微分積分学 I　小テスト：担当 向谷　博明

2016 年 7 月 15, 18 日

学部	学籍番号	氏名

以下の定積分を計算せよ．

(1) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{\pi}{\boxed{}\sqrt{\boxed{}}}$

(2) $\displaystyle\int_0^{2\pi} \sin 3x \cos x dx = \boxed{}$

(3) $\displaystyle\int_0^2 x^2\sqrt{2x-x^2}dx = \frac{\boxed{}}{\boxed{}}\pi$

(4) $\displaystyle\int_0^{\log 2} \frac{dx}{e^{-x}+1} = \log\frac{\boxed{}}{\boxed{}}$

(5) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x\sin x}{\cos^3 x}dx = \frac{\pi}{\boxed{}} - \frac{\boxed{}}{\boxed{}}$

(6) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x^6+x^4} = \frac{\pi}{\boxed{}} + \frac{\boxed{}}{\boxed{}}\sqrt{\boxed{}} - \frac{\boxed{}}{\boxed{}}$

(7) $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}dx = \boxed{}\sqrt{\boxed{}} - \frac{\boxed{}}{\boxed{}}\pi$

(8) $\displaystyle\int_0^1 (x^3+x)\sqrt{x^2+1}dx = \frac{\boxed{}}{\boxed{}}\sqrt{\boxed{}} - \frac{\boxed{}}{\boxed{}}$

(9) $\displaystyle\int_0^1 (x-1)\sqrt{x^2+1}dx = -\frac{\boxed{}}{\boxed{}} + \frac{\sqrt{\boxed{}}}{\boxed{}}$

$$-\frac{\boxed{}}{\boxed{}}\log\left(\boxed{} + \sqrt{\boxed{}}\right)$$

(10) $\displaystyle\int_0^1 x^2(1-x)^5 dx = \frac{\boxed{}}{\boxed{}}$

以下の定積分を計算せよ．

(1) $$\int_{-\frac{3}{4}}^{-\frac{1}{4}} \frac{dx}{\sqrt{-x-x^2}} = \frac{\pi}{\boxed{}}$$

(2) $$\int_0^{2\pi} \sin 3x \cos^2 x dx = \boxed{}$$

(3) $$\int_0^1 \frac{5x^4}{x^{10}+3} dx = \frac{\pi}{\boxed{}\sqrt{\boxed{}}}$$

(4) $$\int_0^{\log 2} \frac{dx}{e^x+1} = \log \frac{\boxed{}}{\boxed{}}$$

(5) $$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x} = \boxed{}$$

(6) $$\int_0^1 \frac{dx}{x^4+x^2+1} = \frac{\log \boxed{}}{\boxed{}} + \frac{\sqrt{\boxed{}}}{\boxed{}}\pi$$

(7) $$\int_{\frac{\sqrt{3}}{2}}^1 \frac{\sqrt{1-x^2}}{x^3} dx = \frac{\boxed{}}{\boxed{}} - \frac{\log \boxed{}}{\boxed{}}$$

(8) $$\int_0^1 (x^2+1)\sqrt{1-x^2} dx = \frac{\boxed{}}{\boxed{}}\pi$$

(9) $$\int_0^1 (x+1)\sqrt{x^2+1} dx = -\frac{\boxed{}}{\boxed{}} + \frac{\boxed{}}{\boxed{}}\sqrt{\boxed{}} + \frac{\boxed{}}{\boxed{}} \log\left(\boxed{} + \sqrt{\boxed{}}\right)$$

(10) $$\int_0^1 x(1-x^2)^5 dx = \frac{\boxed{}}{\boxed{}}$$

以下の定積分を計算せよ．

(1) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{\pi}{\boxed{4}\sqrt{\boxed{2}}}$

(2) $\displaystyle\int_0^{2\pi} \sin 3x \cos x dx = \boxed{0}$

(3) $\displaystyle\int_0^2 x^2\sqrt{2x-x^2}dx = \frac{\boxed{5}}{\boxed{8}}\pi$

(4) $\displaystyle\int_0^{\log 2} \frac{dx}{e^{-x}+1} = \log\frac{\boxed{3}}{\boxed{2}}$

(5) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x\sin x}{\cos^3 x}dx = \frac{\pi}{\boxed{4}} - \frac{\boxed{1}}{\boxed{2}}$

(6) $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x^6+x^4} = \frac{\pi}{\boxed{1\ 2}} + \frac{\boxed{8}}{\boxed{2\ 7}}\sqrt{\boxed{3}} - \frac{\boxed{2}}{\boxed{3}}$

(7) $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}dx = \boxed{2}\sqrt{\boxed{3}} - \frac{\boxed{2}}{\boxed{3}}\pi$

(8) $\displaystyle\int_0^1 (x^3+x)\sqrt{x^2+1}dx = \frac{\boxed{4}}{\boxed{5}}\sqrt{\boxed{2}} - \frac{\boxed{1}}{\boxed{5}}$

(9) $\displaystyle\int_0^1 (x-1)\sqrt{x^2+1}dx = -\frac{\boxed{1}}{\boxed{3}} + \frac{\sqrt{\boxed{2}}}{\boxed{6}}$

$$-\frac{\boxed{1}}{\boxed{2}}\log\left(\boxed{1} + \sqrt{\boxed{2}}\right)$$

(10) $\displaystyle\int_0^1 x^2(1-x)^5dx = \frac{\boxed{1}}{\boxed{1\ 6\ 8}}$

以下の定積分を計算せよ.

(1) $\displaystyle\int_{-\frac{3}{4}}^{-\frac{1}{4}} \frac{dx}{\sqrt{-x-x^2}} = \frac{\pi}{\boxed{3}}$

(2) $\displaystyle\int_0^{2\pi} \sin 3x \cos^2 x dx = \boxed{0}$

(3) $\displaystyle\int_0^1 \frac{5x^4}{x^{10}+3}dx = \frac{\pi}{\boxed{6}\sqrt{\boxed{3}}}$

(4) $\displaystyle\int_0^{\log 2} \frac{dx}{e^x+1} = \log\frac{\boxed{4}}{\boxed{3}}$

(5) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x} = \boxed{1}$

(6) $\displaystyle\int_0^1 \frac{dx}{x^4+x^2+1} = \frac{\log\boxed{3}}{\boxed{4}} + \frac{\sqrt{\boxed{3}}}{\boxed{1\ 2}}\pi$

(7) $\displaystyle\int_{\frac{\sqrt{3}}{2}}^1 \frac{\sqrt{1-x^2}}{x^3}dx = \frac{\boxed{1}}{\boxed{3}} - \frac{\log\boxed{3}}{\boxed{4}}$

(8) $\displaystyle\int_0^1 (x^2+1)\sqrt{1-x^2}dx = \frac{\boxed{5}}{\boxed{1\ 6}}\pi$

(9) $\displaystyle\int_0^1 (x+1)\sqrt{x^2+1}dx = -\frac{\boxed{1}}{\boxed{3}} + \frac{\boxed{7}}{\boxed{6}}\sqrt{\boxed{2}}$

$$+\frac{\boxed{1}}{\boxed{2}}\log\left(\boxed{1}+\sqrt{\boxed{2}}\right)$$

(10) $\displaystyle\int_0^1 x(1-x^2)^5 dx = \frac{\boxed{1}}{\boxed{1\ 2}}$

《 表面 》

(1) $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\int_0^1 \frac{dx}{\sqrt{2-x^2}} = \left[\frac{1}{\sqrt{2}}\arcsin\frac{x}{\sqrt{2}}\right]_0^1 = \frac{\pi}{4\sqrt{2}}$

(2) $\displaystyle\int_0^{2\pi} \sin 3x\cos x dx = \frac{1}{2}\int_0^{2\pi}(\sin 4x+\sin 2x)dx = \frac{1}{2}\left[-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right]_0^{2\pi} = 0$

【 別解 】 $\displaystyle\int_0^{2\pi} \sin 3x\cos x dx = \int_0^{2\pi}(3\sin x - 4\sin^3 x)\cos x dx = \left[\frac{3}{2}\sin^2 x - \sin^4 x\right]_0^{2\pi} = 0$

(3) $\displaystyle\int_0^2 x^2\sqrt{2x-x^2}dx = \int_0^2 x^2\sqrt{1-(x-1)^2}dx$

ここで $x-1=\sin t$ とおく．このとき $dx=\cos t dt$ であり, x が $0\to 2$ まで動けば t は $-\frac{\pi}{2}\to\frac{\pi}{2}$ まで動くので,

$\displaystyle\int_0^2 x^2\sqrt{2x-x^2}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin t+1)^2\cos^2 t dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^2 t + 2\sin t + 1)(1-\sin^2 t)dt$

$\displaystyle = 2\int_0^{\frac{\pi}{2}}(1-\sin^4 t)dt = 2\left(1-\frac{3}{4}\cdot\frac{1}{2}\right)\cdot\frac{\pi}{2} = \frac{5}{8}\pi$

(4) $\displaystyle\int_0^{\log 2}\frac{dx}{e^{-x}+1} = \int_0^{\log 2}\frac{e^x}{e^x+1}dx = \Big[\log(e^x+1)\Big]_0^{\log 2} = \log\frac{3}{2}$

(5) $\displaystyle\int_0^{\frac{\pi}{4}}\frac{x\sin x}{\cos^3 x}dx = \left[x\frac{1}{2\cos^2 x}\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{dx}{\cos^2 x} = \frac{\pi}{4} - \frac{1}{2}\Big[\tan x\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}$

(6) $\displaystyle\int_1^{\sqrt{3}}\frac{dx}{x^6+x^4} = \int_1^{\sqrt{3}}\frac{dx}{x^4(x^2+1)} = \int_0^1\left(-\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^2+1}\right)dx = \left[\frac{1}{x}-\frac{1}{3x^3}+\arctan x\right]_1^{\sqrt{3}}$

$\displaystyle = \frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{\pi}{3} - 1 + \frac{1}{3} - \frac{\pi}{4} = \frac{8}{9\sqrt{3}} - \frac{2}{3} + \frac{\pi}{12}$

(7) $\sqrt{x^2-4}=t$ とおく．このとき, $x=\sqrt{t^2+4}$, $xdx=tdt$ であり, x が $2\to 4$ まで動けば t は $0\to 2\sqrt{3}$ まで動くので,

$\displaystyle\int_2^4\frac{\sqrt{x^2-4}}{x}dx = \int_0^{2\sqrt{3}}\frac{t}{\sqrt{t^2+4}}\cdot\frac{t}{x}dt = \int_0^{2\sqrt{3}}\frac{t^2}{t^2+4}dt = \int_0^{2\sqrt{3}}\left(1-\frac{4}{t^2+4}\right)dt = \left[t-2\arctan\frac{t}{2}\right]_0^{2\sqrt{3}} = 2\sqrt{3}-\frac{2}{3}\pi$

(8) $\displaystyle\int_0^1(x^3+x)\sqrt{x^2+1}dx = \int_0^1 x(x^2+1)^{\frac{3}{2}}dx = \left[\frac{1}{5}(x^2+1)^{\frac{5}{2}}\right]_0^1 = \frac{4\sqrt{2}-1}{5}$

【 別解 】$x=\tan t$ とおく．このとき $dx=\sec^2 t dt$ であり, x が $0\to 1$ まで動けば t は $0\to\frac{\pi}{4}$ まで動くので,

$\displaystyle\int_0^1(x^3+x)\sqrt{x^2+1}dx = \int_0^{\frac{\pi}{4}}(\tan^3 t+\tan t)\sec t\sec^2 t dt = \int_0^{\frac{\pi}{4}}\left(\frac{\sin^3 t}{\cos^6 t}+\frac{\sin t}{\cos^4 t}\right)dt$

$\displaystyle = \int_0^{\frac{\pi}{4}}\frac{\sin^3 t + \sin t\cos^2 t}{\cos^6 t}dt = \int_0^{\frac{\pi}{4}}\frac{\sin t}{\cos^6 t}dt = \left[\frac{1}{5}\cdot\frac{1}{\cos^5 t}\right]_0^{\frac{\pi}{4}} = \frac{4\sqrt{2}-1}{5}$

(9) $\displaystyle\int_0^1(x-1)\sqrt{x^2+1}dx = \int_0^1 x\sqrt{x^2+1}dx - \int_0^1\sqrt{x^2+1}dx$

$\displaystyle = \left[\frac{1}{3}(x^2+1)^{\frac{3}{2}}\right]_0^1 - \frac{1}{2}\left[x\sqrt{x^2+1}+\log(x+\sqrt{x^2+1})\right]_0^1 = \frac{\sqrt{2}}{6} - \frac{1}{3} - \frac{1}{2}\log(1+\sqrt{2})$

(10) 公式を利用すれば,

$\displaystyle\int_0^1 x^2(1-x)^5 dx = \frac{2!\cdot 5!}{(2+5+1)!} = \frac{2!\cdot 5!}{8!} = \frac{2}{6\cdot 7\cdot 8} = \frac{1}{168}$

《 裏面 》

(1) $\displaystyle\int_{-\frac{3}{4}}^{-\frac{1}{4}}\frac{dx}{\sqrt{-x-x^2}} = \int_{-\frac{3}{4}}^{-\frac{1}{4}}\frac{dx}{\sqrt{\frac{1}{4}-\left(x+\frac{1}{2}\right)^2}} = 2\Big[\arcsin(2x+1)\Big]_{-\frac{3}{4}}^{-\frac{1}{4}} = \frac{\pi}{3}$

(2) $\displaystyle\int_0^{2\pi}\sin 3x\cos^2 x dx = \frac{1}{2}\int_0^{2\pi}\sin 3x(1+\cos 2x)dx = \frac{1}{2}\int_0^{2\pi}\left(\sin 3x + \frac{1}{2}\sin 5x + \frac{1}{2}\sin x\right)dx$

$\displaystyle = \left[-\frac{1}{6}\cos 3x - \frac{1}{20}\cos 5x - \frac{1}{4}\cos x\right]_0^{2\pi} = 0$

(3) $t=x^5$ とおく．このとき $dt=5x^4dx$ であり, x が $0\to 1$ まで動けば t も $0\to 1$ まで動くので,

$\displaystyle\int_0^1\frac{5x^4}{x^{10}+3}dx = \int_0^1\frac{dt}{t^2+3} = \left[\frac{1}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}}\right]_0^1 = \frac{\pi}{6\sqrt{3}}$

(4) $\displaystyle\int_0^{\log 2}\frac{dx}{e^x+1} = \int_0^{\log 2}\frac{e^{-x}}{e^{-x}+1}dx = \Big[-\log(e^{-x}+1)\Big]_0^{\log 2} = -\log\frac{3}{2} + \log 2 = \log\frac{4}{3}$

(5) $t=\tan\dfrac{x}{2}$ とおく．このとき $dx=\dfrac{2dt}{t^2+1}$ であり, x が $0\to\dfrac{\pi}{2}$ まで動けば t は $0\to 1$ まで動くので,

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+\cos x}=\int_0^1\frac{1}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{t^2+1}=\int_0^1 dt=1$$

【 別解 】$\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{1+\cos x}=\int_0^{\frac{\pi}{2}}\frac{(1-\cos x)}{(1+\cos x)(1-\cos x)}dx=\int_0^{\frac{\pi}{2}}\frac{(1-\cos x)}{\sin^2 x}dx$

$$=\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^2 x}-\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}dx=\Big[-\cot x\Big]_0^{\frac{\pi}{2}}+\left[\frac{1}{\sin x}\right]_0^{\frac{\pi}{2}}=1$$

【 注意 】本来, 次章で扱う広義積分になっていることに注意されたい．すなわち, $1-\cos x$ を分子・分母にかけた時点で, 分母に $\sin^2 x$ が現れ, これが元凶となっている．したがって, 解答の詳細は以下のとおりである．

$$\int_0^{\frac{\pi}{2}}\frac{dx}{1+\cos x}=\int_0^{\frac{\pi}{2}}\frac{(1-\cos x)}{(1+\cos x)(1-\cos x)}dx=\lim_{\varepsilon\to+0}\int_\varepsilon^{\frac{\pi}{2}}\frac{(1-\cos x)}{\sin^2 x}dx$$

$$=\lim_{\varepsilon\to+0}\int_\varepsilon^{\frac{\pi}{2}}\left(\frac{1}{\sin^2 x}-\frac{\cos x}{\sin^2 x}\right)dx=\lim_{\varepsilon\to+0}\left[-\cot x+\frac{1}{\sin x}\right]_\varepsilon^{\frac{\pi}{2}}=1-\lim_{\varepsilon\to+0}\left(-\cot\varepsilon+\frac{1}{\sin\varepsilon}\right)$$

$$=1-\lim_{\varepsilon\to+0}\frac{1-\cos\varepsilon}{\sin\varepsilon}\cdot\frac{1+\cos\varepsilon}{1+\cos\varepsilon}=1-\lim_{\varepsilon\to+0}\frac{\sin\varepsilon}{1+\cos\varepsilon}=1$$

或は, 先の結果を利用すれば,

$\displaystyle\int\frac{dx}{1+\cos x}=\frac{\sin x}{1+\cos x}$ より, 計算することも可能である．ちなみに,

$$\frac{d}{dx}\frac{\sin x}{1+\cos x}=\frac{\cos x(1+\cos x)-\sin x(-\sin x)}{(1+\cos x)^2}=\frac{1}{1+\cos x}$$

より, 不定積分が正しいことが分かる．

(6) $\displaystyle\int_0^1\frac{dx}{x^4+x^2+1}=\int_0^1\frac{dx}{(x^2+1)^2-x^2}=\int_0^1\frac{dx}{(x^2-x+1)(x^2+x+1)}$

$$=\frac{1}{2}\int_0^1\left(\frac{-x+1}{x^2-x+1}+\frac{x+1}{x^2+x+1}\right)dx=\frac{1}{4}\int_0^1\left(\frac{-2x+1+1}{x^2-x+1}+\frac{2x+1+1}{x^2+x+1}\right)dx$$

$$=\frac{1}{4}\left[\log\frac{x^2+x+1}{x^2-x+1}\right]_0^1+\frac{1}{4}\int_0^1\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{4}\int_0^1\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{\log 3}{4}+\frac{1}{4}\left[\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}\right]_0^1+\frac{1}{4}\left[\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right]_0^1=\frac{\log 3}{4}+\frac{\pi}{4\sqrt{3}}$$

(7) $x=\sin t$ とおく．このとき $dx=\cos t dt$ であり, x が $\dfrac{\sqrt{3}}{2}\to 1$ まで動けば t は $\dfrac{\pi}{3}\to\dfrac{\pi}{2}$ まで動くので,

$$\int_{\frac{\sqrt{3}}{2}}^1\frac{\sqrt{1-x^2}}{x^3}dx=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\cos t}{\sin^3 t}\cdot\cos t dt=\left[-\frac{1}{2}\cdot\frac{1}{\sin^2 t}\cdot\cos t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}+\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{\sin^2 t}(-\sin t)dt$$

$$=\frac{1}{3}-\frac{1}{2}\left[\log\tan\frac{t}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}=\frac{1}{3}+\frac{1}{2}\log\frac{1}{\sqrt{3}}=\frac{1}{3}-\frac{\log 3}{4}$$

(8) $x=\sin t$ とおく．このとき $dx=\cos t dt$ であり, x が $0\to 1$ まで動けば t は $0\to\dfrac{\pi}{2}$ まで動くので,

$$\int_0^1(x^2+1)\sqrt{1-x^2}dx=\int_0^{\frac{\pi}{2}}(\sin^2 t+1)\cos t\cos t dt=\int_0^{\frac{\pi}{2}}(\sin^2 t+1)(1-\sin^2 t)dt=\int_0^{\frac{\pi}{2}}(1-\sin^4 t)dt$$

$$=\left(1-\frac{3}{4}\cdot\frac{1}{2}\right)\cdot\frac{\pi}{2}=\frac{5}{16}\pi$$

(9) $\displaystyle\int_0^1(x+1)\sqrt{x^2+1}dx=\int_0^1 x\sqrt{x^2+1}dx+\int_0^1\sqrt{x^2+1}dx$

$$=\left[\frac{1}{3}(x^2+1)^{\frac{3}{2}}\right]_0^1+\frac{1}{2}\left[x\sqrt{x^2+1}+\log(x+\sqrt{x^2+1})\right]_0^1=\frac{7\sqrt{2}}{6}-\frac{1}{3}+\frac{1}{2}\log(1+\sqrt{2})$$

(10) $t=\sin x$ とおく．このとき $dt=\cos x dx$ であり, x が $0\to 1$ まで動けば t は $0\to\dfrac{\pi}{2}$ まで動くので,

$$\int_0^1 x(1-x^2)^5dx=\int_0^{\frac{\pi}{2}}\sin x(1-\sin^2 x)^5\cos x dx=\int_0^{\frac{\pi}{2}}\cos^{11}x\sin x dx=\left[-\frac{1}{12}\cos^{12}x\right]_0^{\frac{\pi}{2}}=\frac{1}{12}$$