CONTROL OF LINEAR MULTIPARAMETER SINGULARITY PERTURBED SYSTEMS

Hiroaki Mukaidani * Koich Mizukami **

* Faculty of Information Sciences, Hiroshima City University,
  Asaminami-ku, Hiroshima, 731–3194 Japan
  E–mail: mukaida@im.hiroshima-cu.ac.jp

** Faculty of Engineering, Hiroshima Kokusai Gakuin University,
  Nakano Aki-ku, Hiroshima, 739–0321 Japan

Abstract: In this paper, the linear quadratic optimal control for multiparameter singularly perturbed systems (MSPS) is studied in a different approach from the existing methods. The attention is focused on the design of a near–optimal controller which does not depend on the values of the small unknown parameters. The resulting controller achieves $O(|\mu|^2)$ near–optimal cost compared with previously proposed result in the literature. To obtain such a controller, the existence of a unique and bounded solution of a multiparameter algebraic Riccati equation (MARE) is newly proven.

Keywords: Multiparameter singularly perturbed systems, Linear quadratic optimal control, Near–optimal control, Algebraic Riccati equation

1. INTRODUCTION

The deterministic and stochastic multimodeling stability, control, filtering and dynamic games have been investigated extensively by several researchers (see e.g., Khalil and Kokotović, 1978, 1979; Coumarbatch and Gajić, 2000; Gajić, 1988; Wang et al., 1994). The multimodeling problems arise in large scale dynamic systems. For example, these multimodel situations in practice are illustrated by the multiarea power system (Khalil and Kokotović, 1978) and the passenger car model (Coumarbatch and Gajić, 2000). In order to obtain the optimal solution to the multimodeling problems, we must solve the multiparameter algebraic Riccati equation (MARE), which are parameterized by the small positive same order parameters $\varepsilon_j$, $j = 1, 2, \cdots$. Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., Laub, 1979). One of the approaches is the invariant subspace approach based on the Hamiltonian matrix (Laub, 1979). However, when the ARE is known to be ill–conditioned (Laub, 1979) such an approach is not ad–equate to the multiparameter singularly perturbed systems (MPS) since there is no guarantee of symmetry for the solution of the ARE. Note that it is very hard to solve directly the MARE due to high dimension and numerical stiffness (Coumarbatch and Gajić, 2000). More recently, the exact slow–fast decomposition method for solving the MARE has been proposed in Coumarbatch and Gajić (2000). However, a limitation of these approaches is that the small parameters are assumed to be known. In practice, the small perturbation parameters $\varepsilon_j$ are often not known. Thus, it is not applicable to a large class of problems where the parameters represent small unknown perturbations whose values are not known exactly.

A popular approach to deal with the MSPS is the two–time–scale design method (see e.g., Khalil and Kokotović, 1978, 1979; Gajić, 1988; Wang et al., 1994). For example, optimal control of a class of MSPS, being only on the slow variable, has been studied by Khalil and Kokotović (1979), where the design of the $\varepsilon_j$–independent reduced–order controller has been suggested. However, in order to obtain the slow subsystem, the nonsingularity of
the fast state matrices are needed. In Wang et al. (1994), by making use of the descriptor variable approach, the main results of Khalil and Kokotović (1979) have been extended to the nonstandard MSPS such that at least one of the fast state matrices is singular. However, the proposed controller only achieves \( O(\|\mu\|) \) (where \( \|\mu\| \) denotes the norm of the vector \( \|\varepsilon_1 \cdots \varepsilon_N\| \) ) approximation of the optimal cost.

In this paper, we study the linear quadratic optimal control problems for MSPS. We first investigate the unique and bounded solution of the MARE and establish its asymptotic structure. Thus, this paper presents an improvement on some of the results of Gajić (1988) in the sense that some assumptions are relaxed. Using the asymptotic structure, a new near–optimal controller which does not depend on the values of the small parameters is obtained. This is done by eliminating the parameters \( \varepsilon_j \) for the full–order controller. We emphasize that structure of the resulting controller achieves \( O(\|\mu\|^2) \) near–optimal cost compared with the previously proposed in the literature. Even if the parameters are unknown, when the parameters are sufficiently small, the near–optimal controller can be used reliably for the MSPS.

2. MSPS

We consider a specific structure of \( N \)–lower level multi–fast subsystems interconnected through the dynamics of a higher level slow subsystem.

\[
\begin{align*}
\dot{x}_0 &= A_{00}x_0 + \sum_{j=1}^{N} A_{0j}x_j + \sum_{j=1}^{N} B_{0j}u_j, \quad (1a) \\
\varepsilon_j \dot{x}_j &= A_{j0}x_0 + A_{jj}x_j + B_{jj}u_j, \quad (1b) \\
y_0 &= C_{00}x_0, \quad (1c) \\
y_j &= C_{j0}x_0 + C_{jj}x_j, \quad j = 1, 2, \cdots, N, \quad (1d) \\
x_j(0) &= x_j^0, \quad j = 0, 1, 2, \cdots, N, \quad (1e)
\end{align*}
\]

where \( x_j \in \mathbb{R}^{n_j}, \ j = 0, 1, \cdots, N \) are the state vectors, \( u_j \in \mathbb{R}^{m_j}, \ j = 1, 2, \cdots, N \) are the control inputs, \( y_j \in \mathbb{R}^{l_j}, \ j = 0, 1, \cdots, N \) are the outputs. We assume that the ratios of the small positive parameter \( \varepsilon_j > 0, \ j = 1, 2, \cdots, N \) are bounded by some positive constants \( \kappa_{ij}, \bar{k}_{ij} \) (see e.g., Khalil and Kokotović, 1978, 1979),

\[
0 < \bar{k}_{ij} \leq \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \leq \kappa_{ij} < \infty. \quad (2)
\]

Note that at least one of the fast state matrices \( A_{jj}, \ j = 1, 2, \cdots, N \) may be singular. The performance criterion is given by

\[
J = \frac{1}{2} \int_0^\infty \left( y^T y + \sum_{j=1}^{N} u_j^T R_j u_j \right) dt, \quad (3)
\]

where \( y^T = [y_0^T \cdots y_m^T]^T \in \mathbb{R}^{l}, \ l = \sum_{j=0}^{N} l_j. \)

In order to find a near–optimal control without the knowledge of the small perturbation parameters \( \varepsilon_j \), let the optimal control for the regulator problem (1) and (3) be

\[
u_{\text{opt}} = -R^{-1} B_{e}^T P_e x,
\]

where \( P_e \) satisfies the MARE

\[
A_{e}^T P_e + P_e A_{e} - P_e S_{e} P_e + Q = 0, \quad (5)
\]

with

\[
A_{e} := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_{e}^{-1} A_{f0} & \Pi_{e}^{-1} A_{ff} \end{bmatrix}, \\
\Pi_{e} := \text{block} - \text{diag} \left( \varepsilon_1 I_{n_1}, \cdots, \varepsilon_N I_{n_N} \right), \\
A_{0f} := [A_{01} \cdots A_{0N}], \quad A_{f0} := [A_{f01} \cdots A_{f0N}]^T, \\
A_f := \text{block} - \text{diag} \ (A_{f11} \cdots A_{fNN}), \\
S_{e} := B_{e} R^{-1} B_{e}^T = \begin{bmatrix} S_{00} & \Pi_{e}^{-1} S_{0f} \\ S_{0f}^T \Pi_{e}^{-1} & \Pi_{e}^{-1} S_{f} \end{bmatrix}, \\
S_{00} := \sum_{j=1}^{N} B_{0j} R^{-1} B_{0j}^T, \quad S_{0f} := [S_{01} \cdots S_{0N}], \\
S_{f} := \text{block} - \text{diag} \ (S_{11} \cdots S_{NN}), \\
B_{e} := \begin{bmatrix} B_0 \\ \Pi_{e}^{-1} B_f \end{bmatrix}, \quad B_0 := [B_{01} \cdots B_{0N}], \\
B_f := \text{block} - \text{diag} \ (B_{f11} \cdots B_{fNN}), \\
R := \text{block} - \text{diag} \ (R_{1} \cdots R_{N}), \\
Q := \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix}, \quad Q_{00} := \sum_{j=0}^{N} C_{j0}^T C_{j0}, \\
Q_{0f} := [Q_{01} \cdots Q_{0N}] = [C_{10}^T C_{11} \cdots C_{N0}^T C_{NN}], \\
Q_f := \text{block} - \text{diag} \ (Q_{11} \cdots Q_{NN}) = \text{block} - \text{diag} \ (C_{1f}^T C_{11} \cdots C_{Nf}^T C_{NN}).
\]

In the following analysis, we need some assumptions.

**Assumption 1:** The triples \((A_{jj}, B_{jj}, C_{jj}), \ j = 1, 2, \cdots, N\) are stabilizable and detectable.

**Assumption 2:**

\[
\begin{align*}
\text{rank} \begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & B_0 \\ -A_{f0} & -A_{f} & B_f \end{bmatrix} &= \bar{n}, \quad (6a) \\
\text{rank} \begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{0f} & C_0^T \\ -A_{f0}^T & -A_{f} & C_f \end{bmatrix} &= \bar{n}, \quad (6b)
\end{align*}
\]
where \( \bar{n} := \sum_{j=0}^{N} n_j \),
\[
C_0 := \begin{bmatrix} C_{00} \\ C_{10} \\ \vdots \\ C_{N0} \end{bmatrix}, \quad C_f := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{NN} \end{bmatrix},
\]
with \( \text{Re}[s] \geq 0, \ s \in \mathbb{C} \).

**Assumption 3:** The Hamiltonian matrices \( T_{jj}, \ j = 1, 2, \cdots, N \) are nonsingular, where
\[
T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix}.
\]

Before investigating the optimal control problem, we investigate the asymptotic structure of the MARE (5). Let us introduce the scaling matrices
\[
\Phi_e := \text{block} - \text{diag} \left( I_{n_0} \varepsilon_1 I_{n_1} \cdots \varepsilon_N I_{n_N} \right) = \begin{bmatrix} J_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix}.
\]
In order to avoid the ill-conditioned caused by the large parameter \( \varepsilon_j^{-1} \) which is included in the MARE (5), we introduce the following useful lemma.

**Lemma 1:** The MARE (5) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (7)
\[
G(P) = A^T P + P^T A - P^T S P + Q = 0, \quad (7)
\]
where
\[
A := \begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix}, \quad S := \begin{bmatrix} S_{00} & S_{0f} \\ S_{f0}^T & S_f \end{bmatrix},
\]
\[
P := \begin{bmatrix} P_{00} & P_{0f} \\ P_{f0}^T & P_f \end{bmatrix}, \quad P_{f0} := \begin{bmatrix} P_{f0} \\ \vdots \\ P_{N0} \end{bmatrix},
\]
\[
P_{0f} = P_{f0}^T \Pi_e := \begin{bmatrix} \varepsilon_1 P_{10}^T & \cdots & \varepsilon_N P_{N0}^T \end{bmatrix},
\]
\[
P_f := \begin{bmatrix} P_{11} & \alpha_{12} P_{21}^T & \cdots & \alpha_{1N} P_{N1}^T \\ \vdots & \vdots & \ddots & \vdots \\ P_{N-11} & \cdots & \cdots & \alpha_{N-1N} P_{NN}^T \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix}.
\]

**Proof:** Firstly, by direct calculation we verify that \( P_e = \Phi_e P \). Secondly, it is easy to verify that \( A = \Phi_e A_e, \ S = \Phi_e S_e \Phi_e \). Hence,
\[
A^T P = A^T \Phi_e \Phi_e^{-1} P_e = A^T P_e.
\]
By using the similar calculation, we can immediately rewrite (5) as (7).

The GMARE (7) can be partitioned into
\[
f_1 = P_{00}^T A_{00} + A_{00}^T P_{00} + P_{f0}^T A_f + A_f^T P_{f0}
- P_{00}^T S_{00} P_{00} - P_{f0}^T S_{f0} P_{f0} - P_{00}^T S_{0f} P_{f0}
- P_{f0}^T S_{f0} P_{00} + Q_{00} = 0,
\]
\[
f_2 = A_{00}^T P_{f0}^T \Pi_e + A_{f0}^T P_f + P_{00}^T A_{0f} + P_{f0}^T A_f
- P_{00}^T S_{f0} P_{00} + P_{f0}^T S_{0f} P_{00} + P_{f0}^T S_{0f} P_{f0}
- P_{00}^T S_{0f} P_{f0} - P_{f0}^T S_f P_f + Q_{0f} = 0,
\]
\[
f_3 = P_f^T A_f + A_f^T P_f + \Pi_f P_f A_{0f} + A_{0f}^T P_f
- P_f^T S_f P_f - P_f^T S_{0f} P_{f0} \Pi_e - \Pi_e P_{0f} S_{0f} P_{f0}
- \Pi_e P_{0f} S_{0f} P_{f0} + Q_f = 0.
\]

It is assumed that the limit of \( \alpha_{ij} \) exists as \( \varepsilon_1 \rightarrow 0 \) and \( \varepsilon_j \) tend to zero (see e.g., Khalil and Kokotović, 1978, 1979), that is
\[
\alpha_{ij} = \lim_{\varepsilon_j \rightarrow +0} \alpha_{ij}.
\]

Let \( P_{00}, P_{f0} \) and \( P_f \) be the limiting solutions of the above equation (8) as \( \varepsilon_j \rightarrow +0, \ j = 1, \cdots, N \), then we obtain the following equations
\[
P_{00}^T A_{00} + A_{00}^T P_{00} + P_{f0}^T A_f + A_f^T P_{f0}
- P_{00}^T S_{00} P_{00} - P_{f0}^T S_{f0} P_{f0} - P_{00}^T S_{0f} P_{f0}
- P_{f0}^T S_{0f} P_{00} + Q_{00} = 0,
\]
\[
P_{f0}^T A_f + A_f^T P_f + \Pi_f P_f A_{0f} + A_{0f}^T P_f
- P_f^T S_f P_f - P_f^T S_{0f} P_{f0} \Pi_e - \Pi_e P_{0f} S_{0f} P_{f0}
- \Pi_e P_{0f} S_{0f} P_{f0} + Q_f = 0,
\]
\[
P_f^T A_f + A_f^T P_f - P_f^T S_f P_f + Q_f = 0.
\]

where
\[
P_{fj} := \begin{bmatrix} P_{11} & \alpha_{12} P_{21} & \cdots & \alpha_{1N} P_{N1} \\ P_{21} & \alpha_{22} P_{22} & \cdots & \alpha_{2N} P_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix},
\]
\[
P_{jj} := \begin{bmatrix} \varepsilon_1 P_{11} & \cdots & \alpha_{1N} P_{N1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \varepsilon_1 P_{N1} & \cdots & \alpha_{1N} P_{N1} \end{bmatrix}.
\]

The ARE (10c) obtained is nonsymmetric. However, it is easy to verify that the ARE (10c) is an ARE which admits at least a unique symmetric positive semidefinite stabilizing solution.

**Theorem 1:** Under the assumption 1, the ARE (10c) admits a unique symmetric positive semidefinite stabilizing solution \( P_f \) which can be written as
\[
P_f := \text{block} - \text{diag} \left( \bar{P}_{11}^* \cdots \bar{P}_{NN}^* \right),
\]
where \( \bar{P}_{jj}^* \) is a unique symmetric positive semidefinite stabilizing solution for the following AREs, respectively
$A_{j}^T \tilde{P}_{j}^* + P_{j}^* A_{j} - P_{j}^* S_{j} \tilde{P}_{j}^* + Q_{j} = 0,$  
$j = 1, 2, \ldots, N.$

**Proof:** Substituting (11) into the ARE (10c) as $\tilde{P}_j^* \rightarrow \bar{P}_j$, it is easy to verify that $\bar{P}_j^* A_{j} + A_{j}^T \bar{P}_j^* - \bar{P}_j^* S_{j} \bar{P}_j^* + Q_{j} = 0$. Furthermore, it can be seen that $\bar{P}_j^* = \bar{P}_j^{T^*} \geq 0$ and $A_{j} - S_{j} \bar{P}_j^*$ is stable from (12) under the assumption 1.

$$A_{j} - S_{j} \bar{P}_j^* = \text{block - diag} \left( A_{11} - S_{11} \bar{P}_{11}^* \cdots A_{NN} - S_{NN} \bar{P}_{NN}^* \right) \quad (12)$$

Consequently, there exists a unique solution of the ARE (10c) and its solution is (11) itself. \[ \square \]

The assumption 1 ensures that $A_{j} - S_{j} \bar{P}_j^*$, $j = 1, 2, \ldots, N$ are nonsingular. Substituting the solution of (10c) into (10b) and substituting $\bar{P}_0^*$ into (10a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following 0-order equations (13)

$$\begin{align*}
\bar{P}_{00}^* A + A^T \bar{P}_{00}^* - \bar{P}_{00}^* S \bar{P}_{00}^* + Q &= 0, \quad (13a) \\
\bar{P}_0^* &= -N_2^T + N_1^T \bar{P}_{00}^*, \quad (13b) \\
\Rightarrow \bar{P}_j^* &= -[\bar{P}_{00}^* D_{0j} + (A_{j}^T \bar{P}_j^* + Q_{0j})]D_{j1}^{-1} \\
\bar{P}_j^* A_{j} + A_{j}^T \bar{P}_j^* - \bar{P}_j^* S_{j} \bar{P}_j^* + Q_{j} &= 0, \quad (13c) \\
\Rightarrow \bar{P}_j^* A_{j} + A_{j}^T \bar{P}_j^* - \bar{P}_j^* S_{j} \bar{P}_j^* + Q_{j} &= 0,
\end{align*}$$

where

$$\begin{align*}
A &:= A_{00} + N_1 A_{j0} + S_{0j} N_2^T + N_1 S_{j} N_2^T , \\
S &:= S_{00} + N_1 S_{0j} + S_{0j} N_1^T + N_1 S_{j} N_1^T , \\
Q &:= Q_{11} - N_2 A_{j0} - A_{0j} N_2^T - S_{j} N_2^T , \\
N_2^T &:= \tilde{A}_j^T \tilde{Q}_{0j}^T, \\
N_1^T &:= -\tilde{A}_j^T \tilde{A}_{0j}^T , \\
\tilde{A}_{0j} &:= A_{0j} - S_{0j} \bar{P}_j^*, \\
\tilde{Q}_{0j} &:= Q_{0j} + \tilde{A}_j^T \tilde{P}_0 \bar{P}_j^*, \\
D_{0j} &:= A_{0j} - S_{0j} \bar{P}_j^*, \\
D_{j1} &:= A_{j} - S_{j} \bar{P}_j^*, \quad j = 1, 2, \ldots, N.
\end{align*}$$

In the following we established the relation between the GMARE (7) and the 0-order equations (13). Before doing that, we give the results for the AREs (13).

**Lemma 2:** Under the assumptions 1–3, the following results hold.

(i) The matrices $A$, $S$ and $Q$ do not depend on $\bar{P}_{jj}$, $j = 1, 2, \ldots, N$.

(ii) There exist a matrix $B \in \mathbb{R}^{m \times m}$, $m := \sum_{j=0}^{N} m_j$ and a matrix $C$ with the same dimension as $C_0$ such that $S = B R^{-1} B^T$, $Q = C^T C$. Moreover, the triple $(A, B, C)$ is stabilizable and detectable.

**Proof:** Firstly, we introduce the following coordinate matrix $I_n := \begin{bmatrix} I_{n_1} & 0 & 0 & \cdots & 0 \\
0 & I_{n_2} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I_{n_N} \end{bmatrix}$.

$$\Omega = \begin{bmatrix} I_{n_1} & 0 & 0 & \cdots & 0 \\
0 & I_{n_2} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & I_{n_N} \end{bmatrix} \quad . \quad (14)$$

Furthermore, let us define four partitioned matrices $(\text{Wang et al., 1994})$

$$H_1 = T_{00} := \begin{bmatrix} A_{00} & -S_{00} \\
-Q_{00} & -A_{00}^T \end{bmatrix},$$

$$H_2 := \begin{bmatrix} A_{0j} & -S_{0j} \\
-Q_{0j} & -A_{0j}^T \end{bmatrix},$$

$$H_3 := \begin{bmatrix} A_{j0} & -S_{j0} \\
-Q_{j0} & -A_{j0}^T \end{bmatrix}, \quad H_4 := \begin{bmatrix} A_{j} & -S_{j} \\
-Q_{j} & -A_{j}^T \end{bmatrix}.$$
Taking into consideration the fact that the solution \( \bar{P}_{00}^* \) is positive semidefinite, we have \( P_e \geq 0 \). Secondly, using (16), we obtain

\[
A_e - S_e P_e = \Phi_e^{-1} \left( \begin{bmatrix} \bar{A}_{00} & \bar{A}_{0f} \\ \bar{A}_{f0} & \bar{A}_f \end{bmatrix} + O(\|\mu\|) \right).
\]

The matrix \( \bar{A}_f \) and \( \bar{A}_0 \) are stable since the assumptions 1 and 2 holds. Therefore, if parameter \( \|\mu\| \) is very small, \( A_e - S_e P_e \) is stable by applying the Theorem 1 in Khalil and Kokotović (1979).

3. NEAR-OPTIMAL CONTROL FOR THE MSPO

The required solution of the MARE (5) exists under the assumptions 1–3. Our attention is focused on the specific linear state feedback controller which does not depend on the values of the small parameters. Such the linear state feedback controller is obtained by eliminating \( O(\|\mu\|) \) item of the linear state feedback controller (4). If \( \|\mu\| \) is very small, it is obvious that the linear state feedback controller (4) can be approximated as

\[
u_{\text{app}} = \begin{bmatrix} u_{1\text{app}}^T & \cdots & u_{N\text{app}}^T \end{bmatrix} = -R^{-1}B^T P_{\text{app}} x = -R^{-1}B^T \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0} & \bar{P}_f \end{bmatrix} x, \]

\[
\begin{bmatrix} \bar{P}_{00}^* & 0 & & 0 \\ 0 & \bar{P}_{20} & \cdots & 0 \\ & & \ddots & \vdots \\ 0 & & & \bar{P}_{NN}^* \end{bmatrix} x, \]

where \( B = \Phi_e B_e \).

Remark 1: Even though our control design is quite different from the composite controller design (Khalil and Kokotović, 1979; Wang et al., 1994; Xu et al., 1997), we can be shown that the resulting controller (19) is similar to the existing one.

When \( \|\mu\| \) is sufficiently small, we know from Theorem 2 that the resulting controller (19) will be close to the optimal controller (4). In an optimization problem it is of interest to check whether the resulting value of the cost function will be near its optimal value. The optimal value \( J_{\text{opt}} \) is obtained with the controller (4) which optimizes the cost for the actual system (1).

**Theorem 3:** Under the assumptions 1–3, the use of the reduced–order controller (19) results in \( J_{\text{app}} \) satisfying

\[
J_{\text{app}} = J_{\text{opt}} + O(\|\mu\|^2),
\]

where

\[
J_{\text{opt}} = \frac{1}{2} x(0)^T P_e x(0).
\]

Before proving this theorem, we introduce the following useful lemma (Mukaidani et al., 2001).
Lemma 3: Consider the iterative algorithm which is based on the Kleinman algorithm

\[(A - S P^{(i)})(A - S P^{(i)}) + P^{(i+1)}T(A - S P^{(i)})
+ P^{(i)}TSP^{(i)} + Q = 0, \quad i = 0, 1, \ldots, \]

\[P^{(i)} = \begin{bmatrix}
P^{(i)}_{00} & P^{(i)}_{01} \n
P^{(i)}_{10} & P^{(i)}_{11}
\end{bmatrix}, \quad (21b)

with the initial condition obtained from

\[P^{(0)} = P_{\text{app}} = \begin{bmatrix}
P^{*}_{00} & 0 

P^{*}_{01} & P^{*}_{11}
\end{bmatrix}. \quad (22)

Under the assumptions 1–3, there exists a small \(\delta\) such that for all \(|\mu| \in (0, \delta)\), \(\delta < \sigma\) the iterative algorithm (21) converges to the exact solution of \(P = \Phi e P = P^{T}T\Phi e\) with the rate of quadratic convergence, where \(P^{(i)} = \Phi e P^{(i)} = P^{(i)T}T\Phi e\) is positive semidefinite.

\[|P^{(i)} - P| = O(|\mu|^{2^{i}}), \quad i = 0, 1, 2, \ldots, \]

where

\[\gamma = 2|S| < \infty, \quad \beta = \frac{1}{2}||A||^{-1}, \n
\eta = \beta \cdot ||A|| \cdot||G(P^{(0)}||, \n
\theta = \beta \eta \gamma, \quad \nabla G(P) = \frac{\partial G(P)}{\theta(|P^{(0)}|)}.

Proof: When \(u_{\text{app}}\) is used, the value of the performance index is

\[J_{\text{app}} = \frac{1}{2}(x(0))TW_{e}x(0), \quad (24)

where \(W_{e}\) is a positive semidefinite solution of the multiparameter algebraic Lyapunov equation (MALE)

\[(A - S_{e}P_{\text{app}})W_{e} = W_{e}(A - S_{e}P_{\text{app}})
+ P_{\text{app}}S_{e}P_{\text{app}} + Q = 0, \quad (25)

where \(P_{\text{app}} = \Phi e P_{\text{app}}\). Substituting (5) from (25) we find that \(V_{e} = W_{e} - P_{e}\) satisfies the following MALE

\[(A - S_{e}P_{\text{app}})V_{e} + V_{e}(A - S_{e}P_{\text{app}})
+ (P_{e} - P_{\text{app}})S_{e}(P_{e} - P_{\text{app}}) = 0. \quad (26)

Similarly, subtracting (5) from (21a) we also get the MALE

\[(A - S_{e}P^{(i)}_{e})(P^{(i+1)}_{e} - P_{e})
+ (P^{(i+1)}_{e} - P_{e})(A - S_{e}P^{(i)}_{e})
+ (P_{e} - P^{(i)}_{e})S_{e}(P_{e} - P^{(i)}_{e}) = 0, \quad (27)

where \(P^{(i)}_{e} = \Phi e P^{(i)}_{e}\). When \(i = 0\), we have

\[(A - S_{e}P^{(0)}_{e})(P^{(1)}_{e} - P_{e})
+ (P^{(1)}_{e} - P_{e})(A - S_{e}P^{(0)}_{e})
+ (P_{e} - P^{(0)}_{e})S_{e}(P_{e} - P^{(0)}_{e})
= (A - S_{e}P_{\text{app}})(P^{(1)}_{e} - P_{e})
+ (P^{(1)}_{e} - P_{e})(A - S_{e}P_{\text{app}})
+ (P_{e} - P_{\text{app}})S_{e}(P_{e} - P_{\text{app}}) = 0.

Therefore, it is easy to verify that \(V_{e} = P^{(1)}_{e} - P_{e}\) because \(A_{e} = S_{e}P_{\text{app}}\) is stable from Theorem 1 in Khalil and Kokotović (1979). Using Lemma 3 we obtain that

\[|V_{e}| = \|W_{e} - P_{e}\| = \|P^{(1)}_{e} - P_{e}||\n
\leq \|\Phi e \cdot P^{(1)} - P\| \leq \|P^{(1)} - P\| = O(|\mu|^{2}). \quad (28)

Hence, we have \(V_{e} = W_{e} - P_{e} = O(|\mu||^{2})\), which implies (20).

4. CONCLUSION

In this paper, we have studied the optimal control problem associated with the MSPS. The main contribution of this paper is to propose the new design method of the \(\varepsilon_{1}\)-independent reduced–order controller. Note that our design method is quite different from the existing method such as the two–time–scale design method and the descriptor approach. Furthermore, we have shown that the resulting controller achieves \(O(|\mu||^{2})\) approximation of the optimal cost compared with the previously proposed controller.

References


Wang, Y-Y., M. Paul and N. E. Wu (1994). Near–independent reduced–order controller. Note that our