive computation ! (need Log. and Exp. convolution)

To avoid this difficulty, we use hermitian matrix instead of unitary matrix in our simulation.

→ Cayley Transformation

Cayley Transformation for GWW model

- **Cayley Transformation** S. Mizoguchi, 2004 Nuclear Physics B 716(3).

Unitary matrix : U ⟷ Hermitian matrix : ϕ

$$UU^\dagger = I \qquad\qquad \phi^\dagger = \phi$$

$$U = (I + i\kappa\phi)(I - i\kappa\phi)^{-1}$$

κ : Cayley transformation Parameter

$$UU^\dagger = (I + i\kappa\phi)(I - i\kappa\phi)^{-1}(I + i\kappa\phi^\dagger)^{-1}(I - i\kappa\phi^\dagger) = (I + i\kappa\phi)(I + i\kappa\phi)^{-1}(I - i\kappa\phi)^{-1}(I - i\kappa\phi)$$

$$= I_{N_c \times N_c}$$

We should preserve hermiticity instead of unitarity in some steps.

Under this transformation GWW model become,

$$S(U) = -\frac{1}{2g^2}\mathrm{Tr}(U + U^\dagger) \quad , \quad Z(g) = \int dU \exp[-S(U)]$$

$$\rightarrow \quad S_{CT}(\phi) = N_c \mathrm{Tr}\log(I_{N\times N} + \kappa^2\phi^2) - \frac{N_c}{\lambda}\mathrm{Tr}\left[(I_{N\times N} - \kappa^2\phi^2)(I_{N\times N} + \kappa^2\phi^2)^{-1}\right]$$

from Jacobian : $Z(g) = \int dU e^{-S[U]} = \int d\phi \, \det[\frac{\partial U}{\partial \phi}] \exp[-S'(\phi)]$

$$Z(g) = \int d\phi \exp[-S_{CT}(\phi)]$$

Cayley Transformation for GWW model

- **HSPT formulation for transformed GWW model**
  - Parameters

Expansion parameter s : $sh^2 = \lambda$ $\quad (s=1 \,,\, h^2=\lambda) \,,\, (s=N_c \,,\, h=g)$

The Cayley Transformation parameter : $\kappa = \sqrt{\frac{\lambda}{2}} \left(= \sqrt{\frac{s}{2}} h \right) = \sqrt{\frac{N_c}{2}} g$

Introduce canonical momentum P which is canonical conjugate with ϕ.

$$H(U,P) = \frac{N_c}{2}\mathrm{Tr}(P^2) + N_c \mathrm{Tr}\log(I_{N\times N} + \kappa^2\phi^2) - \frac{N_c}{\lambda}\mathrm{Tr}\left[(I_{N\times N} - \kappa^2\phi^2)(I_{N\times N} + \kappa^2\phi^2)^{-1}\right]$$

Molecular Dynamics eq.

$$\frac{d}{dt}\phi(t) = P(t)$$

$$\frac{d}{dt}P(t) = -\phi\left\{sh^2 + 2(I + \frac{s}{2}\phi^2)^{-1}\right\}(I + \frac{s}{2}\phi^2)^{-1}$$

Cf) Ordinary GWW MD eq.

$$\frac{d}{dt}U(t) = iP(t)U(t)$$

$$\frac{d}{dt}P(t) = \frac{i}{g^2}(U - U^\dagger)$$

Expand ϕ and P in terms of general coupling h.

$$\phi = \sum_{k=1}^{N_t} h^k\phi^{(k)} = h\phi^{(1)} + h^2\phi^{(2)} + \cdots + h^{N_t}\phi^{(N_t)} \,, \qquad P = \sum_{k=1}^{N_t} h^k P^{(k)} = hP^{(1)} + h^2P^{(2)} + \cdots + h^{N_t}P^{(N_t)}$$

Generate lowest order momentum from Gaussian distribution.

Then, MD equation order by order is,

$$\phi^{(k)}(t+dt) = \phi^{(k)}(t) + dtP^{(k)}(t) \qquad (k = 1,\cdots,N_t)$$

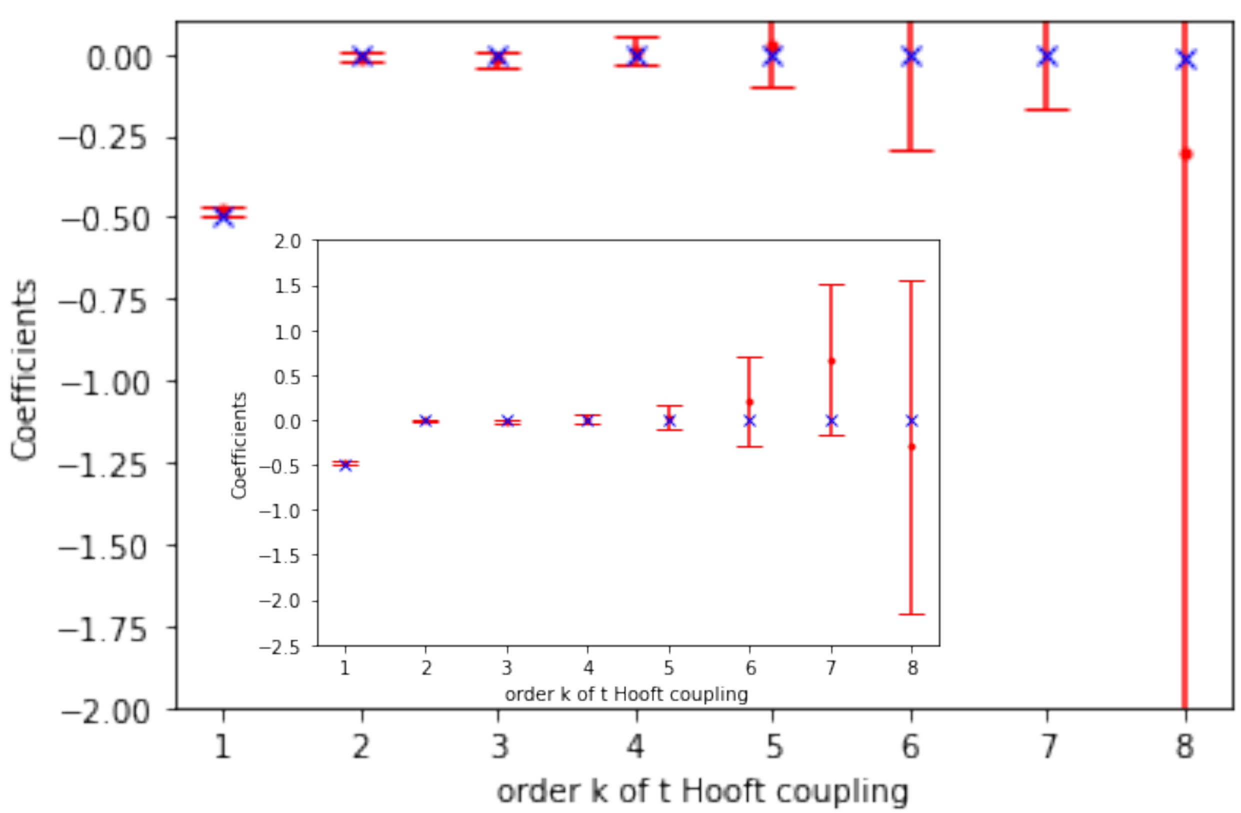
$$\rightarrow \quad P^{(k)}(t+dt) = P^{(k)}(t) - dt\left[\phi * \left\{sh^2 + 2(I + \frac{s}{2}\phi)^{-1}\right\} * (I + \frac{s}{2}\phi)^{-1}\right]^{(k)}$$

Cayley Transformation for GWW model

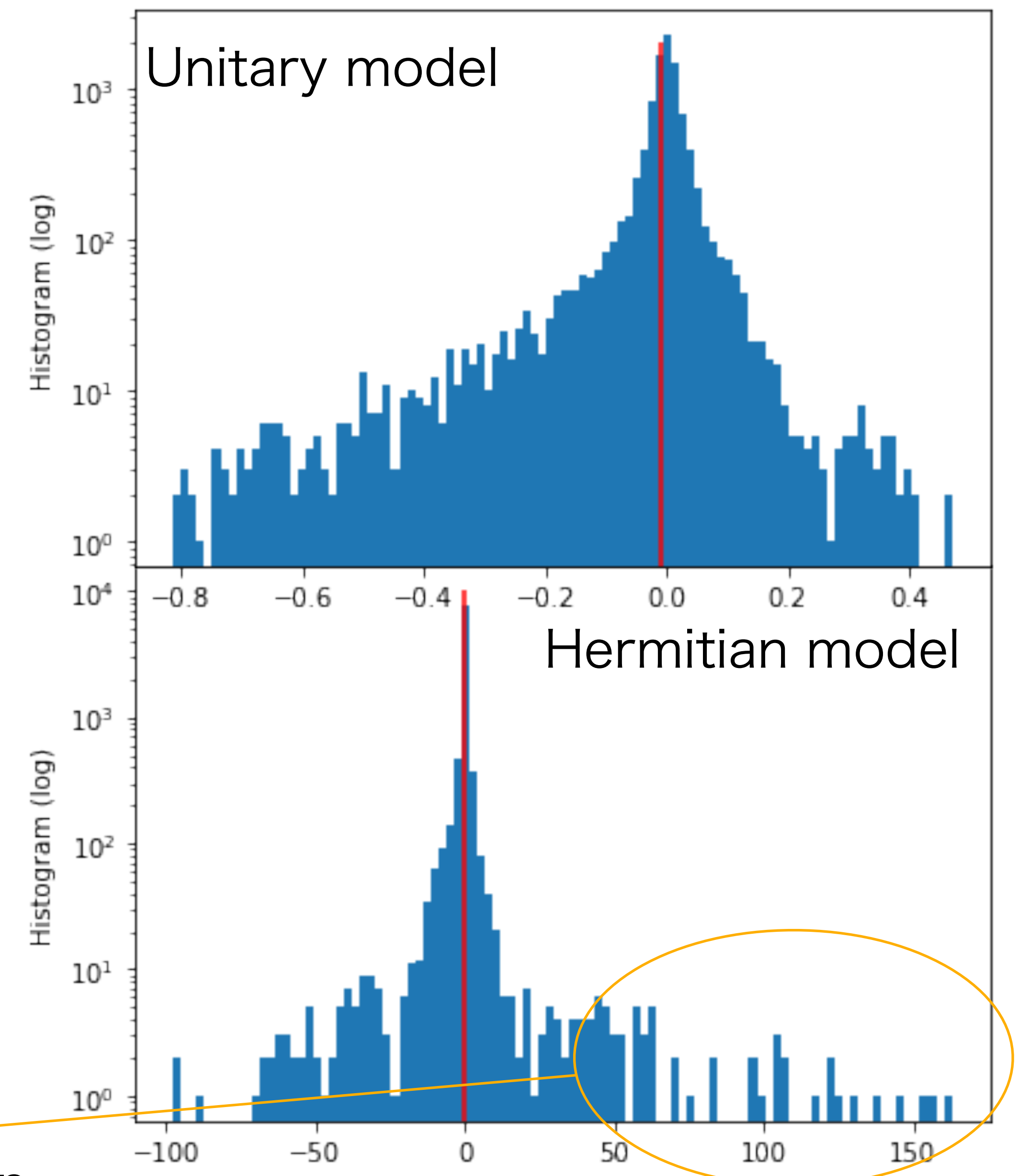
- **The Numerical results for transformed GWW model**
  - HMD Setup

    Matrix size : $N_c = 5$, trajectory length : $\tau = 0.01$, # of MD steps : 5
  - Coefficients

$N_{sample} = 50$



Order of 't Hooft coupling and Coefficients of W.L. expanded in 't Hooft coupling λ.



The coefficient take the value that far from median.

Variance is larger than unitary model

Histogram (log) for 7 order W.L. coefficient C_7.

Red line is exact value $< C_7 > = -0.007358.$

Cayley Transformation for GWW model

- **Some candidates for the Pepe effect (Large variance)**
  - At Large-N_c the eigenvalue distribution can be solved

$$p(\eta) = \begin{cases} \frac{4g}{\pi\lambda(1+g^2\eta^2)^{3/2}}\left(\frac{\lambda}{2} - \frac{g^2\eta^2}{1+g^2\eta^2}\right)^{\frac{1}{2}}, & \lambda \le 2,\ |\eta| < \frac{1}{g}\sqrt{\frac{\lambda}{2-\lambda}}\ . \\ \frac{g}{\pi(1+g^2\eta^2)}\left(1 + \frac{2}{\lambda}\frac{1-g^2\eta^2}{1+g^2\eta^2}\right), & \lambda \ge 2\ . \end{cases}$$

η : Eigenvalue for hermitian matrix

→ Small λ $\lambda \le 2,\ |\eta| < \frac{1}{g}\sqrt{\frac{\lambda}{2-\lambda}}\ .$ $Var(\eta)$: finite

Large λ $\lambda \ge 2$ $Var(\eta)$: Infinite

Center limit Theorem does not hold.

The expectation value does not converge.

- Gauge fixing   F. Di Renzo Nucl. Phys. B 53(1997) 819-822.

  Gauge transformation

$$U \to U' = V^\dagger U V$$

$$= V^\dagger(1 + i\kappa\phi)VV^\dagger(1 - i\kappa\phi)^{-1}V = (1 + i\kappa V^\dagger\phi V)(1 - i\kappa V^\dagger\phi V)^{-1}$$

$$\phi \to \phi' = V^\dagger\phi V$$

To solve large variance, we need finite N_c analysis and improved NSPT.

Summary and Outlook

- **Summary**

1. The behavior of higher-order perturbation coefficients is important from the perspective of resurgence theory.
2. To obtain the higher-order coefficients, we use NSPT which integrate hierarchy partial differential equation on computer.
3. We apply NSPT to GWW model and Cayley transformed GWW.
4. First few order of W.L. are obtained with good accuracy.
5. Because of the Pepe effects, the variance of higher order is large.

- **Outlook**

- We will investigate the causes of the large variance in hermitian model.

→ We try to further analyze finite N_c hermitian GWW and implement a gauge fixed-NSPT in Cayley transformed GWW model.

Backup

- **Borel summation and Resurgence structure.**

Perturbative series $\Phi(g) = \sum_{n=0} C_n g^n$ Coefficients proportional to $n!$: $C_n = Aa^n n!$

・Borel transformation

$$\mathscr{B}\Phi(t) \equiv \sum_{k=0} \frac{C_k}{k!} t^k = \sum_{k=0} \frac{Aa^k k!}{k!} t^k = \sum_{k=0} Aa^k t^k = \frac{A}{1-at}$$

・Borel summation

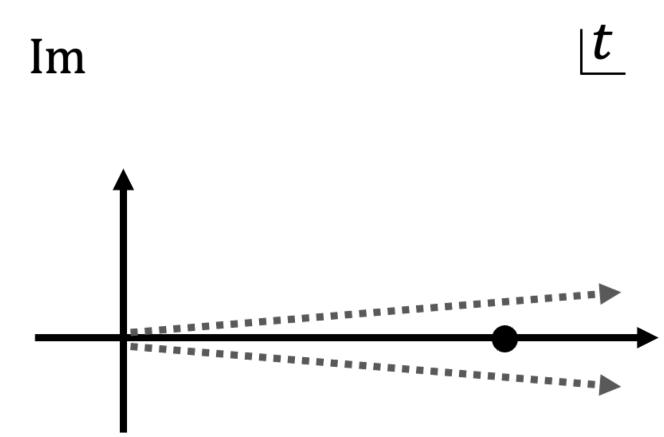
$$S\Phi(g) \equiv \int_0^\infty dt \mathscr{B}\Phi(t) \mathrm{e}^{-gt}$$

This is the identity transformation if $\mathscr{B}\Phi(t)$ have no pole on $[0,\infty]$

If Borel transformation have singular on the counter $[0,\infty]$, we cannot integrate

→ We need to avoid this singular by changing counter (or Complexification of parameter)

This integral have ambiguity due to the avoidance of singularity (Stokes phenomenon)



This ambiguity include the non-perturbative effects

$$(S_+ - S_-)\Phi(\lambda) = \mathrm{e}^{-S_{inst}} S\Phi_1(\lambda)$$

Backup

- **History of Unitarity**
  - The stability of unitarity.

    Use Bruckmann-Puhr delta.

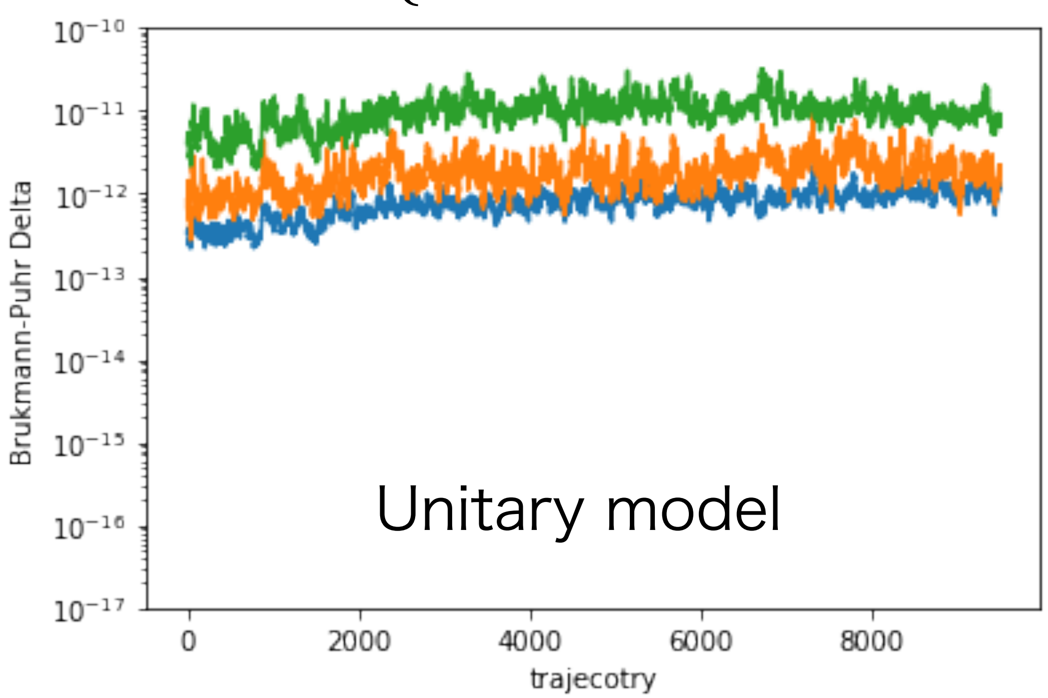
$$\sum_{k=0}^{N_t} V_k \lambda^k \equiv UU^\dagger = \sum_{k=0}^{N_t} \left(\sum_{l=0}^{k} U_{(k-l)} U^\dagger_{(l)} \right) \lambda^k$$

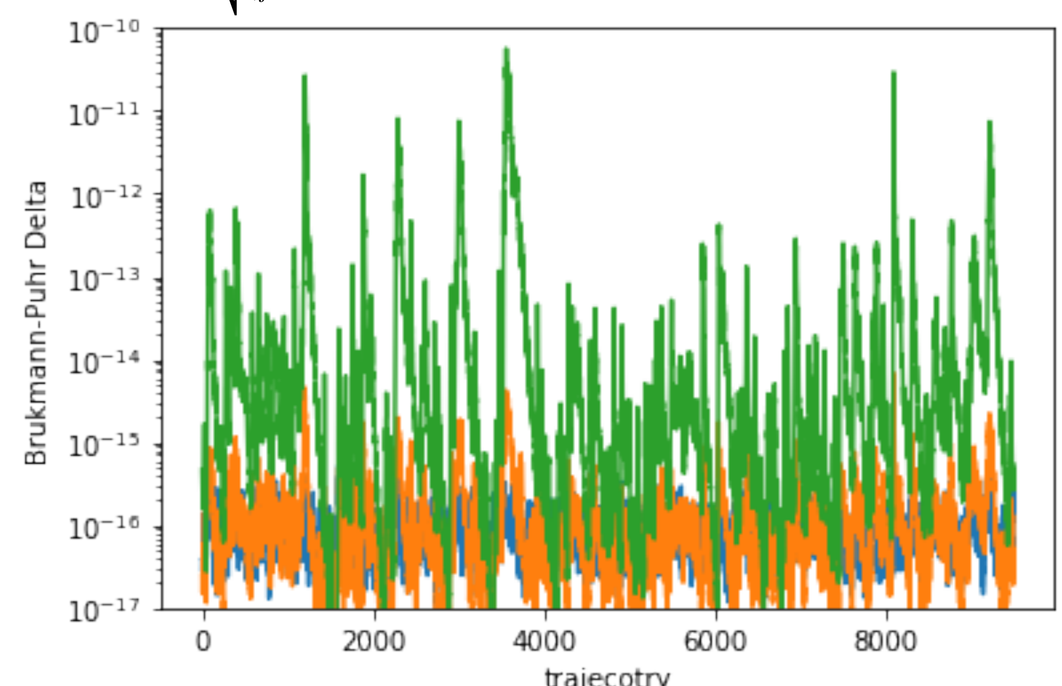
$$\Delta_k = \begin{cases} \frac{1}{N_c} \left| \sqrt{N_c} - ||V_k||_F \right|, & (k=0), \\ ||V_k||_F & (k>0). \end{cases}$$

$||A||_F$ is Frobenius matrix norm.

$$||A||_F \equiv \sqrt{\sum_{i,j=1}^{N_c} |A_{ij}|^2}$$

F. Bruckmann, M. Puhr. Phys. Rev.D **101**, 034513 (2020).





Hermitian model

The history of unitarity $V_{(k)}$ at $k=2$, $k=5$, $k=14$

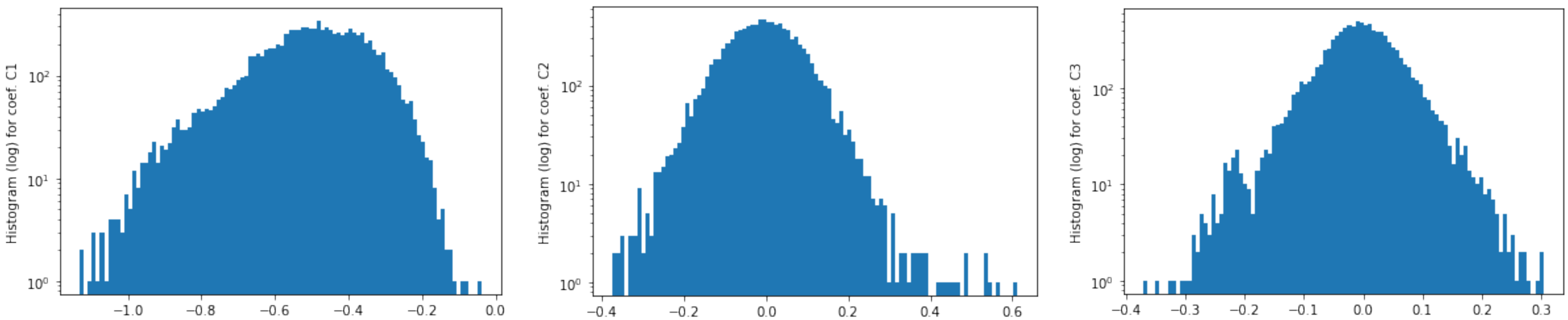
The Bruckmann-Puhr delta in ordinary GWW model is larger than Cayley transformed GWW model.

Hermiticity is easer to maintain than unitarity.

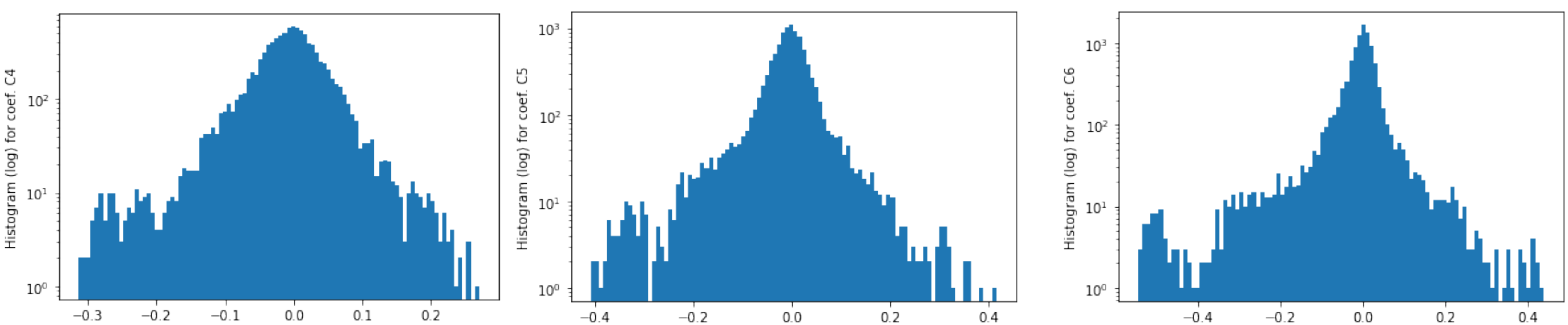
Backup

Unitary Model

- History and Histogram for all order $\mathcal{O}(\lambda^{1\sim6})$ in Unitary GWW model 10000 samples (5000000 trajectory , skip = 500.)



Histogram for W.L. Coefficients λ^k , $k = 1,2,3$

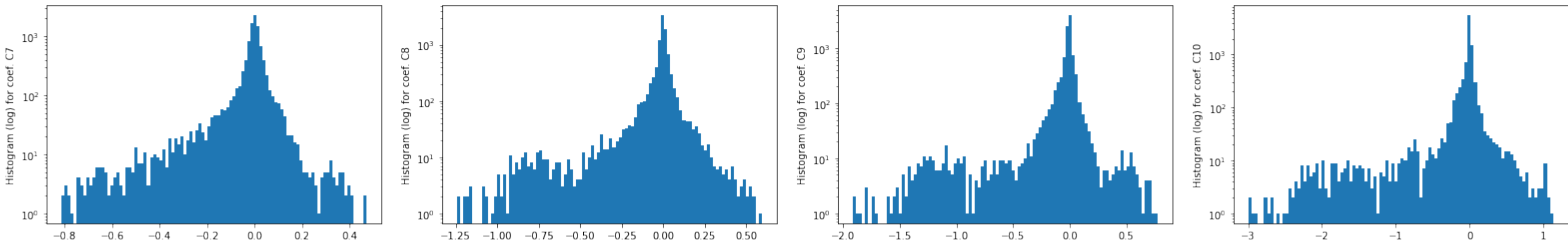


Histogram for W.L. Coefficients λ^k , $k = 4,5,6$

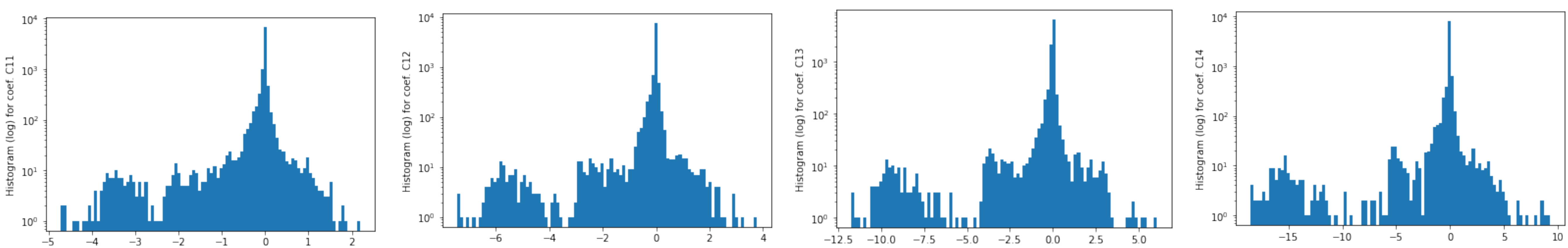
Backup

Unitary Model

- History and Histogram for all order $\mathcal{O}(\lambda^{7\sim14})$ in Unitary GWW model
  10000 samples (5000000 trajectory , skip = 500.)



Histogram for W.L. Coefficients λ^k , $k = 7,8,9,10$

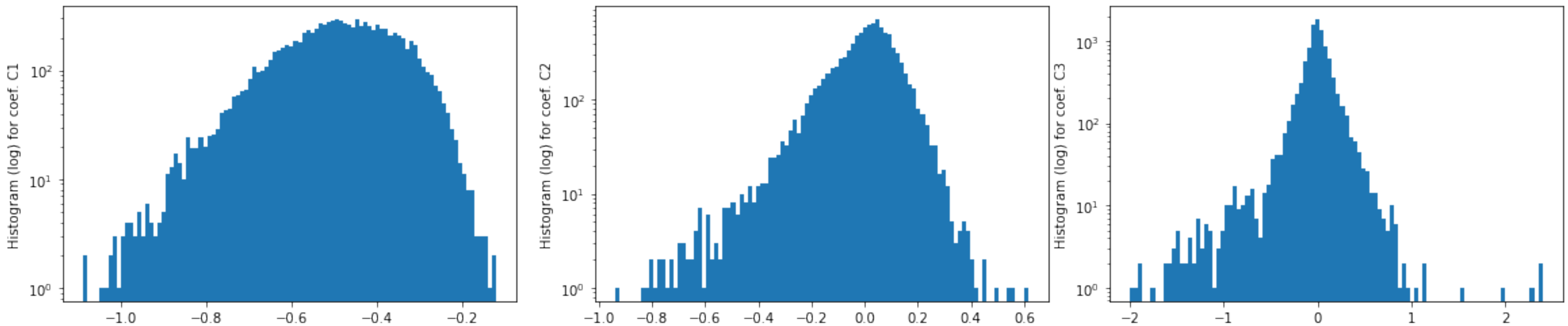


Histogram for W.L. Coefficients λ^k , $k = 11,12,13,14$

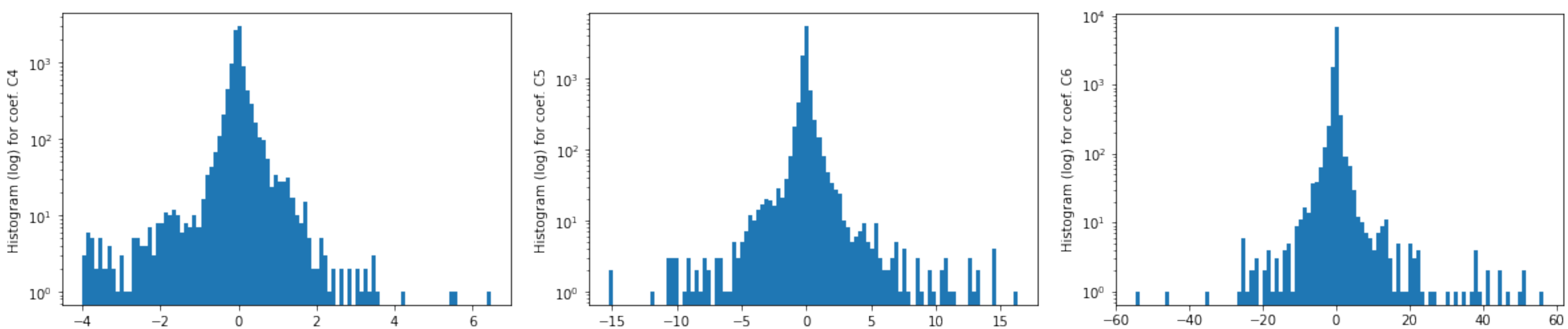
Backup

Hermitian Model

- History and Histogram for all order $\mathcal{O}(\lambda^{1\sim6})$ in Hermitian GWW model
  10000 samples (5000000 trajectory , skip = 500.)



Histogram for W.L. Coefficients λ^k , $k = 1,2,3$

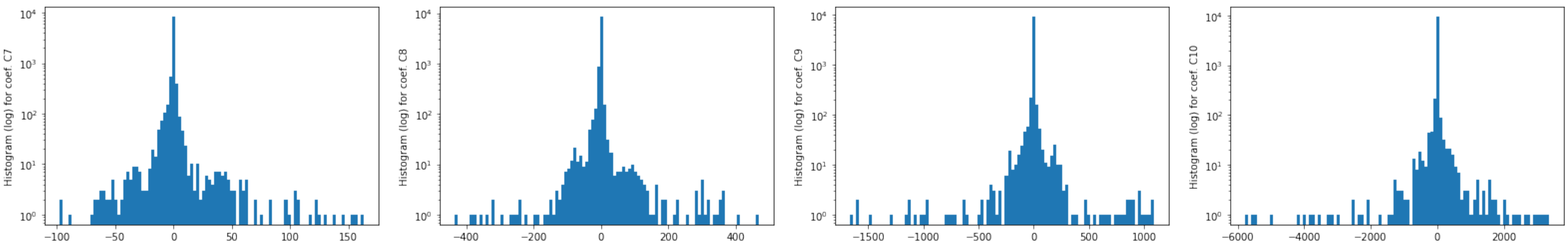


Histogram for W.L. Coefficients λ^k , $k = 4,5,6$

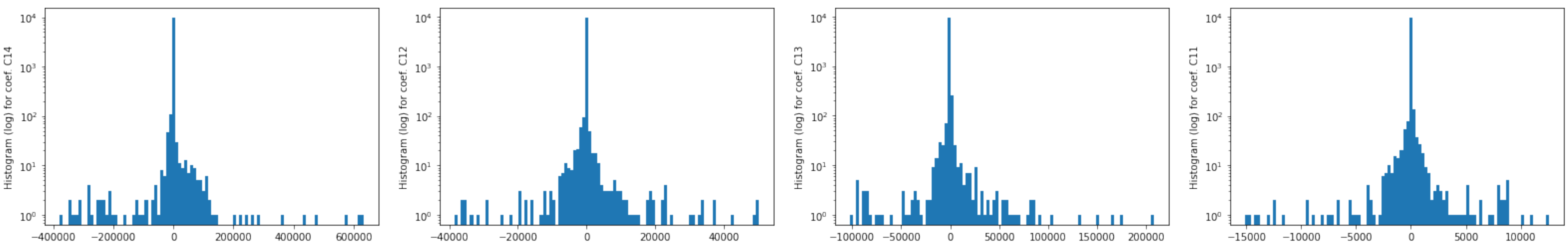
Backup

Hermitian Model

- History and Histogram for all order $\mathcal{O}(\lambda^{7\sim14})$ in Hermitian GWW model
  10000 samples (5000000 trajectory , skip = 500.)



Histogram for W.L. Coefficients λ^k , $k = 7{,}8{,}9{,}10$



Histogram for W.L. Coefficients λ^k , $k = 11{,}12{,}13{,}14$