# Non-Abelian Discrete Groups and Neutrino Flavor Symmetry

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**Abstract**

We discuss the recent progress of flavor models with the non-Abelian discrete symmetry in the lepton sector focusing on the $CP$ violating Dirac phase. It is emphasized that the flavor symmetry with the generalised $CP$ symmetry can predict the $CP$ violating phases.

# 1 Introduction

The neutrino oscillation experiments have determined precisely the two neutrino mass differences and the three neutrino mixing angles. Especially, the recent data of both T2K and NO$\nu$A show that the atmospheric neutrino mixing angle $\theta_{23}$ is near the maximal angle 45°. Indeed, the NuFit 3.2 present the best fit $\theta_{23} = 47.2°$ for the normal hierarchy of neutrino masses [1]. The closer maximal mixing $\theta_{23} = 45°$, the more likely that some symmetry behind it. The recent experimental data of T2K and NO$\nu$A also strongly indicate the CP violation in the neutrino oscillation [2, 3]. We are in the era to develop the flavor structure of Yukawa couplings by focusing on the leptonic CP violation.

These experimental data give us a big hint of the flavor symmetry. Before the reactor experiments reported the non-zero value of $\theta_{13}$, there was a paradigm of "tri-bimaximal mixing" (TBM) [4, 5], which is a simple mixing pattern for leptons and can be easily derived from flavor symmetries. Some authors succeeded to realize the TBM in the $A_4$ models [6, 7, 8, 9]. After those successes, the non-Abelian discrete groups are center of attention at the flavor symmetry [10, 11, 12, 13]. The observation of the non-vanishing $\theta_{13}$ accelerate the study of flavor models deviating from the TBM [14]. In this talk, we summarize the recent progress of the flavor models with the non-Abelian discrete symmetry.

# 2 Tri-bimaximal mixing and Flavor symmetry

## 2.1 Tri-bimaximal mixing

After discovering two large mixing angles of neutrino flavors, Harrison-Perkins-Scott proposed a simple form of the mixing pattern, so-called the tri-bimaximal mixing (TBM) [4, 5] as follows:

$$V_{\rm TBM} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{1}$$

The TBM is given in the neutrino mass matrix $m_{\nu LL}$

$$m_{\nu LL} = \frac{m_1+m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2-m_1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{2}$$

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where $m_1$, $m_2$ and $m_3$ are neutrino masses, in the diagonal basis of the charged lepton. It is remarked that the mixing angles are independent of neutrino masses. It is also noticed that this mass matrix is given in terms of integer matrix elements. The non-Abelian symmetry connects different families by taking a doublet or a triplet irreducible representation for three families. The discrete symmetry gives us the definite meaning of three family. Therefore, the non-Abelian discrete symmetry is appropriate for lepton families in the standpoint of the TBM. The third matrix of the r.h.s in Eq.(2) is $A_4$ symmetric, on the other hand, the first one and the second one are well known as $S_3$ symmetric. Actually, Ma and Rajasekaran presented a simple model with the $A_4$ flavor symmetry [6]. After that, many flavor models with the symmetry $A_4$, $A_5$, $S_3$, $S_4$, $D_4$, $D_6$, $T'$, $Q_4$, $Q_6$, $\Delta(27)$ or $\Delta(54)$ were proposed.

In 2012, the reactor angle was measured by Daya Bay, Reno and Double Chooz as well as the long-baseline neutrino experiments T2K and MINOS. The mixing angle $\theta_{13}$ was found to be of order of the Cabibbo angle, $\theta_c/\sqrt{2} \simeq 9^\circ$, which ruled out the TBM scheme completely. Then, many people worked to explain the deviation from the TBM. In those works, the non-Abelian discrete symmetries have been still active for building flavor models.

## 2.2 $A_4$ and $S_4$ symmetry

The most predictive models involve favor symmetry groups which admit triplet representations. The typical non-Abelian discrete symmetries are $A_4$ and $S_4$, which are adopted in some neutrino models.

Let us introduce these groups briefly [11, 12]. All even permutations of four objects form a group, which is called $A_4$. The order of this group, that is the number of elements, is 12. These elements $g$ are generated by the generators $S$ and $T$, which satisfy $S^2 = T^3 = (ST)^3 = 1$. They are classified by the conjugacy classes as:

$$\begin{array}{lclr} C_1 & : & \{1\}, & h=1, \\ C_3 & : & \{S, T^2ST, TST^2\}, & h=2, \\ C_4 & : & \{T, ST, TS, STS\}, & h=3, \\ C_4' & : & \{T^2, ST^2, T^2S, ST^2S\}, & h=3, \end{array} \tag{3}$$

where we have also shown the orders of each element in the conjugacy class by $h$ with $g^h = 1$. There are four conjugacy classes and there must be four irreducible representations, $\mathbf{1}$, $\mathbf{1}'$, and $\mathbf{1}''$, and a single triplet $\mathbf{3}$.

Next, we present the $S_4$ group, which consists of all permutations among four objects. The order of $S_4$ is equal to $4! = 24$. These elements are generated by the generators $S$, $T$ and $U$, which satisfy $S^2 = T^3 = U^2 = 1$ and $ST^3 = (SU)^2 = (TU)^2 = 1$. They are classified by the conjugacy classes as:

$$\begin{array}{lclr} C_1 & : & \{1\}, & h=1, \\ C_3 & : & \{S, T^2ST, TST^2\}, & h=2, \\ C_6 & : & \{U, TU, SU, T^2U, STSU, ST^2SU\}, & h=2, \\ C_8 & : & \{T, ST, TS, STS, T^2, ST^2, T^2S, ST^2S\}, & h=3, \\ C_6' & : & \{STU, TSU, T^2SU, ST^2U, TST^2U, T^2STU\}, & h=4. \end{array} \tag{4}$$

The group $S_4$ includes five conjugacy classes, that is, there are five irreducible representations, two singlets $\mathbf{1}$ and $\mathbf{1}'$, one doublet $\mathbf{2}$, and two triplets $\mathbf{3}$ and $\mathbf{3}'$.

## 2.3 Flavor models with $A_4$ or $S_4$ symmetry

The model building of the flavor symmetries is not straightforward since the flavor symmetry group $G$ must be broken. The key of the model building is how the symmetry $G$ is broken. The predictions depend crucially on the breaking pattern of $G$. Although $G$ is completely broken in the full theory, there are some relic symmetries of $G$ in the neutrino sector and the charged lepton sector, respectively. These relic symmetries are different in the neutrino sector and the charged lepton sector. This difference is crucial to predict the flavor mixing angles. If no relic symmetries survive, there is no predictive power of the flavor group $G$.

There are two approaches, the direct one and the indirect one [13]. In the direct approach, the different subgroups of the flavor symmetry survive in the neutrino sector or charged lepton sector. Then, the survival symmetry in the neutrino sector is $Z_2 \times Z_2$ (the Klein symmetry), while the symmetry in the charged lepton sector is $Z_3$. However, it is not true that the relevant Klein symmetry is a subgroup of the underlying family symmetry $G$. Indeed, the $S_4$ group has the relevant subgroup $Z_2 \times Z_2$ which is generated by $S$ and $U$, but the $A_4$ symmetry does not.

On the other hand, in the indirect approach, no subgroup of the flavor symmetry survives. Instead, the flavons have special vacuum alignments whose alignment is assisted by the flavor symmetry. These flavons are different ones in the neutrino sector and the charged lepton sector by an additional $Z_n$ symmetry.

### 2.3.1 Direct Approach of $S_4$ symmetry

We discuss the direct approach of the $G = S_4$ group. The $S_4$ group has subgroups, which are nine $Z_3$, four $Z_3$, three $Z_4$ and four $Z_2 \times Z_2$ (Klein four group).

Suppose $S_4$ is spontaneously broken to one of subgroups, in which

$$K_4 : \{1, S, U, SU\} \text{ for neutrinos }, \qquad Z_3 : \{1, T, T^2\} \text{ for charged leptons }, \tag{5}$$

are preserved. The neutrino mass matrix $m_{\nu LL}$ respects the $S$ and $U$ generators, on the other hand, the charged lepton mass matrix $m_\ell$ respects the $T$ generator. Then, these mass matrices satisfy the following relations:

$$S^T m_{\nu LL} S = m_{\nu LL} \ , \qquad U^T m_{\nu LL} U = m_{\nu LL} \ , \qquad T^\dagger m_\ell m_\ell^\dagger T = m_\ell m_\ell^\dagger \ , \tag{6}$$

which turn to

$$[S, m_{\nu LL}] = 0 \ , \qquad [U, m_{\nu LL}] = 0 \ , \qquad [T, m_\ell m_\ell^\dagger] = 0 \ . \tag{7}$$

For the triplets, which are two ones in $S_4$, the representation of $U$ is taken to be:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} , \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} , \qquad U = \mp \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \tag{8}$$

where $\omega^3 = 1$ and the signs $\mp$ in $U$ correspond to the different triplets. The mixing matrix which diagonalizes both $S$ and $U$ is fixed as

$$\begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} , \tag{9}$$

which is just the TBM mixing matrix. Thus, the TBM is derived from the direct approach of the $S_4$ group.

There is another possibility of the breaking pattern of the $S_4$ group. Suppose $S_4$ is spontaneously broken to

$$Z_2 : \{1, SU\} \text{ for neutrinos }, \qquad Z_3 : \{1, T, T^2\} \text{ for charged leptons }, \tag{10}$$

preserved. This case is called as the semi-direct approach.

Then, these mass matrices satisfy the following relations

$$[SU, m_{\nu LL}] = 0 \ , \qquad [T, m_\ell m_\ell^\dagger] = 0 \ , \tag{11}$$

which give us another mixing pattern as follows:

$$V_\nu = \begin{pmatrix} 2/\sqrt{6} & c/\sqrt{3} & s/\sqrt{3} \\ -1/\sqrt{6} & c/\sqrt{3} - s/\sqrt{2} & -s/\sqrt{3} - c/\sqrt{2} \\ -1/\sqrt{6} & c/\sqrt{3} + s/\sqrt{2} & -s/\sqrt{3} + c/\sqrt{2} \end{pmatrix} \ , \tag{12}$$

where $c = \cos\phi$ and $s = \sin\phi$ including a CP violating phase. This mixing matrix is also the tri-maximal mixing which is called $\mathrm{TM}_1$.

# 3 CP symmetry and Flavor Symmetry

Let us start with discussing the generalised $CP$ symmetry [15, 16]. The CP is a discrete symmetry which involves both complex conjugation of the fields and inversion of spatial coordinates,

$$\varphi(x) \to \mathbf{X}_{\mathbf{r_i}} \varphi^*(x') \ , \tag{13}$$

where $x' = (t, -\mathbf{x})$ and $\mathbf{X}_{\mathbf{r_i}}$ is a matrix of transformations of $\varphi(x)$ in the irreducible representation $\mathbf{r_i}$ of the discrete flavor symmetry $G$. If $\mathbf{X}_{\mathbf{r_i}}$ is the unit matrix, the $CP$ transformation is the trivial one. This is the case for the continuous flavor symmetry [16]. However, in the framework of the discrete family symmetry, non-trivial choices of $\mathbf{X}_{\mathbf{r_i}}$ are possible. The unbroken $CP$ transformation $\mathbf{X}_{\mathbf{r_i}}$s form the group $H_{CP}$. Then, $\mathbf{X}_{\mathbf{r_i}}$s must be consistent with the flavor symmetry transformation,

$$\varphi(x) \to \rho_{\mathbf{r_i}}(g)\varphi(x) \ , \quad g \in G \ , \tag{14}$$

where $\rho_{\mathbf{r_i}}(g)$ is the representation matrix for $g$ in the irreducible representation $\mathbf{r_i}$.

The consistent condition is obtained as follows. At first, perform a $CP$ transformation $\varphi(x) \to \mathbf{X}_{\mathbf{r_i}}\varphi^*(x')$, then apply a flavor symmetry transformation, $\varphi(x')^* \to \rho_{\mathbf{r_i}}(g)\varphi(x')^*$, and finally an inverse CP transformation. The whole transformation is written as $\varphi(x) \to \mathbf{X}_{\mathbf{r_i}}\rho^*(g)\mathbf{X}_{\mathbf{r_i}}^{-1}\varphi(x)$, which must be equivalent to some flavor symmetry $\varphi(x) \to \rho_{\mathbf{r_i}}(g')\varphi(x)$. Thus, one obtains the consistent condition [17, 18]

$$\mathbf{X}_{\mathbf{r_i}}\rho^*_{\mathbf{r_i}}(g)\mathbf{X}_{\mathbf{r_i}}^{-1} = \rho_{\mathbf{r_i}}(g') \ , \qquad g, g' \in G \ . \tag{15}$$

The full symmetry of the unbroken flavor symmetry and generalised $CP$ symmetry is the semi-direct product of $G$ and $H_{CP}$, that is $G \otimes H_{CP}$, where $G$ and $H_{CP}$ do not commute in general for the case of the non-Abelian discrete symmetries.

Suppose the full symmetry including the $CP$ symmetry and the flavor symmetry is broken to the subgroups in the neutrino sector and the charged lepton sector, respectively. The $CP$ symmetry gives us the relations as to the neutrino mass matrix and the charged lepton mass matrix as follows:

$$\mathbf{X}_{\mathbf{r_i}}^{\nu T} m_{\nu LL}\mathbf{X}_{\mathbf{r_i}}^{\nu} = m_{\nu LL} \ , \qquad \mathbf{X}_{\mathbf{r_i}}^{\ell\dagger} m_\ell m_\ell^\dagger \mathbf{X}_{\mathbf{r_i}}^{\ell} = m_\ell m_\ell^\dagger \ . \tag{16}$$

Once the subgroups of $G$ and $H_{CP}$ are chosen to satisfy the conditions of Eqs. (15) and (16) for the neutrino sector and the charged lepton sector, respectively, one can predict the $CP$ phase, $\delta_{CP}$.

In this talk, we present an example of the $S_4$ symmetry [19, 20]. Suppose the full symmetry is broken to $G_\nu$ and $H^\nu_{CP}$ in the neutrino sector, while the charged lepton sector respects $T$, that is the diagonal charged lepton mass matrix:

$$G_\nu = \{1, S\}, \qquad X^\nu_{\mathbf{3}} = U, \qquad G_\ell = \{1, T, T^2\}, \qquad X^\ell_{\mathbf{3}} = 1 \ , \tag{17}$$

which satisfy the consistency condition Eq.(15). Then, the neutrino mass matrix, which respects $S$, is given as

$$m_{\nu LL} = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \ , \tag{18}$$

where $\alpha$, $\beta$, $\gamma$ and $\epsilon$ are arbitrary complex parameters. Imposing the CP symmetry $S^T m_{\nu LL} S = m^*_{\nu LL}$ in Eq.(16), one finds $\alpha$, $\beta$ and $\gamma$ to be real, and $\epsilon$ to be pure imaginary. Then, the neutrino mass in Eq.(18) is diagonalised by the unitary matrix:

$$V_\nu = \begin{pmatrix} 2c/\sqrt{6} & 1/\sqrt{3} & 2s/\sqrt{6} \\ -c/\sqrt{6} + is/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} - ic/\sqrt{2} \\ -c/\sqrt{6} + is/\sqrt{2} & 1/\sqrt{3} & -s/\sqrt{6} + ic/\sqrt{2} \end{pmatrix} \ , \tag{19}$$

where $c = \cos\phi$ and $s = \sin\phi$. Since the charged lepton mass matrix is diagonal, we obtain

$$\sin^2\theta_{13} = \frac{2}{3}\sin^2\phi, \quad \sin^2\theta_{12} = \frac{1}{2+\cos 2\phi}, \quad \sin^2\theta_{23} = \frac{1}{2}, \quad |\sin\delta_{CP}| = 1 \ , \tag{20}$$

which correspond to the maximal CP violation, $\delta_{CP} = \pm\pi/2$. The prediction of the $CP$ phase depends on the respected "Generators" of the flavor symmetry and the $CP$ symmetry. Typically, it is simple values 0, $\pm\pi/2$ or $\pi$ for other cases [20, 21].

It is useful to summarize the comprehensive work by Chen et al. [18] in the relation between the discrete symmetries and the physical $CP$ invariance guaranteed by generalized $CP$ transformations. They have studied the $CP$ violation by the automorphisms of G carefully. The origin of the $CP$ violation with a discrete flavor symmetry is categorized into three types: (i) Groups explicitly violate $CP$, which can be related to the complexity of some CG coefficients. An example is the $\Delta(27)$ group. (ii) Groups for which one can find a CP basis in which all the CG coefficients are real. For such groups, imposing $CP$ invariance restricts the phases of coupling coefficients. The examples are $A_4$, $T'$ and $S_4$. (iii) Groups that do not admit real CG coefficients, but can define the generalized CP transformation. An example is $\Sigma(72)$.

# 4 Prospect

The flavor symmetry predicts non-vanishing $\theta_{13}$. The flavor symmetry with the generalised $CP$ symmetry also predicts the $CP$ violating phase. Moreover, the flavor symmetry predicts the mass sum rules. These predictions will be testable by the precise data of the neutrino mixing angles, the $CP$ violating phase and the effective neutrino mass $m_{ee}$.

On the other hand, we have another important question. Can one predict the CKM mixing matrix in the quark sector from the flavor symmetry? We expect challenging works, in which the neutrino mixing angle $\theta_{13}$ is related to the Cabibbo angle and the CP violating phases are related each other in the framework of the unification of quarks and leptons.

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